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Research article

Existence results for a Kirchhoff-type equation involving fractional $p(x)$-Laplacian

Jinguo Zhang*, **Dengyun Yang** and **Yadong Wu**

School of Mathematics and Statistics, Jiangxi Normal University, Nanchang 330022, China

* **Correspondence:** Email: jgzhang@jxnu.edu.cn.

Abstract: The purpose of this paper is to investigate the existence of weak solutions for a Kirchhoff-type problem driven by a non-local integro-differential operator as follows:

$$\begin{cases} M\left(\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy\right)(-\Delta_{p(x)})^s u(x) = f(x,u) & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases}$$

where Ω is a smooth bounded open set in $\mathbb{R}^N$, $s \in (0,1)$ and p is a positive continuous function with $sp(x,y) < N$, M and f are two continuous functions, $(-\Delta_{p(x)})^s$ is the fractional $p(x)$-Laplacian operator. Using variational methods combined with the theory of the generalized Lebesgue Sobolev space, we prove the existence of nontrivial solution for the problem in an appropriate space of functions.

Keywords: fractional $p(x)$-Laplace operator; Kirchhoff-type equation; variational methods
Mathematics Subject Classification: 35R11, 35J35, 35S15

1. Introduction

In this paper we deal with the existence solution to the following Kirchhoff-type problem involving the fractional $p(x)$-Laplace operator:

$$\begin{cases} M\left(\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+sp(x,y)}}dxdy\right)(-\Delta_{p(x)})^s u(x) = f(x,u) & \text{in } \Omega,\\ u=0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $s \in (0,1)$ and $p : \bar{\Omega}\times\bar{\Omega} \to (1,+\infty)$ is a continuous function with $s\,p(x,y) < N$ for any $(x,y) \in \bar{\Omega}\times\bar{\Omega}$, the continuous functions $f : \Omega\times\mathbb{R} \to \mathbb{R}$ and $M : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the following conditions:

(M_0) There exists a constant $a \in [0, 1)$ such that

$$(1 - a)t^{\alpha(x)-1} \leq M(t) \leq (1 + a)t^{\alpha(x)-1}$$

for all $t \in \mathbb{R}^+$, where $\alpha(x) \geq 1$ for all $x \in \Omega$.

(f_1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function and satisfies:

$$|f(x, t)| \leq c(1 + |t|^{\beta(x)-1}), \quad \forall\, (x, t) \in \Omega \times \mathbb{R},$$

where $\beta \in C(\bar{\Omega})$ such that $1 < \beta(x) < p_s^*(x)$ for all $x \in \Omega$, and

$$p_s^*(x) := \frac{Np(x, x)}{N - sp(x, x)}$$

is the critical Sobolev exponent.

(f_2) $f(x, t) = o(|t|^{p_+-1})$ as $t \to 0$ uniformly for $x \in \Omega$.

(f_3) There exist $T > 0$ and $\theta > \frac{1+a}{1-a} \frac{\alpha_+(p_+)^{\alpha_+}}{(p_-)^{\alpha_- -1}}$ such that

$$0 < \theta F(x, t) \leq f(x, t)t, \quad \forall |t| > T, \text{ a.e. } x \in \Omega,$$

where $\alpha_+ = \sup_{\bar{\Omega}} \alpha(x)$, $\alpha_- = \inf_{\bar{\Omega}} \alpha(x)$ and a is a constant given in (M_0).

The nonlocal operator $(-\Delta_{p(x)})^s$ is defined as

$$(-\Delta_{p(x)})^s \varphi(x) = \text{P.V.} \int_{\mathbb{R}^N} \frac{|\varphi(x) - \varphi(y)|^{p(x,y)-2}(\varphi(x) - \varphi(y))}{|x - y|^{N+sp(x,y)}} dy, \ \varphi \in C_0^\infty(\mathbb{R}^N),$$

for all $x \in \mathbb{R}^N$, where P.V. stands for Cauchy principle value and for brevity, p denotes the extensions of the aforementioned continuous function p, in the whole $\mathbb{R}^N \times \mathbb{R}^N$.

Note that the operator $(-\Delta_{p(x)})^s$ is the fractional version of the well known $p(x)$-Laplacian operator $\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2} u)$, which was first introduced by Kaufmann, Rossi and Vidal in [19]. Some results involving fractional $p(x)$-Laplace operator and associated fractional Sobolev spaces with variable exponents are studied in [3,4,18,22,28]. One typical feature of problem (1.1) is the nonlocality, in the sense that the value of $(-\Delta_{p(x)})^s u(x)$ at any point $x \in \Omega$ depends not only on the value of u on Ω, but actually on the entire space $\mathbb{R}^N$. Therefore, the Dirichlet datum is given in $\mathbb{R}^N \backslash \Omega$, which is different from the classical case of the $p(x)$-Laplacian, and not simply on $\partial\Omega$.

When $M \equiv 1$, the (1.1) becomes the fractional $p(x)$-Laplacian equation

$$\begin{cases} (-\Delta_{p(x)})^s u(x) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.2}$$

which can be seen as the fractional form of the following classical stationary elliptic equation

$$-\Delta_{p(x)} u(x) = f(x, u) \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{1.3}$$

In recent years, a great interest has been devoted to Kirchhoff type equation

$$-\Big(a + b \int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(x, u) \text{ in } \Omega, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth domain, $a > 0$, $b \geq 0$, M is a continuous function and u satisfies some boundary conditions. The analogous and classical counterpart of problem (1.4) models several interesting phenomena studied in mathematical physics, even in the one-dimensional case. Detail, Kirchhoff established a model given by the equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = g(x, u), \tag{1.5}$$

where ρ, ρ_0, h, E and L are constants, which extends the classical D'Alambert wave equation by considering the effects of the changes in the length of the strings during the vibrations. In particular, the equation (1.5) contains a nonlocal coefficient $\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$, which depends on the average $\frac{1}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx$ of the kinetic energy $\left| \frac{\partial u}{\partial x} \right|^2$ on $[0, L]$ and therefore the equation is no longer pointwise identity. It is worth pointing out that equation (1.5) received much attention only after Lions [21] proposed an abstract framework to the problem. For example, the Kirchhoff type equations has already been extended to the case involving the p-Laplacian (see [6, 7]), $p(x)$-Laplacian (see [8–12]) and fractional Laplacian (see [14, 20, 25–27]).

Recently, attention has been paid to the study of Kirchhoff-type equation involving the $p(x)$-Laplacian operator, fox example [8–12]. In [10], using variational methods, the authors have investigated nonlocal $p(x)$-Laplacian Dirichlet problem

$$- M\left(\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = f(x, u) \text{ in } \Omega, \tag{1.6}$$

and have showed the existence of a sequence of positive, homoclinic weak solutions of (1.6) under some suitable conditions on f. Dai and Wei [12] studied the problem (1.6) by applying a general variational principle due to Ricceri [23], and prove the existence of infinitely many non-negative solutions of this problem. In [9], by a direct variational approach, the authors establish conditions ensuring the existence and multiplicity of solutions for the $p(x)$-Kirchhoff problem. So the natural question that arises is to see which result we will obtain, if we replace the $p(x)$-Laplacian operator by its fractional version.

Motivated by the papers mentioned above on $p(x)$-Kirchhoff problem and recent results on fractional Sobolev space with variable exponents in [1, 4, 19], we study the existence of weak solutions for problem (1.1) via variational methods. To the best of the authors' knowledge, the present paper seems to be the first to study the existence of weak solutions to the Kirchhoff-type problem with fractional $p(x)$-Laplacian operator. In order to state the main results, we introduce some basic definitions of fractional Sobolev space with variable exponents.

For a smooth bounded domain $\Omega \subset \mathbb{R}^N$, we consider the continuous function $p : \bar{\Omega} \times \bar{\Omega} \to (1, +\infty)$ satisfies the following conditions:

(P_1) $1 < p_- \leq p(x, y) \leq p_+ < +\infty$, where

$$p_- := \min_{(x,y) \in \bar{\Omega} \times \bar{\Omega}} p(x, y) \text{ and } p_+ := \max_{(x,y) \in \bar{\Omega} \times \bar{\Omega}} p(x, y);$$

(P_2) p is symmetric, i.e., $p(x, y) = p(y, x)$ for all $(x, y) \in \bar{\Omega} \times \bar{\Omega}$.

Let us introduce our result. We denote by $X_0 := \{u \in X : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\}$ the Sobolev space defined as the completion of $C_0^\infty(\Omega)$ respect to the norm

$$\|u\|_{X_0} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}}dxdy \leq 1\right\}.$$

For more details on the definitions and properties of spaces X_0 and X, see Section 2.

We are interested in weak solutions of problem (1.1), i.e. $u \in X_0$ such that

$$M\left(\sigma_{s,p(x,y)}(u)\right)\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+sp(x,y)}}dxdy = \int_\Omega f(x,u)\varphi dx, \quad \forall\varphi \in X_0$$

where

$$\sigma_{s,p(x,y)}(u) = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)\,|x - y|^{N+sp(x,y)}}\, dxdy.$$

In order to formulate the variational approach of problem (1.1), we introduce the functional $\Phi : X_0 \to \mathbb{R}$ defined by

$$\begin{aligned}\Phi(u) &= \widehat{M}\left(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p(x,y)}}{p(x,y)|x - y|^{N+sp(x,y)}}dxdy\right) - \int_\Omega F(x,u)dx\\ &= \widehat{M}\left(\sigma_{s,p(x,y)}(u)\right) - \int_\Omega F(x,u)dx,\end{aligned}$$

where $\widehat{M}(t) = \int_0^t M(\tau)d\tau$ and $F(x,u) = \int_0^t f(x,\tau)d\tau$. It is not difficult to prove that the functional Φ is well-defined and $\Phi \in C^1(X_0,\mathbb{R})$. Moreover, for all u, $v \in X_0$, its Fréchet derivative is given by

$$\langle\Phi'(u), v\rangle = M\left(\sigma_{s,p(x,y)}(u)\right)\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp(x,y)}}dxdy - \int_\Omega f(x,u)vdx.$$

Thus, the weak solutions of (1.1) coincide with the critical points of Φ.

The main result of this paper is as follows.

Theorem 1.1. *Assume that* (M_0), (P_1)-(P_2) *and* (f_1)-(f_3) *hold. Then the problem* (1.1) *has at least one nontrivial solution.*

Remark 1.2. *To our best knowledge, Theorem 1.1 is new for the Kirchhoff type problem involving fractional* $p(x)$*-Laplacian operator. We may using the mountain pass theorem to prove our main result.*

This paper is organized as follows. In Section 2, we given some definitions and fundamental properties to the Lebesgue spaces with variable exponents and fractional Sobolev space with variable exponents. Finally, Section 3 deal with the proof of Theorem 1.1.

2. Fractional Sobolev spaces with variable exponent

In this section, we introduce fractional Sobolev spaces with variable exponents and establish the preliminary lemmas and embeddings associated with these spaces. Recalling the definition of the Lebesgue space with variable exponents in [5, 13, 15–17, 22, 24, 28] as follows.

Set

$$C_+(\overline{\Omega}) = \{q \in C(\overline{\Omega}) : q(x) > 1, \quad \forall\, x \in \overline{\Omega}\}.$$

For all $q \in C_+(\overline{\Omega})$, let

$$q_+ := \sup_{x\in\overline{\Omega}} q(x) \quad \text{and} \quad q_- := \inf_{x\in\overline{\Omega}} q(x)$$

such that

$$1 < q_- \le q(x) \le q_+ < +\infty. \tag{2.1}$$

For any $q \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space as

$$L^{q(x)}(\Omega) = \left\{ u : u \text{ is measurable real-valued function, } \int_\Omega |u(x)|^{q(x)} dx < +\infty \right\}.$$

The Luxemburg norm on this space is given by the formula

$$\|u\|_{L^{q(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{q(x)} dx \le 1 \right\}.$$

It is well know that $(L^{q(x)}(\Omega), \|\cdot\|_{L^{q(x)}})$ is a separable, uniformly convex Banach space. Let $\hat{q}(x)$ be the conjugate exponent of $q(x)$, that is, $\frac{1}{q(x)} + \frac{1}{\hat{q}(x)} = 1$. Then, we have the following Hölder-type inequality.

Lemma 2.1. *If $u \in L^{q(x)}(\Omega)$ and $v \in L^{\hat{q}(x)}(\Omega)$, we have*

$$\left| \int_\Omega uv \, dx \right| \le \left(\frac{1}{q_-} + \frac{1}{\hat{q}_-} \right) \|u\|_{L^{q(x)}(\Omega)} \, \|v\|_{L^{\hat{q}(x)}(\Omega)} \le 2 \|u\|_{L^{q(x)}(\Omega)} \, \|v\|_{L^{\hat{q}(x)}(\Omega)}. \tag{2.2}$$

A very important role in manipulating the generalized Lebesgue spaces with variable exponent is played by the modular of the $L^{q(x)}(\Omega)$ space, which is the mapping $\rho_{q(x)} : L^{q(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{q(x)}(u) = \int_\Omega |u(x)|^{q(x)} dx.$$

From [16], the following relations hold true.

Lemma 2.2. (*[16, Theorem 1.3]*) *Let $u \in L^{q(x)}(\Omega)$, then we have*

(1) $\|u\|_{L^{q(x)}(\Omega)} > 1 \ (= 1; < 1)$ *if and only if* $\rho_{q(x)}(u) > 1 \ (= 1; < 1,$ *respectively);*
(2) if $\|u\|_{L^{q(x)}(\Omega)} > 1$, *then* $\|u\|^{q_-}_{L^{q(x)}(\Omega)} \le \rho_{q(x)}(u) \le \|u\|^{q_+}_{L^{q(x)}(\Omega)}$;
(3) if $\|u\|_{L^{q(x)}(\Omega)} < 1$, *then* $\|u\|^{q_+}_{L^{q(x)}(\Omega)} \le \rho_{q(x)}(u) \le \|u\|^{q_-}_{L^{q(x)}(\Omega)}$.

Lemma 2.3. (*[16, Theorem 1.4]*) *If u, $u_n \in L^{q(x)}(\Omega)$, then the following statements are equivalent each other:*

(1) $\lim_{n\to\infty} \|u_n - u\|_{L^{q(x)}(\Omega)} = 0$;
(2) $\lim_{n\to\infty} \rho_{q(x)}(u_n - u) = 0$;
(3) u_n *converges to u in Ω in measure and* $\lim_{n\to\infty} \rho_{q(x)}(u_n) = \rho_{q(x)}(u)$.

We set

$$\bar{p}(x) = p(x, x), \quad \forall \, x \in \bar{\Omega}.$$

The fractional variable exponent Sobolev space $W^{s,p(x,y)}(\Omega)$ is defined by

$$W := W^{s,p(x,y)}(\Omega) = \left\{ u \in L^{\bar{p}(x)}(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}} dxdy < +\infty, \text{for some} \lambda > 0 \right\}$$

with the norm

$$\|u\|_W = [u]_W + \|u\|_{L^{\bar{p}(x)}(\Omega)},$$

where

$$[u]_W = \inf \left\{ \lambda > 0 : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}} dxdy \leq 1 \right\}$$

be the corresponding variable exponent Gagliardo seminorm.

From [29] we have the following embedding theorem.

Theorem 2.4. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $s \in (0, 1)$. Assume $p : \overline{\Omega} \times \overline{\Omega} \to (1, +\infty)$ be a continuous variable exponent with $N > sp(x, y)$ for all $(x, y) \in \overline{\Omega} \times \overline{\Omega}$ and (P_1)-(P_2) hold. Let $r : \overline{\Omega} \to (1, +\infty)$ be a continuous variable exponent such that*

$$1 < r_- \leq r(x) < p_s^*(x) := \frac{N\bar{p}(x)}{N - s\bar{p}(x)}, \quad \forall x \in \overline{\Omega}.$$

Then there exists a constant $C = C(N, s, p, r) > 0$ such that

$$\|u\|_{L^{r(x)}(\Omega)} \leq C\|u\|_W, \quad \forall u \in W.$$

Thus, the space W is continuously embedded in $L^{r(x)}(\Omega)$ with $1 < r(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$. Moreover, this embedding is compact.

For studying nonlocal elliptic problems involving the fractional operator $(-\Delta_{p(x)})^s$ with Dirichlet boundary data $u = 0$ in $\mathbb{R}^N \backslash \Omega$ via variational methods, we need to work in a suitable Sobolev space with variable exponents. Set $Q = \mathbb{R}^N \times \mathbb{R}^N \backslash (\Omega^c \times \Omega^c)$ and define

$$X := X^{s,p(x,y)}(\Omega) = \left\{ \begin{array}{l} u : \mathbb{R}^N \to \mathbb{R}, \quad u|_\Omega \in L^{\bar{p}(x)}(\Omega), \\ \displaystyle\int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}} dxdy < +\infty, \text{for some} \lambda > 0 \end{array} \right\}.$$

The space X is endowed with the following norm

$$\|u\|_X = [u]_X + \|u\|_{L^{\bar{p}(x)}(\Omega)},$$

where

$$[u]_X = \inf \left\{ \lambda > 0 : \int_Q \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}} dxdy \leq 1 \right\}$$

is the Gagliardo seminorm with variable exponent. Similar to the space $(W, \|\cdot\|_W)$, we have that $(X, \|\cdot\|_X)$ is a separable reflexive Banach space.

Remark 2.5. *We notice that the norms $\| \cdot \|_W$ and $\| \cdot \|_X$ are not the same, because $\Omega \times \Omega$ is strictly contained in Q, this make the fractional variable Sobolev space W not sufficient for studying the problem* (1.1).

Since p is continuous on $\bar{\Omega}\times\bar{\Omega}$, and $\bar{p}$, r are continuous on $\bar{\Omega}$, by using Tietze extension theorem, we can extend p to $\mathbb{R}^N\times\mathbb{R}^N$ and $\bar{p}$, r to $\mathbb{R}^N$ continuously as above such that $sp(x,y) < N$ for all $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$ and $\bar{p}(x) = p(x,x)$, $r(x) < p_s^*(x)$ for all $x\in\mathbb{R}^N$.

Now, we define the following linear subspace of X as

$$X_0 := X_0^{s,p(x,y)}(\Omega) = \left\{u\in X : u(x) = 0 \text{ a.e. in } \mathbb{R}^N\backslash\Omega\right\}$$

with the norm

$$\begin{aligned}\|u\|_{X_0} &= \inf\left\{\lambda>0 : \int_Q \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}dxdy \le 1\right\}\\ &= \inf\left\{\lambda>0 : \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x-y|^{N+sp(x,y)}}dxdy \le 1\right\}.\end{aligned}$$

For any $u\in X_0$, we define the following modular function $\rho_{X_0} : X_0 \to \mathbb{R}$:

$$\rho_{X_0}(u) = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy.$$

Similarly to the discussion of space W in [29], we could get the following results.

Lemma 2.6. *Let $p : \overline{\Omega}\times\overline{\Omega} \to (1,+\infty)$ be a continuous variable exponent and $s\in(0,1)$. For all $u\in X_0$, we have the following results:*

(i) If $\|u\|_{X_0} > 1$, then $\|u\|_{X_0}^{p_-} \le \rho_{X_0}(u) \le \|u\|_{X_0}^{p_+}$;
(ii) If $\|u\|_{X_0} < 1$, then $\|u\|_{X_0}^{p_+} \le \rho_{X_0}(u) \le \|u\|_{X_0}^{p_-}$.

Proof. We first prove the pair of inequalities. Indeed, it is easy to see that, for all $\lambda\in(0,1)$, we get

$$\begin{aligned}\lambda^{p_+}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy &\le \int_{\mathbb{R}^{2N}} \frac{|\lambda(u(x)-u(y))|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\\ &\le \lambda^{p_-}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy,\end{aligned}$$

that is,

$$\lambda^{p_+}\rho_{X_0}(u) \le \rho_{X_0}(\lambda u) \le \lambda^{p_-}\rho_{X_0}(u). \tag{2.3}$$

Now, if $\|u\|_{X_0} > 1$, we have $0 < \frac{1}{\|u\|_{X_0}} < 1$ and $\rho_{X_0}(\frac{1}{\|u\|_{X_0}}u) = 1$. Then, taking $\lambda = \frac{1}{\|u\|_{X_0}}$ in (2.3), we get

$$\frac{\rho_{X_0}(u)}{\|u\|_{X_0}^{p_+}} \le 1 \le \frac{\rho_{X_0}(u)}{\|u\|_{X_0}^{p_-}}.$$

This completes the proof of Lemma 2.6 (i). The proof of the second is essentially the same □

Lemma 2.7. *Let u, $u_n\in X_0$. Then the following statements are equivalent:*

(1) $\lim\limits_{n\to\infty}\|u_n-u\|_{X_0} = 0$;
(2) $\lim\limits_{n\to\infty}\rho_{X_0}(u_n-u) = 0$.

Proof. Similarly to the discussion of the norm in variable exponent space, we could get the above results. Here we omit the proof of Lemma 2.7. □

Theorem 2.8. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$. Assume $p : \overline{\Omega} \times \overline{\Omega} \to (1, +\infty)$ be a continuous variable exponent with $N > sp(x,y)$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$ and (P_1)-(P_2) hold. Then, for any continuous variable exponent $r : \overline{\Omega} \to (1, +\infty)$ such that $1 < r_- \leq r(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$, there exists a constant $C = C(N, s, p, r) > 0$ such that*

$$\|u\|_{L^{r(x)}(\Omega)} \leq C\|u\|_{X_0}, \quad \forall u \in X_0. \tag{2.4}$$

Moreover, this embedding is compact.

Proof. First, we claim that there exists a constant $C_0 > 0$ such that

$$C_0\|u\|_{L^{q(x)}(\Omega)} \leq \|u\|_{X_0}, \quad \forall u \in X_0, \tag{2.5}$$

where $q \in C_+(\bar{\Omega})$ satisfies $\bar{p}(x) \leq q(x) < p_s^*(x)$ for all $x \in \bar{\Omega}$.

Let $\mathcal{A} = \{u \in X_0 : \|u\|_{L^{q(x)}(\Omega)} = 1\}$. Taking a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\lim_{n\to\infty} \|u_n\|_{X_0} = \inf_{u\in\mathcal{A}} \|u\|_{X_0}$. So, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^{q(x)}(\Omega)$ and X_0. Hence, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in W. Up to a subsequence, there exist a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, still denote by $\{u_n\}_{n\in\mathbb{N}}$, and $u_0 \in W$ such that $u_n \rightharpoonup u_0$ weakly in W. By Theorem 2.4, we get that $u_n \to u_0$ strongly in $L^{q(x)}(\Omega)$ and $\|u_0\|_{L^{q(x)}(\Omega)} = 1$.

Now, we extend u_0 to $\mathbb{R}^N$ be setting $u_0 = 0$ in $\mathbb{R}^N \backslash \Omega$. This implies $u_n(x) \to u_0$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Hence, by Fatou's Lemma, we have

$$\int_{\mathbb{R}^{2N}} \frac{|u_0(x) - u_0(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy \leq \liminf_{n\to\infty} \int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} dxdy,$$

which implies that $\|u_0\|_{X_0} \leq \liminf_{n\to\infty} \|u_n\|_{X_0}$ and $u_0 \in X_0$. This and $\|u_0\|_{L^{q(x)}(\Omega)} = 1$ given that $u_0 \in \mathcal{A}$. Therefore, we obtain $\|u_0\|_{X_0} = \inf_{u\in\mathcal{A}} \|u\|_{X_0} := C_0$ and this proves our claim.

From (2.5), it follows that

$$\|u\|_W \leq \|u\|_{L^{q(x)}(\Omega)} + [u]_W \leq \|u\|_{L^{q(x)}(\Omega)} + \|u\|_{X_0} \leq (1 + \frac{1}{C_0})\|u\|_{X_0}, \tag{2.6}$$

which implies that X_0 is continuously embedded in W. From Theorem 2.4, there exists a constant $C = C(N, s, p, r) > 0$ such that

$$\|u\|_{L^{r(x)}(\Omega)} \leq C\|u\|_{X_0}, \quad \forall u \in X_0.$$

To prove the embedding given in (2.4) is compact, let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in X_0. So, (2.6) implies that $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in W, and from Theorem 2.4 we infer that there exists $u \in L^{r(x)}(\Omega)$ such that $u_n \to u$ strongly in $L^{r(x)}(\Omega)$. This completes the proof of Theorem 2.8. □

Remark 2.9. *(i) $(X_0, \|\cdot\|_{X_0})$ is a separable, reflexive and uniformly convex Banach space.*

(ii) Theorem 2.4 remains true if we replace W by X_0.

(iii) Since $1 < p_- \leq \overline{p}(x) < p_s^*(x)$ for all $x \in \bar{\Omega}$, Theorem 2.4 implies that the norm $\|\cdot\|_{X_0}$ and $\|\cdot\|_X$ are equivalent on X_0.

3. Proof of the main result

In this section, we give the proof of Theorem 1.1 by applying mountain pass theorem.

Theorem 3.1 (Mountain Pass Theorem [2])**.** *Let E be a Banach space and $I \in C^1(E)$. Assume that*

(i) $I(0) = 0$*;*
(ii) There exist $r, \rho > 0$ *such that* $I(u) \geq r$ *for all* $u \in E$ *with* $\|u\|_E = \rho$*;*
(iii) There exists $u_0 \in E$ *such that* $\lim_{t\to\infty} I(tu_0) < 0$.

Take $t_0 > 0$ *such that* $\|t_0u_0\|_E > \rho$ *and* $I(t_0u_0) < 0$. *Set*

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0, 1], E) : \gamma(0) = 0$ *and* $\gamma(1) = t_0u_0\}$. *Then there exists a Palais-Smale sequence at level* $c \in \mathbb{R}$ *for* I*, that is, there exists a sequence* $\{u_n\}_{n\in\mathbb{N}} \subset E$ *such that*

$$\lim_{n\to\infty} I(u_n) = c, \quad \lim_{n\to\infty} I'(u_n) = 0 \ \textit{ strongly in } E^*.$$

Let the functional $L : X_0 \to X_0^*$ be defined by

$$\langle L(u), v\rangle = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp(x,y)}} dxdy,$$

where X_0^* is the dual space of X_0. Then,

$$\langle \sigma'_{s,p(x,y)}(u), v\rangle = \langle L(u), v\rangle = \langle(-\Delta_{p(x)})^s u, v\rangle, \quad \forall u, v \in X_0.$$

For operator L, from Lemma 4.2 of [4] we have

Lemma 3.2. *Assume that hypothesis* (P_1) *and* (P_2) *are satisfied,* $s \in (0, 1)$ *and* $N > s\,p(x, y)$ *for all* $(x, y) \in \overline{\Omega} \times \overline{\Omega}$. *Then, the the following assertions hold:*

(1) L is a bounded and strictly monotone operator;
(2) L is a homeomorphism;
(3) L is a mapping of type (S_+)*, namely, if* $u_n \rightharpoonup u$ *weakly in* X_0 *and* $\limsup_{n\to\infty}\langle L(u_n) - L(u), u_n - u\rangle \leq 0$, *then* $u_n \to u$ *strongly in* X_0.

The following result shows that the functional Φ satisfies the geometrical condition of the mountain pass theorem.

Lemma 3.3. *Assume that* (M_0), (P_1)-(P_2) *and* (f_1)-(f_3) *hold. Then, the following statements hold:*

(i) There exist numbers $\rho > 0$ *and* $r > 0$ *such that* $\Phi(u) \geq r$ *for all* $u \in X_0$ *with* $\|u\|_{X_0} = \rho$*;*
(ii) There exists $u_0 \in C_0^\infty(\Omega)$ *with* $\|u_0\|_{X_0} > \rho$ *such that* $\Phi(u_0) < 0$.

Proof. (i) By the assumptions (f_1) and (f_2), we have

$$F(x, t) \leq \varepsilon|t|^{p_+} + c_\varepsilon|t|^{\beta(x)}, \quad \forall\,(x, t) \in \Omega \times \mathbb{R}. \tag{3.1}$$

Since the embeddings $X_0 \hookrightarrow L^{\beta(x)}(\Omega)$ and $X_0 \hookrightarrow L^{p_+}(\Omega)$ are continuous, there exist some positive constants c_1, c_2 such that

$$\|u\|_{L^{\beta(x)}(\Omega)} \leq c_1\|u\|_{X_0} \quad \text{and} \quad \|u\|_{L^{p_+}(\Omega)} \leq c_2\|u\|_{X_0} \tag{3.2}$$

for all $u \in X_0$. In view of (M_0), (3.1) and (3.2), for all $\|u\|_{X_0} < 1$, we have

$$\begin{aligned} \Phi(u) &\geq \frac{1-a}{\alpha_+ (p_+)^{\alpha_+}}\|u\|_{X_0}^{\alpha_+ p_+} - \int_\Omega F(x,u)dx \\ &\geq \frac{1-a}{\alpha_+ (p_+)^{\alpha_+}}\|u\|_{X_0}^{\alpha_+ p_+} - \varepsilon\int_\Omega |u|^{p_+}dx - c_\varepsilon \int_\Omega |u|^{\beta(x)}dx \\ &\geq \frac{1-a}{\alpha_+ (p_+)^{\alpha_+}}\|u\|_{X_0}^{\alpha_+ p_+} - \varepsilon c_1\|u\|_{X_0}^{p_+} - c_\varepsilon c_2\|u\|_{X_0}^{\beta_-} \\ &\geq \left(\frac{1-a}{\alpha_+ (p_+)^{\alpha_+}}\|u\|_{X_0}^{(\alpha_+-1)p_+} - \varepsilon c_1 - c_\varepsilon\|u\|_{X_0}^{\beta_- - p_+}\right)\|u\|_{X_0}^{p_+}. \end{aligned} \tag{3.3}$$

Define the function $g : [0,\infty) \to \mathbb{R}$ by

$$\begin{aligned} g(t) &= \frac{1-a}{\alpha_+ (p_+)^{\alpha_+}} t^{(\alpha_+-1)p_+} - c_\varepsilon t^{\beta_- - p_+} \\ &= \left(\frac{1-a}{\alpha_+ (p_+)^{\alpha_+}} t^{\alpha_+ p_+} - c_\varepsilon t^{\beta_-}\right) t^{-p_+}. \end{aligned}$$

Since $\beta_- > p_+\alpha_+$, it is clear that there exists a $t_0 > 0$ small enough such that $\max_{t\geq 0} g(t) = g(t_0) > 0$. Hence, for a fixed $\varepsilon \in (0, \frac{g(t_0)}{c_1})$ small enough, there exist number $\rho(= t_0) \in (0,1)$ and $r > 0$ such that $\Phi(u) \geq r > 0$ for all $u \in X_0$ with $\|u\|_{X_0} = \rho$.

(ii) From (f_3), it follows that

$$F(x,t) \geq C|t|^\theta, \quad x \in \Omega, \ \ |t| > T. \tag{3.4}$$

For $w \in X_0\backslash\{0\}$ and $t > 1$, we have

$$\begin{aligned} \Phi(tw) &= \widehat{M}\left(\sigma_{s,p(x,y)}(tw)\right) - \int_\Omega F(x,tw)dx \\ &\leq \frac{1+a}{\alpha_- (p_-)^{\alpha_-}} t^{\alpha_+ p_+}\|w\|_{X_0}^{\alpha_+} - Ct^\theta \int_\Omega |w|^\theta dx \\ &\to -\infty, \quad \text{as } \ t \to \infty, \end{aligned} \tag{3.5}$$

due to $\theta > \alpha_+ p_+$. Then, for $t > 1$ large enough, we can take $u_0 = tw$ with $\|u_0\|_{X_0} > \rho$ and $\Phi(u_0) < 0$. This completes the proof of (ii). $\square$

Define

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \Phi(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1], X_0) : \gamma(0) = 0 \text{ and } \gamma(1) = u_0\}$. Then, combining Lemma 3.3 and the mountain pass theorem (see Theorem 3.1), we deduce that there exists a Palais-Smale sequence $\{u_n\}_{n\in\mathbb{N}} \subset X_0$ such that

$$\Phi(u_n) \to c, \quad \Phi'(u_n) \to 0 \ \text{ as } \ n \to \infty.$$

Next, we need to prove that Φ satisfies the $(PS)_c$ condition in X_0. This plays a key role in obtain the existence of nontrivial weak solution for the given problem.

Lemma 3.4. *Under the same assumptions of Theorem 1.1, the functional Φ satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}}$ be a $(PS)_c$-sequence for Φ in X_0, that is,

$$\Phi(u_n) \to c, \quad \Phi'(u_n) \to 0 \text{ as } n \to \infty. \tag{3.6}$$

We first show that the sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X_0. Indeed, from (M_0), we get

$$\widehat{M}(t) = \int_0^t M(\eta)d\eta \geq \frac{1-a}{\alpha(x)}t^{\alpha(x)}, \quad \forall x \in \Omega, \tag{3.7}$$

and from (f_3) there exists $C_0 > 0$ such that

$$\int_{\Omega\cap\{|u_n|\leq T\}} \Big|f(x,u_n)u_n - \theta F(x,u_n)\Big|dx \leq C_0. \tag{3.8}$$

Then, from (3.6), (3.7), (3.8) and Lemma 2.6 (i), we have

$$\begin{aligned}
c + o_n(1)\|u_n\|_{X_0} &= \Phi(u_n) - \frac{1}{\theta}\langle\Phi'(u_n), u_n\rangle \\
&= \widehat{M}(\sigma_{s,p(x,y)}(u_n)) - \frac{1}{\theta}M(\sigma_{s,p(x,y)}(u_n))\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy \\
&\quad + \frac{1}{\theta}\int_\Omega \Big(f(x,u_n)u_n - \theta F(x,u_n)\Big)dx \\
&\geq \frac{1-a}{\alpha(x)}(\sigma_{s,p(x,y)}(u_n))^{\alpha(x)} \\
&\quad - \frac{1+a}{\theta}(\sigma_{s,p(x,y)}(u_n))^{\alpha(x)-1}\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy - \frac{1}{\theta}C_0 \\
&\geq \frac{1-a}{\alpha_+(p_+)^{\alpha_+}}\left(\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\right)^{\alpha(x)} \\
&\quad - \frac{1+a}{\theta(p_-)^{\alpha_--1}}\left(\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\right)^{\alpha(x)} - \frac{1}{\theta}C_0 \\
&= \left(\frac{1-a}{\alpha_+(p_+)^{\alpha_+}} - \frac{1+a}{\theta(p_-)^{\alpha_--1}}\right)\left(\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}dxdy\right)^{\alpha(x)} - \frac{1}{\theta}C_0 \\
&\geq \left(\frac{1-a}{\alpha_+(p_+)^{\alpha_+}} - \frac{1+a}{\theta(p_-)^{\alpha_--1}}\right)\|u_n\|_{X_0}^{\alpha_-p_-} - \frac{1}{\theta}C_0.
\end{aligned}$$

Hence, from (f_3) and $1 < \alpha_-p_-$, it is easy to check that the $(PS)_c$ sequence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X_0.

Next, we prove that $\{u_n\}_{n\in\mathbb{N}}$ contains a subsequence converging strongly in X_0. Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X_0 and X_0 is reflexive, up to a to a subsequence, there exist a subsequence, still denote by $\{u_n\}_{n\in\mathbb{N}}$, and $u_0 \in X_0$ such that $u_n \rightharpoonup u_0$ weakly in X_0. Since $\beta(x) < p_s^*(x)$ for all $x \in \Omega$, by Theorem 2.4 and Remark 2.9, it follows that $\{u_n\}_{n\in\mathbb{N}}$ converge strongly to u_0 in $L^{\beta(x)}(\Omega)$. Using Hölder inequality and embedding theorem, we have

$$\begin{aligned}
&\left|\int_\Omega f(x,u_n)(u_n-u_0)dx\right| \\
&\leq c\int_\Omega |u_n-u_0|dx + c\|u_n-u_0\|_{L^{\beta(x)}(\Omega)}\||u_n|^{\beta(x)-1}\|_{L^{\hat{\beta}(x)}(\Omega)} \\
&\to 0 \text{ as } n\to\infty,
\end{aligned} \tag{3.9}$$

where $\hat{\beta}(x)$ is the conjugate exponent of $\beta(x)$. Hence, $\langle\Phi'(u_n), u_n-u\rangle = o_n(1)$ and (3.9) imply that

$$\lim_{n\to\infty} M\left(\sigma_{s,p(x,y)}(u_n)\right)\langle L(u_n), u_n-u\rangle = 0. \tag{3.10}$$

Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X_0, passing to subsequence, if necessary, we may assume that

$$\sigma_{s,p(x,y)}(u_n)\to t_0\geq 0 \text{ as } n\to\infty.$$

If $t_0=0$, then $\{u_n\}_{n\in\mathbb{N}}$ converge strongly to $u_0=0$ in X_0 and the proof is finished.
If $t_0>0$, since the function M is continuous, we have

$$M\left(\sigma_{s,p(x,y)}(u_n)\right)\to M(t_0)\geq 0 \text{ as } n\to\infty. \tag{3.11}$$

Hence, by (M_0), for n large enough, there exist c_1, $c_2>0$ such that

$$0<c_1<M\left(\sigma_{s,p(x,y)}(u_n)\right)<c_2.$$

Combining (3.10) with (3.11), we deduce

$$\lim_{n\to\infty}\langle L(u_n), u_n-u\rangle = 0. \tag{3.12}$$

Thus, Lemma 3.2 (3) and (3.12) imply that $u_n\to u$ strongly in X_0 and Φ satisfies the $(PS)_c$ condition for all $c\in\mathbb{R}$. This completes the proof. □

Proof of Theorem 1.1 From Lemma 3.3, Lemma 3.4 and the fact that $\Phi(0)=0$, Φ satisfies the Mountain Pass Theorem 3.1. So Φ has at least one critical point u_1 such that $\Phi(u_1)=c\geq r>0$, i.e., problem (1.1) has at least one nontrivial weak solution in X_0.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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