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Research article

Embedding and Volterra integral operators on a class of Dirichlet-Morrey spaces

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Abstract: A class of Dirichlet-Morrey spaces $D_{\beta,\lambda}$ is introduced in this paper. For any positive Borel measure μ, the boundedness and compactness of the identity operator from $D_{\beta,\lambda}$ into the tent space $\mathcal{T}_s^1(\mu)$ are characterized. As an application, the boundedness of the Volterra integral operator T_g : $D_{\beta,\lambda} \to F(1, \beta - s, s)$ is studied. Moreover, the essential norm and the compactness of the operator T_g are also investigated.

Keywords: Dirichlet-Morrey space; Carleson measure; Volterra integral operator
Mathematics Subject Classification: 30H25, 47B91

1. Introduction

Let $\mathbb{D}$ and $\partial\mathbb{D}$ denote the unit disc of complex plane $\mathbb{C}$ and its boundary, respectively. Let $H(\mathbb{D})$ be the class of holomorphic functions on $\mathbb{D}$. For $0 < p < \infty$, the Hardy space H^p consists of those functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Let H^∞ denote the space of bounded analytic functions with the supremum norm $\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)|$.

For $\alpha > -1$ and $0 < p < \infty$, the weight Dirichlet space D_α^p consists of those functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{D_\alpha^p} = |f(0)| + \left(\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^\alpha dA(z) \right)^{\frac{1}{p}} < \infty,$$

where dA denotes the normalized area measure on $\mathbb{D}$. When $\alpha = 1$ and $p = 2$, the space D_α^p is the Hardy space H^2. When $\alpha = p$, D_α^p is just the Bergman space A^p.

Let $f \in H(\mathbb{D})$, $0 < p < \infty$, $-2 < q < \infty$ and $0 \le s < \infty$. We say that $f \in F(p, q, s)$ if

$$\|f\|_{F(p,q,s)} = |f(0)| + \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\sigma_a(z)|^2)^s dA(z) \right)^{1/p} < \infty,$$

where $\sigma_a = \frac{a-z}{1-\bar{a}z}$ is a Möbius map that interchanges 0 and a. The space $F(p, q, s)$ was introduced by Zhao in [25]. For $q + s > -1$, the space $F(p, q, s)$ is nontrivial. When $q = p - 2$, $F(p, p - 2, s)$ are Möbius invariant spaces that contain some classical spaces. For instance, when $s > 1$, $F(p, p - 2, s)$ is the classical Bloch space $\mathcal{B}$. When $p = 2$, $F(p, p - 2, s)$ is the Q_s space. If $p = 2$ and $s = 1$, $F(p, p - 2, s)$ is the $BMOA$ space.

Let $g \in H(\mathbb{D})$. The Volterra integral operator T_g is defined by

$$T_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta, \ z \in \mathbb{D}, \ \ f \in H(\mathbb{D}).$$

The operator T_g has been investigated by many researchers. Pommerenke [13] showed that T_g is bounded on H^2 if and only if $g \in BMOA$. Aleman and Siskakis [2] proved that T_g is bounded on H^p if and only if $g \in BMOA$ when $p \ge 1$. See [1–3, 6, 8, 9, 14, 15, 17–19] and the references therein for more information of the operator T_g.

For any arc $I \subset \partial\mathbb{D}$, let $|I| = \int_I \frac{|d\xi|}{2\pi}$ be the normalized arc length of I and

$$S(I) = \{z = re^{i\theta} \in \mathbb{D} : 1 - |I| \le r < 1, e^{i\theta} \in I\}$$

be the Carleson box based on I. Let $0 < p, s < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. The tent space $\mathcal{T}_s^p(\mu)$ consists of all μ-measure functions f satisfying

$$\|f\|^p_{\mathcal{T}_s^p(\mu)} = \sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^s} \int_{S(I)} |f(z)|^p d\mu(z) < \infty.$$

It was first introduced by Pau and Zhao in [12]. They also showed that $\mathcal{T}_s^p(\mu)$ is a Banach space for $p \ge 1$. In [24], Xiao showed that the Q_p $(0 < p < 1)$ space is continuously contained in $\mathcal{T}_s^2(\mu)$ if and only if

$$\sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^p} \left(\log \frac{2}{|I|} \right)^2 < \infty.$$

Let $0 \le \lambda \le 1$. The analytic Morrey space $\mathcal{L}^{2,\lambda}(\mathbb{D})$, which introduced by Wu and Xie in [22], consists of all functions $f \in H^2(\mathbb{D})$ such that

$$\sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^\lambda} \int_I |f(\xi) - f_I|^2 \frac{|d\xi|}{2\pi} < \infty,$$

where $f_I = \frac{1}{|I|} \int_I f(\xi) \frac{|d\xi|}{2\pi}$. From [8], the equivalent norm of $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$ can be defined as

$$\|f\|_{\mathcal{L}^{2,\lambda}} = |f(0)| + \sup_{a\in\mathbb{D}} (1 - |a|^2)^{\frac{1-\lambda}{2}} \|f \circ \sigma_a - f(a)\|_{H^2}.$$

It is obvious that $\mathcal{L}^{2,1}(\mathbb{D}) = BMOA$, $\mathcal{L}^{2,0}(\mathbb{D}) = H^2$. Moreover,

$$BMOA \subset \mathcal{L}^{2,\lambda} \subset H^2, \quad 0 < \lambda < 1.$$

See [23] for the generalization of the Morrey space.

Recently, Galanopoulos, Merchán and Siskakis [6] defined the Dirichlet-Morrey space $D_p^{2,\lambda}$, which consisting of all functions $f \in D_p^2$ such that

$$\|f\|_{D_p^{2,\lambda}} = |f(0)| + \sup_{a\in\mathbb{D}}(1-|a|^2)^{\frac{p(1-\lambda)}{2}}\|f\circ\sigma_a - f(a)\|_{D_p^2} < \infty,$$

where $0 \le p, \lambda \le 1$. It is easy to check that $D_1^{2,\lambda} = \mathcal{L}^{2,\lambda}, D_p^{2,1} = Q_p, D_p^{2,0} = D_p^2$ and

$$Q_p \subset D_p^{2,\lambda} \subset D_p^2, 0 < \lambda < 1.$$

Recently, Morrey-type spaces have received a lot of attention and many results have been obtained. For example, Li, Liu and Lou proved that T_g is bounded on $\mathcal{L}^{2,\lambda}(\mathbb{D})$ if and only if $g \in BMOA$ when $0 < \lambda < 1$ in [8]. In [6], Galanopoulos, Merchán and Siskakis proved that if T_g is bounded on $D_p^{2,\lambda}$, then $g \in Q_p$, while if $g \in W_p$, then T_g is bounded on $D_p^{2,\lambda}$. Here the space W_p is the space consisting of all functions $g \in H(\mathbb{D})$ such that

$$\int_{\mathbb{D}} |f(z)|^2|g'(z)|^2(1-|z|^2)^p dA(z) \le C\|f\|_{D_p^2}^2, \quad f \in D_p^2.$$

Clearly, the necessary and sufficient condition for the boundedness of T_g on $D_p^{2,\lambda}$ are not obtained. See [4, 10, 16, 27] and references therein for more information on other Morrey-type spaces.

Motivated by the definitions of the Morrey space $\mathcal{L}^{2,\lambda}$ and the Dirichlet-Morrey space $D_p^{2,\lambda}$, we introduce a class of Dirichlet-Morrey spaces as follows. Assume that $-1 < \beta < 0$, $0 \le \lambda \le 1$ and $f \in D_\beta^1$. We say that f belongs to the Dirichlet-Morrey space $D_{\beta,\lambda}$ if

$$\|f\|_{D_{\beta,\lambda}} = |f(0)| + \sup_{a\in\mathbb{D}}(1-|a|^2)^{(\beta+1)(1-\lambda)}\|f\circ\sigma_a - f(a)\|_{D_\beta^1} < \infty.$$

It is obvious that $D_{\beta,\lambda}$ is a linear space. Under the above norm, it is easy to check that $D_{\beta,\lambda}$ is a Banach space. By a simple calculation, we have that $D_{\beta,0} = D_\beta^1$, $D_{\beta,1} = F(1,-1,\beta+1)$ and

$$F(1,-1,\beta+1) \subset D_{\beta,\lambda} \subset D_\beta^1, \quad 0 < \lambda < 1.$$

In this paper, we first state some basic properties for the Dirichlet-Morrey space $D_{\beta,\lambda}$ and then investigate the boundedness and compactness of the identity operator $I_d : D_{\beta,\lambda} \to \mathcal{T}_s^1(\mu)$. Using the embedding theorem, we give a necessary and sufficient condition for the boundedness of the operator $T_g : D_{\beta,\lambda} \to F(1,\beta-s,s)$ when $-1 < \beta < 0$, $0 < \lambda, s < 1$ such that $s \ge \lambda(\beta+1)$. Moreover, the essential norm and compactness of $T_g : D_{\beta,\lambda} \to F(1,\beta-s,s)$ are also investigated. In particular, we will prove that $T_g : D_{\beta,\lambda} \to D_{\beta,\lambda}$ is bounded (compact) if and only if $g \in F(1,-1,\beta+1)$ ($g \in F_0(1,-1,\beta+1)$).

In this paper, we say that $f \preceq g$ if there exists a constant C such that $f \le Cg$. If both $f \preceq g$ and $g \preceq f$ are valid, we write $f \approx g$.

2. Some basic properties

In this section, we characterize some basic properties of the space $D_{\beta,\lambda}$. These properties play an important role in the proof of our main results. We first recall the definition of α-Carleson measure.

Suppose that $0 < \alpha < \infty$ and μ is a positive Borel measure on $\mathbb{D}$. We say that μ is a α-Carleson measure if (see [12])

$$\|\mu\|_{CM_\alpha} = \sup_{I\subset\partial\mathbb{D}} \frac{\mu(S(I))}{|I|^\alpha} < \infty.$$

When $\alpha = 1$, μ is called the Carleson measure. We say that μ is a vanishing α-Carleson measure if

$$\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^\alpha} = 0.$$

The Carleson measure is a very important tool in the theory of function spaces and operator theory (see [5, 7, 11, 24]).

Lemma 1. *[12] Let $\alpha, q > 0$ and μ be a positive Borel measure on $\mathbb{D}$. Then μ is α-Carleson measure if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^q}{|1-\bar{a}z|^{q+\alpha}} d\mu(z) < \infty.$$

Proposition 1. *Let $-1 < \beta < 0$, $0 < \lambda < 1$ and $f \in H(\mathbb{D})$. Then $f \in D_{\beta,\lambda}$ if and only if*

$$\sup_{I\subset\partial\mathbb{D}} \left(\frac{1}{|I|^{\lambda(\beta+1)}} \int_{S(I)} |f'(z)|(1-|z|^2)^\beta dA(z) \right) < \infty. \tag{2.1}$$

Proof. First, suppose that $f \in D_{\beta,\lambda}$. For any arc $I \subset \partial\mathbb{D}$, let $a = (1-|I|)\xi$, where ξ is the center of arc I. Then

$$|1-\bar{a}z| \approx 1-|a|^2 \approx |I| = 1-|a|, \ z \in S(I).$$

Changing the variable $z = \sigma_a(w)$, we have

$$\begin{aligned}
\|f\|_{D_{\beta,\lambda}} &\geq (1-|a|^2)^{(\beta+1)(1-\lambda)} \|f\circ\sigma_a - f(a)\|_{D^1_\beta} \\
&= (1-|a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |(f\circ\sigma_a)'(z)|(1-|z|^2)^\beta dA(z) \\
&= (1-|a|^2)^{(\beta+1)(1-\lambda)} \int_{\mathbb{D}} |f'(\sigma_a(z))| \frac{(1-|z|^2)^\beta(1-|a|^2)}{|1-\bar{a}z|^2} dA(z) \\
&= (1-|a|^2)^{(\beta+1)(2-\lambda)} \int_{\mathbb{D}} |f'(w)| \frac{(1-|w|^2)^\beta}{|1-\bar{a}w|^{2\beta+2}} dA(w) \\
&\geq \frac{1}{|I|^{\lambda(\beta+1)}} \int_{S(I)} |f'(w)|(1-|w|^2)^\beta dA(w),
\end{aligned}$$

which implies the desired result by the arbitrariness of I.

Conversely, assume that (2.1) holds. Let $d\mu_f(z) = |f'(z)|(1-|z|^2)^\beta dA(z)$. Then

$$\sup_{I\subset\partial\mathbb{D}} \frac{\mu_f(S(I))}{|I|^{\lambda(\beta+1)}} = \sup_{I\subset\partial\mathbb{D}} \left(\frac{1}{|I|^{\lambda(\beta+1)}} \int_{S(I)} |f'(z)|(1-|z|^2)^\beta dA(z) \right) < \infty.$$

So μ_f is a $\lambda(\beta+1)$-Carleson measure. Then for each $a \in \mathbb{D}$,

$$\|f\circ\sigma_a - f(a)\|_{D^1_\beta} = \int_{\mathbb{D}} |f'(z)| \frac{(1-|a|^2)^{\beta+1}(1-|z|^2)^\beta}{|1-\bar{a}z|^{2\beta+2}} dA(z)$$

$$= \int_{\mathbb{D}} \frac{(1-|a|^2)^{\beta+1}}{|1-\bar{a}z|^{2\beta+2}} d\mu_f(z).$$

Therefore, by Lemma 1 we have

$$\begin{aligned}\sup_{a\in\mathbb{D}}(1-|a|^2)^{(\beta+1)(1-\lambda)}\|f\circ\sigma_a - f(a)\|_{D^1_\beta} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^{(\beta+1)(2-\lambda)}}{|1-\bar{a}z|^{2\beta+2}} d\mu_f(z) \\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^q}{|1-\bar{a}z|^{\alpha+q}} d\mu_f(z) < \infty,\end{aligned}$$

where $\alpha = \lambda(\beta+1) > 0$, $q = (\beta+1)(2-\lambda) > 0$. The proof is complete. □

Proposition 2. *Let* $-1 < \beta < 0$, $0 < \lambda < 1$. *Then the following statements hold.*

(i) For any $f \in D_{\beta,\lambda}$,

$$|f(z)| \leq \frac{\|f\|_{D_{\beta,\lambda}}}{(1-|z|^2)^{(\beta+1)(1-\lambda)}},\ z \in \mathbb{D}.$$

(ii) The function $f_{\beta,\lambda}(z) = \frac{1}{(1-z)^{(\beta+1)(1-\lambda)}}$ *belongs to* $D_{\beta,\lambda}$.

Proof. (i) Suppose that $f \in D_{\beta,\lambda}$. For each $a \in \mathbb{D}$, applying the Lemma 4.12 in [26], we get

$$\begin{aligned}|f'(a)|(1-|a|^2) &\leq (\beta+1)\int_{\mathbb{D}} |(f\circ\sigma_a)'(z)|(1-|z|^2)^\beta dA(z) \\ &= \frac{(\beta+1)}{(1-|a|^2)^{(\beta+1)(1-\lambda)}}(1-|a|^2)^{(\beta+1)(1-\lambda)}\|f\circ\sigma_a - f(a)\|_{D^1_\beta} \\ &\leq \frac{\|f\|_{D_{\beta,\lambda}}}{(1-|a|^2)^{(\beta+1)(1-\lambda)}}.\end{aligned}$$

So

$$|f'(a)| \leq \frac{\|f\|_{D_{\beta,\lambda}}}{(1-|a|^2)^{(\beta+1)(1-\lambda)+1}},\ \ a \in \mathbb{D}.$$

Since $f(z) - f(0) = \int_0^z f'(w)dw$, by integrating both sides of the last inequality, we obtain the desired result.

(ii) By Proposition 1, it suffices to show that

$$\sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^{\lambda(\beta+1)}} \int_{S(I)} |f'_{\beta,\lambda}(z)|(1-|z|^2)^\beta dA(z) < \infty. \tag{2.2}$$

Set

$$S(a,r) = \{z \in \mathbb{D} : |a-z| < r, a \in \partial\mathbb{D}, 0 < r < 1\}.$$

Then the inequality (2) is equivalent to

$$\sup_{a\in\partial\mathbb{D},0<r<1} \frac{1}{r^{\lambda(\beta+1)}} \int_{S(a,r)} |f'_{\beta,\lambda}(z)|(1-|z|^2)^\beta dA(z) < \infty. \tag{2.3}$$

Since

$$\int_{S(a,r)} |f'_{\beta,\lambda}(z)|(1-|z|^2)^\beta dA(z)$$

$$
\begin{aligned}
&=(\beta+1)(1-\lambda)\int_{S(a,r)}\frac{(1-|z|^2)^{\beta}}{|1-z|^{(\beta+1)(1-\lambda)+1}}dA(z)\\
&\approx\int_{S(a,r)}\frac{1}{|1-z|^{(\beta+1)(1-\lambda)+1-\beta}}dA(z)\le\int_{S(1,r)}\frac{1}{|1-z|^{(\beta+1)(1-\lambda)+1-\beta}}dA(z)\\
&\le\int_{|w|<r}\frac{1}{|w|^{(\beta+1)(1-\lambda)+1-\beta}}dA(w)=\int_0^r h^{\beta-(\beta+1)(1-\lambda)}dh\approx r^{\lambda(\beta+1)},
\end{aligned}
$$

we see that the inequality (2.3) holds. The proof is complete. □

3. Embedding theorem from $D_{\beta,\lambda}$ to tent spaces

In this section, we study the boundedness and compactness of the identity operator $I_d : D_{\beta,\lambda}\to\mathcal{T}_s^1(\mu)$. We say that I_d is compact if

$$
\lim_{n\to\infty}\frac{1}{|I|^s}\int_{S(I)}|f_n(z)|d\mu(z)=0,
$$

where $I\subset\partial\mathbb{D}$, $\{f_n\}$ is a bounded sequence in $D_{\beta,\lambda}$ and converges to zero uniformly on every compact subset of $\mathbb{D}$.

We begin this section with several lemmas.

Lemma 2. *[12, Corollary 2.5] Let $a,b\in\mathbb{D}$ and $r>-1$, $s,t>0$ such that $0<s+t-r-2<s$. Then*

$$
\int_{\mathbb{D}}\frac{(1-|z|^2)^r}{|1-\bar{a}z|^s|1-\bar{b}z|^t}dA(z)\le\frac{1}{(1-|a|^2)^{s+t-r-2}}.
$$

Lemma 3. *Let $-1<\beta<0$, $0<\lambda<1$, $q\ge\lambda(\beta+1)$ and $f\in D_{\beta,\lambda}$. Then*

$$
\sup_{a\in\mathbb{D}}(1-|a|^2)^{(\beta+1)(1-\lambda)}\int_{\mathbb{D}}|(f\circ\sigma_a)(z)-f(a)|\frac{(1-|z|^2)^{q+(\beta+1)(1-\lambda)-1}}{|1-\bar{a}z|^{q-\lambda(\beta+1)+1}}dA(z)\le\|f\|_{D_{\beta,\lambda}}.
$$

Proof. From Lemma 1 in [11], for any $a\in\mathbb{D}$ and $f\in H(\mathbb{D})$,

$$
(f\circ\sigma_a)(z)-f(a)=\int_{\mathbb{D}}(f\circ\sigma_a)'(t)(1-|t|^2)^2\frac{h_z(t)}{\bar{t}(1-\bar{t}z)^3}dA(t),
$$

where $h_z(t)=1-(1-\bar{z}t)^3$ is uniformly bounded on $\mathbb{D}$ and satisfies $h_z(0)=0$. Employing Schwarz's Lemma, we have $|h_z(t)|\le|t|$. Using this, we deduce that

$$
|(f\circ\sigma_a)(z)-f(a)|\le\int_{\mathbb{D}}|(f\circ\sigma_a)'(t)|\frac{(1-|t|^2)^2}{|1-\bar{t}z|^3}dA(t).
$$

According to the fact that $1-|t|\le|1-\bar{a}t|$, Lemma 2 and Fubini's Theorem, we get

$$
(1-|a|^2)^{(\beta+1)(1-\lambda)}\int_{\mathbb{D}}|(f\circ\sigma_a)(z)-f(a)|\frac{(1-|z|^2)^{q+(\beta+1)(1-\lambda)-1}}{|1-\bar{a}z|^{q-\lambda(\beta+1)+1}}dA(z)
$$

$$
\begin{aligned}
&\lesssim(1-|a|^2)^{(\beta+1)(1-\lambda)}\int_{\mathbb{D}}\int_{\mathbb{D}}|(f\circ\sigma_a)'(t)|\frac{(1-|t|^2)^2}{|1-\bar{t}z|^3}dA(t)\frac{(1-|z|^2)^{q+(\beta+1)(1-\lambda)-1}}{|1-\bar{a}z|^{q-\lambda(\beta+1)+1}}dA(z)\\
&\lesssim(1-|a|^2)^{(\beta+1)(1-\lambda)}\int_{\mathbb{D}}|(f\circ\sigma_a)'(t)|(1-|t|^2)^2\int_{\mathbb{D}}\frac{(1-|z|^2)^{q+(\beta+1)(1-\lambda)-1}}{|1-\bar{t}z|^3|1-\bar{a}z|^{q-\lambda(\beta+1)+1}}dA(z)dA(t)\\
&\lesssim(1-|a|^2)^{(\beta+1)(1-\lambda)}\int_{\mathbb{D}}|(f\circ\sigma_a)'(t)|(1-|t|^2)^2\frac{1}{(1-|t|^2)^{2-\beta}}dA(t)\\
&\lesssim(1-|a|^2)^{(\beta+1)(1-\lambda)}\int_{\mathbb{D}}|(f\circ\sigma_a)'(t)|(1-|t|^2)^{\beta}dA(t)\\
&\lesssim\|f\|_{D_{\beta,\lambda}}.
\end{aligned}
$$

The proof is complete. □

Lemma 4. *[21] Let $0<\alpha<1$ and μ be a positive Borel measure on $\mathbb{D}$. Then the identity operator $I_d: D^1_{\alpha-1}\to L^1(\mu)$ is bounded if and only if μ is α-Carleson measure.*

The following theorem is the main result in this section.

Theorem 1. *Let $-1<\beta<0$, $0<\lambda<1$, $s\geq\lambda(\beta+1)$ and μ be a positive Borel measure on $\mathbb{D}$. Then the identity operator $I_d: D_{\beta,\lambda}\to\mathcal{T}^1_s(\mu)$ is bounded if and only if the measure μ is a $s+(\beta+1)(1-\lambda)$-Carleson measure.*

Proof. First, assume that $I_d: D_{\beta,\lambda}\to\mathcal{T}^1_s(\mu)$ is bounded. For any arc $I\subset\partial\mathbb{D}$, let $a=(1-|I|)\xi$, where ξ is the center of arc I. Then

$$|1-\bar{a}z|\approx 1-|a|^2\approx|I|,\ \ z\in S(I).$$

Set

$$f_a(z)=\frac{1-|a|^2}{(1-\bar{a}z)^{1+(\beta+1)(1-\lambda)}},\ \ z\in\mathbb{D}.$$

Using Proposition 2, we obtain that $f_a\in D_{\beta,\lambda}$ with $\|f_a\|_{D_{\beta,\lambda}}\lesssim 1$. Moreover $|f_a(z)|\approx\frac{1}{|I|^{(\beta+1)(1-\lambda)}},\ z\in S(I)$. Hence,

$$\frac{\mu(S(I))}{|I|^{s+(\beta+1)(1-\lambda)}}\approx\frac{1}{|I|^s}\int_{S(I)}|f_a(z)|d\mu(z)\lesssim\|f_a\|_{D_{\beta,\lambda}}<\infty,$$

which implies that μ is a $s+(\beta+1)(1-\lambda)$-Carleson measure.

Conversely, suppose that μ is a $s+(\beta+1)(1-\lambda)$-Carleson measure. Let $f\in D_{\beta,\lambda}$. For any arc $I\subset\partial\mathbb{D}$, let $a=(1-|I|)\xi$, where ξ is the center of arc I.

$$
\begin{aligned}
&\frac{1}{|I|^s}\int_{S(I)}|f(z)|d\mu(z)\\
\leq&\frac{1}{|I|^s}\int_{S(I)}|f(a)|d\mu(z)+\frac{1}{|I|^s}\int_{S(I)}|f(z)-f(a)|d\mu(z)\\
:=&A+B.
\end{aligned}
$$

From Proposition 2, we have that

$$A\lesssim\|f\|_{D_{\beta,\lambda}}\frac{\mu(S(I))}{|I|^{s+(\beta+1)(1-\lambda)}}\lesssim\|f\|_{D_{\beta,\lambda}}.$$

Since

$$\frac{d}{dz}\frac{f(z)-f(a)}{(1-\bar{a}z)^{s+(\beta+1)(2-\lambda)}}=\frac{f'(z)}{(1-\bar{a}z)^{s+(\beta+1)(2-\lambda)}}+\frac{(s+(\beta+1)(2-\lambda))\bar{a}(f(z)-f(a))}{(1-\bar{a}z)^{s+(\beta+1)(2-\lambda)+1}},$$

by Lemma 4 we obtain

$$\begin{aligned}
B =&\frac{1}{|I|^s}\int_{S(I)}|f(z)-f(a)|d\mu(z)\\
\approx&(1-|a|^2)^{(\beta+1)(2-\lambda)}\int_{S(I)}\left|\frac{f(z)-f(a)}{(1-\bar{a}z)^{s+(\beta+1)(2-\lambda)}}\right|d\mu(z)\\
\leq&(1-|a|^2)^{(\beta+1)(2-\lambda)}\int_{\mathbb{D}}\left|\frac{f(z)-f(a)}{(1-\bar{a}z)^{s+(\beta+1)(2-\lambda)}}\right|d\mu(z)\\
\leq&(1-|a|^2)^{(\beta+1)(2-\lambda)}\int_{\mathbb{D}}\left|\frac{d}{dz}\frac{f(z)-f(a)}{(1-\bar{a}z)^{s+(\beta+1)(2-\lambda)}}\right|(1-|z|^2)^{s+(\beta+1)(1-\lambda)-1}dA(z)\\
\leq&B_1+B_2,
\end{aligned}$$

where

$$B_1=(1-|a|^2)^{(\beta+1)(2-\lambda)}\int_{\mathbb{D}}\frac{|f'(z)|}{|1-\bar{a}z|^{s+(\beta+1)(2-\lambda)}}(1-|z|^2)^{s+(\beta+1)(1-\lambda)-1}dA(z)$$

and

$$B_2=(1-|a|^2)^{(\beta+1)(2-\lambda)}\int_{\mathbb{D}}\frac{|f(z)-f(a)|}{|1-\bar{a}z|^{s+(\beta+1)(2-\lambda)+1}}(1-|z|^2)^{s+(\beta+1)(1-\lambda)-1}dA(z).$$

Changing the variable $z=\sigma_a(w)$, we have

$$\begin{aligned}
B_1 &= (1-|a|^2)^{(\beta+1)(2-\lambda)}\int_{\mathbb{D}}|f'(\sigma_a(w))|\frac{(1-|w|^2)^{s+(\beta+1)(1-\lambda)-1}}{|1-\bar{a}w|^{s-\lambda(\beta+1)+2}(1-|a|^2)^{\beta}}dA(w)\\
&\leq (1-|a|^2)^{(\beta+1)(1-\lambda)}\int_{\mathbb{D}}|(f\circ\sigma_a)'(w)|(1-|w|^2)^{\beta}dA(w)\\
&\leq \|f\|_{D_{\beta,\lambda}}.
\end{aligned}$$

Changing the variable $z=\sigma_a(w)$ and using Lemma 3, we obtain

$$\begin{aligned}
B_2 &= (1-|a|^2)^{(\beta+1)(2-\lambda)}\int_{\mathbb{D}}\frac{|f(z)-f(a)|}{|1-\bar{a}z|^{s+(\beta+1)(2-\lambda)+1}}(1-|z|^2)^{s+(\beta+1)(1-\lambda)-1}dA(z)\\
&= (1-|a|^2)^{(\beta+1)(1-\lambda)}\int_{\mathbb{D}}|(f\circ\sigma_a)(w)-f(a)|\frac{(1-|w|^2)^{s+(\beta+1)(1-\lambda)-1}}{|1-\bar{a}w|^{s-\lambda(\beta+1)+1}}dA(w)\\
&\leq \|f\|_{D_{\beta,\lambda}}.
\end{aligned}$$

So the identity operator $I_d : D_{\beta,\lambda}\to\mathcal{T}_s^1(\mu)$ is bounded. The proof is complete. □

Theorem 2. *Let $-1<\beta<0$, $0<\lambda<1$, $s\geq\lambda(\beta+1)$ and μ be a positive Borel measure on $\mathbb{D}$ such that point evaluation is a bounded functional on $\mathcal{T}_s^1(\mu)$. Then the identity operator $I_d : D_{\beta,\lambda}\to\mathcal{T}_s^1(\mu)$ is compact if and only if the measure μ is a vanishing $s+(\beta+1)(1-\lambda)$-Carleson measure.*

Proof. First, we suppose that $I_d : D_{\beta,\lambda} \to \mathcal{T}_s^1(\mu)$ is compact. Let $\{I_n\}$ be a sequence of subarcs of $\partial\mathbb{D}$ with $\lim\limits_{n\to\infty} |I_n| = 0$. Set $a_n = (1 - |I_n|)\xi_n$, where ξ_n is the midpoint of I_n. By simple calculation we have that, for any $z \in S(I_n)$, $1 - |a_n|^2 \approx |1 - \bar{a}_n z| \approx |I_n|$. Set

$$f_n(z) = \frac{1 - |a_n|^2}{(1 - \bar{a}_n z)^{1+(\beta+1)(1-\lambda)}}, \ z \in \mathbb{D}.$$

By Proposition 2, we see that $\{f_n\}$ is a bounded sequence in $D_{\beta,\lambda}$ and converges to zero uniformly on every compact subset of $\mathbb{D}$. Then

$$\frac{\mu(S(I_n))}{|I_n|^{s+(\beta+1)(1-\lambda)}} \approx \frac{1}{|I_n|^s} \int_{S(I_n)} |f_n(z)| d\mu(z) \leq \|f_n\|_{\mathcal{T}_s^1} \to 0,$$

as $n \to \infty$, which implies that μ is a vanishing $s + (\beta + 1)(1 - \lambda)$-Carleson measure.

Conversely, suppose that μ is a vanishing $s + (\beta + 1)(1 - \lambda)$-Carleson measure. Then μ is a $s + (\beta + 1)(1 - \lambda)$-Carleson measure. Therefore, the identity operator $I_d : D_{\beta,\lambda} \to \mathcal{T}_s^1(\mu)$ is bounded. From [9] we have $\|\mu - \mu_r\|_{CM_{s+(\beta+1)(1-\lambda)}} \to 0$, as $r \to 1$, where $\mu_r(z) = 0$ for $r \leq |z| < 1$ and $\mu_r(z) = \mu(z)$ for $|z| < r$. Let $\{f_n\}$ be a bounded sequence in $D_{\beta,\lambda}$ with $\sup_{n\in\mathbb{N}} \|f_n\|_{D_{\beta,\lambda}} \leq 1$ and converges to zero uniformly on every compact subset of $\mathbb{D}$. We obtain

$$\begin{aligned}
\frac{1}{|I|^s} \int_{S(I)} |f_n(z)| d\mu(z) \leq& \frac{1}{|I|^s} \int_{S(I)} |f_n(z)| d\mu_r(z) + \frac{1}{|I|^s} \int_{S(I)} |f_n(z)| d(\mu - \mu_r)(z) \\
\leq& \frac{1}{|I|^s} \int_{S(I)} |f_n(z)| d\mu_r(z) + \|\mu - \mu_r\|_{CM_{s+(\beta+1)(1-\lambda)}} \|f_n\|_{D_{\beta,\lambda}} \\
\leq& \frac{1}{|I|^s} \int_{S(I)} |f_n(z)| d\mu_r(z) + \|\mu - \mu_r\|_{CM_{s+(\beta+1)(1-\lambda)}}.
\end{aligned}$$

Letting $n \to \infty$ and $r \to 1$, we get $\lim_{n\to\infty} \|f_n\|_{\mathcal{T}_s^1} = 0$. So the identity operator $I_d : D_{\beta,\lambda} \to \mathcal{T}_s^1(\mu)$ is compact. The proof is complete. □

4. Boundedness of integral operator

In this section, we characterize the boundedness of the operator $T_g : D_{\beta,\lambda} \to F(1, \beta - s, s)$ when $-1 < \beta < 0$, $0 < \lambda, s < 1$ such that $s \geq \lambda(\beta + 1)$.

Theorem 3. *Let $g \in H(\mathbb{D})$, $-1 < \beta < 0$, $0 < \lambda, s < 1$ such that $s \geq \lambda(\beta + 1)$. Then $T_g : D_{\beta,\lambda} \to F(1, \beta - s, s)$ is bounded if and only if*

$$g \in F(1, \beta - s - (\beta + 1)(1 - \lambda), s + (\beta + 1)(1 - \lambda)).$$

Proof. First, suppose that $T_g : D_{\beta,\lambda} \to F(1, \beta - s, s)$ is bounded. For any fixed arc $I \subset \partial\mathbb{D}$, let $e^{i\theta}$ denote the center of I and $a = (1 - |I|)e^{i\theta}$. Set

$$f_a(z) = \frac{1}{(1 - \bar{a}z)^{(\beta+1)(1-\lambda)}}, \ z \in \mathbb{D}.$$

Then we get $\|T_g f_a\|_{F(1,\beta-s,s)} \leq \|T_g\|_{D_{\beta,\lambda}\to F(1,\beta-s,s)}\|f_a\|_{D_{\beta,\lambda}} < \infty$, by the assumption and Proposition 2. Since $(T_g f_a)'(z) = f_a(z)g'(z)$, we have

$$
\begin{aligned}
\infty > \|T_g f_a\|_{F(1,\beta-s,s)} &\geq \frac{1}{|I|^s}\int_{S(I)} |(T_g f_a)'(z)|(1-|z|^2)^\beta dA(z)\\
&= \frac{1}{|I|^s}\int_{S(I)} |f_a(z)||g'(z)|(1-|z|^2)^\beta dA(z)\\
&= \frac{1}{|I|^s}\int_{S(I)} \frac{1}{|1-\bar{a}z|^{(\beta+1)(1-\lambda)}}|g'(z)|(1-|z|^2)^\beta dA(z)\\
&\approx \frac{1}{|I|^{s+(\beta+1)(1-\lambda)}}\int_{S(I)} |g'(z)|(1-|z|^2)^\beta dA(z) = \frac{\mu_g(S(I))}{|I|^{s+(\beta+1)(1-\lambda)}},
\end{aligned}
$$

where $d\mu_g(z) = |g'(z)|(1-|z|^2)^\beta dA(z)$. Hence μ_g is a $s+(\beta+1)(1-\lambda)$ -Carleson measure. Employing Lemma 1, we obtain that

$$
\begin{aligned}
\infty &> \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^{s+(\beta+1)(1-\lambda)}}{|1-\bar{a}z|^{2[s+(\beta+1)(1-\lambda)]}} d\mu_g(z)\\
&\approx \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^{s+(\beta+1)(1-\lambda)}}{|1-\bar{a}z|^{2[s+(\beta+1)(1-\lambda)]}} |g'(z)|(1-|z|^2)^\beta dA(z)\\
&\approx \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g'(z)|(1-|z|^2)^{\beta-s-(\beta+1)(1-\lambda)}(1-|\sigma_a(z)|^2)^{s+(\beta+1)(1-\lambda)} dA(z),
\end{aligned}
$$

which implies that $g \in F(1,\beta-s-(\beta+1)(1-\lambda), s+(\beta+1)(1-\lambda))$.

Conversely, assume that $g \in F(1,\beta-s-(\beta+1)(1-\lambda), s+(\beta+1)(1-\lambda))$. Then we see that μ_g is a $s+(\beta+1)(1-\lambda)$ -Carleson measure. For each $f \in D_{\beta,\lambda}$, by Theorem 1, we have

$$
\begin{aligned}
&\sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^s}\int_{S(I)} |(T_g f)'(z)|(1-|z|^2)^\beta dA(z)\\
=&\sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^s}\int_{S(I)} |f(z)||g'(z)|(1-|z|^2)^\beta dA(z)\\
=&\sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^s}\int_{S(I)} |f(z)| d\mu_g(z)\\
\leq&\|f\|_{D_{\beta,\lambda}} < \infty.
\end{aligned}
$$

Therefore, $T_g : D_{\beta,\lambda} \to F(1,\beta-s,s)$ is bounded. The proof is complete. □

In particular, taking $s = \lambda(\beta+1)$, we get the following result.

Theorem 4. *Let $-1 < \beta < 0$, $0 < \lambda < 1$ and $g \in H(\mathbb{D})$. Then $T_g : D_{\beta,\lambda} \to D_{\beta,\lambda}$ is bounded if and only if $g \in F(1,-1,\beta+1)$.*

5. Essential norm of integral operator

In this section, we investigate the essential norm of the operator $T_g : D_{\beta,\lambda} \to F(1,\beta-s,s)$. We first recall some definitions. The essential norm of $T : X \to Y$ is defined by

$$\|T\|_{e,X\to Y} = \inf_K\{\|T-K\|_{X\to Y} : K \text{ is a compact operator from } X \text{ to } Y\},$$

where $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces, $\|T-K\|_{X\to Y}$ is the operator norm of the operator $T-K$ from X to Y. It is easy to see that $T: X \to Y$ is compact if $\|T\|_{e,X\to Y} = 0$.

For a closed subspace A of X, given $f \in X$, the distance from f to A is defined by $\text{dist}_X(f, A) = \inf_{g\in A} \|f - g\|_X$.

Let $F_0(1, \beta - s - (\beta+1)(1-\lambda), s + (\beta+1)(1-\lambda))$ denote the space of all functions $f \in F(1, \beta - s - (\beta+1)(1-\lambda), s + (\beta+1)(1-\lambda))$ such that

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|(1-|z|^2)^{\beta-s-(\beta+1)(1-\lambda)}(1-|\sigma_a(z)|^2)^{s+(\beta+1)(1-\lambda)} dA(z) = 0.$$

The following lemma gives the distance from $F(1, \beta - s - (\beta+1)(1-\lambda), s + (\beta+1)(1-\lambda))$ to $F_0(1, \beta - s - (\beta+1)(1-\lambda), s + (\beta+1)(1-\lambda))$.

Lemma 5. *Let $-1 < \beta < 0$, $0 < \lambda < 1$. If $g \in F(1, m, t)$, then*

$$\limsup_{|a|\to 1} \int_{\mathbb{D}} |g'(z)|(1-|z|^2)^m (1-|\sigma_a(z)|^2)^t dA(z)$$
$$\approx \text{dist}_{F(1,m,t)}(g, F_0(1, m, t)) \approx \limsup_{r\to 1^-} \|g - g_r\|_{F(1,m,t)}.$$

Here $m = \beta - s - (\beta+1)(1-\lambda)$, $t = s + (\beta+1)(1-\lambda)$, $g_r(z) = g(rz)$, $0 < r < 1$, $z \in \mathbb{D}$.

Proof. Given any $g \in F(1, m, t)$, then $g_r \in F_0(1, m, t)$ and $\|g_r\|_{F(1,m,t)} \le \|g\|_{F(1,m,t)}$. For any $\xi \in (0, 1)$, there exists a $a \in (0, \xi)$ such that $\sigma_a(z)$ lies in a compact subset of $\mathbb{D}$. Then $\lim_{r\to 1^-} \sup_{z\in\mathbb{D}} |g'(\sigma_a(z)) - rg'(r\sigma_a(z))| = 0$. Changing the variable $z = \sigma_a(w)$, we have

$$\begin{aligned}
&\lim_{r\to 1^-} \sup_{|a|\le\xi} \int_{\mathbb{D}} |g'(z) - g_r'(z)|(1-|z|^2)^m (1-|\sigma_a(z)|^2)^t dA(z) \\
&= \lim_{r\to 1^-} \sup_{|a|\le\xi} \int_{\mathbb{D}} |g'(\sigma_a(w)) - g_r'(\sigma_a(w))|(1-|\sigma_a(w)|^2)^m (1-|w|^2)^t |\sigma_a'(w)|^2 dA(w) \\
&\le \lim_{r\to 1^-} \sup_{|a|\le\xi} \sup_{w\in\mathbb{D}} |g'(\sigma_a(w)) - g_r'(\sigma_a(w))| \int_{\mathbb{D}} \frac{(1-|w|^2)^{m+t}}{(1-|a|^2)^{2+m}} dA(w) \\
&\le \lim_{r\to 1^-} \sup_{w\in\mathbb{D}} |g'(\sigma_a(w)) - g_r'(\sigma_a(w))| \frac{1}{(m+t+1)(1-\xi^2)^{2+m}} = 0.
\end{aligned}$$

By the definition of distance mentioned above, we obtain that

$$\begin{aligned}
&dist_{F(1,m,t)}(g, F_0(1, m, t)) = \inf_{f\in F_0(1,m,t)} \|g - f\|_{F(1,m,t)} \\
&\le \lim_{r\to 1^-} \|g - g_r\|_{F(1,m,t)} \\
&\approx \lim_{r\to 1^-} \sup_{|a|>\xi} \int_{\mathbb{D}} |g'(z) - g_r'(z)|(1-|z|^2)^m (1-|\sigma_a(z)|^2)^t dA(z) \\
&+ \lim_{r\to 1^-} \sup_{|a|\le\xi} \int_{\mathbb{D}} |g'(z) - g_r'(z)|(1-|z|^2)^m (1-|\sigma_a(z)|^2)^t dA(z) \\
&\le \sup_{|a|>\xi} \int_{\mathbb{D}} |g'(z)|(1-|z|^2)^m (1-|\sigma_a(z)|^2)^t dA(z)
\end{aligned}$$

$$+\limsup_{r\to 1^-}\sup_{|a|>\xi}\int_{\mathbb{D}}|g_r'(z)|(1-|z|^2)^m(1-|\sigma_a(z)|^2)^t dA(z).$$

We write $\Phi_{r,a}(z)=\sigma_{ra}\circ r\sigma_a(z)$. Then $\Phi_{r,a}$ is an analytic self-map of $\mathbb{D}$ and $\Phi_{r,a}(0)=0$. Changing the variable $z=\sigma_a(w)$, and using the Littlewood's Subordination Theorem (see Theorem 1.7 of [5]), we get

$$\begin{aligned}
&\int_{\mathbb{D}}|g_r'(z)|(1-|z|^2)^m(1-|\sigma_a(z)|^2)^t dA(z)\\
=&\int_{\mathbb{D}}|g_r'(\sigma_a(w))|(1-|\sigma_a(w)|^2)^m(1-|w|^2)^t|\sigma_a'(w)|^2 dA(w)\\
\leq&\int_{\mathbb{D}}|g'\circ\sigma_{ra}\circ\Phi_{r,a}(w)|(1-|\sigma_{ra}\circ\Phi_{r,a}(w)|^2)^m(1-|w|^2)^t|\sigma_a'(w)|^2 dA(w)\\
\leq&\int_{\mathbb{D}}|g'(\sigma_{ra}(w))|(1-|\sigma_{ra}(w)|^2)^m(1-|w|^2)^t|\sigma_a'(w)|^2 dA(w)\\
\leq&\int_{\mathbb{D}}|g'(w)|(1-|w|^2)^m(1-|\sigma_{ra}(w)|^2)^t dA(w).
\end{aligned}$$

Take the supremum on the above inequality over $w\in\mathbb{D}$. Because of the arbitrariness of ξ, we obtain

$$\mathrm{dist}_{F(1,m,t)}(g,F_0(1,m,t))\leq\limsup_{|a|\to 1}\int_{\mathbb{D}}|g'(z)|(1-|z|^2)^m(1-|\sigma_a(z)|^2)^t dA(z).$$

For each $g\in F(1,m,t)$, it is easy to get

$$\begin{aligned}
dist_{F(1,m,t)}(g,F_0(1,m,t))&=\inf_{f\in F_0(1,m,t)}\|g-f\|_{F(1,m,t)}\\
&\geq\limsup_{|a|\to 1}\int_{\mathbb{D}}|g'(z)|(1-|z|^2)^m(1-|\sigma_a(z)|^2)^t dA(z).
\end{aligned}$$

The proof is complete. □

Lemma 6. *Let $-1<\beta<0$, $0<\lambda,s<1$ such that $s\geq\lambda(\beta+1)$. If $f\in F(1,\beta-s-(\beta+1)(1-\lambda),s+(\beta+1)(1-\lambda))$, then*

$$|f'(a)|\leq\frac{\|f\|_{F(1,\beta-s-(\beta+1)(1-\lambda),s+(\beta+1)(1-\lambda))}}{(1-|a|^2)^{s-\lambda(\beta+1)+1}},\ a\in\mathbb{D}.$$

Proof. For any $a\in\mathbb{D}$, by a change of variable argument, we have

$$\begin{aligned}
&\int_{\mathbb{D}}|f'(z)|(1-|z|^2)^{\beta-s-(\beta+1)(1-\lambda)}(1-|\sigma_a(z)|^2)^{s+(\beta+1)(1-\lambda)}dA(z)\\
=&\int_{\mathbb{D}}|f'(\sigma_a(z))|(1-|\sigma_a(z)|^2)^{\beta-s-(\beta+1)(1-\lambda)}(1-|z|^2)^{s+(\beta+1)(1-\lambda)}|\sigma_a'(z)|^2 dA(z)\\
\geq&\int_{\mathbb{D}}|(f\circ\sigma_a)'(z))|(1-|a|^2)^{s-\lambda(\beta+1)}(1-|z|^2)^{\beta}dA(z)\\
\geq&|f'(a)|(1-|a|^2)^{s-\lambda(\beta+1)+1}.
\end{aligned}$$

The last inequality used the Lemma 4.12 in [26]. The proof is complete.

□

Lemma 7. *Let* $-1 < \beta < 0$, $0 < \lambda, s < 1$ *such that* $s \geq \lambda(\beta + 1)$. *If* $g \in F(1, \beta - s - (\beta + 1)(1 - \lambda), s + (\beta + 1)(1 - \lambda))$ *and* $0 < r < 1$, *then* $T_{g_r} : D_{\beta,\lambda} \to F(1, \beta - s, s)$ *is compact.*

Proof. Let $\{f_n\}$ be a bounded sequence in $D_{\beta,\lambda}$ such that $\{f_n\}$ converges to zero uniformly on any compact subset of $\mathbb{D}$. Changing the variable $z = \sigma_a(w)$, for any $a \in \mathbb{D}$, from Lemma 6 and Proposition 1 we have

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(T_{g_r} f_n)'(z)|(1 - |z|^2)^{\beta-s}(1 - |\sigma_a(z)|^2)^s dA(z) \\
=& \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_n(z)||g_r'(z)|(1 - |z|^2)^{\beta-s}(1 - |\sigma_a(z)|^2)^s dA(z) \\
\leq& \frac{\|g\|_{F(1,m,t)}}{(1 - r^2)^{s-\lambda(\beta+1)+1}} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_n(z)|(1 - |z|^2)^{\beta-s}(1 - |\sigma_a(z)|^2)^{\lambda(\beta+1)} dA(z) \\
\leq& \frac{\|g\|_{F(1,m,t)}}{(1 - r^2)^{s-\lambda(\beta+1)+1}} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_n'(z)|(1 - |z|^2)^{\beta+1-s}(1 - |\sigma_a(z)|^2)^{\lambda(\beta+1)} dA(z) \\
\leq& \frac{\|g\|_{F(1,m,t)}}{(1 - r^2)^{s-\lambda(\beta+1)+1}} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_n'(z)|(1 - |z|^2)^{\beta-\lambda(\beta+1)}(1 - |\sigma_a(z)|^2)^{\lambda(\beta+1)} dA(z) \\
\leq& \frac{\|g\|_{F(1,m,t)}}{(1 - r^2)^{s-\lambda(\beta+1)+1}} \|f_n\|_{D_{\beta,\lambda}},
\end{aligned}$$

where $m = \beta - s - (\beta + 1)(1 - \lambda)$, $t = s + (\beta + 1)(1 - \lambda)$. Using the Dominated Convergence Theorem we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|T_{g_r} f_n\|_{D_{\beta,\lambda}} &\leq \lim_{n\to\infty} \int_{\mathbb{D}} |f_n'(z)|(1 - |z|^2)^{\beta+1-s} dA(z) \\
&\leq \int_{\mathbb{D}} \lim_{n\to\infty} |f_n'(z)|(1 - |z|^2)^{\beta+1-s} dA(z) = 0,
\end{aligned}$$

which implies that $T_{g_r} : D_{\beta,\lambda} \to F(1, \beta - s, s)$ is compact. The proof is complete. □

The following conclusion is important for studying the essential norm of operators on some analytic function spaces, see [20].

Lemma 8. *Let* X, Y *be two Banach spaces of analytic functions on* $\mathbb{D}$. *Suppose that*

(i) The point evaluation functionals on Y *are continuous.*
(ii) The closed unit ball of X *is a compact subset of* X *in the topology of uniform convergence on compact sets.*
(iii) $T : X \to Y$ *is continuous when* X *and* Y *are given the topology of uniform convergence on compact sets.*

Then, T *is a compact operator if and only if for any bounded sequence* $\{f_n\}$ *in* X *such that* $\{f_n\}$ *converges to zero uniformly on every compact set of* $\mathbb{D}$, *then the sequence* $\{T f_n\}$ *converges to zero in the norm of* Y.

Theorem 5. *Let* $g \in H(\mathbb{D})$, $-1 < \beta < 0$, $0 < \lambda, s < 1$ *such that* $s \geq \lambda(\beta + 1)$. *If* $T_g : D_{\beta,\lambda} \to F(1, \beta - s, s)$ *is bounded, then*

$$\|T_g\|_{e, D_{\beta,\lambda}\to F(1,\beta-s,s)} \approx \mathrm{dist}_{F(1,m,t)}(g, F_0(1, m, t)).$$

Here $m = \beta - s - (\beta + 1)(1 - \lambda)$, $t = s + (\beta + 1)(1 - \lambda)$.

Proof. Assume that $\{I_n\}$ is a sequence of subarcs of $\partial\mathbb{D}$ with $\lim_{n\to\infty} |I_n| = 0$. Let $a_n = (1 - |I_n|)\xi_n$, where ξ_n is the center of arc I_n. Then $\{a_n\}$ is a bounded sequence in $\mathbb{D}$ such that $\lim_{n\to\infty} |a_n| = 1$. Set

$$f_n(z) = \frac{1 - |a_n|^2}{|1 - \bar{a}_n z|^{(\beta+1)(1-\lambda)+1}}, z \in \mathbb{D}.$$

Then $\{f_n\}$ is a bounded sequence in $D_{\beta,\lambda}$ and converges to zero uniformly on every compact subset of $\mathbb{D}$. Moreover,

$$|f_n(z)| \approx \frac{1}{(1 - |a_n|^2)^{(\beta+1)(1-\lambda)}}, \ z \in S(I_n).$$

For any compact operator $K : D_{\beta,\lambda} \to F(1, \beta - s, s)$, by Lemma 8, we have $\lim_{n\to\infty} \|K f_n\|_{F(1,\beta-s,s)} = 0$. Hence

$$\begin{aligned}
\|T_g - K\|_{D_{\beta,\lambda}\to F(1,\beta-s,s)} &\geq \limsup_{n\to\infty} \|(T_g - K)(f_n)\|_{F(1,\beta-s,s)} \\
&\geq \limsup_{n\to\infty} (\|T_g f_n\|_{F(1,\beta-s,s)} - \|K f_n\|_{F(1,\beta-s,s)}) \\
&= \limsup_{n\to\infty} \|T_g f_n\|_{F(1,\beta-s,s)} \\
&\geq \limsup_{n\to\infty} \int_{\mathbb{D}} |f_n(z)||g'(z)|(1 - |z|^2)^{\beta-s}(1 - |\sigma_{a_n}(z)|^2)^s dA(z) \\
&\geq \limsup_{n\to\infty} \int_{S(I_n)} |g'(z)| \frac{(1 - |z|^2)^\beta (1 - |a_n|^2)^{s-(\beta+1)(1-\lambda)}}{|1 - \bar{a}_n z|^{2s}} dA(z) \\
&\geq \limsup_{n\to\infty} \int_{S(I_n)} |g'(z)|(1 - |z|^2)^{\beta-s-(\beta+1)(1-\lambda)}(1 - |\sigma_{a_n}(z)|^2)^{s+(\beta+1)(1-\lambda)} dA(z).
\end{aligned}$$

Then it is obvious that

$$\|T_g\|_{e,D_{\beta,\lambda}\to F(1,\beta-s,s)} \geq \limsup_{n\to\infty} \int_{\mathbb{D}} |g'(z)|(1 - |z|^2)^m (1 - |\sigma_{a_n}(z)|^2)^t dA(z).$$

Since $\{a_n\}$ is arbitrary, using Lemma 5, we have

$$\|T_g\|_{e,D_{\beta,\lambda}\to F(1,\beta-s,s)} \geq \text{dist}_{F(1,m,t)}(g, F_0(1, m, t)).$$

Conversely, by Lemma 7 and Theorem 3,

$$\begin{aligned}
\|T_g\|_{e,D_{\beta,\lambda}\to F(1,\beta-s,s)} &\leq \|T_g - T_{g_r}\|_{D_{\beta,\lambda}\to F(1,\beta-s,s)} \\
&= \|T_{g-g_r}\|_{D_{\beta,\lambda}\to F(1,\beta-s,s)} \approx \|g - g_r\|_{F(1,m,t)}.
\end{aligned}$$

Using Lemma 5 again, we obtain that

$$\|T_g\|_{e,D_{\beta,\lambda}\to F(1,\beta-s,s)} \preceq \lim_{r\to 1^-} \|g - g_r\|_{F(1,m,t)} \approx \text{dist}_{F(1,m,t)}(g, F_0(1, m, t)).$$

The proof is complete. □

From the last theorem, we get the following corollary.

Corollary 1. *Let* $-1 < \beta < 0$, $0 < \lambda, s < 1$ *such that* $s \geq \lambda(\beta+1)$. *If* $g \in H(\mathbb{D})$, *then* $T_g : D_{\beta,\lambda} \to F(1, \beta - s, s)$ *is compact if and only if*

$$g \in F_0(1, \beta - s - (\beta+1)(1-\lambda), s + (\beta+1)(1-\lambda)).$$

In particular, when $s = \lambda(\beta+1)$, we get the following result.

Corollary 2. *Let* $-1 < \beta < 0$, $0 < \lambda < 1$. *If* $g \in H(\mathbb{D})$, *then* $T_g : D_{\beta,\lambda} \to D_{\beta,\lambda}$ *is compact if and only if* $g \in F_0(1, -1, \beta+1)$.

6. Conclusions

In this paper, we mainly prove that the identity operator $I_d : D_{\beta,\lambda} \to \mathcal{T}_s^1(\mu)$ is bounded(compact) if and only if the measure μ is a $s+(\beta+1)(1-\lambda)$-Carleson measure(vanishing $s+(\beta+1)(1-\lambda)$-Carleson measure). As an application, we prove that Volterra integral operator $T_g : D_{\beta,\lambda} \to F(1, \beta - s, s)$ is bounded(compact) if and only if

$$g \in F(1, \beta - s - (\beta+1)(1-\lambda), s + (\beta+1)(1-\lambda))(g \in F_0(1, \beta - s - (\beta+1)(1-\lambda), s + (\beta+1)(1-\lambda))).$$

In particular, $T_g : D_{\beta,\lambda} \to D_{\beta,\lambda}$ is bounded(compact) if and only if $g \in F(1, -1, \beta+1)(g \in F_0(1, -1, \beta+1))$.

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Conflict of interest

We declare that we have no conflict of interest.

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