Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(7): 7895–7908.
DOI:10.3934/math.2021458
Received: 20 March 2021
Accepted: 12 May 2021
Published: 18 May 2021

Research article

A new type of Kannan's fixed point theorem in strong b- metric spaces

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Abstract: In this paper, we prove some generalizations of Kannan-type fixed point theorems for singlevalued and multivalued mappings defined on a complete strong b- metric space in terms of a Suzuki-type contraction. Our results extend a result of Górnicki [10]. Furthermore, after each theorem are exemples and corollaries respectively.

Keywords: fixed point; complete strong b- metric space; strong b- metric spaces; Kannan mapping; Kannan type mapping
Mathematics Subject Classification: 47H10, 54H25

1. Introduction and preliminaries

We know that most of the theorems such as Banach's [1], Benavides's et al. [2], Caristi's [3], Ciric's [5], Ekeland's [8, 9], Kirk's [14, 15], Meir's et al. [17], Nadler's et al. [18], Subrahmanyam's [22], Suzuki's [23–25] belong to Leader type, i.e. mapping T has a unique fixed point and $\{T^n x\}$ converges to the fixed point for all $x \in X$. Notice that such a mapping is called a Picard operator in [20]. That are the pivotal results in nonlinear analysis and has many useful applications and generalizations, but every contraction mapping is a continuous function. In 1968, Kannan [12] was the first proved the following result.

Theorem 1.1. *[12] Let (X, d) be a complete metric space and T be a self-mapping on X satisfying*

$$d(Tx, Ty) \leq r\{d(x, Tx) + d(y, Ty)\},$$

for all $x, y \in X$ and $r \in [0, \frac{1}{2})$. Then, T has a unique fixed point $\bar{x} \in X$ and for any $x \in X$, the sequence of iterates $\{T^n x\}$ converges to $\bar{x}$.

The mapping satisfying the contraction conditions of the above theorem is called Kannan mapping and which is not necessarily continuous. Another important meaning of Kannan mapping is being able to describe the completeness of space in terms of the fixed point property of the mapping. This was proved by Subrahmanyam [27] in 1975, this is a metric space (X, d) is complete if and only if every Kannan mapping has a unique fixed point in X. Contractions (in the sense of Banach) do not have this property. Also, several mathematicians have studied the metric completeness. For example, Kirk [13] proved that Caristi's fixed point theorem [3,4] characterizes the metric completeness. For other results in this setting, see [6,11,19,21] and others. In 2018, Górnicki [10] proved the following result.

Let $\mathcal{S}$ denote the class of functions which satisfy the simple condition

$$\mathcal{S} = \{f : (0, \infty) \to [0, \frac{1}{2}) : f(t_n) \to \frac{1}{2} \text{ implies } t_n \to 0 \text{ as } n \to \infty\}.$$

We do not assume that f is continuous in any sense.

Theorem 1.2. *[10] Let (X, d) be a complete metric spase, let $T : X \to X$, and suppose there exists $f \in \mathcal{S}$ such that for each $x, y \in X$ with $x \neq y$,*

$$d(Tx, Ty) \leq f(d(x, y))\{d(x, Tx) + d(y, Ty)\}.$$

Then, T has a unique fixed point $\bar{x} \in X$ and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to $\bar{x}$.

Another view of Suzuki [26] in 2007, his has proved the following fixed point theorem.

Theorem 1.3. *[26] Let (X, d) be a complete metric space and let $T : X \to X$. Define a nonincreasing function $\theta : [0, 1) \to (\frac{1}{2}, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ (1-r)r^{-2} & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq 2^{-\frac{1}{2}}, \\ (1+r)r^{-1} & \text{if } 2^{-\frac{1}{2}} \leq r < 1. \end{cases}$$

Assume that there exists $r \in [0, 1)$ such that

$$\theta(r)d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq rd(x, y)$$

for all $x, y \in X$. Then, T has a unique fixed point $\bar{x} \in X$ and for any $x \in X$, the sequence of iterates $\{T^n x\}$ converges to $\bar{x}$.

In this article, our idea comes from the results in [12] to extend the result in [10] for a class of contractive mappings in strong b- metric spaces. Moreover, we prove new version fixed point theorems for singlevalued and multivalued mappings as combining the results in [12] and [26]. We first recall some concepts in strong b- metric spaces.

Definition 1.4. [16] Let X be a nonempty set and $K \geq 1$. A mapping $D : X \times X \to [0; +\infty)$ is called a strong b-metric on X if

(D1) $D(x, y) = 0$ if and only if $x = y$;

(D2) $D(x, y) = D(y, x)$ for all $x, y \in X$;

(D3) $D(x, y) \leqslant D(x, z) + KD(z, y)$ for all $x, y, z \in X$.

Then (X, D, K) is called a strong b- metric space.

Definition 1.5. [16] Let (X, D, K) be a strong b- metric spase. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) A sequence $\{x_n\}$ is called convergent to x if $\lim_{n\to\infty} D(x_n, x) = 0$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) A sequence $\{x_n\}$ is called Cauchy sequence in X if $\lim_{n,m\to\infty} D(x_n, x_m) = 0$.

(iii) The strong b- metric space (X, D, K) is called complete if every Cauchy sequence in X is converges.

Proposition 1.6. *[16] Let (X, D, K) be a strong b- metric spase and $\{x_n\}$ be a sequence in X. Then*

(1) *If $\{x_n\}$ converges to $x \in X$ and $\{x_n\}$ converges to $y \in X$, then $x = y$.*

(2) *If $\lim_{n\to\infty} x_n = x \in X$ and $\lim_{n\to\infty} y_n = y \in X$, then $\lim_{n\to\infty} D(x_n, y_n) = D(x, y)$.*

Proposition 1.7. *[16] Let $\{x_n\}$ be a sequence in a strong b- metric spase and suppose*

$$\sum_{n=1}^{\infty} D(x_n, x_{n+1}) < +\infty.$$

Then $\{x_n\}$ is a Cauchy sequence.

2. Control function for mappings singlevalue

Using a Kannan-type contraction, we obtain the following generalization of Theorem 1.2.

Theorem 2.1. *Let (X, D, K) be a complete strong b- metric space, let $T : X \to X$ be a mapping and suppose there exists $f \in \mathcal{S}$ such that for each $x, y \in X$ with $x \neq y$,*

$$D(Tx, Ty) \leq f(D(x, y))\{D(x, Tx) + D(y, Ty)\}.$$

Then, T has a unique fixed point $\bar{x} \in X$ and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to $\bar{x}$.

Proof. Fix $x_0 \in X$ and define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. Assume that there exists n such that $x_{n+1} = x_n$ then x_n is the fixed point of T. Therefore, suppose that $x_{n+1} \neq x_n$ for all $n \geq 0$. Set $D_n = D(x_n, x_{n+1})$ for all $n \geq 0$. By hypothesis, we have

$$\begin{aligned} D_{n+1} &= D(x_{n+1}, x_{n+2}) \\ &= D(Tx_n, Tx_{n+1}) \\ &\leq f(D(x_n, x_{n+1}))\{D(x_n, Tx_n) + D(x_{n+1}, Tx_{n+1})\} \\ &< \frac{1}{2}\{D(x_n, Tx_n) + D(x_{n+1}, Tx_{n+1})\} \\ &= \frac{1}{2}\{D_n + D_{n+1}\}, \end{aligned}$$

so $D_{n+1} < D_n$ for all $n \geq 0$. Hence $\{D_n\}$ is monotonic decreasing and bounded below, so there exists $\eta \geq 0$ such that

$$\lim_{n\to\infty} D_n = \eta.$$

Assume $\eta > 0$. Then by hypothesis, we have

$$D(x_{n+1}, x_{n+2}) \leq f(D(x_n, x_{n+1}))\{D(x_n, x_{n+1}) + D(x_{n+1}, x_{n+2})\} \text{ for all } n \geq 0,$$

which deduces

$$\frac{D_{n+1}}{D_n + D_{n+1}} \leq f(D_n) \text{ for all } n \geq 0.$$

Letting $n \to \infty$, we obtain $\lim_{n\to\infty} f(D_n) \geq \frac{1}{2}$, and since$f \in \mathcal{S}$ this in turn implies $\eta = 0$. So $\lim_{n\to\infty} D_n = 0$. On the other hand, with $m \neq n$ we have

$$\begin{aligned} D(x_{n+1}, x_{m+1}) &\leq f(D(x_n, x_m))\{D(x_n, x_{n+1}) + D(x_m, x_{m+1})\} \\ &< \frac{1}{2}\{D_n + D_m)\} \to 0, \end{aligned}$$

as $n, m \to \infty$, so $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there is $\bar{x} \in X$ such that $\lim_{n\to\infty} x_n = \bar{x}$. Since

$$\begin{aligned} D(T\bar{x}, \bar{x}) &\leq D(Tx_n, T\bar{x}) + KD(Tx_n, \bar{x}) \\ &\leq f(D(x_n, \bar{x}))\{D(x_n, Tx_n) + D(\bar{x}, T\bar{x})\} + KD(x_{n+1}, \bar{x}) \end{aligned}$$

implies

$$D(T\bar{x}, \bar{x}) \leq \frac{f(D(x_n, \bar{x}))}{1 - f(D(x_n, \bar{x}))} D_n + \frac{K}{1 - f(D(x_n, \bar{x}))} D(x_{n+1}, \bar{x}) \to 0$$

as $n \to \infty$. Hence, $T\bar{x} = \bar{x}$. Suppose $\bar{y}$ is another fixed point of T. By hypothesis, we have

$$D(\bar{x}, \bar{y}) = D(T\bar{x}, T\bar{y}) \leq f(D(\bar{x}, \bar{y}))\{D(\bar{x}, T\bar{x}) + D(\bar{y}, T\bar{y})\} = 0.$$

So $D(\bar{x}, \bar{y}) = 0$ implies $\bar{x} = \bar{y}$. Hence, T has a unique fixed point $\bar{x} \in X$. □

If in Theorem 2.1 we take $K = 1$ then strong b- metric space is a usual metric spase, then we obtain the following corollaries.

Corollary 2.2. (Theorem 5.1, [10]) *Let (X, d) be a complete metric spase, let $T : X \to X$ be a mapping and suppose there exists $f \in \mathcal{S}$ such that for each $x, y \in X$ with $x \neq y$,*

$$d(Tx, Ty) \leq f(d(x, y))\{d(x, Tx) + d(y, Ty)\}.$$

Then, T has a unique fixed point $\bar{x} \in X$ and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to $\bar{x}$.

Example 2.3. Let $X = \{0, 1, 2\}$ and let $D : X \times X \to [0, +\infty)$ by

$$D(0, 0) = D(1, 1) = D(2, 2) = 0,$$

$$D(0, 1) = D(1, 0) = \frac{1}{2},$$

$$D(0, 2) = D(2, 0) = 6,$$

$$D(1,2) = D(2,1) = 5.$$

Then $(X, D, K = 2)$ is a strong b-metric space, but it is not metric space since $6 = D(2,0) > D(2,1) + D(1,0) = \frac{11}{2}$. Hence, Theorem 1.2 can't be applied. Let $T : X \to X$ by $T0 = 0, T1 = 0, T2 = 1$ and the function $f \in \mathcal{S}$ give by $f(t) = \frac{1}{2}e^{\frac{-t}{6}}, t > 0$ and $f(0) \in [0, \frac{1}{2})$. Then

$$D(T0,T1) = D(0,0) = 0 < \frac{1}{4}e^{\frac{-1}{12}} = f(D(0,1))\{D(0,T0) + D(1,T1)\},$$

$$D(T1,T2) = D(0,1) = \frac{1}{2} < \frac{11}{4}e^{\frac{-5}{6}} = f(D(1,2))\{D(1,T1) + D(2,T2)\},$$

$$D(T2,T0) = D(1,0) = \frac{1}{2} < \frac{5}{2e} = f(D(2,0))\{D(2,T2) + D(0,T0)\},$$

Therefore T satisfies all the conditions of Theorem 2.1. It is see that T has a unique fixed point $\bar{x} = 0$.

For the use in strong b- metric spaces we will consider the class of functions

$$\mathcal{F}_q = \{\psi : (0,\infty) \to [0,q) : \psi(t_n) \to q \text{ implies } t_n \to 0 \text{ as } n \to \infty\},$$

where $q \in (0, \frac{1}{2})$. We do not assume that ψ is continuous in any sense.

Theorem 2.4. *Let (X, D, K) be a complete strong b- metric space and let $T : X \to X$ be a mapping. Suppose there exists $\psi \in \mathcal{F}_q$ satisfying*

$$\frac{1}{K+1}D(x,Tx) \le D(x,y)$$

implies

$$D(Tx,Ty) \le \psi(D(x,y))\{D(x,Tx) + D(y,Ty)\}.$$

for all $x, y \in X$ with $x \neq y$. Then, T has a unique fixed point $\bar{x} \in X$ and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to $\bar{x}$.

Proof. Fix $x_0 \in X$ and define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \ge 0$. Assume that there exists n such that $x_{n+1} = x_n$ then x_n is the fixed point of T. Therefore, suppose that $x_{n+1} \neq x_n$ for all $n \ge 0$. Set $D_n = D(x_n, x_{n+1})$ for all $n \ge 0$. Since

$$\frac{1}{K+1}D(x_n,Tx_n) = \frac{1}{K+1}D(x_n,x_{n+1}) \le D(x_n,x_{n+1}),$$

and by hypothesis, we have

$$\begin{aligned} D_{n+1} &= D(x_{n+1},x_{n+2}) \\ &= D(Tx_n,Tx_{n+1}) \\ &\le \psi(D(x_n,x_{n+1}))\{D(x_n,Tx_n) + D(x_{n+1},Tx_{n+1})\} \\ &< q\{D(x_n,Tx_n) + D(x_{n+1},Tx_{n+1})\} \\ &= q\{D_n + D_{n+1}\}, \end{aligned}$$

so

$$D_{n+1} < \frac{q}{1-q}D_n = hD_n, \text{ where } h = \frac{q}{1-q} \in (0,1).$$

Thus,

$$D_n < h^n D_0 \text{ for all } n \geq 1.$$

Hence,

$$\sum_{n=1}^{\infty} D_n \leq D_0 \sum_{n=1}^{\infty} h^n < +\infty.$$

By Proposition 1.7, we have $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $\bar{x} \in X$ such that $\lim_{n\to\infty} x_n = \bar{x} \in X$. Now, we show that for any $n \geq 0$, either

$$\frac{1}{K+1} D(x_n, Tx_n) \leq D(x_n, \bar{x}) \text{ or } \frac{1}{K+1} D(Tx_n, Tx_{n+1}) \leq D(Tx_n, \bar{x}). \tag{2.1}$$

Arguing by contradiction, we suppose that for some $n \geq 0$,

$$D(x_n, \bar{x}) < \frac{1}{K+1} D(x_n, Tx_n)$$

and

$$D(Tx_n, \bar{x}) < \frac{1}{K+1} D(Tx_n, Tx_{n+1}).$$

Then, by the triangle inequality, we have

$$\begin{aligned} D_n &= D(x_n, Tx_n) \\ &\leq D(x_n, x^*) + KD(Tx_n, x^*) \\ &< \frac{1}{K+1} D(x_n, Tx_n) + \frac{K}{K+1} D(Tx_n, Tx_{n+1}) \\ &= \frac{1}{K+1} D_n + \frac{K}{K+1} D_{n+1} \\ &\leq D_n. \end{aligned}$$

This is a contradiction. Hence, from Equation (2.1) for any $n \geq 0$ we have, either

$$D(x_{n+1}, T\bar{x}) \leq \psi(D(x_n, \bar{x}))\{D(x_n, Tx_n) + D(\bar{x}, T\bar{x})\}, \tag{2.2}$$

or

$$D(x_{n+2}, T\bar{x}) \leq \psi(D(x_{n+1}, \bar{x}))\{D(x_{n+1}, Tx_{n+1}) + D(\bar{x}, T\bar{x})\}. \tag{2.3}$$

Then, either (2.2) holds for infinity natural numbers n or (2.3) holds for infinity natural number n. Suppose (2.2) holds for infinity natural numbers n. We can choose in that infinity set the sequence $\{n_k\}$ is monotone strictly increasing sequence of natural numbers. Therefore, sequence $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and

$$\begin{aligned} D(x_{n_k+1}, T\bar{x}) &\leq \psi(D(x_{n_k}, \bar{x}))\{D(x_{n_k}, Tx_{n_k}) + D(\bar{x}, T\bar{x})\} \\ &< q\{D(x_{n_k}, \bar{x}) + 2KD(x_{n_k+1}, \bar{x}) + D(x_{n_k+1}, T\bar{x})\}. \end{aligned}$$

This is equivalent with

$$D(x_{n_k+1}, T\bar{x}) < \frac{q}{1-q}\{D(x_{n_k}, \bar{x}) + 2KD(x_{n_k+1}, \bar{x})\}.$$

Letting $k \to \infty$ and because x_{n_k+1} is converge $\bar{x}$ we have $\lim_{k\to\infty} x_{n_k+1} = T\bar{x}$ thus $T\bar{x} = \bar{x}$. If (2.3) holds for infinity natural numbers n, by using an argument similar to that of above we have $\bar{x}$ is a fixed point of T. Suppose $\bar{y}$ is another fixed point of T. Then

$$0 = \frac{1}{K+1} D(\bar{x}, T\bar{x}) \leq D(\bar{x}, \bar{y}),$$

and by hypothesis, we have

$$D(\bar{x}, \bar{y}) = D(T\bar{x}, T\bar{y}) \quad \leq \quad \psi(D(\bar{x}, \bar{y}))\{D(\bar{x}, T\bar{x}) + D(\bar{y}, T\bar{y})\} = 0.$$

So $D(\bar{x}, \bar{y}) = 0$ implies $\bar{x} = \bar{y}$. Hence, T has a unique fixed point $\bar{x} \in X$. □

Example 2.5. Let $X = \{0, 1, 2\}$ and let $D : X \times X \to [0, +\infty)$ be defined by $D(x, y) = (x - y)^2$. Then $(X, D, K = 3)$ is a complete strong b- metric space.

Let $T : X \to X$ be defined by $T0 = 1, T1 = 1, T2 = 0$ and the function $\psi(t) = \frac{1}{3} e^{\frac{-t}{8}}, t > 0$, and $\psi(0) \in [0, \frac{1}{3})$. Then $\psi \in \mathcal{F}_{\frac{1}{3}}$. Since

$$\frac{1}{4} = \frac{1}{4} D(0, T0) \leq D(0, y)$$

holds for any $y \in X \backslash \{0\}$ and

$$D(T0, T1) = D(1, 1) = 0 < \frac{1}{3} e^{-\frac{1}{8}} = \psi(D(0, 1))\{D(0, T0) + D(1, T1)\},$$

$$D(T0, T2) = D(1, 0) = 1 < \frac{5}{3} \cdot \frac{1}{\sqrt{e}} = \psi(D(0, 2))\{D(0, T0) + D(2, T2)\},$$

we have

$$\frac{1}{4} D(0, T0) \leq D(0, y) \text{ implies } D(T0, Ty) \leq \psi(D(x, y))\{D(0, T0) + D(y, Ty)\},$$

for all $y \in X \backslash \{0\}$. Again, since $0 = \frac{1}{4} D(1, T1) \leq D(1, y)$ holds for any $y \in X \backslash \{1\}$ and

$$D(T1, T0) = D(1, 1) = 0 < \frac{1}{3} e^{-\frac{1}{8}} = \psi(D(1, 0))\{D(1, T1) + D(0, T0)\},$$

$$D(T1, T2) = D(1, 0) = 1 < \frac{4}{3} e^{-\frac{1}{8}} = \psi(D(1, 2))\{D(1, T1) + D(2, T2)\},$$

then

$$\frac{1}{4} D(1, T1) \leq D(1, y) \text{ implies } D(T1, Ty) \leq \psi(D(x, y))\{D(1, T1) + D(y, Ty)\},$$

for all $y \in X \backslash \{1\}$. Finally, by $\frac{1}{4} D(2, T2) = 1 \leq D(2, y)$ if and only if $y \in X \backslash \{2\}$ and

$$D(T2, T0) = D(0, 1) = 1 < \frac{5}{3} \cdot \frac{1}{\sqrt{e}} = \psi(D(2, 0))\{D(2, T2) + D(0, T0)\},$$

$$D(T2, T1) = D(0, 1) = 1 < \frac{4}{3} e^{-\frac{1}{8}} = \psi(D(2, 1))\{D(2, T2) + D(1, T1)\},$$

then

$$\frac{1}{4} D(2, T2) \leq D(2, y) \text{ implies } D(T2, Ty) \leq \psi(D(x, y))\{D(2, T2) + D(y, Ty)\},$$

for all $y \in X \backslash \{2\}$. Therefore T satisfies all the conditions of Theorem 2.4. Hence, T has a unique fixed point $\bar{x} = 1$.

Question 2.6. Does there exist $q = \frac{1}{2}$ such that mapping T in Theorem 2.4 has a fixed point free?

Let $\mathcal{H}$ denote the class of functions which satisfy the simple condition

$$\mathcal{H} = \{\varphi : (0, \infty) \to [0, \frac{1}{3}) : \varphi(t_n) \to \frac{1}{3} \text{ implies } t_n \to 0 \text{ as } n \to \infty\}.$$

We do not assume that φ is continuous in any sense.

Theorem 2.7. *Let (X, D, K) be a complete strong b- metric space, let $T : X \to X$, and suppose there exists $\varphi \in \mathcal{H}$ such that for each $x, y \in X$ with $x \neq y$,*

$$D(Tx, Ty) \leq \varphi(D(x, y))\{D(x, Tx) + D(y, Ty) + D(x, y)\}.$$

Then, T has a unique fixed point $\bar{x} \in X$ and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to $\bar{x}$.

Proof. Fix $x_0 \in X$ and define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. Assume that there exists n such that $x_{n+1} = x_n$ then x_n is the fixed point of T. Therefore, suppose that $x_{n+1} \neq x_n$ for all $n \geq 0$. Set $D_n = D(x_n, x_{n+1})$ for all $n \geq 0$. By hypothesis, we have

$$\begin{aligned}
D_{n+1} &= D(x_{n+1}, x_{n+2}) \\
&= D(Tx_n, Tx_{n+1}) \\
&\leq \varphi(D(x_n, x_{n+1}))\{D(x_n, Tx_n) + D(x_{n+1}, Tx_{n+1}) + D(x_n, x_{n+1})\} \\
&< \frac{1}{3}\{D(x_n, Tx_n) + D(x_{n+1}, Tx_{n+1}) + D(x_n, x_{n+1})\} \\
&= \frac{1}{3}\{2D_n + D_{n+1}\},
\end{aligned}$$

so $D_{n+1} < D_n$ for all n. Hence $\{D_n\}$ is monotonic decreasing and bounded below. So there exists $\eta \geq 0$ such that

$$\lim_{n\to\infty} D_n = \eta.$$

Assume $\eta > 0$. By hypothesis, we have

$$D(x_{n+1}, x_{n+2}) \leq \varphi(D(x_n, x_{n+1}))\{2D(x_n, x_{n+1}) + D(x_{n+1}, x_{n+2})\} \text{ for all } n \geq 0,$$

which deduces

$$\frac{D_{n+1}}{2D_n + D_{n+1}} \leq \varphi(D_n) \text{ for all } n \geq 0.$$

Letting $n \to \infty$, we obtain $\lim_{n\to\infty} \varphi(D_n) \geq \frac{1}{3}$, and since $\varphi \in \mathcal{H}$ this in turn implies $\eta = 0$. So $\lim_{n\to\infty} D_n = 0$. On the other hand, with $m \neq n$ and by hypothesis, we have

$$\begin{aligned}
D(x_{n+1}, x_{m+1}) &\leq \varphi(D(x_n, x_m))\{D(x_n, x_{n+1}) + D(x_m, x_{m+1}) + D(x_n, x_m)\} \\
&\leq \frac{1}{3}\{D(x_n, x_{n+1}) + D(x_m, x_{m+1}) + KD(x_n, x_{n+1}) \\
&\quad + D(x_{n+1}, x_{m+1}) + KD(x_m, x_{m+1})\},
\end{aligned}$$

we deduce

$$D(x_{n+1}, x_{m+1}) \leq \frac{K+1}{2}\Big\{D_n + D_m\Big\} \to 0,$$

as $n, m \to \infty$, so $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, then we have $\lim_{n\to\infty} x_n = \bar{x} \in X$. Then

$$\begin{aligned} D(T\bar{x}, \bar{x}) &\leq D(Tx_n, T\bar{x}) + KD(Tx_n, \bar{x}) \\ &\leq \varphi(D(x_n, \bar{x}))\{D(x_n, Tx_n) + D(\bar{x}, T\bar{x}) + D(x_n, \bar{x})\} + KD(x_{n+1}, \bar{x}). \end{aligned}$$

This implies that

$$\begin{aligned} D(T\bar{x}, \bar{x}) &\leq \frac{\varphi(D(x_n, \bar{x}))}{1-\varphi(D(x_n, \bar{x}))} D_n + \frac{\varphi(D(x_n, \bar{x}))}{1-\varphi(D(x_n, \bar{x}))} D(x_n, \bar{x}) \\ &+ \frac{K}{1-\varphi(D(x_n, \bar{x}))} D(x_{n+1}, \bar{x}) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence, $T\bar{x} = \bar{x}$. Suppose $\bar{y}$ is another fixed point of T. Then

$$D(\bar{x}, \bar{y}) = D(T\bar{x}, T\bar{y}) \leq \frac{1}{3}\{D(\bar{x}, T\bar{x}) + D(\bar{y}, T\bar{y}) + D(\bar{x}, \bar{y})\},$$

and

$$\frac{2}{3} D(\bar{x}, \bar{y}) \leq \frac{1}{3}\{D(\bar{x}, T\bar{x}) + D(\bar{y}, T\bar{y})\} = 0,$$

so $D(\bar{x}, \bar{y}) = 0$. Hence, T has a unique fixed point $\bar{x} \in X$, so for each $x \in X$ the sequence of iterates $\{T^n x\}$ converges to $\bar{x}$. □

If in Theorem 2.7 we take $K = 1$ then strong b- metric space is a usual metric spase, then we obtain the following corollaries.

Corollary 2.8. (Theorem 5.2, [10]) *Let (X, d) be a complete metric spase, let $T : X \to X$, and suppose there exists $\varphi \in \mathcal{H}$ such that for each $x, y \in X$ with $x \neq y$,*

$$d(Tx, Ty) \leq \varphi(d(x, y))\{d(x, Tx) + d(y, Ty) + d(x, y)\}.$$

Then, T has a unique fixed point $\bar{x} \in X$ and for any $x \in X$ the sequence of iterates $\{T^n x\}$ converges to $\bar{x}$.

3. A Dube-singh type fixed point theorem for multivalued mappings

Let (X, D, K) be a strong b- metric space. Let $CB(X)$ be the collection of all nonempty bounded closed subsets of X. Let $T : X \to CB(X)$ be a multivalued mapping on X. Let H be the Hausdorff metric on $CB(X)$ induced by D, that is,

$$H(A, B) := \max\{\sup_{x\in B} d(x, A); \sup_{x\in A} d(x, B)\},$$

where $A, B \in CB(X)$ and $d(x, A) := \inf_{y\in A} D(x, y)$.

In 1970, Dube and Singh [7] prove result following.

Theorem 3.1. *[7] Let (X, d) be a complete metric space. If $T : X \to CB(X)$ is a continuous multivalued mapping satisfying the relation*

$$H(Tx, Ty) \leq s\{d(x, Tx) + d(y, Ty)\}, \text{ for all } x, y \in X$$

(where $0 \leq s < \frac{1}{2}$), then T has at least one fixed point.

Lemma 3.2. *Let (X, D, K) is a strong b- metric space and $A, B \in CB(X)$. If $H(A, B) > 0$ then for each $h > 1$ and $a \in A$ there exists $b \in B$ such that*

$$D(a, b) < h \cdot H(A, B).$$

Proof. Using characterized of infimum, with $\varepsilon = (h - 1) \cdot H(A, B) > 0$ there exists $b \in B$ such that

$$D(a, b) < d(a, B) + \varepsilon.$$

On the other hand, by the definition of $H(A, B)$ we have

$$d(a, B) \leq H(A, B).$$

This which deduces

$$D(a, b) < h \cdot H(A, B).$$

□

Now, we extend above result for a class of contractive mappings in strong b- metric spaces.

Theorem 3.3. *Let (X, D, K) is a complete strong b- metric space and let $T : X \to CB(X)$ be an multivalued mapping. Suppose there exists $s \in (0, k)$ with $0 < k < \frac{1}{2}$ satisfying*

$$\frac{1}{K+1} d(x, Tx) \leq D(x, y) \text{ implies } H(Tx, Ty) \leq s\{d(x, Tx) + d(y, Ty)\},$$

for all $x, y \in X$. Then T has a unique fixed point $\bar{x} \in X$. Moreover, for each $x \in X$, the sequence of iterates $\{T^n x\}$ converges to $\bar{x}$.

Proof. Let $x_0 \in X$ and choose $x_1 \in Tx_0$.
Step 1. If $H(Tx_0, Tx_1) = 0$ then $Tx_0 = Tx_1$. Thus, x_1 is a fixed point of T. If $H(Tx_0, Tx_1) > 0$, by Lemma 3.2 then for each $h_1 > 1$, there exists $x_2 \in Tx_1$ such that

$$D(x_1, x_2) < h_1 H(Tx_0, Tx_1).$$

Step 2. Similarly, if $H(Tx_1, Tx_2) = 0$ then $Tx_1 = Tx_2$. Thus, x_2 is a fixed point of T. If $H(Tx_1, Tx_2) > 0$, by Lemma 3.2 then for each $h_2 > 1$, there exists $x_3 \in Tx_2$ such that

$$D(x_2, x_3) < h_2 H(Tx_1, Tx_2).$$

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Step n. Continuing in this manner, if $H(Tx_{n-1}, Tx_n) = 0$ then $Tx_{n-1} = Tx_n$. Thus, x_n is a fixed point of T. If $H(Tx_{n-1}, Tx_n) > 0$, by Lemma 3.2 then for each $h_n > 1$, there exists $x_{n+1} \in Tx_n$ such that

$$D(x_n, x_{n+1}) < h_n H(Tx_{n-1}, Tx_n).$$

The above process continues, if at step k satisfy $H(Tx_{k-1}, Tx_k) = 0$ then x_k is a fixed point of T. If not, we get obtain two sequences $\{x_n\}$ and $\{h_n\}_{n\geq 1}$ such that $x_n \in Tx_{n-1}, h_n > 1$ and

$$D(x_n, x_{n+1}) < h_n H(Tx_{n-1}, Tx_n), \text{ for all } n \geq 1. \tag{3.1}$$

Since $\frac{1}{K+1} d(x_{n-1}, Tx_{n-1}) \leq \frac{1}{K+1} D(x_{n-1}, x_n) \leq D(x_{n-1}, x_n)$ and by hypothesis, we have

$$\begin{aligned} H(Tx_{n-1}, Tx_n) &\leq s\{d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)\} \\ &\leq s\{D(x_{n-1}, x_n) + D(x_n, x_{n+1})\}. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2), we get

$$D(x_n, x_{n+1}) < h_n s\{D(x_{n-1}, x_n) + D(x_n, x_{n+1})\}.$$

We can choose $h_n = \frac{k}{s} > 1$ with $s \in (0, k)$ and $0 < k < \frac{1}{2}$. Then we obtain

$$D_n < \frac{k}{1-k} D_{n-1}, \text{ where } \frac{k}{1-k} < 1 \text{ and } D_n = D(x_n, x_{n+1}).$$

Thus,

$$D_n < \Big(\frac{k}{1-k}\Big)^n D_0 \text{ for all } n \geq 1.$$

Hence,

$$\sum_{n=1}^{\infty} D_n \leq D_0 \sum_{n=1}^{\infty} \Big(\frac{k}{1-k}\Big)^n < +\infty.$$

By Proposition 1.7, we have $\{x_n\}$ is a Cauchy sequence in X. By X is complete, there exists $\bar{x} \in X$ such that $\lim_{n\to\infty} x_n = \bar{x}$. Now, we show that for any $n \geq 0$, either

$$\frac{1}{K+1} d(x_n, Tx_n) \leq D(x_n, \bar{x}) \text{ or } \frac{1}{K+1} d(x_{n+1}, Tx_{n+1}) \leq D(x_{n+1}, \bar{x}). \tag{3.3}$$

Arguing by contradiction, we suppose that for some $n \geq 0$ such that

$$D(x_n, \bar{x}) < \frac{1}{K+1} d(x_n, Tx_n) \text{ and } D(x_{n+1}, \bar{x}) < \frac{1}{K+1} d(x_{n+1}, Tx_{n+1}).$$

Then, by the triangle inequality, we have

$$\begin{aligned} D_n = D(x_n, x_{n+1}) &\leq D(x_n, \bar{x}) + KD(x_{n+1}, \bar{x}) \\ &< \frac{1}{K+1} d(x_n, Tx_n) + \frac{K}{K+1} d(x_{n+1}, Tx_{n+1}) \end{aligned}$$

$$\leq \frac{1}{K+1}D(x_n, x_{n+1}) + \frac{K}{K+1}D(x_{n+1}, x_{n+2})$$
$$\leq D_n.$$

This is a contradiction. Hence, from (3.3) and by hypotheses for each $n \geq 0$, either

$$H(Tx_n, T\bar{x}) \leq s\{d(x_n, Tx_n) + d(\bar{x}, T\bar{x})\}, \tag{3.4}$$

or

$$H(Tx_{n+1}, T\bar{x}) \leq s\{d(x_{n+1}, Tx_{n+1}) + d(\bar{x}, T\bar{x})\}. \tag{3.5}$$

Then, either (3.4) holds for infinity natural numbers n or (3.5) holds for infinity natural numbers n. Suppose (3.4) holds for infinity natural numbers n. We can choose in that infinity set the sequence $\{n_k\}$ is a monotone strictly increasing sequence of natural numbers. Therefore, sequence $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and

$$\begin{aligned} d(\bar{x}, T\bar{x}) &\leq d(Tx_{n_k}, \bar{x}) + KH(Tx_{n_k}, T\bar{x}) \\ &\leq D(x_{n_k+1}, \bar{x}) + Ks\{d(x_{n_k+1}, Tx_{n_k+1}) + d(\bar{x}, T\bar{x})\} \end{aligned}$$

this is equivalent with

$$d(\bar{x}, T\bar{x}) \leq \frac{1+Ks}{1-Ks}D(x_{n_k+1}, \bar{x}) + \frac{K^2 s}{1-Ks}D(x_{n_k+2}, \bar{x}).$$

On taking limit on both sides of above inequality, we have $d(\bar{x}, T\bar{x}) = 0$. It means that $\bar{x} \in T\bar{x}$. If (3.5) holds for infinity natural numbers n, by using an argument similar to that of above we have $\bar{x}$ is a fixed point of T. Suppose $\bar{y}$ is another fixed point of T. Then $0 = \frac{1}{K+1}d(\bar{x}, T\bar{x}) \leq D(\bar{x}, \bar{y})$ and by hypothesis, we have

$$\begin{aligned} H(T\bar{x}, T\bar{y}) &\leq s\{d(\bar{x}, T\bar{x}) + d(\bar{y}, T\bar{y})\} \\ &\leq s\{D(\bar{x}, \bar{x}) + D(\bar{y}, \bar{y})\} = 0. \end{aligned}$$

This implies $H(T\bar{x}, T\bar{y}) = 0$ implies $T\bar{x} = T\bar{y}$ means $\bar{x} = \bar{y}$. Hence, T has a unique fixed point $\bar{x} \in X$. □

Example 3.4. Let $X = \{1, 2, 3\}, K = 3$. A mapping $D : X \times X \to [0, \infty)$ defined by

$$D(1, 2) = 1, D(1, 3) = 4, D(2, 3) = 2 \text{ and } D(1, 1) = D(2, 2) = D(3, 3) = 0.$$

Then (X, D, K) is a complete strong b- metric space.

Define the mapping $T : X \to CB(X)$ by $T1 = \{2\}, T2 = \{2\}, T3 = \{1, 2\}$. We have

$$H(T1, T2) = H(\{2\}, \{2\}) = D(2, 2) = 0,$$

$$H(T2, T3) = H(\{2\}, \{1, 2\}) = D(2, 2) = 0,$$

$$H(T1, T3) = H(\{2\}, \{1, 2\}) = D(2, 2) = 0.$$

On the other hand, since
$$\frac{1}{4} = \frac{1}{4}d(1, T1) \leq D(1, y)$$
holds for any $y \in X \backslash \{1\}$ and
$$0 = H(T1, T2) \leq s\{d(1, T1) + d(2, T2)\} = s,$$
$$0 = H(T1, T3) \leq s\{d(1, T1) + d(3, T3)\} = 3s,$$
then
$$\frac{1}{4}d(1, T1) \leq D(1, y) \text{ implies } H(T1, Ty) \leq s\{d(1, T1) + d(y, Ty)\},$$
for all $y \in X$. Again, since $0 = \frac{1}{4}d(2, T2) \leq D(2, y)$ holds for all $y \in X$ and
$$0 = H(T2, T1) \leq s\{d(2, T2) + d(1, T1)\} = s,$$
$$0 = H(T2, T3) \leq s\{d(2, T2) + d(3, T3)\} = 2s,$$
then
$$\frac{1}{4}d(2, T2) \leq D(2, y) \text{ implies } H(T2, Ty) \leq s\{d(2, T2) + d(y, Ty)\},$$
for all $y \in X$. Finally, by $\frac{1}{2} = \frac{1}{4}d(3, T3) \leq D(3, y)$ if and only if $y \in X \backslash \{3\}$ and
$$0 = H(T3, T2) \leq s\{d(3, T3) + d(2, T2)\} = 2s,$$
$$0 = H(T3, T1) \leq s\{d(3, T3) + d(1, T1)\} = 3s,$$
then
$$\frac{1}{4}d(3, T3) \leq D(3, y) \text{ implies } H(T3, Ty) \leq s\{d(3, T3) + d(y, Ty)\},$$
for all $y \in X$. Thus all the hypothesis of Theorem 3.3 are satisfied. Hence $\bar{x} = 2$ is a unique fixed point of T.

Question 3.5. Does there exist $k = \frac{1}{2}$ such that mapping T in Theorem 3.3 has a fixed point free?

Conflict of interest

The author declare no conflict of interest.

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