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Research article

Dynamical behaviors to the coupled Schrödinger-Boussinesq system with the beta derivative

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Abstract: In this paper, the modified auxiliary expansion method is used to construct some new soliton solutions of coupled Schrödinger-Boussinesq system that includes beta derivative. The new exact solution is obtained have a hyperbolic function, trigonometric function, exponential function, and rational function. These solutions might appreciate in laser and plasma sciences. It is shown that this method, provides a straightforward and powerful mathematical tool for solving the nonlinear problems. Moreover, the linear stability of this nonlinear system is analyzed.

Keywords: Schrödinger-Boussinesq system; beta derivative; modulation instability analysis
Mathematics Subject Classification: 35Q41, 35Q60

1. Introduction

It is clear that most of the events that occur in mathematical physics and engineering areas can be described by partial differential equations. The physical phenomena of nonlinear partial differential equations (NLPDEs) can connect to a lot of areas of sciences, for example, plasma physics, optical fibers, nonlinear optics, fluid mechanics, chemistry, biology, geochemistry, and engineering sciences [1].

Scientists have been used and improved many methods to obtain the analytic, semi analytic and numerical solution of (NLPDEs), such as shooting and Runga-Kutta fourth order technique [2, 3], Adomian decomposition method [4], homotopy perturbation method [5], Adams-Bashforth-Moulton method [6], sine-Gordon expansion method [7, 8], sinh-Gordon expansion method [9, 10], an extended trial equation method [11], the degenerate Darboux transformation [12, 13], the multiplier approach [14], the improved Bernoulli sub-equation function method [15–17], a modified simple equation method [18], method of undetermined coefficients [19], a functional variable method [20], the trial equation method [21], couple of integration schemes [22], lie symmetries along with $\left({}^{G'}/_{G}\right)$-expansion method [23], improved tan $(\phi(\xi)/2)$-expansion method [24, 25], inverse mapping method [26], Wronskian determinants [27], the simple equation method [28], tanh function method [29], the extended homoclinic test method [30], Jacobi elliptic function anzätz method [31], the decomposition-Sumudu-like-integral-transform method [32], hypothetical method [33], $\exp(-\varphi(\xi))$-expansion method [34, 35], symbolic computational method [36–39], the Thomson scattering [40], the Lie group analysis method [41], and Darboux covariant Lax pairs and Bäcklund transformations [42].

The fractional differential equation is a generalization of the classical integer differential equation that has many distinct advantages. In contrast to the classic integer derivative model, the fractional derivative model has more precise mathematical physical structure simulations [43]. Wang et al. [44] studied a wick-type stochastic fractional nonlinear Schrödinger equation and constructed its fractional optical solitons. Nabti and Ghanbari [45] presented a global stability analysis of a fractional SVEIR epidemic model. Ismael et al. [46] investigated the Lakshmanan-Porsezian-Daniel model include conformable fractional derivative. Lu et al. [47] used the fractional Riccati method and fractional bifunction method to study the fractional complex Ginzburg-Landau equation. Fang et al. [48] found discrete fractional soliton solutions of conformable fractional discrete complex cubic Ginzburg-Landau equation. Ghanbari and Kumar [49] offered the existence of chaos in a fractional predator-prey-pathogen model. Ghanbari [50] explored the dynamics of an eco-epidemiological system using a nonlinear fractional differential equation system. Yu et al. [51] used the fractional mapping equation method and fractional bi-function method to investigate a space-time fractional nonlinear Schrödinger equation, and exact to a suggested equation was driven by using the Mittag-Leffler function.

The Boussinesq equation was introduced by Boussinesq [52] to describe two-dimensional irrotational flows of an inviscid liquid in a uniform rectangular channel as well as it was the first equation proposed in the research paper to describe a large range of physical phenomena. The Schrödinger-Boussinesq system raised in laser and plasma physics also has been attracted by many mathematicians and physicists. Manafian and Aghdaei [53] used the improved tan $\left({}^{\phi(\xi)}/_{2}\right)$ -expansion method and reported some exact solutions to the Schrödinger-Boussinesq system. Osman et al. [54] studied the variable-coefficients coupled Schrödinger-Boussinesq equation by using a unified method. MU and QIN [55] constructed rational solutions, breather, and the second-order rational solution by employing the Hirota technique. Bai and Wang [56] used the time-splitting Fourier spectral method for the coupled Schrödinger-Boussinesq equations and Ray in Ref. [57] used the time-Splitting Spectral Technique for the suggested system include Riesz fractional derivative. Banquet et al. [58] found the existence of local and global solutions for coupled Schrödinger-Boussinesq systems involving singular initial data. Liang [59] studied modulational instability and reported some stationary waves for the coupled generalized Schrodinger-Boussinesq system. Kılıcman and Reza Abazari [60] addressed travelling wave solutions of the Schrödinger-Boussinesq System via $\left({}^{G'}/_{G}\right)$-expansion method.

In this paper, we use the modified auxiliary expansion method to seek novel soliton solutions of the coupled Beta derivative of the Schrödinger-Boussinesq system that occur during the stationary propagation of coupled nonlinear magnetosonic waves and upper-hybrids in magnetized plasmas. The new solutions are presented as the family solution and expressed in hyperbolic, trigonometric, exponential and fractional function. Finally, the linear stability analysis and instability modulation Schrödinger-Boussinesq system are also presented.

This paper is organized as follows. In section 2, some basic definitions, properties, and the theorem about the Beta derivative are given. The structures of the modified auxiliary expansion method are given in section 3. Soliton solutions are constructed for coupled Schrödinger-Boussinesq system with Beta derivative in section 4. Linear Stability Analysis of coupled Schrödinger-Boussinesq system is presented in section 5. In section 6, we provide a conclusion to the studied system.

2. The beta derivative

In this section, we introduce some basic definitions, properties, and the theorem about the beta derivative of a function of order α [61].

Definition 1. Let f be a function, then the beta derivative of a function f of order α is defined as
${}_0^{A}D_t^{\alpha}\left(f\left(t\right)\right)=\lim\limits_{\Delta\to 0}\frac{f\left(t+\Delta\left(t+\frac{1}{\Gamma(\alpha)}\right)\right)-f(t)}{\Delta}, \qquad \text{for all} \quad t>0, \quad 0<\alpha<1.$

Theorem 1. Suppose that $0<\beta\leq 1,\ \beta>0$, and the function $f,\ g$ are α -differentiable at a point $t>0$, then

I. ${}_0^{A}D_t^{\alpha}\left(af\left(t\right)+bg\left(t\right)\right)=a{}_0^{A}D_t^{\alpha}f\left(t\right)+b{}_0^{A}D_t^{\alpha}g\left(t\right),\ \ a,b\in\mathbb{R}.$

II. ${}_0^{A}D_t^{\alpha}\left(f\left(t\right).g\left(t\right)\right)=f\left(t\right){}_0^{A}D_t^{\alpha}g\left(t\right)+g\left(t\right){}_0^{A}D_t^{\alpha}f\left(t\right).$

III.

$$ {}_0^{A}D_t^{\alpha}\left\{\frac{f\left(t\right)}{g\left(t\right)}\right\}=\frac{g\left(t\right){}_0^{A}D_t^{\alpha}f\left(t\right)-f\left(t\right){}_0^{A}D_t^{\alpha}g\left(t\right)}{\left(g\left(t\right)\right)^2}. $$

IV. ${}_0^{A}D_t^{\alpha}\left(C\right)=0$, where C is a constant function.

V. Suppose that f is differentiable function, $\Delta=\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha-1}h,\ \ h\to 0$ when $\Delta\to 0$ then

$$ {}_0^{A}D_t^{\alpha}f\left(t\right)=\left(t+\frac{1}{\Gamma\left(\alpha\right)}\right)^{1-\alpha}\frac{df\left(t\right)}{dt}. $$

VI.

$$ {}_0^{A}D_t^{\alpha}\left(\frac{f\left(t\right)}{g\left(t\right)}\right)=l\frac{df\left(\eta\right)}{d\eta}, $$

where l is a constant and $\eta=\frac{l}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}$.

3. General form of method

Assume, we have the following nonlinear partial differential equation (NLPDE)

$$ P\left({}_0^{A}D_x^{\alpha}u,{}_0^{A}D_t^{\alpha}u,{}_0^{A}D_t^{\alpha}{}_0^{A}D_x^{\alpha}u,{}_0^{A}D_x^{2\alpha}u,\ \ldots\right)=0. \tag{3.1} $$

To find explicit exact solutions of coupled Schrödinger-Boussinesq system, we use the following transformation

$$ u\left(x,y,t\right)=U\left(\xi\right),\xi=\frac{1}{\alpha}\left(x+\frac{1}{\Gamma\left(\alpha\right)}\right)^{\alpha}-\frac{\nu}{\alpha}\left(t+\frac{1}{\Gamma\left(\alpha\right)}\right)^{\alpha}, \tag{3.2} $$

where ν is arbitrary constant and ξ is the symbol of the wave variable. Inserting Eq (2) to Eq (1), we get a nonlinear ordinary differential equation (NLODE)

$$N\left(U, U', U'', \ldots\right) = 0. \tag{3.3}$$

The trial equation of solution for Eq (3) is given by

$$U\left(\xi\right) = a_0 + \sum_{i=1}^{n} a_i K^{i\,\Phi(\xi)} + \sum_{i=1}^{n} b_i K^{-i\,\Phi(\xi)}, \tag{3.4}$$

where a_0, a_i and b_i are non-zero constants and $\Phi\left(\xi\right)$ is the auxiliary ODE given by

$$\Phi'\left(\xi\right) = \frac{K^{-\Phi(\xi)} + \mu\, K^{\Phi(\xi)} + \lambda}{\ln\left(K\right)}, \tag{3.5}$$

where μ, λ are constants and $K > 0,\ K \neq 1$. The auxiliary ODE has the general solution:

I. When $\lambda^2 - 4\mu > 0$, then $f\left(\xi\right) = \log_K\left(-\lambda - \sqrt{\lambda^2 - 4\mu}\tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left(C + \xi\right)\right)\right)$.

II. When $\lambda^2 - 4\mu < 0$, then

$$f\left(\xi\right) = \log_K\left(-\lambda + \sqrt{-\lambda^2 + 4\mu}\tan\left(\frac{1}{2}\sqrt{-\lambda^2 + 4\mu}\left(C + \xi\right)\right)\right).$$

III. When $\lambda^2 - 4\mu \neq 0$, $\lambda = 0$ and$\mu < 0$, then

$$f\left(\xi\right) = \log_K\left(\sqrt{-4\mu}\coth\left(\frac{1}{2}\sqrt{-4\mu}\left(C + \xi\right)\right)\right).$$

IV. When $\lambda^2 - 4\mu \neq 0$, $\lambda = 0$ and$\mu > 0$, then

$$f\left(\xi\right) = \log_K\left(\sqrt{4\mu}\cot\left(\frac{1}{2}\sqrt{4\mu}\left(C + \xi\right)\right)\right).$$

V. When $\lambda^2 - 4\mu > 0$ and $\mu = 0$, then

$$f\left(\xi\right) = \log_K\left(\frac{-1 + \cosh\left(\lambda\left(C + \xi\right)\right) + \sinh\left(\lambda\left(C + \xi\right)\right)}{\lambda}\right).$$

VI. When $\lambda^2 - 4\mu = 0, \lambda \neq 0$ and $\mu \neq 0$, then

$$f\left(\xi\right) = \log_K\left(\frac{-2 - \lambda\left(C + \xi\right)}{2\mu\left(C + \xi\right)}\right).$$

VII. When $\lambda^2 - 4\mu = 0, \lambda = 0$ and $\mu = 0$, then

$$f\left(\xi\right) = \log_K\left(\xi + C\right).$$

4. The coupled Schrödinger-Boussinesq system with the beta derivative

Consider the coupled Schrödinger-Boussinesq system [62] with the Beta Derivative as follows:

$$i{}_0^{A}D_t^{\alpha}E + {}_0^{A}D_x^{2\alpha}E + \beta E - EN = 0, \tag{4.1}$$

$$3{}_0^{A}D_t^{2\alpha}N - {}_0^{A}D_x^{4\alpha}N + 3{}_0^{A}D_x^{2\alpha}N^2 + \gamma{}_0^{A}D_x^{2\alpha}N - {}_0^{A}D_x^{2\alpha}|E|^2 = 0, \tag{4.2}$$

where β and γ are real constants, $E(x,t)$ is a complex function, and $N(x,t)$ is a real function. To find the explicit exact solutions of coupled S-B system, we use the following transformation

$$E(x,t) = U(\xi)e^{i\theta},\ N(x,t) = V(\xi),\theta = \frac{\kappa}{\alpha}\left(x + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} + \frac{\omega}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} + l, \xi = \frac{1}{\alpha}\left(x + \frac{1}{\Gamma(\alpha)}\right)^{\alpha} + \frac{\delta}{\alpha}\left(t + \frac{1}{\Gamma(\alpha)}\right)^{\alpha}, \tag{4.3}$$

where δ is arbitrary constant and κ is the symbol of the soliton wave number, ω represents the soliton frequency and l symbolize the phase constant. Substituting Eq (4.3) into Eqs (4.1) and (4.2), we obtain

$$i(\delta + 2\kappa)U' + \left(\beta - \omega - \kappa^2\right)U + U'' - UV = 0, \tag{4.4}$$

$$\left(3\delta^2 + \gamma\right)V + 3V^2 - U^2 - V'' - A = 0, \tag{4.5}$$

where Eq (4.5) is found by integrating twice with respect to ξ and A is the arbitrary constant of integration. By Separate Eq (4.4) into real and imaginary parts, we get

$$\left(\beta - \omega - \kappa^2\right)U + U'' - UV = 0, \tag{4.6}$$

and

$$\delta = -2\kappa. \tag{4.7}$$

Inserting Eq (4.7) into Eqs (4.5) and (4.6), we get

$$\left(\beta - \omega - \kappa^2\right)U + U'' - UV = 0, \tag{4.8}$$

$$\left(12\kappa^2 + \gamma\right)V + 3V^2 - U^2 - V'' - A = 0. \tag{4.9}$$

From Eq (4.8), we have

$$V(\xi) = \frac{U''(\xi)}{U(\xi)} + \left(\beta - \omega - \kappa^2\right).$$

By using the homogeneous balance principle between the highest-order derivatives and nonlinear terms appearing in Eqs (4.9) and (4.10), we get $m = 2$ and $n = 2$ for U and V, respectively. We assume that the solutions of Eqs (4.9) and (4.10) have the following form

$$U(\xi) = a_0 + a_1\mathrm{K}^{f(\xi)} + a_2\mathrm{K}^{2f(\xi)} + b_1\mathrm{K}^{-f(\xi)} + b_2\mathrm{K}^{-2f(\xi)}, \tag{4.10}$$

$$V(\xi) = c_0 + c_1\mathrm{K}^{f(\xi)} + c_2\mathrm{K}^{2f(\xi)} + d_1\mathrm{K}^{-f(\xi)} + d_2\mathrm{K}^{-2f(\xi)}. \tag{4.11}$$

By inserting Eqs (4.10) and (4.11) into Eqs (4.8) and (4.9) and collecting all terms with the same order of $\mathrm{K}^{-f(\xi)}$ together, putting each coefficient of each polynomial to zero, we conclude the following cases:

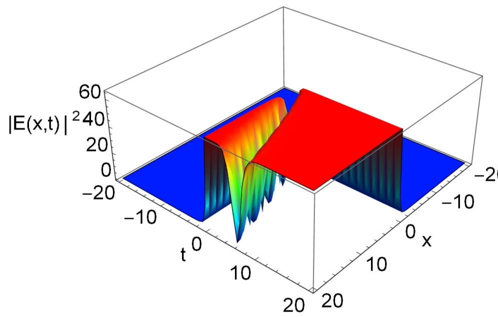
Case One. When $a_0 = \frac{\lambda\sqrt{2\lambda^2\mu^2-8\mu^3+\sqrt{\mu^4\left(12A+\beta_2^2+24\beta_2\kappa^2+144\kappa^4+3\lambda^4-24\lambda^2\mu+48\mu^2\right)}}}{\sqrt{2}\mu}$, $a_2 = 0$, $a_1 = \sqrt{2}\sqrt{2\lambda^2\mu^2-8\mu^3+\sqrt{\mu^4\left(12A+\beta_2{}^2+24\beta_2\kappa^2+144\kappa^4+3\lambda^4-24\lambda^2\mu+48\mu^2\right)}}$, $b_1 = 0$, $b_2 = 0$, $c_0 = \frac{-\beta_2\mu^2-12\kappa^2\mu^2+3\lambda^2\mu^2+\sqrt{\mu^4\left(12A+\beta_2{}^2+24\beta_2\kappa^2+144\kappa^4+3\lambda^4-24\lambda^2\mu+48\mu^2\right)}}{6\mu^2}$, $c_1 = 2\lambda\mu$, $c_2 = 2\mu^2$, $d_1 = 0$, $d_2 = 0$, We obtain the following families of solutions.

Family 1. When $\Delta = \lambda^2 - 4\mu > 0$, $2\lambda^2\mu^2 - 8\mu^3 + \mathrm{H}_1 > 0$, $-\frac{\lambda^2}{2} + 2\mu > 0$, $\left(12A + \beta_2{}^2 + 24\beta_2\kappa^2 + 3\left(48\kappa^4 + \left(\lambda^2 - 4\mu\right)^2\right)\right)\mu^4 > 0$ and $\mu \neq 0$, then

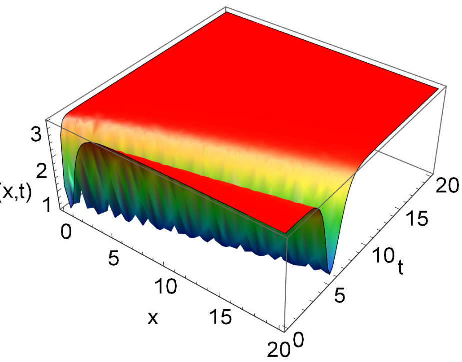
$$E(x,t) = -\frac{e^{\frac{i\left(l\alpha+\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa+\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\omega\right)}{\alpha}}\sqrt{\Delta\left(2\lambda^2\mu^2-8\mu^3+\mathrm{H}_1\right)}\tanh\left(\frac{\sqrt{\Delta}\left(C\alpha+\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}-2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)}{2\alpha}\right)}{\sqrt{2}\mu}, \tag{4.12}$$

$$N(x,t) = -\frac{\beta_2}{6} - 2\kappa^2 + \frac{H_1}{6\mu^2} + \frac{1}{2}\Delta\tanh^2\left(\frac{\sqrt{\Delta}\left(C\alpha+\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}-2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)}{2\alpha}\right), \tag{4.13}$$

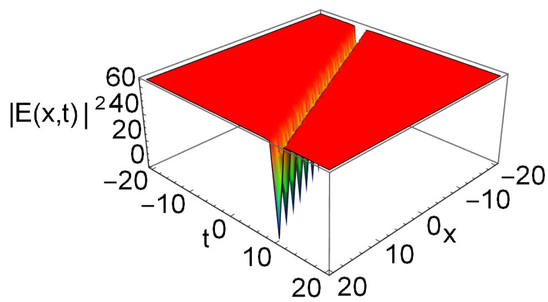
where $\mathrm{H}_1 = \sqrt{\left(12A + \beta_2{}^2 + 24\beta_2\kappa^2 + 3\left(48\kappa^4 + \Delta^2\right)\right)\mu^4}$. Eq (4.12) and Eq (4.13) are dark soliton solutions as shown in Figure (1).



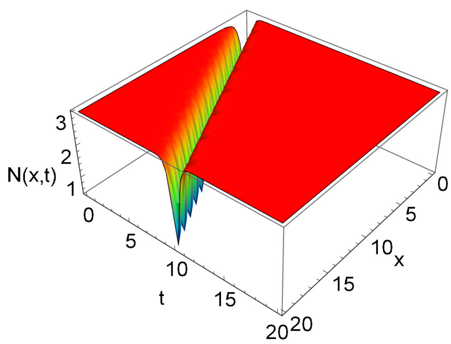
(a) $\alpha = 1/2$



(b) $\alpha = 1/2$



(c) $\alpha = 1$



(d) $\alpha = 1$

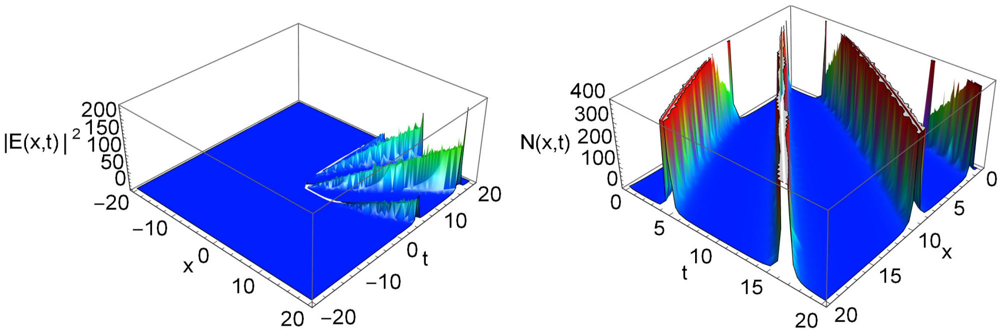
Figure 1. 3-D figure of soliton solutions for Eq (4.12) and Eq (4.13) plotted when $\lambda = 3, \mu = 1, C = 1, A = 1, \beta_2 = 1, \kappa = 1, \beta_1 = 0.1, l = 1, \omega = 1$.

Family 2. When $\Delta = \lambda^2 - 4\mu < 0$, $2\lambda^2\mu^2 - 8\mu^3 + H_1 > 0$, $-\frac{\Delta}{2} > 0$, $\left(12A + {\beta_2}^2 + 24\beta_2\kappa^2 + 3\left(48\kappa^4 + \Delta^2\right)\right)\mu^4 > 0$ and $\mu \neq 0$, then

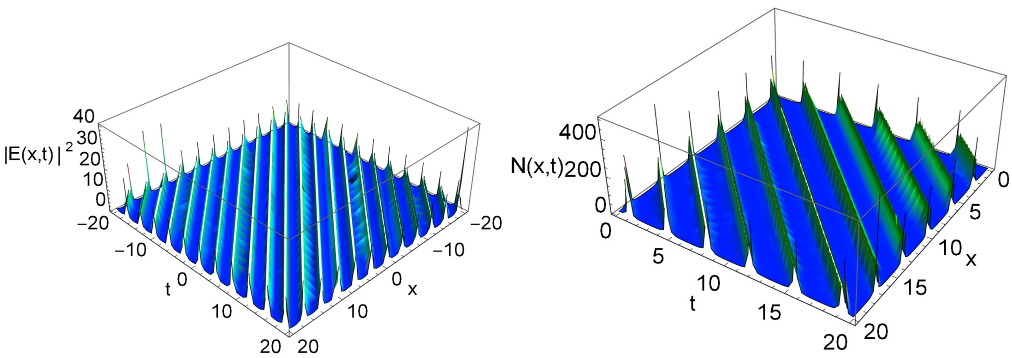
$$E(x,t) = \frac{e^{\frac{i\left(l\alpha + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^\alpha \kappa + \left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^\alpha \omega\right)}{\alpha}} \sqrt{-\frac{\Delta}{2}\left(2\lambda^2\mu^2 - 8\mu^3 + H_1\right)} \tan\left(\frac{\left(C\alpha + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^\alpha - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^\alpha \kappa\right)\sqrt{-\Delta}}{2\alpha}\right)}{\mu}, \tag{4.14}$$

$$N(x,t) = -\frac{\beta_2}{6} - 2\kappa^2 + \frac{H}{6\mu^2} - \frac{1}{2}\Delta \tan^2\left(\frac{\left(C\alpha + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^\alpha - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^\alpha \kappa\right)\sqrt{-\Delta}}{2\alpha}\right). \tag{4.15}$$

Equations (4.14) and (4.15) are singular soliton solutions as seen in Figure (2).



(a) $\alpha = 1/2$ **(b)** $\alpha = 1/2$



(c) $\alpha = 1$ **(d)** $\alpha = 1$

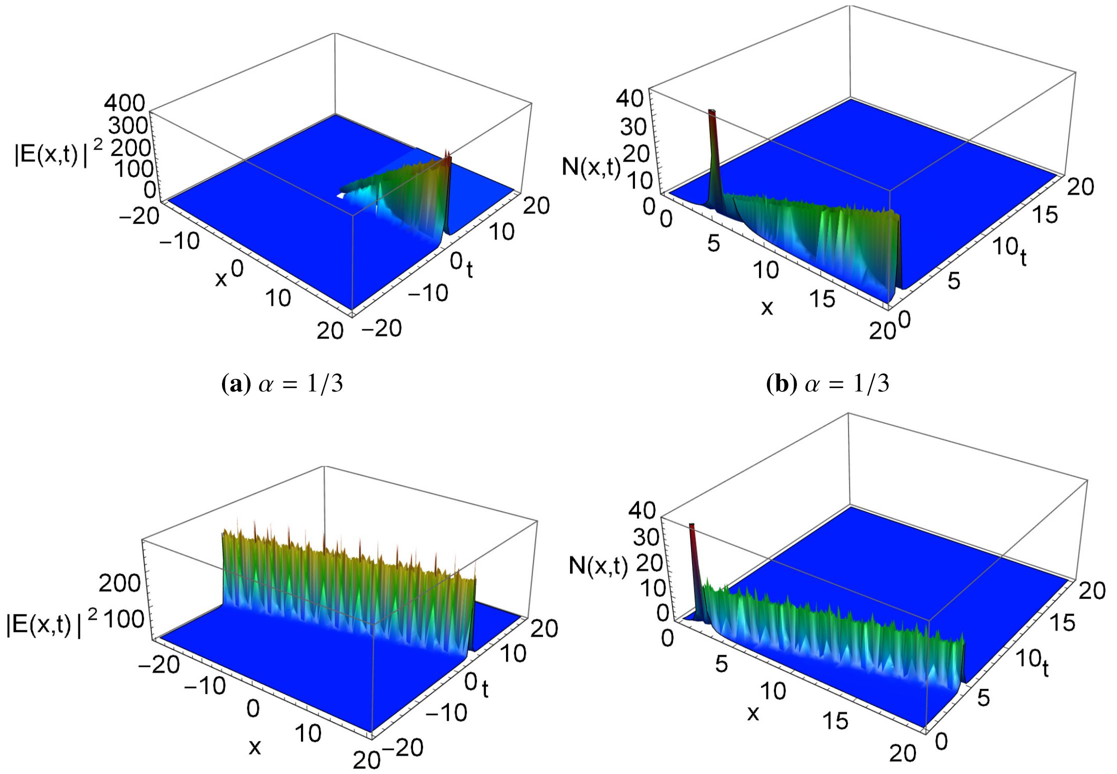
Figure 2. 3-D figure of singular soliton solutions for Eq (4.14) and Eq (4.15) plotted when $\lambda = 1, \mu = 1, c = 1, A = 0.1, \beta_2 = 0.2, \kappa = 0.5, \beta_1 = 0.3, l = 2, \omega = 4$.

Family 3. When $\lambda = 0$, $\mu < 0$, $H_2 > \mu$ and $\mu^4\left(12A + {\beta_2}^2 + 24\beta_2\kappa^2 + 48\left(3\kappa^4 + \mu^2\right)\right) > 0$, then

$$E(x,t) = \frac{\sqrt{2\left(H_2 - 8\mu^3\right)}}{\sqrt{-\mu}} e^{\frac{i\left(l\alpha + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^\alpha \kappa + \left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^\alpha \omega\right)}{\alpha}} \coth\left(\frac{\sqrt{-\mu}\left(C\alpha - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^\alpha \kappa + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^\alpha\right)}{\alpha}\right), \tag{4.16}$$

$$N(x,t) = \frac{1}{6\mu^2}\left(H_2 - \beta_2\mu^2 - 12\kappa^2\mu^2 - 12\mu^3 \coth^2\left(\frac{\sqrt{-\mu}\left(C\alpha - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^\alpha \kappa + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^\alpha\right)}{\alpha}\right)\right), \tag{4.17}$$

where $H_2 = \sqrt{\mu^4\left(12A + \beta_2{}^2 + 24\beta_2\kappa^2 + 48\left(3\kappa^4 + \mu^2\right)\right)}$.
These solutions are singular soliton solutions as presented in Figure (3).



(a) $\alpha = 1/3$ **(b)** $\alpha = 1/3$

(c) $\alpha = 1$ **(d)** $\alpha = 1$

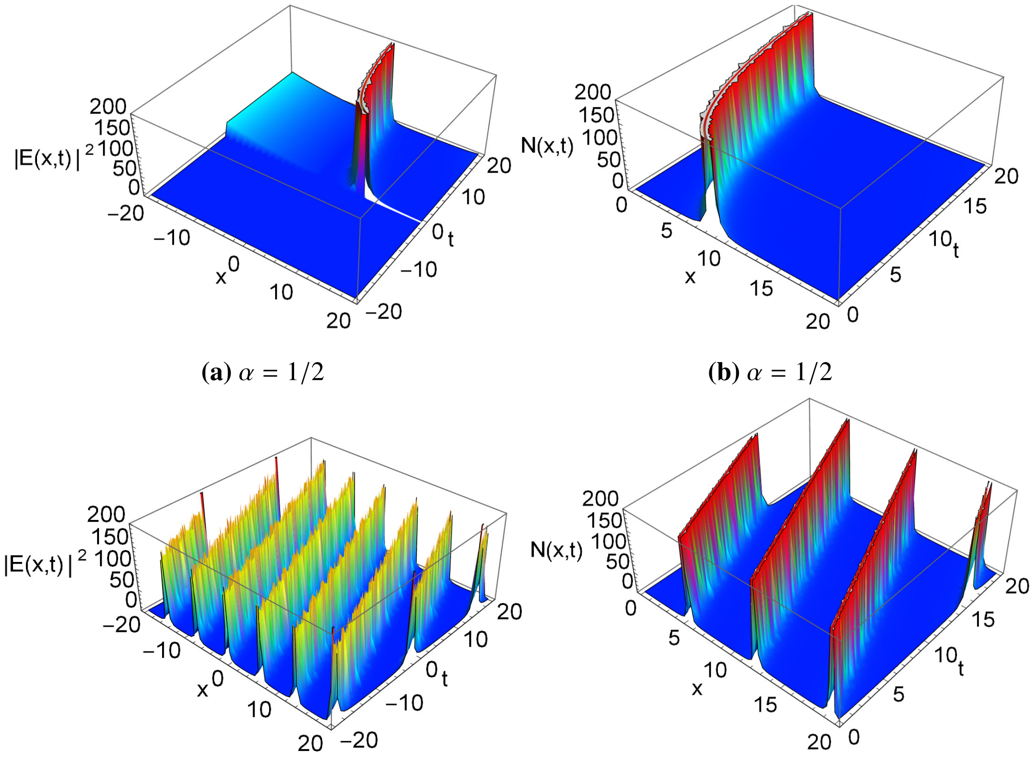
Figure 3. 3-D plot of soliton solutions for Eq (4.16) and Eq (4.17) plotted when $\lambda = 0, \mu = -0.01, C = 0.1, A = 1, \beta_2 = -0.2, \kappa = -0.2, \beta_1 = 0.3, l = 0.2, \omega = 4$.

Family 4. When $\lambda = 0$, $\mu > 0$, $H_2 > \mu$ and $\mu^4\left(12A + \beta_2{}^2 + 24\beta_2\kappa^2 + 48\left(3\kappa^4 + \mu^2\right)\right) > 0$, then

$$E(x,t) = \frac{\sqrt{2\left(H_2 - 8\mu^3\right)}}{\sqrt{\mu}} e^{\frac{i\left(l\alpha + \left(x + \frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa + \left(t + \frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\omega\right)}{\alpha}} \cot\left(\frac{\sqrt{\mu}\left(C\alpha + \left(x + \frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t + \frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)}{\alpha}\right), \tag{4.18}$$

$$N(x,t) = \frac{H_2 - \beta_2\mu^2 - 12\kappa^2\mu^2 + 12\mu^3\cot^2\left(\frac{\left(C\alpha + \left(x + \frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t + \frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\sqrt{\mu}}{\alpha}\right)}{6\mu^2}. \tag{4.19}$$

These solutions are singular soliton solutions as presented in Figure (4).



(a) $\alpha = 1/2$ **(b)** $\alpha = 1/2$

(c) $\alpha = 1$ **(d)** $\alpha = 1$

Figure 4. 3-D plot of singular soliton solutions for Eq (4.18) and Eq (4.19) plotted when $\lambda = 0, \mu = 0.2, C = 0.4, A = 1, \beta_2 = 3, \kappa = -0.2, \beta_1 = 0.3, l = 0.2, \omega = 4$.

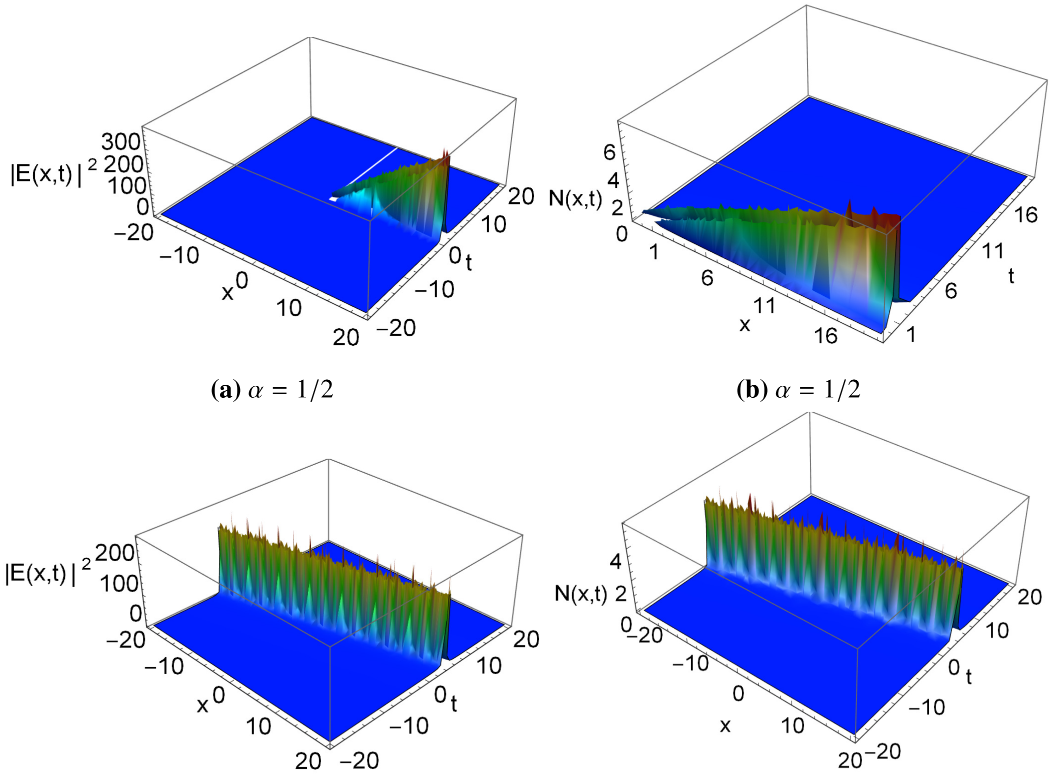
Family 5. When $\lambda^2 - 4\mu > 0$ and $\mu = 0$, then the solution of this family could not be found because is located at the denominator of a_0, c_0.

Family 6. When $\lambda^2 - 4\mu = 0, \lambda \neq 0, 2\lambda^2\mu^2 - 8\mu^3 + H_1 > 0$ and $\mu \neq 0$, then

$$E(x,t) = -\frac{\alpha\sqrt{2\left(2\lambda^2\mu^2 - 8\mu^3 + H_1\right)}\, e^{\frac{i\left(l\alpha + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa + \left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\omega\right)}{\alpha}}}{\left(C\alpha + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\mu}, \tag{4.20}$$

$$N(x,t) = \frac{\left(\left(\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\lambda + \alpha\left(2 + C\lambda\right)\right)^2}{2\left(C\alpha + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)^2} - \frac{2\alpha\lambda}{C\alpha + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa} - \lambda^2 + \frac{H_1 - \beta_2\mu^2 - 12\kappa^2\mu^2 + 3\lambda^2\mu^2}{6\mu^2}. \tag{4.21}$$

As shown in Figure (5), Eq (4.20) and Eq (4.21) are singular solutions, too.



(a) $\alpha = 1/2$ **(b)** $\alpha = 1/2$

(c) $\alpha = 1$ **(d)** $\alpha = 1$

Figure 5. 3-D plot of soliton solutions for Eq (4.20) and Eq (4.21) plotted when $\lambda = 2, \mu = 1, C = 1, A = 0.1, \beta_2 = 3, \kappa = 2, \beta_1 = 3, l = 0.2, \omega = 4$.

Family 7. When $\lambda^2 - 4\mu = 0, \lambda = 0$ and $\mu = 0$, then the solution of this family could not be found because $\mu = 0$ is located at the denominator of a_0, c_0.

Case Two. When $a_0 = \frac{\lambda\sqrt{2\Delta\mu^2}}{\mu}, a_1 = -2\sqrt{2\mu^2\Delta}$, $a_2 = 0, b_1 = 0$, $b_2 = 0$, $c_0 = \frac{1}{6}\left(5\lambda^2 - 8\mu - \sqrt{\lambda^4 - 8\lambda^2\mu + 16\mu^2 - 12A}\right)$, $c_1 = 2\lambda\mu$, $c_2 = 2\mu^2$, $d_1 = 0$, $d_2 = 0$, $\omega = \frac{1}{12}\left(12\beta_1 + \beta_2 - 10\lambda^2 + 40\mu + \sqrt{\lambda^4 - 8\lambda^2\mu + 16\mu^2 - 12A}\right)$, $\kappa = -\frac{\sqrt{\sqrt{\lambda^4 - 8\lambda^2\mu + 16\mu^2 - 12A} - \beta_2}}{2\sqrt{3}}$, we obtain the following families of solutions.

Family 1. When $\Delta = \lambda^2 - 4\mu > 0, \beta 2 > 0, \beta 2 > H_3$ and $\Delta^2 - 12A > 0$, then

$$E(x,t) = \frac{\sqrt{2}\Delta^{3/2}\mu e^{\frac{i\left(12l\alpha - 2\sqrt{3}\left(x + \frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\sqrt{-(\beta 2 + H_3)} + \left(t + \frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\left(12\beta_1 + \beta_2 - 10\lambda^2 - H_3 + 40\mu\right)\right)}{12\alpha}}}{\sqrt{\Delta\mu^2}} \times \tanh\left(\frac{\left(3\left(x + \frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} + \sqrt{3}\left(t + \frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\sqrt{-(\beta_2 + H_3)}\right)\sqrt{\Delta}}{6\alpha}\right), \tag{4.22}$$

$$N(x,t) = \frac{1}{6}\left(2\lambda^2 + H_3 - 8\mu + 3\Delta\tanh^2\left(\frac{\left(3\left(x + \frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} + \sqrt{3}\left(t + \frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\sqrt{-(\beta_2 + H_3)}\right)\sqrt{\Delta}}{6\alpha}\right)\right), \tag{4.23}$$

where $H_3 = \sqrt{\Delta^2 - 12A}$.

Family 2. When$\Delta < 0, \beta_2 > 0, \beta 2 > H_3$ and $\Delta^2 - 12A > 0$, then

$$E(x,t) = -\frac{\sqrt{2}\mathrm{e}^{\frac{\mathrm{i}\left(12l\alpha - 2\sqrt{3}\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\sqrt{-\beta 2 - H_3} + \left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\left(12\beta_1 + \beta_2 - 10\lambda^2 - H_3 + 40\mu\right)\right)}{12\alpha}}\sqrt{\Delta\mu^2}\sqrt{-\Delta}}{\mu} \tan\left(\frac{\left(3C\alpha + 3\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} + \sqrt{3}\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\sqrt{-(\beta_2 + H_3)}\right)\sqrt{-\Delta}}{6\alpha}\right), \tag{4.24}$$

$$N(x,t) = \frac{1}{6}\left(2\lambda^2 + H_3 - 8\mu - 3\Delta\tan^2\left(\frac{\left(3C\alpha + 3\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} + \sqrt{3}\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\sqrt{-(\beta_2 + H_3)}\right)\sqrt{-\Delta}}{6\alpha}\right)\right). \tag{4.25}$$

Family 3. When $\lambda = 0, \mu < 0, A > 0$ and $-\beta_2 - 2\sqrt{4\mu^2 - 3A} > 0$, then

$$E(x,t) = -4\sqrt{2\mu^2}\mathrm{e}^{\frac{\mathrm{i}\left(12l\alpha - 2\sqrt{3}\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}H_4 + \left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\left(12\beta_1 + \beta_2 + 40\mu - 2\sqrt{4\mu^2 - 3A}\right)\right)}{12\alpha}} \times \coth\left(\frac{\sqrt{-\mu}\left(3\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} + \sqrt{3}\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}H_4\right)}{3\alpha}\right), \tag{4.26}$$

$$N(x,t) = \frac{1}{3}\left(\sqrt{4\mu^2 - 3A} - 4\mu - 6\mu\coth^2\left(\frac{\sqrt{-\mu}\left(3\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} + \sqrt{3}\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}H_4\right)}{3\alpha}\right)\right), \tag{4.27}$$

where $H_4 = \sqrt{-\beta_2 - 2\sqrt{4\mu^2 - 3A}}$.

Family 4. When $\lambda = 0, \mu < 0, A < 0$ and $-\beta_2 - 2\sqrt{4\mu^2 - 3A} > 0$, then

$$E(x,t) = \frac{4\sqrt{2}\mu^{5/2}}{\sqrt{-\mu^3}}\mathrm{e}^{\frac{\mathrm{i}\left(12l\alpha - 2\sqrt{3}\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}H_4 + \left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\left(12\beta_1 + \beta_2 + 40\mu - 2\sqrt{4\mu^2 - 3A}\right)\right)}{12\alpha}} \cot\left(\sqrt{\mu}\left(C + \frac{\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha} + \frac{\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}H_4}{\sqrt{3}\alpha}\right)\right), \tag{4.28}$$

$$N(x,t) = \frac{\sqrt{2}}{3}\left(\sqrt{4\mu^2 - 3A} - 4\mu + 6\mu\cot^2\left(\sqrt{\mu}\left(C + \frac{\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}}{\alpha} + \frac{\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}H_4}{\sqrt{3}\alpha}\right)\right)\right). \tag{4.29}$$

Family 5. When $\Delta > 0$ and $\mu = 0$, then the solution of $E(x,t)$could not be found because $\mu = 0$ is located at the denominator of a_0.

$$N(x,t) = \frac{1}{6}\left(5\lambda^2 + \sqrt{\lambda^4 - 12A}\right). \tag{4.30}$$

Family 6. When $\Delta = 0, \lambda \neq 0, -\beta 2 - H > 0$ and $\mu \neq 0$,

$$E(x,t) = \frac{6\alpha\sqrt{2\Delta\mu^2}\mathrm{e}^{\frac{\mathrm{i}\left(12l\alpha - 2\sqrt{3}\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\sqrt{-\beta_2 - H_3} + \left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\left(12\beta_1 + \beta_2 - 10\lambda^2 - H_3 + 40\mu\right)\right)}{12\alpha}}}{3C\alpha + 3\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} + \sqrt{3}\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\sqrt{-(\beta_2 + H_3)\mu}}, \tag{4.31}$$

$$N(x,t)=\frac{1}{6}\left(\begin{array}{l}\dfrac{3\left(6\alpha+3C\alpha\lambda+3\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\lambda+\sqrt{3}\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\lambda\sqrt{-\beta_2-H_3}\right)^2}{\left(3C\alpha+3\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}+\sqrt{3}\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\sqrt{-\beta_2-H_3}\right)^2}+H_3-8\mu-\lambda^2\\ -\dfrac{36\alpha\lambda}{3C\alpha+3\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}+\sqrt{3}\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\sqrt{-\beta_2-H_3}}\end{array}\right). \tag{4.32}$$

Family 7. When $\Delta=0,\lambda=0$ and $\mu=0$, then the solution $E(x,y,t)$ could not be found because $\mu=0$ is located at the denominator of a_0.

$$N(x,t)=\frac{\sqrt{-A}}{\sqrt{3}}. \tag{4.33}$$

Case Three. When $a_0=\frac{\lambda\sqrt{\lambda^2c_0+8\mu c_0-3c_0^2-A-2\lambda^2\mu-4\mu^2}}{\sqrt{\lambda^2-2c_0}}$, $a_1=0$, $a_2=0$, $b_1=\frac{2\sqrt{\lambda^2c_0+8\mu c_0-3c_0^2-A-2\lambda^2\mu-4\mu^2}}{\sqrt{\lambda^2-2c_0}}$, $b_2=0$, $c_1=0$, $c_2=0$, $d_1=2\lambda$, $d_2=2$, $\omega=\frac{2A+12\beta_1\lambda^2+\beta_2\lambda^2-\lambda^4+20\lambda^2\mu+8\mu^2-24\beta_1c_0-2\beta_2c_0-6\lambda^2c_0-48\mu c_0+18c_0^2}{12\left(\lambda^2-2c_0\right)}$, $\kappa=\frac{\sqrt{\lambda^4+4\lambda^2\mu-8\mu^2+2\beta_2c_0-6\lambda^2c_0+6c_0^2-2A-\beta_2\lambda^2}}{2\sqrt{3\left(\lambda^2-2c_0\right)}}$, we obtain the following families of solutions.

Family 1. When $\Delta=\lambda^2-4\mu>0$ and $\left(\lambda^2+8\mu\right)c_0-3c_0^2-A-2\mu\left(\lambda^2+2\mu\right)>0$, then

$$E(x,t)=\frac{H_5\left(\lambda^2-4\mu+\lambda\sqrt{\Delta}\tanh\left(\frac{\left(\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}-2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\sqrt{\Delta}}{2\alpha}\right)\right)}{\sqrt{\lambda^2-2c_0}\left(\lambda+\sqrt{\Delta}\tanh\left(\frac{\left(\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}-2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\sqrt{\Delta}}{2\alpha}\right)\right)}\times e^{\frac{i\left(l\alpha+\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa+\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\omega\right)}{\alpha}}, \tag{4.34}$$

$$N(x,t)=c_0-\frac{4\mu\left(\lambda^2-2\mu+\lambda\sqrt{\Delta}\tanh\left(\frac{\left(\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}-2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\sqrt{\Delta}}{2\alpha}\right)\right)}{\left(\lambda+\sqrt{\Delta}\tanh\left(\frac{\left(\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}-2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\sqrt{\Delta}}{2\alpha}\right)\right)^2}, \tag{4.35}$$

where $H_5=\sqrt{\left(\lambda^2+8\mu\right)c_0-3c_0^2-A-2\mu\Delta}$.

Family 2. When $\Delta<0$ and $\left(\lambda^2+8\mu\right)c_0-3c_0^2-A-2\mu\left(\lambda^2+2\mu\right)>0$, then

$$E(x,t)=\frac{H_5}{\sqrt{\lambda^2-2c_0}}e^{\frac{i\left(l\alpha+\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa+\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\omega\right)}{\alpha}}\times\left(\lambda-\frac{4\mu}{\lambda-\sqrt{-\Delta}\tan\left(\frac{\left(C\alpha+\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}-2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\sqrt{-\Delta}}{2\alpha}\right)}\right), \tag{4.36}$$

$$N(x,t)=\frac{4\mu\left(2\mu-\lambda^2+\lambda\sqrt{-\Delta}\tan\left(\frac{\left(C\alpha+\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}-2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\sqrt{-\Delta}}{2\alpha}\right)\right)}{\left(\lambda-\sqrt{-\Delta}\tan\left(\frac{\left(C\alpha+\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}-2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\sqrt{-\Delta}}{2\alpha}\right)\right)^2}+c_0. \tag{4.37}$$

Family 3. $\lambda = 0, \mu < 0, c_0 < 0$ and $8\mu c_0 - 3c_0^2 - A - 4\mu^2 > 0$

$$E(x,t) = \frac{\sqrt{2}\mathrm{e}^{\frac{\mathrm{i}\left(l\alpha+\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa+\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\omega\right)}{\alpha}}\sqrt{-\mu}H_6}{\sqrt{-c_0}}\tanh\left(\frac{\left(\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\sqrt{-\mu}}{\alpha}\right), \tag{4.38}$$

$$N(x,t) = c_0 - 2\mu\tanh\left[\frac{\left(\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\sqrt{-\mu}}{\alpha}\right]^2, \tag{4.39}$$

where $H_6 = \sqrt{8\mu c_0 - 3c_0^2 - A - 4\mu^2}$.

Family 4. When $\lambda = 0, \mu > 0$ and $8\mu c_0 - 3c_0^2 - A - 4\mu^2 > 0$, then

$$E(x,t) = \frac{\sqrt{2\mu}\,\mathrm{e}^{\frac{\mathrm{i}\left(l\alpha+\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa+\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\omega\right)}{\alpha}}H_6}{\sqrt{-c_0}}\tan\left(\frac{\left(C\alpha + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\sqrt{\mu}}{\alpha}\right), \tag{4.40}$$

$$N(x,t) = c_0 + 2\mu\tan^2\left(\frac{\left(C\alpha + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\sqrt{\mu}}{\alpha}\right). \tag{4.41}$$

Family 5. When $\Delta > 0, \lambda^2 - 2c_0 > 0, \lambda^2 c_0 - 3c_0^2 - A > 0$ and $\mu = 0$, then

$$E(x,t) = \frac{\lambda\sqrt{\lambda^2 c_0 - 3c_0^2 - A}}{\sqrt{\lambda^2 - 2c_0}}\mathrm{e}^{\frac{\mathrm{i}\left(l\alpha+\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa+\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\omega\right)}{\alpha}}\coth\left(\frac{\left(C\alpha + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\lambda}{2\alpha}\right), \tag{4.42}$$

$$\begin{aligned} N(x,t) = {} & \frac{1}{2}csch^2\left(\frac{\left(C\alpha + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\lambda}{2\alpha}\right)\times \\ & \left(\lambda^2 + \left(\cosh\left(\frac{\left(C\alpha + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\lambda}{\alpha}\right) - 1\right)c_0\right). \end{aligned} \tag{4.43}$$

Family 6. When $\Delta = 0, \lambda \neq 0, \lambda^2 - 2c_0 > 0, \left(\lambda^2 + 8\mu\right)c_0 - 3c_0^2 - A - 2\mu\left(\lambda^2 + 2\mu\right) > 0$ and $\mu \neq 0$, then

$$\begin{aligned} E(x,t) = {} & \frac{\left(\left(\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\Delta + \alpha\left(2\lambda + C\lambda^2 - 4C\mu\right)\right)H_5}{\left(\left(\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\lambda + \alpha(2 + C\lambda)\right)\sqrt{\lambda^2 - 2c_0}}\times \\ & \mathrm{e}^{\frac{\mathrm{i}\left(l\alpha+\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa+\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\omega\right)}{\alpha}}, \end{aligned} \tag{4.44}$$

$$\begin{aligned} N(x,t) = {} & \frac{8\left(C\alpha + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)^2\mu^2}{\left(\left(\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\lambda + \alpha(2 + C\lambda)\right)^2} - \\ & \frac{4\left(C\alpha + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\lambda\mu}{\left(\left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\lambda + \alpha(2 + C\lambda)} + c_0. \end{aligned} \tag{4.45}$$

Family 7. When $\Delta = 0, \lambda = 0, \mu = 0, c_0 > 0$, and $-A - 3c_0^2 > 0$, then

$$E(x,t) = \frac{\sqrt{2} e^{\frac{i\left(l\alpha + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa + \left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\omega\right)}{\alpha}} \alpha \sqrt{-A-3c_0^2}}{\left(C\alpha + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)\sqrt{-c_0}}, \tag{4.46}$$

$$N(x,t) = \frac{2\alpha^2}{\left(C\alpha + \left(x+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha} - 2\left(t+\frac{\alpha}{\Gamma(\alpha)}\right)^{\alpha}\kappa\right)^2} + c_0. \tag{4.47}$$

5. Linear stability analysis

In this section, we construct the modulation instability (MI) of the stationary solutions of Eqs (4.1) and (4.2) via the virtue of linear stability analysis. The MI may consist of exponential growth of small disturbances in the amplitude or optical wave phase [63]. It is essential that we can observe MI in the nonlinear physics of the wave. Suppose that Eqs (4.1) and (4.2) have the following stationary solutions [64]:

$$E(x,t) = a\, e^{i\varphi \frac{1}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}}, \qquad N(x,t) = b, \tag{5.1}$$

where a and b are arbitrary real constants. Putting Eq (5.1) into Eqs (4.1) and (4.2), we get $\varphi = (\beta_1 - b)$. Suppose that the perturbed stationary solution has the form:

$$E(x,t) = (a + \varepsilon U(x,t))\, e^{i(\beta_1 - b)\frac{1}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}},\ N(x,t) = b + \varepsilon V(x,t), \tag{5.2}$$

where $U(x,t)$ is complex fractional function, and $V(x,t)$ is real fractional function. Putting Eq (5.2) into Eqs (4.1) and (4.2), the results satisfy the following linear equations.

$$aV - iU_t - U_{xx} = 0, \tag{5.3}$$

$$3V_{tt}(x,t) - V_{xxxx}(x,t) + \beta_2 V_{xx}(x,t) - \left(U_{xx}(x,t) + U^*_{xx}(x,t)\right) + 6bV_{xx}(x,t) = 0. \tag{5.4}$$

Where $*$ is the symbol of the conjugate and so, Eqs (5.3) and (5.4) can be written as

$$\mathrm{U}(x,t) = U_1 e^{i\left(-W\frac{1}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha} + M\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)} + U_2 e^{-i\left(-W\frac{1}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha} + M\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)}, \tag{5.5}$$

$$\mathrm{V}(x,t) = V_1 e^{i\left(-W\frac{1}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha} + M\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)} + V_2 e^{-i\left(-W\frac{1}{\alpha}\left(t+\frac{1}{\Gamma(\alpha)}\right)^{\alpha} + M\frac{1}{\alpha}\left(x+\frac{1}{\Gamma(\alpha)}\right)^{\alpha}\right)}. \tag{5.6}$$

Where W denotes the complex frequency, M is real disturbance wave-number, and U_1, U_2, V_1, V_2 are the coefficients of the linear combination. Substituting Eqs (5.5) and (5.6), we get the following homogeneous equations

$$\begin{aligned}
&M^2U_1 + aV_1 - WU_1 = 0,\\
&M^2U_2 + aV_2 + WU_2 = 0,\\
&M^2U_1 + M^2U_2 - 6bM^2V_1 - M^4V_1 - 3W^2V_1 - M^2\beta_2V_1 = 0,\\
&M^2U_1 + M^2U_2 - 6bM^2V_2 - M^4V_2 - 3W^2V_2 - M^2\beta_2V_2 = 0.
\end{aligned} \tag{5.7}$$

Evaluating the determinant and equaling to zero, we get the following relationship:

$$\left(6bM^2 + M^4 + 3W^2 + M^2\beta_2\right)\left(2aM^4 + \left(M^4 - W^2\right)\left(6bM^2 + M^4 + 3W^2 + M^2\beta_2\right)\right) = 0. \tag{5.8}$$

According to Eq (5.8), we can discuss the following cases of the MI for Eqs (4.1) and (4.2) as follows

Case 1. In case

$$W = \mp \frac{iM\sqrt{6b + M^2 + \beta_2}}{\sqrt{3}}, \tag{5.9}$$

we observe that the modulation instability of the Eqs (4.1) and (4.2) occurs when the wave number contains an imaginary value, therefor

$$6b + M^2 + \beta_2 > 0. \tag{5.10}$$

Case 2. In case

$$w = \mp\sqrt{\frac{M^2\left(2M^2 - 6b - \beta_2\right) \mp M^2\sqrt{24a + 36b^2 + M^2\left(48b + 16M^2\right) + \beta_2\left(12b + 8M^2 + \beta_2\right)}}{6}}, \tag{5.11}$$

we observe that the modulation instability of the Eqs (4.1) and (4.2) occurs when either

$$24a + 36b^2 + M^2\left(48b + 16M^2\right) + \beta_2\left(12b + 8M^2 + \beta_2\right) > 0, \tag{5.12}$$

or

$$M^2\left(2M^2 - 6b - \beta_2\right) \mp M^2\sqrt{24a + 36b^2 + M^2\left(48b + 16M^2\right) + \beta_2\left(12b + 8M^2 + \beta_2\right)} < 0. \tag{5.13}$$

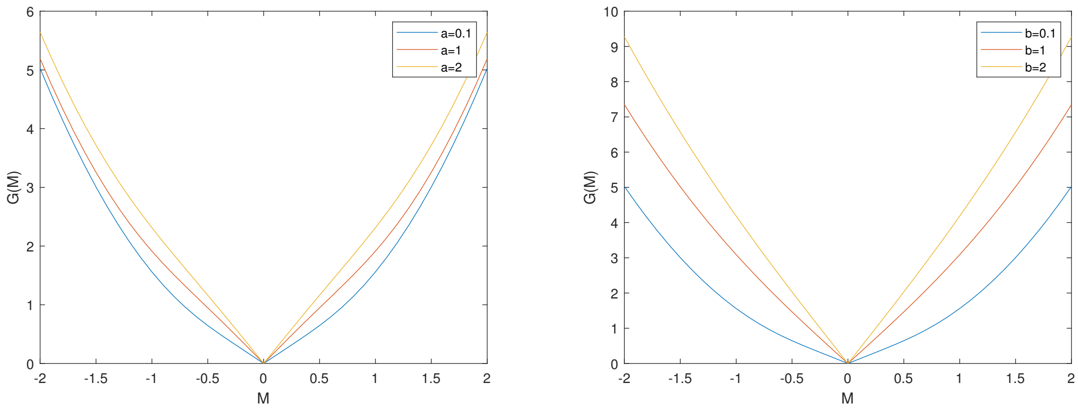
Moreover, we investigate the modulation Instability gain spectrum $G(W)$, which is determined by the maximum absolute value for the imaginary part of the wave number and defined as

$$G(M) = 2\,\mathrm{Im}(w) = \frac{i2M\sqrt{6b + M^2 + \beta_2}}{\sqrt{3}}, \tag{5.14}$$

and

$$G(M) = 2\,\mathrm{Im}(w) = \frac{2}{\sqrt{6}}\,\mathrm{Im}\left(\sqrt{M^2\left(2M^2 - \beta_2 - 6b\right) - M^2\sqrt{24a + 36b^2 + M^2\left(48b + 16M^2\right) + \beta_2\left(12b + 8M^2 + \beta_2\right)}}\right). \tag{5.15}$$

The effect of the arbitrary constants a and b are illustrated graphically as seen in Figures (6) and (7).



(a) **(b)**

Figure 6. 2-D figure of gain spectrum $G(M)$ for different value of parameters.

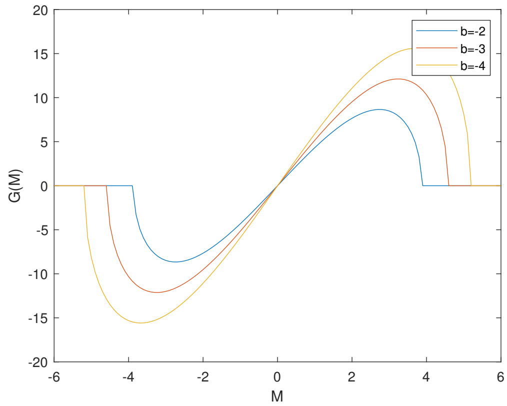


Figure 7. 2-D figure of gain spectrum $G(M)$, when $\beta_2 = -3$.

6. Conclusions

In this research, we constructed the new periodic, singular solutions of the coupled Schrödinger-Boussinesq system with beta derivative via a modified auxiliary expansion method. We found and investigated the several new family's solutions and one family are shown graphically in 2-D and 3-D; to more understands their physical characteristics. The novel solutions included hyperbolic function, trigonometric function, rational function, and constant function. The linear stability analysis of coupled Schrödinger-Boussinesq are studied and the modulation instability of two cases are analyzed. Moreover, the two cases of instability modulation and its gain spectrum are illustrated graphically. These new solutions and results might appreciate in laser and plasma sciences.

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Conflict of interest

The authors declare that they have no conflict of interest.

References

1. O. A. Ilhan, A. Esen, H. Bulut, H. M. Baskonus, Singular solitons in the pseudo-parabolic model arising in nonlinear surface waves, *Results Phys.*, **12** (2019), 1712–1715.
2. H. F. Ismael, Carreau-Casson fluids flow and heat transfer over stretching plate with internal heat source/sink and radiation, *Int. J. Adv. Appl. Sci. J.*, **4** (2017), 11–15.
3. A. Zeeshan, H. F. Ismael, M. A. Yousif, T. Mahmood, S. U. Rahman, Simultaneous effects of slip and wall stretching/shrinking on radiative flow of magneto nanofluid through porous medium, *J. Magn.*, **23** (2018), 491–498.

4. H. F. Ismael, K. K. Ali, MHD casson flow over an unsteady stretching sheet, *Adv. Appl. Fluid Mech.,* **20** (2017), 533–541.
5. M. A. Yousif, B. A. Mahmood, K. K. Ali, H. F. Ismael, Numerical simulation using the homotopy perturbation method for a thin liquid film over an unsteady stretching sheet, *Int. J. Pure Appl. Math.,* **107** (2016), 289–300.
6. H. M. Baskonus, H. Bulut, On the numerical solutions of some fractional ordinary differential equations by fractional Adams-Bashforth-Moulton method, *Open Math.,* **13** (2015), 4255–4270.
7. K. K. Ali, R. Yilmazer, H. M. Baskonus, H. Bulut, New wave behaviors and stability analysis of the Gilson-Pickering equation in plasma physics, *Indian J. Phys.,* **95** (2021), 1003–1008.
8. H. Ismael, H. Bulut, On the wave solutions of (2+1)-dimensional time-fractional Zoomeron equation, *Konuralp J. Math.,* **8** (2020), 410–418.
9. C. Cattani, T. A. Sulaiman, H. M. Baskonus, H. Bulut, On the soliton solutions to the Nizhnik-Novikov-Veselov and the Drinfel'd-Sokolov systems, *Opt. Quant. Electron.,* **50** (2018), 1–11.
10. H. Bulut, T. A. Sulaiman, H. M. Baskonus, Dark, bright optical and other solitons with conformable space-time fractional second-order spatiotemporal dispersion, *Optik,* **163** (2018), 1–7.
11. J. Manafian, M. Foroutan, A. Guzali, Applications of the ETEM for obtaining optical soliton solutions for the Lakshmanan-Porsezian-Daniel model, *Eur. Phys. J. Plus,* **132** (2017), 1–22.
12. W. Liu, D. Q. Qiu, Z. W. Wu, J. S. He, Dynamical behavior of solution in integrable nonlocal Lakshmanan-Porsezian-Daniel equation, *Commun. Theor. Phys.,* **65** (2016), 671.
13. X. B. Wang, B. Han, Vector nonlinear waves in a two-component Bose-Einstein condensate system, *J. Phys. Soc. Japan,* **89** (2020), 124003.
14. A. Biswas, A. H. Kara, R. T. Alqahtani, M. Z. Ullah, H. Triki, M. Belic, Conservation laws for optical solitons of lakshmanan-porsezian-daniel model, *Proc. Roman. Acad. Ser. A,* **19** (2018), 39–44.
15. H. M. Baskonus, G. Yel, H. Bulut, Novel wave surfaces to the fractional Zakharov-Kuznetsov-Benjamin-Bona-Mahony equation, *AIP Conf. Proc.,* **1863** (2017), 560084.
16. H. M. Baskonus, H. Bulut, Exponential prototype structures for (2+1)-dimensional Boiti-Leon-Pempinelli systems in mathematical physics, *Waves Random Complex Media,* **26** (2016), 189–196.
17. H. M. Baskonus, H. Bulut, On the complex structures of Kundu-Eckhaus equation via improved Bernoulli sub-equation function method, *Waves Random Complex Media,* **25** (2015), 720–728.
18. A. Javid, N. Raza, Singular and dark optical solitons to the well posed Lakshmanan-Porsezian-Daniel model, *Optik,* **171** (2018), 120–129.
19. J. Vega-Guzman, A. Biswas, M. F. Mahmood, Q. Zhou, S. P. Moshokoa, M. Belic, Optical solitons with polarization mode dispersion for Lakshmanan-Porsezian-Daniel model by the method of undetermined coefficients, *Optik,* **171** (2018), 114–119.
20. Z. Hammouch, T. Mekkaoui, Traveling-wave solutions of the generalized Zakharov equation with time-space fractional derivatives, *J. MESA,* **5** (2014), 489–498.
21. A. Biswas, Y. Yıldırım, E. Yaşar, R. T. Alqahtani, Optical solitons for Lakshmanan-Porsezian-Daniel model with dual-dispersion by trial equation method, *Optik,* **168** (2018), 432–439.

22. A. Biswas, M. Ekicid, A. Sonmezoglud, H. Trikie, F. B. Majida, Q. Zhou, et al., Optical solitons with Lakshmanan-Porsezian-Daniel model using a couple of integration schemes, *Optik,* **158** (2018), 705–711.

23. C. M. Khalique, I. E. Mhlanga, Travelling waves and conservation laws of a (2+1)-dimensional coupling system with Korteweg-de Vries equation, *Appl. Math. Nonlinear Sci.,* **3** (2018), 241–254.

24. J. Manafian, M. Lakestani, A. Bekir, Study of the analytical treatment of the (2+1)-dimensional zoomeron, the duffing and the SRLW equations via a new analytical approach, *Int. J. Appl. Comput. Math.,* **2** (2016), 243–268.

25. Z. Hammouch, T. Mekkaoui, P. Agarwal, Optical solitons for the Calogero-Bogoyavlenskii-Schiff equation in (2+1) dimensions with time-fractional conformable derivative, *Eur. Phys. J. Plus,* **133** (2018), 1–6.

26. M. Dewasurendra, K. Vajravelu, On the method of inverse mapping for solutions of coupled systems of nonlinear differential equations arising in nanofluid flow, heat and mass transfer, *Appl. Math. Nonlinear Sci.,* **3** (2018), 1–14.

27. X. B. Wang, S. F. Tian, T. T. Zhang, Characteristics of the breather and rogue waves in a (2+1)-dimensional nonlinear Schrödinger equation, *Proc. Am. Math. Soc.,* **146** (2018), 3353–3365.

28. L. D. Moleleki, T. Motsepa, C. M. Khalique, Solutions and conservation laws of a generalized second extended (3+1)-dimensional Jimbo-Miwa equation, *Appl. Math. Nonlinear Sci.,* **3** (2018), 459–474.

29. A. J. M. Jawad, M. J. Abu-AlShaeer, A. Biswas, Q. Zhou, S. Moshokoa, M. Belic, Optical solitons to Lakshmanan-Porsezian-Daniel model for three nonlinear forms, *Optik,* **160** (2018), 197–202.

30. X. B. Wang, S. F. Tian, L. L. Feng, T. T. Zhang, On quasi-periodic waves and rogue waves to the (4+1)-dimensional nonlinear Fokas equation, *J. Math. Phys.,* **59** (2018), 073505.

31. H. Yépez-Martínez, J. F. Gómez-Aguilar, M-derivative applied to the soliton solutions for the Lakshmanan-Porsezian-Daniel equation with dual-dispersion for optical fibers, *Opt. Quant. Electron.,* **51** (2019), 31.

32. X. Yang, Y. Yang, C. Cattani, C. M. Zhu, A new technique for solving the 1-D Burgers equation, *Therm. Sci.,* **21** (2017), 129–136.

33. J. J. Mao, S. F. Tian, L. Zou, T. T. Zhang, Stability analysis, optical solitons and complexitons of the two-dimensional complex Ginzburg-Landau equation, *J. Electromagn. Waves Appl.,* **33** (2019), 1224–1238.

34. K. Khan, M. A. Akbar, The $\exp(-\Phi(\xi))$-Expansion method for finding travelling wave solutions of Vakhnenko-Parkes equation, *Int. J. Dyn. Syst. Differ. Equations,* **5** (2014), 72–83.

35. C. Cattani, T. A. Sulaiman, H. M. Baskonus, H. Bulut, Solitons in an inhomogeneous Murnaghan's rod, *Eur. Phys. J. Plus,* **133** (2018), 1–11.

36. J. J. Mao, S. F. Tian, L. Zou, T. T. Zhang, X. J. Yan, Bilinear formalism, lump solution, lumpoff and instanton/rogue wave solution of a (3+1)-dimensional B-type Kadomtsev-Petviashvili equation, *Nonlinear Dyn.,* **95** (2019), 3005–3017.

37. D. Liu, X. Ju, O. A. Ilhan, J. Manafian, H. F. Ismael, Multi-waves, breathers, periodic and cross-kink solutions to the (2+1)-dimensional variable-coefficient Caudrey-Dodd-Gibbon-Kotera-Sawada equation, *J. Ocean Univ. China,* **20** (2020), 35–44.

38. X. W. Yan, S. F. Tian, M. J. Dong, L. Zhou, T. T. Zhang, Characteristics of solitary wave, homoclinic breather wave and rogue wave solutions in a (2+1)-dimensional generalized breaking soliton equation, *Comput. Math. Appl.,* **76** (2018), 179–186.

39. X. W. Yan, S. F. Tian, M. J. Dong, L. Zou, Bäcklund transformation, rogue wave solutions and interaction phenomena for a (3+1)-dimensional B-type Kadomtsev-Petviashvili-Boussinesq equation, *Nonlinear Dyn.,* **92** (2018), 709–720.

40. S. F. Tian, H. Q. Zhang, On the integrability of a generalized variable-coefficient Kadomtsev-Petviashvili equation, *J. Phys. A: Math. Theor.,* **45** (2012), 055203.

41. X. B. Wang, S. F. Tian, C. Y. Qin, T. T. Zhang, Lie symmetry analysis, conservation laws and exact solutions of the generalized time fractional Burgers equation, *Europhys. Lett.,* **114** (2016), 20003.

42. S. Tian, H. Zhang, On the integrability of a generalized variable-coefficient forced Korteweg-de Vries equation in fluids, *Stud. Appl. Math.,* **132** (2014), 212–246.

43. B. H. Wang, Y. Y. Wang, C. Q. Dai, Y. X. Chen, Dynamical characteristic of analytical fractional solitons for the space-time fractional Fokas-Lenells equation, *Alex. Eng. J.,* **59** (2020), 4699–4707.

44. B. H. Wang, Y. Y. Wang, C. Q. Dai, Fractional optical solitons with stochastic properties of a wick-type stochastic fractional NLSE driven by the Brownian motion, *Waves Random Complex Media,* (2021), 1–14. Available from: `https://doi.org/10.1080/17455030.2021.1905910`.

45. A. Nabti, B. Ghanbari, Global stability analysis of a fractional SVEIR epidemic model, *Math. Methods Appl. Sci.,* 2021. Available from: `https://doi.org/10.1002/mma.7285`.

46. H. F. Ismael, H. M. Baskonus, H. Bulut, Abundant novel solutions of the conformable Lakshmanan-Porsezian-Daniel model, *Discrete Contin. Dyn. Syst.-S,* 2020. Available from: `https://doi.org/10.3934/dcdss.2020398`.

47. P. Lu, B. Wang, C. Dai, Fractional traveling wave solutions of the (2+1)-dimensional fractional complex Ginzburg-Landau equation via two methods, *Math. Methods Appl. Sci.,* **43** (2020), 8518–8526.

48. J. J. Fang, D. S. Mou, Y. Y. Wang, H. C. Zhang, C. Q. Dai, Y. X. Chen, Soliton dynamics based on exact solutions of conformable fractional discrete complex cubic Ginzburg-Landau equation, *Results Phys.,* **20** (2021), 103710.

49. B. Ghanbari, S. Kumar, A study on fractional predator-prey-pathogen model with Mittag-Leffler kernel-based operators, *Numer. Meth. Part. D. E.,* 2020. Available from: `https://doi.org/10.1002/num.22689`.

50. B. Ghanbari, On the modeling of an eco-epidemiological model using a new fractional operator, *Results Phys.,* **21** (2021), 103799.

51. L. J. Yu, G. Z. Wu, Y. Y. Wang, Y. X. Chen, Traveling wave solutions constructed by Mittag-Leffler function of a (2+1)-dimensional space-time fractional NLS equation, *Results Phys.,* **17** (2020), 103156.

52. J. Boussinesq, Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond., *J. Math. Pures Appl.,* (1872), 55–108. Available from: `http://eudml.org/doc/234248`.
53. J. Manafian, M. F. Aghdaei, Abundant soliton solutions for the coupled Schrödinger-Boussinesq system via an analytical method, *Eur. Phys. J. Plus,* **131** (2016), 1–29.
54. M. S. Osman, J. A. T. Machado, D. Baleanu, On nonautonomous complex wave solutions described by the coupled Schrödinger-Boussinesq equation with variable-coefficients, *Opt. Quant. Electron.,* **50** (2018), 1–11.
55. G. Mu, Z. Qin, Rogue waves for the coupled Schrödinger-Boussinesq equation and the coupled Higgs equation, *J. Phys. Soc. Japan,* **81** (2012), 084001.
56. D. Bai, J. Wang, The time-splitting Fourier spectral method for the coupled Schrödinger-Boussinesq equations, *Commun. Nonlinear Sci. Numer. Simul.,* **17** (2012), 1201–1210.
57. S. S. Ray, A novel approach with time-splitting spectral technique for the coupled Schrödinger-Boussinesq equations involving Riesz fractional derivative, *Commun. Theor. Phys.,* **68** (2017), 301.
58. C. Banquet, L. C. F. Ferreira, E. J. Villamizar-Roa, On the Schrödinger-Boussinesq system with singular initial data, *J. Math. Anal. Appl.,* **400** (2013), 487–496.
59. Z. F. Liang, Modulational instability and stationary waves for the coupled generalized Schrödinger-Boussinesq system, *Z. Naturforsch. A,* **66** (2011), 143–150.
60. A. Kılıcman, R. Abazari, *Travelling wave solutions of the Schrödinger-Boussinesq system*, In: *Abstract and Applied Analysis,* Hindawi, 2012.
61. A. Atangana, D. Baleanu, New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model, *Therm. Sci.,* **20** (2016), 763–769.
62. J. Manafian, M. F. Aghdaei, Abundant soliton solutions for the coupled Schrödinger-Boussinesq system via an analytical method, *Eur. Phys. J. Plus,* **131** (2016), 1–29.
63. X. Wen, Construction of new exact rational form non-travelling wave solutions to the (2+1)-dimensional generalized Broer-Kaup system, *Appl. Math. Comput.,* **217** (2010), 1367–1375.
64. D. Guo, S. F. Tian, T. T. Zhang, J. Li, Modulation instability analysis and soliton solutions of an integrable coupled nonlinear Schrödinger system, *Nonlinear Dyn.,* **94** (2018), 2749–2761.