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Research article

A fractional Laplacian problem with critical nonlinearity

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Abstract: In this paper, using the mountain pass theorem we obtain a positive solution to the fractional Laplacian problem

$$\begin{cases} (-\Delta)^s u = g(x)(u-k)_+^{q-1} + u^{\frac{n+2s}{n-2s}} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $0 < s < 1, 2 \leq q < 2n/(n-2s)$ and $k \in (0, \infty)$ is an arbitrary number. The function $g : \Omega \to \mathbb{R}$ is a nonnegative continuous function satisfying some integrability condition.

Keywords: fractional Laplacian; critical Sobolev growth; positive solutions; variational methods
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1. Introduction

In recent years, motivated by problems that appear in anomalous diffusions in plasmas, flames propagation and chemical reactions in liquids etc, fractional Laplacian problems have received extensive attention in the literature. Also, due to its rich multi-origins, fractional Laplacian operators turn out to have many different definitions. When studying fractional problems in the whole space $\mathbb{R}^n$, the fractional Laplacian $(-\Delta)^s$ is usually defined via the Fourier transformation $(-\Delta)^s f(x) = (|2\pi i\xi|^{2s}\hat{f}(\xi))^\vee$, see e.g. Frank and Lenzmann [12] and Di Nezza, Palatucci and Valdinoci [11]. Equivalently, this operator can be written as the difference quotient

$$(-\Delta)^s f(x) = c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2s}} dy \tag{1.1}$$

for sufficiently regular f, where $c_{n,s}$ is a normalization constant, for a proof see e.g. [11]. An advantage of the difference quotient type definition is that it can be directly extended to define the so-called fractional p-Laplacian operator

$$(-\Delta)_p^s f(x) = c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+sp}} dy,$$

and the so called regional fractional Laplacian operator

$$(-\Delta)_\Omega^s f(x) = c_{n,s} P.V. \int_{\Omega} \frac{f(x) - f(y)}{|x-y|^{n+2s}} dy,$$

in case the problem under consideration is restricted to a bounded domain $\Omega \subset \mathbb{R}^n$, see e.g. Chen [10] and the references therein. It is beyond the size of this paper to present all the other definitions for fractional Laplacian operators in the literature. We refer the interested readers to the references mentioned above.

In this paper, we are concerned with the fractional Laplacian operator $(-\Delta)^s$ on a bounded domain defined via the spectral of the Laplacian operator $-\Delta$. This type of fractional Laplacian operator was first defined by Cabré and Tan [7] for $s = 1/2$ and then extended by Brändle, Colorado, de Pablo and Sánchez [4] to the whole range $s \in (0, 1)$. This type of definition can be viewed as a discrete version of Fourier transformation for functions on a bounded domain. More precisely, let $s \in (0, 1)$ and $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Let $\{\lambda_k, \varphi_k\}_{k\geq 1}$ be the corresponding eigenvalues and eigenfunctions of $-\Delta$ on Ω with Dirichlet boundary value $\varphi_k|_{\partial\Omega} = 0$ and normalization $\|\varphi_k\|_{L^2(\Omega)} = 1$. Then, for $u = \sum_{k\geq 1} c_k \varphi_k$ satisfying $\sum_{k\geq 1} c_k^2 \lambda_k^{2s} < \infty$, we define

$$(-\Delta)^s u = \sum_{k\geq 1} \lambda_k^s c_k \varphi_k. \tag{1.2}$$

Therefore, a natural function space of $(-\Delta)^s$ in the sense of functional analysis is given by

$$H_0^s(\Omega) = \left\{ u = \sum c_k \varphi_k \in L^2(\Omega) : \|u\|^2_{H_0^s(\Omega)} \equiv \sum c_k^2 \lambda_k^s < \infty \right\}$$

such that

$$\langle (-\Delta)^s u, v \rangle \equiv \int_\Omega (-\Delta)^{s/2} u (-\Delta)^{s/2} v \mathrm{d}x = \sum_{k\geq 1} c_k d_k \lambda_k^s$$

is well defined for every $u = \sum c_k \varphi_k, v = \sum d_k \varphi_k \in H_0^s(\Omega)$. Observe that a remarkable feature of the operator $(-\Delta)^s$ is that it is nonlocal, in the sense that for any point $x \in \Omega$, the value of $(-\Delta)^s u$ at point x can be obtained only if one knows the distribution of u in the whole domain, so that it is possible derive all of the coefficients $c_k \equiv \int_\Omega u \varphi_k \mathrm{d}x$. For a systematical study on this type of fractional Laplacian operators, we refer to e.g. Cabré and Tan [7] for the case $s = 1/2$ and Brändle et al. [4] for the general case $s \in (0, 1)$.

Now our problem under consideration can be stated as below. Fix $s \in (0, 1)$ and consider the following fractional Laplacian problem

$$\begin{cases} (-\Delta)^s u = g(x)(u-k)_+^{q-1} + u^{2_s^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $2_s^* = 2n/(n-2s)$, $2 \le q < 2_s^*$ and $k \in (0,\infty)$ is an arbitrary positive number. A positive function $u \in H_0^s(\Omega)$ is called a (weak) solution of problem (1.3) if for every $v \in H_0^s(\Omega)$, there holds

$$\int_\Omega (-\Delta)^{s/2}u(-\Delta)^{s/2}v\mathrm{d}x = \int_\Omega \left(g(x)(u-k)_+^{q-1} + u^{2_s^*-1}\right)v\mathrm{d}x.$$

Precise assumptions on the coefficient function g will be given soon. Our aim is to prove that under appropriate conditions on the function g and the parameters q, n, there does exist at least one positive solution to the above problem.

Problem (1.3) is not totally new. Problems of type (1.3) have been extensively studied in the literature with different leading operators. In the local case (i.e., $s = 1$) with $k = 0$, problem of type (1.3) dates back to the famous work Brézis and Nirenberg [6], where positive solutions to the critical growth problem was obtained:

$$\begin{cases} -\Delta u = \lambda u + u^{2^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this case, the function g is the constant function λ. Since then, there are numerous extensions and important variants which have been studied. In [1], Ambrosetti, Brezis and Cerami considered the problem combined with concave and convex nonlinearities

$$\begin{cases} -\Delta u = \lambda u^q + u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < q < 1 < p$ and $\lambda > 0$. They proved among many other results that when λ is sufficiently small, this problem has a positive solution, see [1, Theorem 2.1] for more results. Quite recently, this problem was extended by Barrios, Colorado, Servadei,and Soria [3] to the nonlocal setting

$$\begin{cases} (-\Delta)^s u = \lambda u^q + u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \backslash \Omega, \end{cases}$$

where $0 < s < 1$, $0 < q < p = \frac{n+2s}{n-2s}$ and $\lambda > 0$, and $(-\Delta)^s u$ is defined via the difference quotient (1.1). Note that in this problem the boundary condition is in fact assumed on the complement of Ω. They obtained similar results [3, Theorem 1.1, 1.2] as that of [1, Theorem 2.1]. In [15] Servadei and Valdinoci considered the even more general integrodifferential problem of Brézis and Nirenberg type

$$\begin{cases} \mathcal{L}_K u = \lambda u + f(x,u) + |u|^{\frac{4s}{n-2s}} u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^n \backslash \Omega, \end{cases}$$

where $s \in (0,1)$ and $\mathcal{L}_K$ is a nonlocal operator with kernel function K such that $(-\Delta)^s$ plays a model case. Under various natural assumptions they proved several existence results to this problem, see [15, Theorem 1–3]. In particular, as an application, they obtained a positive solution in the model case

$\mathcal{L}_K = (-\Delta)^s$ and $f \equiv 0$, see [15, Theorem 4]. Similar to [3], but with the leading operator $(-\Delta)^s$ defined in the spectral way (1.2), Barrios, Colorado, de Pablo and Sánchez [2] considered the problem

$$\begin{cases} (-\Delta)^s u = \lambda u^q + u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $0 < q < p = \frac{n+2s}{n-2s}$. Then also obtained existence and nonexistence results under various assumptions on λ, see [2, Theorem 1.1, 1.2, 1.3] for details. For the case $p < \frac{n+2s}{n-2s}$, see Brändle, Colorado, de Pablo and Sánchez [4].

In the case $g \not\equiv$ constant, Gazzola [13] studied this type of problem with p-Laplacian operator $-\Delta_p$ as the leading term,

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = g(x)(u-k)_+^{q-1} + u^{p^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $1 < p < n$, $p^* = np/(n-p)$ and positive solutions were obtained among other results, see [13, Theorem 2]. Inspired by the work of Gazzola [13] and Cabré and Tan [7], Wang studied problem (1.3) with $s = 1/2$ and obtained a positive solution to her problem. We mention that Li and Xiang [14] extended Wang's work [17] to the setting of regional fractional Laplacion problems.

Inspired by Wang's work [17], in this paper we study problem (1.3) with s belonging to the full range $(0, 1)$. Throughout we assume that $2 \le q < 2_s^*$ and g satisfies

(G1) $g \in C(\Omega)$ is a nonnegative nontrivial function, and

(G2) $g \in L^{2n/(2n-(n-2s)q)}(\Omega)$. In addition, if $q = 2$, then $\|g\|_{L^{n/2s}(\Omega)} < S$, where S is defined as in (1.6) in the below.

Our main result reads as follows.

Theorem 1.1. *Assume that $s \in (0, 1)$ and g satisfies (G1) (G2). Then problem (1.3) admits a positive solution for every $k \in (0, \infty)$, provided either $q \ge 2$ and $n > 2s(1 + 2/q)$, or $q = 2$ and $n = 4s$.*

It is clear that the nonlocality of $(-\Delta)^s$ makes it difficult to deal with problem (1.3) directly. Our strategy is to use a localization method. In the global case $\Omega = \mathbb{R}^n$, a localization principle for fractional Laplacian problem was first systematically developed by Caffarelli and Silvestre [9]. In our local case, a localization principle was also developed by Cabré and Tan [7] for the case $s = 1/2$ and by Brändle et al. [4] for the general case $s \in (0, 1)$. To state the localization principle, we need to introduce some notations. Denote

$$C = \Omega \times (0, \infty) \quad \text{and} \quad \partial_L C = \partial\Omega \times (0, \infty)$$

with coordinates $(x, y) \in C$ for $x \in \Omega$ and $y > 0$. According to Brändle et al. [4], for every $u \in H_0^s(\Omega)$, there exists a unique $2s$-harmonic extension $\bar{u}$ which is equal to u on $\Omega \times \{y = 0\}$ in the sense of trace with $\bar{u} = 0$ on the lateral boundary $\partial_L C$, and satisfies the equation

$$\mathrm{div}(y^{1-2s}\nabla \bar{u}) = 0 \qquad \text{in } C.$$

Following Brändle et al. [4], introduce the function space

$$X_0^{2s}(C) = \overline{C_0^\infty(\Omega \times [0,\infty))}^{\|\cdot\|_{X_0^{2s}(C)}} \quad \text{with} \quad \|z\|_{X_0^{2s}(C)} = \left(\kappa_s \int_C y^{1-2s}|\nabla z|^2 \mathrm{d}x\mathrm{d}y\right)^{1/2},$$

where $\kappa_s = 2^{1-2s}\Gamma(1-s)/\Gamma(s)$ is a normalization constant such that the $2s$-harmonic extension $\bar{u}$ of u satisfies $\|\bar{u}\|_{X_0^{2s}(C)} = \|u\|_{H_0^s(\Omega)}$. Then, problem (1.3) is equivalent to

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla \bar{u}) = 0, \quad \bar{u} > 0 & \text{in } C, \\ \bar{u} = 0 & \text{on } \partial_L C, \\ \frac{\partial \bar{u}}{\partial \nu^{2s}} = g(x)(\bar{u}-k)_+^{q-1} + \bar{u}^{2_s^*-1} & \text{on } \Omega \times \{y=0\}, \end{cases} \tag{1.4}$$

where

$$\frac{\partial \bar{u}}{\partial \nu^{2s}}(x,0) = -\lim_{y \to 0+} \frac{1}{\kappa_s} y^{1-2s} u_y(x,y).$$

That is, $u \in H_0^s(\Omega)$ is a solution to problem (1.3) if and only if $\bar{u} \in X_0^{2s}(C)$ is a solution to problem (1.4) which means that

$$\kappa_s \int_C y^{1-2s}\nabla \bar{u} \cdot \nabla \Phi \mathrm{d}x\mathrm{d}y = \int_\Omega \left(g(x)(\bar{u}-k)_+^{q-1} + \bar{u}^{2_s^*-1}\right)\Phi \mathrm{d}x$$

holds for all $\Phi \in X_0^{2s}(C)$. This explains the localization method.

So, to prove Theorem 1.1, we turn to the equivalent problem (1.4). Note that problem (1.4) is variational with the energy functional $\mathcal{J} : X_0^{2s}(C) \to \mathbb{R}$ being given by

$$\mathcal{J}(\bar{u}) = \frac{1}{2}\|\bar{u}\|_{X_0^{2s}(C)}^2 - \int_\Omega \left(\frac{1}{q} g(x)(\bar{u}-k)_+^q + \frac{1}{2_s^*}\bar{u}_+^{2_s^*}\right)\mathrm{d}x$$

for $\bar{u} \in X_0^{2s}(C)$. To find a critical point of $\mathcal{J}$, we will use the mountain pass theorem (see e.g. Struwe [16, Chapter 6]).

Before ending the introduction, we record the following inequality (see [4, Theorem 2.1]) for later use: there exists $C = C(n,s) > 0$ such that, for every $w \in X^{2s}(\mathbb{R}_+^{n+1})$, the closure of $\overline{C_0^\infty(\mathbb{R}_+^{n+1})}$ under the seminorm $\|u\|_{X^{2s}(\mathbb{R}_+^{n+1})}^2 = \int_{\mathbb{R}_+^{n+1}} \kappa_s y^{1-2s}|\nabla w(x,y)|^2 \mathrm{d}x\mathrm{d}y$, there holds

$$C\left(\int_{\mathbb{R}^n} |w(x,0)|^{2_s^*}\mathrm{d}x\right)^{2/2_s^*} \le \int_{\mathbb{R}_+^{n+1}} \kappa_s y^{1-2s}|\nabla w(x,y)|^2 \mathrm{d}x\mathrm{d}y. \tag{1.5}$$

Let

$$S = \inf\left\{\frac{\int_{\mathbb{R}_+^{n+1}} \kappa_s y^{1-2s}|\nabla w(x,y)|^2 \mathrm{d}x\mathrm{d}y}{\left(\int_{\mathbb{R}^n} |w(x,0)|^{2_s^*}\mathrm{d}x\right)^{2/2_s^*}} : w \in X^{2s}(\mathbb{R}_+^{n+1}), w \not\equiv 0\right\} \tag{1.6}$$

be the best constant for inequality (1.5). It is known [4, Theorem 2.1] that S is attained by the functions

$$U_{\epsilon,x_0}(x,y) = (P_y^{1-2s} * u_{\epsilon,x_0})(x), \qquad (x,y) \in \mathbb{R}^n \times [0,\infty) \tag{1.7}$$

for all $\epsilon > 0$ and $x_0 \in \mathbb{R}^n$, where $P_y^{1-2s}(x) = k_{1-2s} y^{-n}(1 + |x|/y)^{-(n+2s)/2}$ is the so-called s-Poisson kernel and

$$u_{\epsilon,x_0}(x) = c_n \left(\frac{\epsilon}{|x - x_0|^2 + \epsilon^2} \right)^{(n-2s)/2}.$$

The constant $c_n > 0$ is chosen such that

$$\int_{\mathbb{R}^{n+1}_+} \kappa_s y^{1-2s} |\nabla U_{\epsilon,x_0}|^2 \mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^n} |U_{\epsilon,x_0}(x,0)|^{2^*_s} \mathrm{d}x = S^{n/2s}.$$

Our notations are standard. We use C to denote positive constants that are different from line to line. For simplicity, we will use $\|\cdot\|$ to denote the norm $\|\cdot\|_{X_0^{2s}(C)}$ throughout.

2. Proof of the main result

This section is devoted to the proof of Theorem 1.1. Our method is to use the well known mountain pass theorem, see e.g. Struwe [16, Theorem 6.1]. The first lemma points out that

Lemma 2.1. *$\mathcal{J}$ satisfies the mountain pass geometry.*

The proof is standard, and we omit the details. We shall also need

Lemma 2.2. *Under the assumptions (G_1) and (G_2), there hold*

$$g(u_m - k)_+^{q-1} \to g(u - k)_+^{q-1} \qquad \text{in } L^{\frac{2n}{n+2s}}(\Omega)$$

and

$$g(u_m - k)_+^{q} \to g(u - k)_+^{q} \qquad \text{in } L^{2s}(\Omega)$$

for every $(u_m) \subset X_0^{2s}(C)$ that converges weakly to $u \in X_0^{2s}(C)$.

Lemma 2.2 is a consequence of Vitali's convergence theorem. We omit the details, see also Wang [17, Lemma 2.1] and Gazzola [13, Lemma 1].

We say that $(u_m) \in X_0^{2s}(C)$ is a $(PS)_c$ sequence, if $\mathcal{J}(u_m) \to c \in \mathbb{R}$ and $\mathcal{J}'(u_m) \to 0$ as $m \to \infty$. Say that the functional $\mathcal{J}$ satisfies $(PS)_c$ compactness condition if $(u_m) \in X_0^{2s}(C)$ is a $(PS)_c$ sequence, then it contains a convergent subsequence in $X_0^{2s}(C)$. Next lemma says that this happens when c is below a critical value, as that was observed by Brezis and Nirenberg [6].

Lemma 2.3. *$\mathcal{J}$ satisfies the $(PS)_c$ condition provided $c < \frac{s}{n} S^{\frac{n}{2s}}$.*

Proof. Let $(u_m) \subset X_0^{2s}(C)$ be a $(PS)_c$ sequence. First we claim that (u_m) is a bounded sequence in $X_0^{2s}(C)$. This will follow from a simple combination of the $(PS)_c$ assumption. Let $\beta \geq 2$. Then

$$\mathcal{J}(u_m) - \frac{1}{\beta} \langle \mathcal{J}'(u_m), u_m \rangle = c + o(1) + o(1)\|u_m\|.$$

Taking $\beta = q$ when $q > 2$ yields

$$\left(\frac{1}{2} - \frac{1}{q}\right)\|u_m\|^2 + \frac{k}{q} \int_\Omega g(x)(u_m - k)_+^{q-1} dx + \left(\frac{1}{q} - \frac{1}{2^*_s}\right) \int_\Omega (u_m)_+^{2^*_s} dx = c + o(1) + o(1)\|u_m\|.$$

Then the assumption $g \geq 0$ implies that

$$\left(\frac{1}{2} - \frac{1}{q}\right)\|u_m\|^2 \leq c + o(1) + o(1)\|u_m\|. \tag{2.1}$$

When $q = 2$, take $2_s^* > \beta > 2$. Then using the assumption (G_2) and Hölder's inequality gives

$$\left(\frac{1}{2} - \frac{1}{\beta}\right)\left(1 - \frac{\|g\|_{L^{\frac{n}{2s}}}}{S}\right)\|u_m\|^2 \leq c + o(1) + o(1)\|u_m\|. \tag{2.2}$$

Now the boundedness of $(u_m) \subset X_0^{2s}(C)$ follows from (2.1) and (2.2).

Next we prove that (u_m) has a convergent subsequence in $X_0^{2s}(C)$. Since we have proved the boundedness of (u_m) in $X_0^{2s}(C)$, we may assume without loss of generality that, up to a subsequence,

$$\begin{aligned}
&u_m \rightharpoonup u \text{ in } X_0^{2s}(C),\\
&u_m(\cdot, 0) \to u(\cdot, 0) \text{ in } L^2(\Omega),\\
&u_m(\cdot, 0) \to u(\cdot, 0) \text{ a.e. in } \Omega,
\end{aligned}$$

for some $u \in X_0^{2s}(C)$. Put $v_m = u_m - u$. It suffices to prove that $\|v_m\| \to 0$.

It is standard to find that $\mathcal{J}'(u) = 0$. To examine the sequence closer, note that

$$\|u_m\|^2 = \|v_m + u\|^2 = \|v_m\|^2 + \|u\|^2 + o(1).$$

Using Lemma 2.2 we find that

$$\int_\Omega g(u_m - k)_+^q \to \int_\Omega g(u - k)_+^q,$$

By Lemma 2 of Brézis and Lieb [5],

$$\int_\Omega (u_m)_+^{2_s^*} = \int_\Omega (v_m)_+^{2_s^*} + \int_\Omega u_+^{2_s^*} + o(1).$$

As a result, we obtain the first decomposition

$$\begin{aligned}
\mathcal{J}(u_m) &= \frac{1}{2}\|u_m\|^2 - \int_\Omega \left(\frac{1}{q} g(x)(u_m - k)_+^q + \frac{1}{2_s^*}(u_m)_+^{2_s^*}\right) dx\\
&= \frac{1}{2}(\|v_m\|^2 + \|u\|^2) - \int_\Omega \frac{1}{q} g(x)(u - k)_+^q - \frac{1}{2_s^*}\int_\Omega \left((v_m)_+^{2_s^*} + u_+^{2_s^*}\right) + o(1)\\
&= \mathcal{J}(u) + \frac{1}{2}\|v_m\|^2 - \frac{1}{2_s^*}\int_\Omega (v_m)_+^{2_s^*} + o(1).
\end{aligned}$$

Similarly, we also have the second decomposition

$$\begin{aligned}
o(1) = \langle \mathcal{J}'(u_m), u_m \rangle &= \|u_m\|^2 - \int_\Omega \left(g(x)(u_m - k)_+^{q-1} + (u_m)_+^{2_s^*-1}\right) \cdot u_m dx\\
&= \|v_m\|^2 + \|u\|^2 - \int_\Omega g(x)(u - k)_+^{q-1} \cdot u - \int_\Omega (v_m)_+^{2_s^*} - \int_\Omega u_+^{2_s^*} + o(1)\\
&= \langle \mathcal{J}'(u), u \rangle + \|v_m\|^2 - \int_\Omega (v_m)_+^{2_s^*} + o(1)\\
&= \|v_m\|^2 - \int_\Omega (v_m)_+^{2_s^*} + o(1),
\end{aligned}$$

where we used used the assumption $\mathcal{J}'(u_m) \to 0$ in the first equality and the fact $\mathcal{J}'(u) = 0$ in the last line.

To continue, note that $\|v_m\| \le C(\|u_m\| + \|u\|)$ is a bounded sequence. So we may assume $\|v_m\|^2 \to b \ge 0$. Then the above second decomposition gives

$$\int_\Omega (v_m)_+^{2_s^*} = \|v_m\|^2 + o(1) \to b.$$

Recall the trace inequality (1.5). This implies $S b^{\frac{2}{2_s^*}} \le b$. As a consequence, we infer that either $b = 0$ or $b \ge S^{\frac{n}{2s}}$ holds.

We have to exclude the case $b \ge S^{\frac{n}{2s}}$. To this end, note that $\mathcal{J}'(u) = 0$ implies $\mathcal{J}(u) \ge 0$. Thus the first decomposition leads to

$$\mathcal{J}(u_m) \ge \frac{1}{2}\|v_m\|^2 - \frac{1}{2_s^*}\int_\Omega (v_m)_+^{2_s^*} + o(1).$$

If $b \ge S^{\frac{n}{2s}}$, then taking limit in this inequality gives $c \ge \frac{s}{n} S^{\frac{n}{2s}}$, which contradicts with our assumption $c < \frac{s}{n} S^{\frac{n}{2s}}$. Hence $b = 0$. That is $v_m \to 0$ in $X_0^{2s}(C)$. The proof is finished. □

To proceed, we may assume that $0 \in \Omega$ and $g(0) > 0$ with no loss of generality. By the assumption $(G1)$ we can assume $B_r(0) \subset \{x \in \Omega : g(x) > g(0)\}$ for some $r > 0$ sufficiently small. For simplicity, write $u_\epsilon = u_{\epsilon,0}, U_\epsilon = U_{\epsilon,0}$, where $u_{\epsilon,x_0}, U_{\epsilon,x_0}$ are defined as in (1.7). Let $0 < \rho < r$ and take a cut-off function $\eta \in C^\infty(\overline{C})$ such that $0 \le \eta \le 1$ and $\eta(x,y) \equiv 1$ for $|x| < \frac{\rho}{2}$ and $y \ge 0$, and $\eta(x,y) \equiv 0$ for $|x| \ge \rho$. These auxiliary functions and parameters are used to construct a special path in $X_0^{2s}(C)$ starting from the origin such that the following lemma holds.

Lemma 2.4. *For $\epsilon > 0$ sufficiently small there holds*

$$\max_{t\ge 0} \mathcal{J}(t\eta U_\epsilon) < \frac{s}{n} S^{\frac{n}{2s}}.$$

Proof. Write $v_\epsilon = \eta U_\epsilon$ and $\gamma(t) = t v_\epsilon$ for $t \ge 0$. Suffices to show that $\max_{t>0} \mathcal{J}(\gamma(t)) < \frac{s}{n} S^{\frac{n}{2s}}$ for ϵ sufficiently small. By a direct computation we have

$$\begin{aligned}\mathcal{J}(\gamma(t)) &= \frac{1}{2}\|t v_\epsilon\|^2 - \int_\Omega \frac{1}{q} g(x)(t v_\epsilon - k)_+^q - \frac{1}{2_s^*}\int_\Omega (t v_\epsilon)^{2_s^*} \\ &= \frac{t^2}{2}\|v_\epsilon\|^2 - \int_\Omega \frac{1}{q} g(x)(t v_\epsilon - k)_+^q - \frac{t^{2_s^*}}{2_s^*}\int_\Omega v_\epsilon^{2_s^*}.\end{aligned}$$

Since the second term on the right hand side is nonnegative, there holds

$$\mathcal{J}(\gamma(t)) \le \frac{t^2}{2}\|v_\epsilon\|^2 - \frac{t^{2_s^*}}{2_s^*}\int_\Omega v_\epsilon^{2_s^*} \to -\infty \qquad \text{as } t \to \infty.$$

Also note that v_ϵ is a bounded function. So for t sufficiently small, we have $(t v_\epsilon - k)_+ \equiv 0$. Hence for t sufficiently small,

$$\mathcal{J}(\gamma(t)) = \frac{t^2}{2}\|v_\epsilon\|^2 - \frac{t^{2_s^*}}{2_s^*}\int_\Omega v_\epsilon^{2_s^*} > 0 \qquad \text{for } t \to 0.$$

So $\mathcal{J}(\gamma(t))$ achieves a positive maximum on $(0,\infty)$.

Let t_ϵ be such that $\mathcal{J}(\gamma(t_\epsilon)) = \max_{t>0}\mathcal{J}(\gamma(t))$. We claim that for ϵ sufficiently small, there exist constants C_1, C_2 independent of ϵ such that

$$0 < C_1 \le t_\epsilon \le C_2. \tag{2.3}$$

Notice that

$$\mathcal{J}(\gamma(t)) \le \frac{t^2}{2}\|v_\epsilon\|^2 - \frac{t^{2_s^*}}{2_s^*}\int_\Omega v_\epsilon^{2_s^*} \le 0 \qquad \text{for } t \ge \bar{t}_\epsilon,$$

where $\bar{t}_\epsilon = (2_s^*\|v_\epsilon\|^2/2\int_\Omega v_\epsilon^{2_s^*})^{(n-2s)/4s}$. We infer that $\mathcal{J}(\gamma(t)) \le 0$ holds for all $t \ge \bar{t}_\epsilon$, which implies

$$t_\epsilon \le \bar{t}_\epsilon \le C_2 \qquad \text{as } \epsilon \to 0$$

for some $C_2 > 0$ independent of ϵ. On the other hand, by Lemma 2.1, t_ϵ cannot converge to zero. Therefore there exists $C_1 > 0$ such that

$$t_\epsilon \ge C_1$$

for ϵ sufficiently small. The claim is proved.

To further compute the maximum $\mathcal{J}(\gamma(t_\epsilon))$, according to [2], there hold

$$\|v_\epsilon\|^2 = S^{\frac{n}{2s}} + O(\epsilon^{n-2s}) \quad \text{and} \quad \int_\Omega v_\epsilon^{2_s^*} = S^{\frac{n}{2s}} + O(\epsilon^n)$$

for ϵ sufficiently small. Thus

$$\begin{aligned}
\frac{t_\epsilon^2}{2}\|v_\epsilon\|^2 - \frac{t_\epsilon^{2_s^*}}{2_s^*}\int_\Omega v_\epsilon^{2_s^*} &= \frac{t_\epsilon^2}{2}(S^{\frac{n}{2s}} + O(\epsilon^{n-2s})) - \frac{t_\epsilon^{2_s^*}}{2_s^*}(S^{\frac{n}{2s}} + O(\epsilon^n))\\
&= \frac{s}{n}S^{\frac{n}{2s}} + \left(\frac{t_\epsilon^2 - 1}{2} - \frac{t_\epsilon^{2_s^*} - 1}{2_s^*}\right)S^{\frac{n}{2s}} + O(\epsilon^{n-2s})\\
&\le \frac{s}{n}S^{\frac{n}{2s}} + O(\epsilon^{n-2s}),
\end{aligned}$$

where we used in the last line the fact that $\frac{t^2-1}{2} - \frac{t^{2_s^*}-1}{2_s^*} \le 0$ for all $t \ge 0$. Therefore we derive the estimate

$$\mathcal{J}(\gamma t_\epsilon) \le \frac{s}{n}S^{\frac{n}{2s}} - \int_\Omega \frac{1}{q}g(x)(t_\epsilon v_\epsilon - k)_+^q + O(\epsilon^{n-2s}). \tag{2.4}$$

We need to estimate $\int_\Omega g(t_\epsilon v_\epsilon - k)_+^q$. For ϵ sufficiently small, we always have

$$\int_\Omega g(t_\epsilon v_\epsilon - k)_+^q \ge g(0)\int_{B_{\rho/2}(0)}(t_\epsilon U_\epsilon - k)_+^q.$$

In the case $q \ge 2$ and $n > 2s(1 + 2/q)$, there holds

$$\int_{B_{\rho/2}(0)}(t_\epsilon U_\epsilon - k)_+^q \ge C\int_{B_\epsilon(0)}\epsilon^{-\frac{n-2s}{2}q} = C\epsilon^{n-\frac{n-2s}{2}q}$$

for some $C > 0$ independent of ϵ, where we have used the estimate (2.3) for t_ϵ. In the case $q = 2, n = 4s$, note that on the set $\{\epsilon < |x| < \epsilon^{3/4}\}$, there holds

$$U_\epsilon(x) \geq \frac{\epsilon^s}{(2|x|^2)^s} \geq \frac{\epsilon^{-\frac{1}{2}s}}{2^s} > 2k$$

for ϵ sufficiently small. Hence,

$$g(0) \int_{B_{\rho/2}(0)} (t_\epsilon u_\epsilon - k)_+^2 \geq C \int_{\{\epsilon<|x|<\epsilon^{3/4}\}} \frac{\epsilon^{2s}}{|x|^{4s}} = C\epsilon^{2s} |\ln \epsilon|$$

for some $C > 0$ independent of ϵ. Thus, there exists $C > 0$ such that

$$\int_\Omega g(t_\epsilon v_\epsilon - k)_+^q \geq \begin{cases} C\epsilon^{n-\frac{n-2s}{2}q}, & \text{if } q \geq 2 \text{ and } n > 2s(1+2/q) \\ C\epsilon^{2s} \ln \epsilon & \text{if } q = 2, n = 4s. \end{cases} \tag{2.5}$$

Combining (2.4) and (2.5) yields

$$\mathcal{J}(\gamma t_\epsilon) \leq \begin{cases} \frac{s}{n} S^{\frac{n}{2s}} - C\epsilon^{n-\frac{n-2s}{2}q} \left(1 - \epsilon^{\frac{n-2s}{2}q-2s}\right) & \text{if } q \geq 2 \text{ and } n > 2s(1+2/q), \\ \frac{s}{n} S^{\frac{n}{2s}} - C\epsilon^{2s}(|\ln \epsilon| - 1) & \text{if } q = 2, n = 4s. \end{cases}$$

In the case $q \geq 2$ and $n > 2s(1 + 2/q)$, it holds $\frac{n-2s}{2}q - 2s > 0$. Therefore, in both cases we can deduce that

$$\mathcal{J}(\gamma t_\epsilon) < \frac{s}{n} S^{\frac{n}{2s}},$$

provided ϵ is sufficiently small. The proof is complete. □

Now we are in the position to prove Theorem 1.1.

Proof of Theorem 1.1. First choose $e = t_0 \eta U_\epsilon$, where ϵ and η are chosen as in Lemma 2.4 and t_0 is sufficiently large such that $\mathcal{J}(t_0 \eta U_\epsilon) < 0$, and then let

$$\Gamma = \left\{\gamma \in C\left([0, 1], X_0^{2s}(C)\right) : \gamma(0) = 0 \text{ and } \gamma(1) = e\right\}$$

and

$$c_0 = \inf_{\gamma\in\Gamma} \max_{t\geq 0} \mathcal{J}(\gamma(t)).$$

By Lemma 2.4, we have $c_0 < \frac{s}{n} S^{n/2s}$. Since $\mathcal{J}$ satisfies the geometry of mountain pass, there exists a sequence $(u_m) \subset X_0^{2s}(C)$ satisfying $\mathcal{J}(u_m) \to c_0$ and $\mathcal{J}'(u_m) \to 0$ as $m \to \infty$. Therefore, Lemma 2.3 implies that problem (1.4) admits a nonnegative nontrivial solution. Finally, a maximum principle of Cabré and Sire [8, Remark 4.2] implies that the solution is positive in Ω. The proof is complete. □

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Conflict of interest

All authors declare no conflict of interest in this paper.

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