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Research article

Positive solutions of IBVPs for q-difference equations with p-Laplacian on infinite interval

Changlong Yu[1,2,*], Jufang Wang[2], Huode Han[2] and Jing Li[1,*]

[1] Interdisciplinary Research Institute, Faculty of Science, Beijing University of Technology, Beijing 100124, China

[2] College of Sciences, Hebei University of Science and Technology, Shijiazhuang, 050018, Hebei, China

* **Correspondence:** Email: changlongyu@126.com, leejing@bjut.edu.cn.

Abstract: Nowadays, many researches have considerable attention to the nonlinear q-difference equations boundary value problems as important and useful tool for modeling of different phenomena in various research fields. In this work, we investigate a class of q-difference equations boundary value problems with integral boundary conditions with p-Laplacian on infinite intervals. By applying the Avery-Peterson fixed point theorem in a cone, we establish the existence of three positive solutions for the above boundary value problem. Finally, the main results is illustrated with the aid of an example.

Keywords: Avery-Peterson fixed point theorem; boundary value problem; p-Laplacian operator; positive solutions; quantum calculus
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1. Introduction

The q-calculus or quantum calculus is an old subject that was initially developed by Jackson [1, 2], and has been confirmed to have numerous applications in a variety of subjects such as quantum mechanics, hypergeometric series, particle physics, complex analysis and so on [3, 4]. As well as known, the nonlinear difference equation can more accurately describe some phenomena in the natural world. In recent decades, the nonlinear q-difference equations boundary value problems have been widely used in various research fields [5, 6]. However, for the existence of positive solutions to boundary value problems of q-difference equation with p-Laplacian operator, few works were done. Especially, there are few works on q-difference equations boundary value problems with p-Laplacian on infinite intervals.

The existence and multiplicity of positive solutions to differential equations with or without the p-Laplacian operator subject to Dirichlet, Sturm-Liouville or nonlinear boundary value conditions have been investigated extensively, see [7–13] and the references therein. In [14], Lian et al. investigated the existence of positive solutions for the boundary value problem with p-Laplacian operator

$$\begin{cases} (\varphi_p(x'(t)))' + \phi(t)f(t, x(t), x'(t)) = 0, \quad 0 < t < +\infty, \\ \alpha x(0) - \beta x'(0) = 0, \quad x'(\infty) = 0, \end{cases}$$

by using the Avery-Peterson fixed point theorem in a cone.

In [15], Guo et al. established the existence of three positive solutions for m-point BVPs on infinite intervals

$$\begin{cases} (\varphi_p(x'(t)))' + \phi(t)f(t, x(t), x'(t)) = 0, \quad 0 < t < +\infty, \\ x(0) = \sum_{i=1}^{m-2} \alpha_i x'(\eta_i), \quad \lim_{t\to+\infty} x'(t) = 0, \end{cases}$$

by applying the Avery-Peterson fixed point theorem in a cone.

Motivated by the above works, we study the following nonlinear q-difference equation with the integral boundary value problem (IBVP) on infinite intervals

$$\begin{cases} (D_q(\varphi_p(D_q x(t))) + \phi(t)f(t, x(t),\ D_q x(t)) = 0, \quad 0 < t < +\infty, \\ x(0) = \int_0^{+\infty} g(s)D_q x(s)d_q s, \quad \lim_{t\to+\infty} D_q x(t) = 0, \end{cases} \tag{1.1}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$, $\phi : R_+ \to R_+$ and $f(t, u, v) : R_+^3 \to R_+$ are continuous functions, $R_+ = [0, +\infty)$, $g \in L_q^1[0, +\infty)$ is nonnegative, $L_q^1[0, +\infty)$ is the set of functions which are q-integrable.

In this article, we use the following conditions:

(H_1) $\phi \in C(R_+, R_+)$, $\phi \not\equiv 0$ on any interval of form $(t_0, +\infty)$ and $\int_0^\infty \phi(s)d_q s < +\infty$, $\int_0^\infty \varphi^{-1}(\int_\tau^\infty \phi(s)d_q s)d_q\tau < +\infty$;

(H_2) $f(t, (1 + t)u, v) \in C(R_+^3, R_+)$, $f(t, 0, 0) \not\equiv 0$ on any subinterval of $(0, +\infty)$ and when u, v are bounded, $f(t, (1 + t)u, v)$ is bounded on $[0, +\infty)$.

2. Preliminary results

In this section, we present some definitions, theorems and lemmas, which will be needed in the proof of the main results.

Definition 2.1 [16] The q-derivative of the function f is defined as

$$D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad x \neq 0,$$

and

$$(D_q f)(0) = \lim_{x\to 0}(D_q f)(x).$$

Note that $\lim_{q\to 1^-} D_q f(x) = f'(x)$.

Definition 2.2 [16] Suppose $0 < a < b$. The q-integral is defined as

$$\int_0^b f(x)d_qx = (1-q)\,b\sum_{j=0}^{\infty} q^j f(q^j b),$$

and

$$\int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx.$$

Definition 2.3 [16] The improper q-integral of $f(x)$ on $[0,+\infty)$ is defined to be

$$\int_0^{\infty} f(x)d_qx = \sum_{j=-\infty}^{+\infty} \int_{q^{j+1}}^{q^j} f(x)d_qx,$$

if $0 < q < 1$, or

$$\int_0^{\infty} f(x)d_qx = \sum_{j=-\infty}^{+\infty} \int_{q^j}^{q^{j+1}} f(x)d_qx,$$

if $q > 1$, where $\int_{q^{j+1}}^{q^j} f(x)d_qx = (1-q)q^j f(q^j)$.

Theorem 2.4 [16] (Fundamental theorem of q-calculus) If $F(x)$ is an antiderivative of $f(x)$ and $F(x)$ is continuous at $x = 0$, we have

$$\int_a^b f(x)d_qx = F(b) - F(a),$$

where $0 \le a < b \le \infty$.

Definition 2.5 [17] Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions:

(i) $x \in P$ and $\lambda \ge 0$ imply $\lambda x \in P$;

(ii)$x \in P$ and $-x \in P$ imply $\lambda x = 0$.

Definition 2.6 [17] Given a cone P in a real Banach space E. A continuous map ψ is called a concave (resp. convex) functional on P if for all $x, y \in P$ and $0 < \lambda < 1$, it holds $\psi(\lambda x + (1-\lambda)y) \ge \lambda\psi(x) + (1-\lambda)\psi(y)$, (resp. $\psi(\lambda x + (1-\lambda)y) \le \lambda\psi(x) + (1-\lambda)\psi(y)$).

Let α, γ, θ, ψ be nonnegative continuous maps on P with α concave, γ, θ convex. Then for positive numbers a, b, c, d, we define the following subsets of P:

$$\begin{aligned}
&P(\gamma^d) = \{x \in P : \gamma(x) < d\};\\
&P(\alpha_b, \gamma^d) = \{x \in \overline{P(\gamma^d)} : b \le \alpha(x)\};\\
&P(\alpha_b, \theta^c, \gamma^d) = \{x \in \overline{P(\gamma^d)} : b \le \alpha(x), \theta(x) \le c\};\\
&R(\psi_a, \gamma^d) = \{x \in \overline{P(\gamma^d)} : a \le \psi(x)\}.
\end{aligned}$$

It is obvious that $P(\gamma^d), P(\alpha_b,\gamma^d), P(\alpha_b,\theta^c,\gamma^d)$ are convex and $R(\psi_a,\gamma^d)$ is closed.

Theorem 2.7 [17] (Avery-Peterson fixed point theorem) Let P be a cone of a real Banach space E. Let γ,θ be non-negative convex functional on P satisfying

$$\psi(\lambda x)\le\lambda\psi(x),\ \forall 0<\lambda<1,\ \alpha(x)\le\psi(x),\ \|x\|\le M\gamma(x),\ \forall x\in\overline{P(\gamma^d)}$$

with M,d some positive numbers. Suppose that $T:\overline{P(\gamma^d)}\to\overline{P(\gamma^d)}$ is completely continuous and there exist positive numbers a,b,c with $a<b$ such that

(1) $\{x\in P(\alpha_b,\theta^c,\gamma^d):\alpha(x)\ge b\}\ne\emptyset$ and $\alpha(Tx)>b$ for $x\in P(\alpha_b,\theta^c,\gamma^d)$;

(2) $\alpha(Tx)>b$ for $x\in P(\alpha_b,\gamma^d)$ with $\theta(Tx)>c$;

(3) $0\ne R(\psi_a,\gamma^d)$ and $\psi(x)<a$ for $x\in R(\psi_a,\gamma^d)$ with $\psi(x)=a$.

Then T has at least three fixed points $x_1,x_2,x_3\in P(\gamma^d)$ such that

$$\gamma(x_i)\le d,\ i=1,2,3;\ \psi(x_1)<a;\ \psi(x_2)>a \text{ with } \alpha(x_2)<b;\ \alpha(x_3)>b.$$

Consider the space

$$X=\{x\in C_q^1[0,+\infty]:\sup_{0\le t<+\infty}\frac{|x(t)|}{1+t}<+\infty,\ \lim_{t\to+\infty}D_qx(t)=0\}$$

with the norm $\|x\|=\max\{\|x\|_1,\|D_qx\|_\infty\}$, where

$$\|x\|_1=\sup_{0\le t<+\infty}\frac{|x(t)|}{1+t},\ \ \|D_qx\|_\infty=\sup_{0\le t<+\infty}|D_qx(t)|.$$

Obviously, we can obtain that $(X,\|\cdot\|)$ is a Banach space. Let $P\subset X$ by

$$P=\{x\in X:x(t)\ge 0,t\in[0,+\infty),x(0)=\int_0^{+\infty}g(s)D_qx(s)d_qs,\ x\text{ is concave on }[0,+\infty)\}.$$

Remark 2.8 If x satisfies (1.1), then $(D_q(\varphi_p(D_qx(t)))=-\phi(t)f(t,x(t),\ D_qx(t))\le 0$ on $[0,+\infty)$, which implies that x is concave on $[0,+\infty)$. Moreover, if $\lim_{t\to+\infty}D_qx(t)=0$, then $D_qx(t)\ge 0,t\in[0,+\infty)$ and so x is monotone increasing on $[0,+\infty)$.

Let $k>1$ be a constant. For $x\in P$, define the nonnegative continuous functionals:

$$\alpha(x)=\frac{k}{k+1}\min_{\frac{1}{k}\le t<k}x(t),\ \gamma(x)=\sup_{0\le t<+\infty}D_qx(t),$$

$$\psi(x)=\theta(x)=\sup_{0\le t<\infty}\frac{|x(t)|}{1+t},\ A=\int_0^{+\infty}g(s)d_qs,$$

and set

$$C=\varphi_p^{-1}\Big(\int_0^{+\infty}\phi(s)d_qs\Big),\ C_1(t)=\int_0^t\varphi_p^{-1}\Big(\int_\tau^{+\infty}\phi(s)d_qs\Big)d_q\tau.$$

Since the Arzela-Ascoli theorem does not apply in the space X, we need a modified compactness criterion to prove T is compact. In the following, we present an explicit one.

Definition 2.9 For $l>0$, let $V=\{x\in X:\|x\|<l\}$, and $V_1:=\{\frac{x(t)}{1+t},x\in V\}\bigcup\{D_qx(t),x\in V\}$, which is called equiconvergent at infinity if for all $\varepsilon>0$, there exists $T=T(\varepsilon)>0$ such that for all $x\in V_1$, $|\frac{x(t_1)}{1+t_1}-\frac{x(t_2)}{1+t_2}|<\varepsilon$, $|D_qx(t_1)-D_qx(t_2)|<\varepsilon$, $\forall t_1,t_2\ge T$.

Lemma 2.10 If $\{\frac{x(t)}{1+t},x\in V\}$ and $\{D_qx(t),x\in V\}$ are both equicontinuous on any compact interval of

$[0, +\infty)$ and equiconvergent at infinity. Then V is relatively compact on X.
Proof. Its proof is similar to the proof of Lemma 2.2 in literature [18].
Lemma 2.11 Let $g \in L_q^1[0, +\infty)$ and g is nonnegative, if $v(t)$ is nonnegative and continuous on $[0, +\infty)$ and $\lim\limits_{t\to+\infty} v(t)$ exists. Then there exists at least one η, $0 \le \eta < +\infty$ such that

$$\int_0^{+\infty} g(s)v(s)d_qs = v(\eta)\int_0^{+\infty} g(s)d_qs.$$

Proof. It is obvious that the function $v(t)$ exists and has maxima and minima which are nonnegative and noted by M^*, m^* on $[0, +\infty)$, then for all $t \in [0, +\infty)$, we have $m^* \le v(t) \le M^*$, so

$$m^*\int_0^{+\infty} g(s)d_qs \le \int_0^{+\infty} g(s)v(s)d_qs \le M^*\int_0^{+\infty} g(s)v(s)d_qs.$$

If $\displaystyle\int_0^{+\infty} g(s)d_qs = 0$, the result is clear; if $\int_0^{+\infty} g(s)d_qs > 0$, there is

$$m^* \le \frac{\int_0^{+\infty} g(s)v(s)d_qs}{\int_0^{+\infty} g(s)d_qs} \le M^*.$$

Therefore, there exists at least one η, $0 \le \eta < +\infty$ such that

$$\int_0^{+\infty} g(s)v(s)d_qs = v(\eta)\int_0^{+\infty} g(s)d_qs.$$

Lemma 2.12 Let $y \in C[R_+, R_+]$, and $\displaystyle\int_0^{+\infty} g(t)d_qt < \infty$, then q-difference IBVP

$$\begin{cases} (D_q(\varphi_p(D_qx(t))) + y(t) = 0, \quad 0 < t < +\infty, \\ x(0) = \displaystyle\int_0^{+\infty} g(s)D_qx(s)d_qs, \lim_{t\to+\infty} D_qx(t) = 0, \end{cases} \tag{2.1}$$

has a unique solution

$$x(t) = \int_0^{+\infty} g(s)\varphi_p^{-1}\Big(\int_s^{+\infty} y(\tau)d_q\tau\Big)d_qs + \int_0^t \varphi_p^{-1}\Big(\int_s^{+\infty} y(\tau)d_q\tau\Big)d_qs.$$

Proof. We integrate the equantion from t to $+\infty$, and get

$$D_qx(t) = \varphi_p^{-1}\Big(\int_t^{+\infty} y(s)d_qs\Big). \tag{2.2}$$

Integrating (2.2) from 0 to t, and

$$x(0) = \int_0^{+\infty} g(s)D_qx(s)d_qs,$$

we can get

$$x(t) = \int_0^{+\infty} g(s)\varphi_p^{-1}\Big(\int_s^{+\infty} y(\tau)d_q\tau\Big)d_qs + \int_0^t \varphi_p^{-1}\Big(\int_s^{+\infty} y(\tau)d_q\tau\Big)d_qs.$$

Define the operator $T : P \to C^1[0, +\infty)$ by

$$Tx(t) = \int_0^{+\infty} g(s)\varphi_p^{-1}\Big(\int_s^{+\infty} \phi(\tau)f(\tau, x(\tau), D_q x(\tau))d_q\tau\Big)d_q s$$
$$+ \int_0^{t} \varphi_p^{-1}\Big(\int_s^{+\infty} \phi(\tau)f(\tau, x(\tau), D_q x(\tau))d_q\tau\Big)d_q s.$$

Lemma 2.13 For $x \in P$, $\|x\|_1 \le M\|x\|_\infty$, where $M = \max\{\int_0^{+\infty} g(s)d_q s, 1\}$.

Proof. Since $x \in P$,

$$\frac{x(t)}{1+t} = \frac{\int_0^t D_q x(s)d_q s + \int_0^{+\infty} g(s)D_q x(s)d_q s}{1+t}$$
$$\le \frac{t + \int_0^{+\infty} g(s)d_q s}{1+t}\|D_q x\|_\infty \le M\|D_q x\|_\infty.$$

Lemma 2.14 For $x \in P$, $\alpha(x) \ge \frac{\theta(x)}{k+1}$.

Proof. Obviously, x is increasing on $[0, +\infty)$. Moreover, since $D_q x(+\infty) = 0$, the function $\frac{x(t)}{1+t}$ achieves its maximum at $\sigma \in [0, +\infty)$, then $\theta(x) = \frac{x(\sigma)}{1+\sigma}$. Furthermore, x is concave, so

$$\alpha(x) = \frac{k}{k+1}x\Big(\frac{1}{k}\Big) = \frac{k}{k+1}x\Big(\frac{k-1+k\sigma}{k+k\sigma}\cdot\frac{1}{k-1+k\sigma} + \frac{\sigma}{k+k\sigma}\Big)$$
$$\ge \frac{1}{k+1}\cdot\frac{x(\sigma)}{1+\sigma} = \frac{1}{k+1}\theta(x).$$

Lemma 2.15 Let $(H_1) - (H_2)$ hold. Then, $T : P \to P$ is completely continuous.

Proof. It is easy to verify that $T : P \to P$ is well defined. Now, we prove that T is continuous and compact respectively.

(i) T is continuous.

Let $x_n \to x$ as $n \to +\infty$ in P, then there exists r_0 such that $\sup_{n\in N\backslash\{0\}}\|x_n\| < r_0$. Set $Br_0 = \sup\{f(t, (1+t)u, v), (t, u, v) \in [0, +\infty) \times [0, r_0]^2\}$. Then we have

$$\int_0^{+\infty} \phi(s)|f(s, x_n, D_q x_n) - f(s, x, D_q x)|d_q s \le 2Br_0\int_0^{+\infty} \phi(s)d_q s.$$

Therefore, by the Lebesgue dominated convergence theorem, we get

$$|\varphi_p((D_qTx_n)(t)) - \varphi_p((D_qTx)(t))| = \Big|\int_t^{+\infty} \phi(s)(f(s, x_n, D_q x_n) - f(s, x, D_q x))d_q s\Big|$$
$$\le \Big|\int_0^{+\infty} \phi(s)(f(s, x_n, D_q x_n) - f(s, x, D_q x))d_q s\Big| \to 0, \quad n \to +\infty.$$

Furthermore, $\|Tx_n - Tx\| \le M\|D_qTx_n - D_qTx\|_\infty \to 0$, as $n \to +\infty$. Hence, T is continuous.

(ii) T is compact.

T is compact provided that it maps bounded sets into relatively compact sets. Let Ω be any bounded subset of P. Then there exists $r > 0$ such that $\|x\| \leq r$ for all $x \in \Omega$. Obviously,

$$\|D_qTx\|_\infty \leq \varphi_p^{-1}\Big(\int_0^{+\infty} \phi(s)f(s, x(s), D_qx(s))d_qs\Big) \leq C\varphi_p^{-1}(Br)$$

for all $x \in \Omega$. Hence, $\|T\Omega\| \leq MC\varphi_p^{-1}(Br)$. So $T\Omega$ is bounded. Moreover, for any $L \in [0, +\infty)$, and $t_1, t_2 \in [0, L]$,

$$\begin{aligned}\left|\frac{(Tx)(t_1)}{1+t_1} - \frac{(Tx)(t_2)}{1+t_2}\right| \leq & \int_0^{+\infty} g(s)\varphi_p^{-1}\Big(\int_s^{+\infty} \phi(\tau)f(\tau, x(\tau), D_qx(\tau))d_q\tau\Big)d_qs\left|\frac{1}{1+t_1} - \frac{1}{1+t_2}\right| \\ & + \int_0^{t_2} \varphi_p^{-1}\Big(\int_s^{+\infty} \phi(\tau)f(\tau, x(\tau), D_qx(\tau))d_q\tau\Big)d_qs\left|\frac{1}{1+t_1} - \frac{1}{1+t_2}\right| \\ & + \frac{1}{1+t_1}\left|\int_{t_1}^{t_2} \varphi_p^{-1}\Big(\int_s^{+\infty} \phi(\tau)f(\tau, x(\tau), D_qx(\tau))d_q\tau\Big)d_qs\right| \\ \leq & \varphi_p^{-1}(Br)(AC + C_1(L))\left|\frac{1}{1+t_1} - \frac{1}{1+t_2}\right| + |C_1(t_1) - C_2(t_2)| \to 0, (t_1 \to t_2),\end{aligned}$$

and

$$\begin{aligned}|\varphi_p((D_qTx)(t_1)) - \varphi_p((D_qTx)(t_2))| &= \left|\int_{t_1}^{t_2} \phi(s)f(s, x(s), D_qx(s))d_qs\right| \\ &\leq Br\left|\int_{t_1}^{t_2} \phi(s)d_qs\right| \to 0, \quad (t_1 \to t_2)\end{aligned}$$

for all $x \in \Omega$. So $T\Omega$ is equicontinuous on any compact interval of $[0, +\infty)$. Finally, for any $x \in \Omega$,

$$\begin{aligned}\lim_{t\to+\infty}\left|\frac{(Tx)(t)}{1+t}\right| &= \lim_{t\to+\infty}\frac{1}{1+t}\int_0^t \varphi_p^{-1}\Big(\int_s^{+\infty} \phi(\tau)f(\tau, x(\tau), D_qx(\tau))d_q\tau\Big)d_qs \\ &\leq M\varphi_p^{-1}(Br)\lim_{t\to+\infty}\varphi_p^{-1}\Big(\int_t^{+\infty} \phi(s)d_qs\Big) = 0,\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to+\infty}|D_qTx)(t)| &= \lim_{t\to+\infty}\varphi_p^{-1}\Big(\int_t^{+\infty} \phi(s)f(s, x(s), D_qx(s))d_qs\Big) \\ &\leq \varphi_p^{-1}(Br)\lim_{t\to+\infty}\varphi_p^{-1}\Big(\int_t^{+\infty} \phi(s)d_qs\Big) = 0.\end{aligned}$$

So $T\Omega$ is equiconvergent at infinity. By using Lemma 2.10, we obtain that $T\Omega$ is relatively compact, that is, T is a compact operator. Hence, $T : P \to P$ is completely continuous. The proof is complete.

3. Main results

For the main result of this paper, we further assume that
(H_3) $f(t, (1+t)u, v) \leq \varphi_p(d/C)$, for $(t, u, v) \in [0, +\infty) \times [0, Md] \times [0, d]$;

(H_4) $f(t,(1+t)u,v) > \varphi_p(b/N)$, for $(t,u,v) \in [\frac{1}{k},k] \times [\frac{b}{k}, \frac{(k+1)^2}{km}b] \times [0,d]$;

(H_5) $f(t,(1+t)u,v) < \varphi_p(a/MC)$, for $(t,u,v) \in [0,+\infty) \times [0,a] \times [0,d]$,

where $m = \min\{A,1\}$, $N = \frac{1}{(k+1)^2}\int_{\frac{1}{k}}^{k}(g(s)+1)\varphi_p^{-1}(\int_s^k \phi(\tau)d_q\tau)d_qs$.

Theorem 3.1 Let $A > 0$. Suppose $(H_1)-(H_5)$ hold. Suppose further that there exist numbers a,b,d such that $0 < ka < bMmkd/(k+1)^2$. Then (1.1) has at least three positive solutions x_1,x_2,x_3 such that

$$\sup_{0\le t<\infty} D_qx_i(t) \le d,\ i=1,2,3;$$

$$\sup_{0\le t<\infty}\frac{|x_1(t)|}{1+t} < a,\ \sup_{0\le t<\infty}\frac{|x_2(t)|}{1+t} < \frac{(k+1)^2b}{km},\ \min_{\frac{1}{k}\le t<k} x_2(t) < \frac{(k+1)b}{k};$$

$$\sup_{0\le t<\infty}\frac{|x_3(t)|}{1+t} < Md,\ \min_{\frac{1}{k}\le t<k} x_3(t) > \frac{(k+1)b}{k}.$$

Proof. Let $X, P, \alpha, \gamma, \theta, \psi$ and T be defined as before respectively. It is easy to prove that the fixed points of T coincide with the solution of BVP (1.1). So it is enough to show that T has three positive fixed points. In fact, for any $x \in \overline{P(\gamma^d)}$, $\sup_{0\le t<\infty} D_qx(t) \le d$ and so $\sup_{0\le t<\infty}\frac{x(t)}{1+t} < Md$. Condition (H_3) implies that $f(t,x(t),D_qx(t)) \le \varphi_p(d/C)$ for all $t \in [0,+\infty)$.

Therefore,

$$\begin{aligned}\gamma(Tx) &= \sup_{0\le t<\infty}(D_qTx)(t) = (D_qTx)(0) = \varphi_p^{-1}\Big(\int_s^{+\infty}\phi(s)f(s,x(s),D_qx(s))d_qs\Big)\\ &\le \frac{d}{c}\varphi_p^{-1}\Big(\int_s^{+\infty}\phi(s)d_qs\Big) = d.\end{aligned}$$

Hence, $T: \overline{P(\gamma^d)} \to \overline{P(\gamma^d)}$ is completely continuous. Obviously, $\alpha,\gamma,\theta,\psi$ satisfy the assumptions in Theorem 2.7. Next, we show that conditions $(1)-(3)$ in Theorem 2.7 hold.

Firstly, choose the function $x(t) = (1-\frac{1}{k+1}e^{-\frac{kt}{A}})\frac{(k+1)^2b}{km}$, $0 \le t < +\infty$. It can be checked that $x \in P(\alpha_b,\theta^c,\gamma^d)$ with $\alpha(x) > b$, where $c = \frac{(k+1)^2b}{km}$, thus $\{x \in P(\alpha_b,\theta^c,\gamma^d) | \alpha(x) > b\} \neq \emptyset$. For any $x \in P(\alpha_b,\theta^c,\gamma^d)$, we obtain

$$\frac{b}{k} \le \frac{1}{k+1}\min_{\frac{1}{k}\le t<k} x(t) < \frac{x(t)}{1+t} \le \frac{(k+1)^2b}{km},\ \ t\in[\frac{1}{k},k],$$

and $0 \le D_qx(t) \le d$, $t \in [0,+\infty)$. In view of assumption (H_4) together with Lemma 2.14, we obtain

$$\begin{aligned}\alpha(Tx) &\ge \frac{1}{k+1}\theta(Tx) = \frac{1}{k+1}\sup_{0\le t<\infty}\frac{(Tx)(t)}{1+t}\\ &= \frac{1}{k+1}\sup_{0\le t<\infty}\frac{1}{1+t}\Big[\int_0^{+\infty} g(s)\varphi_p^{-1}\Big(\int_s^{+\infty}\phi(\tau)f(\tau,x(\tau),D_qx(\tau))d_q\tau\Big)d_qs\\ &\qquad + \int_0^t \varphi_p^{-1}\Big(\int_s^{+\infty}\phi(\tau)f(\tau,x(\tau),D_qx(\tau))d_q\tau\Big)d_qs\Big]\\ &\ge \frac{1}{(k+1)^2}\Big[\int_{\frac{1}{k}}^{k} g(s)\varphi_p^{-1}\Big(\int_s^k\phi(\tau)f(\tau,x(\tau),D_qx(\tau))d_q\tau\Big)d_qs\\ &\qquad + \int_{\frac{1}{k}}^{k}\varphi_p^{-1}\Big(\int_s^k\phi(\tau)f(\tau,x(\tau),D_qx(\tau))d_q\tau\Big)d_qs\Big]\end{aligned}$$

$$> \frac{b}{N}\frac{1}{(k+1)^2}\int_{\frac{1}{k}}^{k}(g(s)+1)\varphi_p^{-1}\Big(\int_s^k \phi(\tau)d_q\tau\Big)d_qs = b.$$

Hence, $\alpha(Tx) > b$ for $x \in P(\alpha_b, \theta^c, \gamma^d)$.

Next, we will verify that the condition (2) of Theorem 2.7 is satisfied. Let $x \in P(\alpha_b, \gamma^d)$ with $\theta(Tx) > c$, it follows from Lemma 2.14 that

$$\alpha(Tx) \ge \frac{1}{k+1}\theta(Tx) > \frac{1}{k+1}c = \frac{1}{k+1}\frac{(k+1)^2 b}{km} = \frac{(k+1)b}{km} > b,$$

Thus, $\alpha(Tx) > b$ for all $x \in P(\alpha_b, \gamma^d)$ with $\theta(Tx) > c$. Finally, we show that condition (3) of Theorem 2.7 is satisfied. It is clear that $0 \in R(\psi_a, \gamma^d)$. Suppose that $x \in R(\psi_a, \gamma^d)$ with $\psi(x) = a$, then by condition (H_5) and Lemma 2.13, we obtain

$$\begin{aligned}\psi(Tx) &\le M\gamma(Tx) = MD_q(Tx)(0)\\ &= M\varphi_p^{-1}\Big(\int_0^{+\infty}\phi(s)f(s,x(s),D_qx(s))d_qs\Big)\\ &\le M\frac{a}{MC}\varphi_p^{-1}\Big(\int_0^{+\infty}\phi(s)d_qs\Big) = a.\end{aligned}$$

Therefore, T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma^d)}$ such that $\psi(x_1) < a, \psi(x_2) > a$ with $\alpha(x_2) < b, \alpha(x_3) > b$. In addition, condition (H_2) guarantees that those fixed points are positive. So (1.1) has at least three positive solutions x_1, x_2, x_3 and the proof is completed.

4. Example

Consider the q-difference equation IBVP

$$\begin{cases} D_q(|D_qx|D_qx) + e_q^{-t}f(t,x(t),\ D_qx(t)) = 0, \quad 0 < t < +\infty,\\ x(0) = \displaystyle\int_0^{+\infty} e_q^{-2s}D_qx(s)d_qs, \quad \lim_{t\to+\infty} D_qx(t) = 0, \end{cases} \tag{4.1}$$

where $e_q^x = \sum_{j=0}^{\infty}\frac{x^j}{[j]!}$, and

$$f(t,u,v) = \begin{cases} \frac{|\sin t|}{100} + 10^4(\frac{u}{1+t})^{10} + \frac{1}{100}(\frac{v}{400}), & u \le 1,\\ \frac{|\sin t|}{100} + 10^4(\frac{1}{1+t})^{10} + \frac{1}{100}(\frac{v}{400}), & u \ge 1. \end{cases}$$

Set $\phi(t) = e_q^{-t}$ and it is easy to verify that (H_1) and (H_2) hold. Choose $k = 3$, $a = \frac{1}{3}$, $b = 2$, $d = 400$. By simple calculations, we can obtain $M = 1$, $m = \frac{1}{2}$, $C = 1$,

$$N = \frac{1}{16}\int_{\frac{1}{3}}^{3}(e_q^{-2s}+1)\sqrt{e_q^{-s}-e^{-3}}d_qs \ge \frac{1}{16}\int_{\frac{1}{3}}^{3}\sqrt{e_q^{-s}-e^{-3}}d_qs > -\frac{1}{8}(e_q^{-\frac{3}{2}} - e_q^{-\frac{1}{6}}).$$

So the nonlinear term f satisfies

(1) $f(t,(1+t)u,v) \le 0.01 + 10^4 + 0.01 < 1.6\times10^5 = \varphi_3(d/C)$, for $(t,u,v) \in [0,+\infty)\times[0,400]\times[0,400]$;

(2) $f(t,(1+t)u,v) \geq 10^4 > \frac{256}{(e_q^{-\frac{3}{2}}-e_q^{-\frac{1}{6}})^2} = \varphi_3(b/N)$, for $(t,u,v) \in [\frac{1}{3},3] \times [\frac{2}{3},\frac{64}{3}] \times [0,400]$;
(3) $f(t,(1+t)u,v) \leq 0.01+10^4 \times (\frac{1}{4})^{10}+0.01 < \frac{1}{9} = \varphi_3(a/MC)$, for $(t,u,v) \in [0,+\infty) \times [0,\frac{1}{3}] \times [0,400]$.
Therefore, the conditions in Theorem 3.1 are all satisfied. So (4.1) has at least three positive solutions x_1, x_2, x_3 such that

$$\sup_{0\leq t<\infty} D_q x_i(t) \leq 400,\ i=1,2,3;$$
$$\sup_{0\leq t<\infty} \frac{|x_1(t)|}{1+t} < \frac{1}{3},\ \ \frac{1}{3} < \sup_{0\leq t<\infty} \frac{|x_2(t)|}{1+t} < \frac{64}{3},\ \min_{\frac{1}{k}\leq t<k} x_2(t) < \frac{8}{3};$$
$$\sup_{0\leq t<\infty} \frac{|x_3(t)|}{1+t} < 400,\ \ \min_{\frac{1}{k}\leq t<k} x_3(t) > \frac{8}{3}.$$

5. Conclusions

This research obtains the existence results of triple positive solutions for a class of q-difference equations boundary value problems with integral boundary conditions with p-Laplacian on infinite intervals by applying the Avery-Peterson fixed point theorem in a cone, which enrich the theories for q-difference equations on infinite intervals, and provide the theoretical guarantee for the application of q-difference equations in every field, such as aerodynamics, electrodynamics of complex medium, capacitor theory, electrical circuits, control theory, and so on. In the future, we will use bifurcation theory, critical point theory, variational method, and other methods to continue our works in this area.

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Conflict of interest

The authors declare that they have no competing interests.

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