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Research article

On comparison results for K-nonnegative double splittings of different K-monotone matrices

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Abstract: The comparison results for K-double splittings of one K-monotone matrix are established in the literatures. As comparison theorems between the spectral radii of different matrices are a useful tool for judging the efficiency of preconditioners, we propose some comparison results for K-nonnegative double splittings of different K-monotone matrices in this note. The obtained results generalize the previous ones.

Keywords: K-monotone matrix; K-nonnegative double splitting; comparison theorem
Mathematics Subject Classification: 15A09, 65F15

1. Introduction

Let $\mathbb{R}^n$ be the set of n-dimensional vectors, K be a proper cone in $\mathbb{R}^n$. Let $\pi(K)$ denote the set of $n \times n$ matrices which leave the proper cone $K \subseteq \mathbb{R}^n$ invariant, then $\pi(K)$ is closed under multiplication and is a proper cone in $\mathbb{R}^{n\times n}$ [1]. It should be noted that both the nonnegative cone $\mathbb{R}^n_+$ and the ice cream cone $\{x \in \mathbb{R}^n | (x_2^2 + x_3^2 + \cdots + x_n^2)^{\frac{1}{2}} < x_1\}$ are particular proper cones.

In this note, we consider the linear system

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is a nonsingular matrix and $A^{-1} \in \pi(K)$, $b \in \mathbb{R}^n$ is given and $x \in \mathbb{R}^n$ is unknown. Woźniki in [18] introduced the double splitting of A as

$$A = P - R - S, \tag{1.2}$$

where P is nonsingular, and the approximate solution x_{i+1} of Eq (1.1) is generated by three successive iterations:

$$x_{i+1} = P^{-1}Rx_i + P^{-1}Sx_{i-1} + P^{-1}b,\ i = 1, 2, \cdots . \tag{1.3}$$

It follows from [6] that the iterative scheme (1.3) can be rewritten in the following equivalent form

$$\begin{pmatrix} x_{i+1} \\ x_i \end{pmatrix} = \begin{pmatrix} P^{-1}R & P^{-1}S \\ I & 0 \end{pmatrix} \begin{pmatrix} x_i \\ x_{i-1} \end{pmatrix} + \begin{pmatrix} P^{-1}b \\ 0 \end{pmatrix}, \; i = 1, 2, \cdots, \tag{1.4}$$

where I denotes the identity matrix with compatible size.

The iterative scheme (1.4) converges to the unique solution $x_\star = A^{-1}b$ of (1.1) if and only if the spectral radius of the iteration matrix

$$W = \begin{pmatrix} P^{-1}R & P^{-1}S \\ I & 0 \end{pmatrix}$$

is less than one, i.e., $\rho(W) < 1$. Hou [7] gave convergence and comparison theorems for K-double splittings of an K-monotone matrix. Wang [19] presented convergence and the comparison results for K-nonnegative double splittings of an K-monotone matrix.

The smaller $\rho(W)$, the faster convergence of the iterative scheme (1.4). One approach for improving the convergence rate of the corresponding iterative method is the preconditioning techniques [2]. More precisely, we may solve the preconditioned linear systems

$$QAx = Qb$$

instead of (1.1), here Q, called the preconditioner, is nonsingular. When there are two or more preconditioners for the linear system (1.1), which one is the most efficient one is worth studying. The comparison theorem between the spectral radii of iteration matrices is a useful tool for judging the efficiency of preconditioner [5]. Therefore, in this note we will present the comparison results between the spectral radii of the corresponding iteration matrices arising from K-nonnegative splittings of different preconditioned matrices. For this purpose, we assume that the preconditioned matrices QA with different preconditioners Q satisfy $(QA)^{-1} \in \pi(K)$. The obtained results of this paper are the generalizations of the corresponding results in [7, 16, 19].

The rest of this article is organized as follows. In Section 2, some definitions and results are reviewed. In Section 3, the main comparison results for K-nonnegative double splittings of different preconditioned matrices are given. Finally, in Section 4, conclusions are drawn.

2. Preliminaries

In this section, some definitions and lemmas, which will be used throughout the paper, will be given.

Definition 2.1. *A vector x in $\mathbb{R}^n$ is called K-nonnegative (K-positive) if x belongs to K (x belongs to* int$K$*, the the interior of K) and is denoted by $x \geq_K 0$ ($x >_K 0$). If $x,\ y \in \mathbb{R}^n$ satisfying $x - y \geq_K 0$ ($x - y >_K 0$), we denote $x \geq_K y$ ($x >_K y$).*

Definition 2.2. *An $n \times n$ real matrix A is called K-nonnegative (K-positive) if $AK \subseteq K$ (respectively, $A(K - \{0\}) \subseteq$* int$K$*) and is denoted as $A \geq_K 0$ ($A >_K 0$). Similarly, for $n \times n$ real matrices A and B we denote $A - B \geq_K 0$ ($A - B >_K 0$) by $A \geq_K B$ ($A >_K B$).*

Clearly, A is K-nonnegative is equivalent to $A \in \pi(K)$ [7]. Basic properties of the K-nonnegative matrix are given in [1, 7, 8]. It should be remarked that the properties of K-nonnegative matrices are very similar to the theory of nonnegative matrices, see for example [1, 3, 4, 8].

Based on the definition of K-nonnegative matrix, we can give the definition of K-monotone matrix.

Definition 2.3. *Let $A \in \mathbb{R}^{n\times n}$ be nonsingular, A is called K-monotone if $A^{-1} \geq_K 0$, i.e., $A^{-1} \in \pi(K)$.*

Hou and Li [8] proposed the definition of the K-nonnegative single splittings, and Wang [19] introduced the K-nonnegative double splittings as:

Definition 2.4. *Let A be a nonsingular matrix. Then,*

(i). the single splitting $A = M - N$ is a K-nonnegative single splitting if $M^{-1}N \geq_K 0$ [8];
(ii). the double splitting $A = P-R-S$ is a K-nonnegative double splitting if $P^{-1}R \geq_K 0$ and $P^{-1}S \geq_K 0$ [19].

It should be remarked that the K-nonnegative single splitting is termed as K-weak splitting in [9] or K-weak splitting of the first type in [8]. Moreover, rewriting (1.2) as

$$A = P - (R + S), \tag{2.1}$$

then it is a single splitting of A. If the double splitting (1.2) is K-nonnegative, then the single splitting (2.1) is also K-nonnegative.

3. Main result

In this section, we will present the comparison results for the K-nonnegative double splittings of different K-monotone matrices. Assume that there are two preconditioners Q_1 and Q_2 for the linear system (1.1), then we have two preconditioned linear systems with coefficient matrices $A_1 = Q_1A$ and $A_2 = Q_2A$, respectively. Moreover, we further assume that the preconditioned matrices A_1 and A_2 are K-monotone matrices. Some excellent comparison results for K-nonnegative single splittings of A_1 and A_2 are given in [9, 11].

Let

$$A_1 = P_1 - R_1 - S_1 \ \text{ and } \ A_2 = P_2 - R_2 - S_2 \tag{3.1}$$

be K-nonnegative double splittings of A_1 and A_2, respectively. Define the corresponding iteration matrices

$$W_1 = \begin{pmatrix} P_1^{-1}R_1 & P_1^{-1}S_1 \\ I & 0 \end{pmatrix} \ \text{and} \ W_2 = \begin{pmatrix} P_2^{-1}R_2 & P_2^{-1}S_2 \\ I & 0 \end{pmatrix}.$$

For $i = 1,\ 2$, if we split

$$\mathbb{A}_i = \begin{pmatrix} A_i & 0 \\ -I & I \end{pmatrix}$$

as

$$\mathbb{A}_i = \mathbb{M}_i - \mathbb{N}_i \tag{3.2}$$

with

$$\mathbb{M}_i = \begin{pmatrix} P_i & S_i \\ 0 & I \end{pmatrix},\ \mathbb{N}_i = \begin{pmatrix} R_i + S_i & S_i \\ I & 0 \end{pmatrix},$$

then $W_i = \mathbb{M}_i^{-1}\mathbb{N}_i$. Therefore, we can get comparison results for the double splitting (3.1) by investigating the splitting (3.2).

Theorem 3.1. *Let A_1 and A_2 be K-monotone matrices, the splittings (3.1) be K-nonnegative and convergent. Suppose $A_2^{-1} \geq_K A_1^{-1}$, $S_2 \geq_K S_1$ and $R_2 + S_2 \geq_K R_1 + S_1$, then*

$$\rho(W_1) \leq \rho(W_2).$$

Proof. Firstly, it is easy to see that the splittings $\mathbb{A}_i = \mathbb{M}_i - \mathbb{N}_i$ are K-nonnegative as the splittings (3.1) are K-nonnegative. Secondly, note that

$$\mathbb{A}_2^{-1} = \begin{pmatrix} A_2^{-1} & 0 \\ A_2^{-1} & I \end{pmatrix} \geq_K \begin{pmatrix} A_1^{-1} & 0 \\ A_1^{-1} & I \end{pmatrix} = \mathbb{A}_1^{-1} \geq_K 0$$

and

$$\mathbb{N}_2 = \begin{pmatrix} R_2 + S_2 & S_2 \\ I & 0 \end{pmatrix} \geq_K \begin{pmatrix} R_1 + S_1 & S_1 \\ I & 0 \end{pmatrix} = \mathbb{N}_1$$

hold under the assumptions. Hence, it follows from [9, Theorem 3.13] that $\rho(W_1) \leq \rho(W_2)$. □

The conditions $S_2 \geq_K S_1$ and $R_2 \geq_K R_1$ imply $S_2 \geq_K S_1$ and $R_2 + S_2 \geq_K R_1 + S_1$, so from Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let A_1 and A_2 be K-monotone matrices, the splittings (3.1) be K-nonnegative and convergent. Suppose $A_2^{-1} \geq_K A_1^{-1}$, $S_2 \geq_K S_1$ and $R_2 \geq_K R_1$, then*

$$\rho(W_1) \leq \rho(W_2).$$

Remark 3.3. *When we pay our attention to the particular proper cone $K = \mathbb{R}_+^n$, then Theorem 3.1 and Corollary 3.2 are Theorem 3.12 and Corollary 3.13 in [16], respectively.*

The following example shows that the conditions that the splittings (3.1) are convergent cannot be dropped in Theorems 3.1.

Example 3.4. *Let $K = \{x \in \mathbb{R}^3 | (x_2^2 + x_3^2)^{\frac{1}{2}} < x_1\}$. Assume that*

$$A = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ -\frac{1}{10} & \frac{5}{4} & 0 \\ 0 & \frac{1}{4} & \frac{5}{4} \end{pmatrix}, \quad P_1 = \begin{pmatrix} -4 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} -2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then we have

$$A_1 = \begin{pmatrix} 2 & 0 & 0 \\ \frac{3}{10} & \frac{5}{2} & 0 \\ 0 & \frac{3}{4} & \frac{15}{4} \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{3}{10} & \frac{5}{2} & 0 \\ 0 & \frac{3}{4} & \frac{15}{4} \end{pmatrix}.$$

If A_1 and A_2 are splitted as

$$A_1 = P_1 - R_1 - S_1 \text{ and } A_2 = P_2 - R_2 - S_2, \tag{3.3}$$

respectively, here

$$R_1 = \begin{pmatrix} -3 & 0 & 0 \\ -\frac{2}{5} & -\frac{1}{4} & 0 \\ 0 & -\frac{3}{8} & -\frac{3}{8} \end{pmatrix}, S_1 = \begin{pmatrix} -3 & 0 & 0 \\ -\frac{9}{10} & -\frac{1}{4} & 0 \\ 0 & -\frac{3}{8} & -\frac{3}{8} \end{pmatrix}$$

and

$$R_2 = \begin{pmatrix} -\frac{3}{2} & 0 & 0 \\ -\frac{2}{5} & -\frac{1}{4} & 0 \\ 0 & -\frac{3}{4} & -\frac{3}{8} \end{pmatrix}, S_2 = \begin{pmatrix} -\frac{3}{2} & 0 & 0 \\ -\frac{9}{10} & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{3}{8} \end{pmatrix}.$$

It is easy to see that

$$A_1^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{3}{50} & \frac{2}{5} & 0 \\ \frac{3}{250} & -\frac{2}{25} & \frac{4}{15} \end{pmatrix}$$

and

$$A_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{25} & \frac{2}{5} & 0 \\ \frac{3}{125} & -\frac{2}{25} & \frac{4}{15} \end{pmatrix},$$

i.e., A_1 and A_2 are K-monotone matrices, although A_1 and A_2 are not monotone matrices. By calculating, we have

$$P_1^{-1}R_1 = \begin{pmatrix} \frac{3}{4} & 0 & 0 \\ \frac{7}{40} & -\frac{1}{8} & 0 \\ 0 & -\frac{1}{8} & -\frac{1}{8} \end{pmatrix}, \quad P_1^{-1}S_1 = \begin{pmatrix} \frac{3}{4} & 0 & 0 \\ -\frac{3}{40} & -\frac{1}{8} & 0 \\ 0 & -\frac{1}{8} & -\frac{1}{8} \end{pmatrix}$$

and

$$P_2^{-1}R_2 = \begin{pmatrix} \frac{3}{4} & 0 & 0 \\ \frac{7}{40} & -\frac{1}{8} & 0 \\ 0 & -\frac{1}{4} & -\frac{1}{8} \end{pmatrix}, \quad P_2^{-1}S_2 = \begin{pmatrix} \frac{3}{4} & 0 & 0 \\ -\frac{3}{40} & -\frac{1}{8} & 0 \\ 0 & 0 & -\frac{1}{8} \end{pmatrix}.$$

Here the splittings (3.3) are not nonnegative double splittings, but K-nonnegative double splittings. It is easy to verify that $A_2^{-1} \geq_K A_1^{-1}$, $S_2 \geq_K S_1$ and $R_2 + S_2 \geq_K R_1 + S_1$, but $\rho(W_1) > 1$ and $\rho(W_2) > 1$. In fact, we have

$$\rho(W_1) = 1.3187 = \rho(W_2).$$

In particular, if we restrict our discussion on the different K-nonnegative double splittings of one K-monotone matrix A, then from Theorem 3.1, Corollary 3.2 and [9, Theorem 3.5], the following corollaries are obtained.

Corollary 3.5. *Let A be K-monotone matrix, the splittings $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be K-nonnegative and convergent. Suppose $S_2 \geq_K S_1$ and $R_2 + S_2 \geq_K R_1 + S_1$, then*

$$\rho(W_1) \leq \rho(W_2).$$

Corollary 3.6. *Let A be K-monotone matrix, the splittings $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be K-nonnegative and convergent. Suppose $S_2 \geq_K S_1$ and $R_2 \geq_K R_1$, then*

$$\rho(W_1) \leq \rho(W_2).$$

Corollary 3.5 and 3.6 are just Theorem 2 and Corollary 1 in [19], respectively. In summary, Theorem 3.1 extends some results in [16] and [19].

In what follows, taking the inverses of P_1 and P_2 into account, we will derive another comparison theorem. Note that for $i = 1,\ 2$,

$$\mathbb{M}_i^{-1} = \begin{pmatrix} P_i^{-1} & -P_i^{-1}S_i \\ 0 & I \end{pmatrix}.$$

Hence, $\mathbb{M}_1^{-1} \geq_K \mathbb{M}_2^{-1}$ if $P_1^{-1} \geq_K P_2^{-1}$ and $P_1^{-1}S_1 \leq_K P_2^{-1}S_2$. If we assume $A_1 \geq_K A_2$ additionally, then the conditions of [9, Theorem 3.15] are satisfied. Therefore, from [9, Theorem 3.15], we have the following comparison result.

Theorem 3.7. *Let A_1 and A_2 be K-monotone matrices, the splittings (3.1) be K-nonnegative and convergent. Suppose $A_1 \geq_K A_2$, $P_1^{-1} \geq_K P_2^{-1}$ and $P_1^{-1}S_1 \leq_K P_2^{-1}S_2$, then*

$$\rho(W_1) \leq \rho(W_2).$$

If we turn our attention to the particular proper cone $K = \mathbb{R}^n_+$, then following conclusion is a direct consequence of Theorem 3.7.

Corollary 3.8. *Let A_1 and A_2 be monotone matrices, the splittings (3.1) be nonnegative and convergent. Suppose $A_1 \geq A_2$, $P_1^{-1} \geq P_2^{-1}$ and $P_1^{-1}S_1 \leq P_2^{-1}S_2$, then*

$$\rho(W_1) \leq \rho(W_2).$$

Moreover, if we consider the different nonnegative double splittings of one matrix A, then from Theorem 3.7, Corollary 3.8, the following corollaries are obtained.

Corollary 3.9. *Let A be K-monotone matrix, the splittings $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be K-nonnegative and convergent. Suppose $P_1^{-1} \geq_K P_2^{-1}$ and $P_1^{-1}S_1 \leq_K P_2^{-1}S_2$, then*

$$\rho(W_1) \leq \rho(W_2).$$

Remark 3.10. *Corollary 3.9 extends [7, Theorem 3.1 (ii)] and [7, Theorem 3.3 (ii)].*

Corollary 3.11. *Let A be monotone matrices, the splittings $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be nonnegative and convergent. Suppose $P_1^{-1} \geq P_2^{-1}$ and $P_1^{-1}S_1 \leq P_2^{-1}S_2$, then*

$$\rho(W_1) \leq \rho(W_2).$$

Remark 3.12. *We assume in Corollary 3.11 that the splittings $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ be nonnegative and convergent, while it assumed that $A = P_1 - R_1 - S_1$ be regular, $A = P_2 - R_2 - S_2$ be nonegative and both be convergent in [16, Theorem 3.9]. So Corollary 3.11 is a new comparison results for different nonnegative double splittings of one monotone matrix, which has weaker conditions than Theorem 3.9 in [16].*

The regular splitting $A = P_1 - R_1 - S_1$ is nonnegative, but not vice versa. The following example shows that the inequality $\rho(W_1) \leq \rho(W_2)$ holds for nonnegative splittings $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ of one monotone matrix A instead of the regular splitting $A = P_1 - R_1 - S_1$ and the nonnegative splitting $A = P_2 - R_2 - S_2$.

Example 3.13. *Let the monotone matrix*

$$A = \begin{pmatrix} 4 & -2 \\ -2 & 5 \end{pmatrix}$$

be splitted as $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ *with*

$$P_1 = \begin{pmatrix} 5 & 0 \\ 1 & 6 \end{pmatrix}, \; R_1 = \begin{pmatrix} 1 & 1 \\ 2 & \frac{3}{4} \end{pmatrix}, \; S_1 = \begin{pmatrix} 0 & 1 \\ 1 & \frac{1}{4} \end{pmatrix}$$

and

$$P_2 = \begin{pmatrix} 5 & 0 \\ 2 & 6 \end{pmatrix}, \; R_2 = \begin{pmatrix} 1 & 1 \\ 2 & \frac{1}{2} \end{pmatrix}, \; S_2 = \begin{pmatrix} 0 & 1 \\ 2 & \frac{1}{2} \end{pmatrix}.$$

Some calculations yield

$$A^{-1} = \begin{pmatrix} \frac{5}{16} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{4} \end{pmatrix}, \; P_1^{-1} = \begin{pmatrix} \frac{1}{5} & 0 \\ -\frac{1}{30} & \frac{1}{6} \end{pmatrix} \text{and } P_2^{-1} = \begin{pmatrix} \frac{1}{5} & 0 \\ -\frac{1}{15} & \frac{1}{6} \end{pmatrix}.$$

It should be noted that both splittings $A = P_1 - R_1 - S_1 = P_2 - R_2 - S_2$ *are nonnegative splittings, not regular splittings. It is easy to see that*

$$P_1^{-1} S_1 = \begin{pmatrix} 0 & \frac{1}{5} \\ \frac{1}{6} & \frac{1}{120} \end{pmatrix} \leq \begin{pmatrix} 0 & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{60} \end{pmatrix} = P_2^{-1} S_2.$$

It follows from Corollary 3.11 that $\rho(W_1) \leq \rho(W_2)$*. In fact, we have*

$$\rho(W_1) = 0.6751 < 0.7172 = \rho(W_2).$$

4. Conclusions

In this paper, we established the comparison results for two K-nonnegative double splittings of different K-monotone matrices, the obtained results generalized the corresponding results in [7,16,19].

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Conflict of interest

The authors declare no conflict of interest.

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