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Research article

Characterizations of Euclidean spheres

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Abstract: We use the tangential component ψ^T of an immersion of a compact hypersurface of the Euclidean space $\mathbf{E}^{m+1}$ in finding two characterizations of a sphere. In first characterization, we use ψ^T as a geodesic vector field (vector field with all its trajectories geodesics) and in the second characterization, we use ψ^T to annihilate the de-Rham Laplace operator on the hypersurface.

Keywords: geodesic vector field; de-Rham Laplace operator; support function; Euclidean space; sphere
Mathematics Subject Classification: 53C20, 53A30

1. Introduction

Geometry of hypersurfaces of a Riemannian manifold is one of the important branches of differential geometry. In that, one of important questions is characterizing spheres among compact hypersurfaces of a Euclidean space [1,3–8]. On a Riemannian manifold (M, g), the Ricci operator S is defined using Ricci tensor Ric, namely $Ric(X, Y) = g(SX, Y)$, $X \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. Similarly, the rough Laplace operator on the Riemannian manifold (M, g), $\Delta : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined by

$$\Delta X = \sum_{i=1}^{m} \left(\nabla_{e_i} \nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X \right), \quad X \in \mathfrak{X}(M), \tag{1.1}$$

where ∇ is the Riemannian connection and $\{e_1, ..., e_m\}$ is a local orthonormal frame on M, $m = \dim M$. Rough Laplace operator is used in finding characterizations of spheres as well as of Euclidean spaces [11, 12]. Recall that de-Rham Laplace operator $\Box : \mathfrak{X}(M) \to \mathfrak{X}(M)$ on a Riemannian manifold (M, g) is defined by ([10], p83)

$$\Box = S + \Delta \tag{1.2}$$

and is used to characterize a Killing vector field on a compact Riemannian manifold. Recall that a vector field $\mathbf{u}$ on a Riemannian manifold (M, g) is said to be a geodesic vector field [9] if

$$\nabla_{\mathbf{u}}\mathbf{u} = 0. \tag{1.3}$$

Let M be an orientable immersed hypersurface of the Euclidean space and $\psi : M \to \mathbf{E}^{m+1}$ be the immersion. We denote the unit normal to the hypersurface M by ξ and support function of the hypersurface by σ defined by $\sigma = \langle \psi, \xi \rangle$, where $\langle , \rangle$ is the Euclidean metric on $\mathbf{E}^{m+1}$. Then treating ψ as position vector field of the hypersurface M, we have $\psi = \psi^T + \sigma\xi$.

Consider the sphere $\mathbf{S}^m(c)$ of constant curvature c as hypersurface of the Euclidean space $\mathbf{E}^{m+1}$ with unit normal ξ and shape operator $A = -\sqrt{c}I$. Now, consider the embedding $\psi : \mathbf{S}^m(c) \to \mathbf{E}^{m+1}$. Then it follows that the tangential component ψ^T for the sphere $\mathbf{S}^m(c)$ satisfies $\Box\psi^T = 0$ as well as ψ^T is a geodesic vector field. These raise two questions: (i) Under what condition on a compact hypersurface M of a Euclidean space $\mathbf{E}^{m+1}$ with immersion $\psi : M \to \mathbf{E}^{m+1}$ such that ψ^T is a geodesic vector field, M is isometric to a sphere? (ii) Under what conditions on a compact hypersurface M of a Euclidean space $\mathbf{E}^{m+1}$ with immersion $\psi : M \to \mathbf{E}^{m+1}$ such that ψ^T satisfying $\Box\psi^T = 0$, M is isometric to a sphere? In this paper, we answer these questions, for first question by showing that under the condition $Ric\left(\psi^T, \psi^T\right) \geq \frac{m-1}{m}\left(\operatorname{div} \psi^T\right)^2$ the hypersurface is isometric to a sphere, where as for the second question, it requires the condition $|\sigma\alpha| \leq 1$, where α is the mean curvature (Theorem 3.1 and Theorem 3.2).

2. Preliminaries

Let M be an orientable immersed hypersurface of of the Euclidean space $\mathbf{E}^{m+1}$ with immersion $\psi : M \to \mathbf{E}^{m+1}$ with unit normal ξ and shape operator A. Then we have the following Gauss-Weingarten formulae

$$D_X Y = \nabla_X Y + g(AX, Y)\xi, \quad D_X\xi = -AX, \quad X, Y \in \mathfrak{X}(M), \tag{2.1}$$

where D, ∇ are Riemannian connections on $\mathbf{E}^{m+1}$, M respectively, g is the induced metric on M and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. The curvature tensor field R and the Ricci curvature Ric of the hypersurface are given by

$$R(X, Y)Z = g(AY, Z)AX - g(AX, X)AY, \quad X, Y, Z \in \mathfrak{X}(M) \tag{2.2}$$

and

$$Ric(X, Y) = m\alpha g(AX, Y) - g(AX, AY), \quad X, Y \in \mathfrak{X}(M), \tag{2.3}$$

where, $\alpha = \frac{1}{m}Tr.A$ is the mean curvature of the hypersurface [2]. Using Eq (2.3), we see that the Ricci operator S of the hypersurface M is given by

$$S(X) = m\alpha AX - A^2X, \quad X \in \mathfrak{X}(M) \tag{2.4}$$

Also, as the Euclidean space $\mathbf{E}^{m+1}$ is space of constant curvature, the Codazzi equation for the hypersurface M is

$$(\nabla A)(X, Y) = (\nabla A)(Y, X), \quad X, Y \in \mathfrak{X}(M), \tag{2.5}$$

where $(\nabla A)(X, Y) = \nabla_X AY - A(\nabla_X Y)$. Using Eq (2.5) and symmetry of the shape operator A, the gradient grad α of the mean curvature α is given by

$$\text{grad}\,\alpha = \frac{1}{m}\sum_{i=1}^{m} (\nabla A)(e_i, e_i), \tag{2.6}$$

where $\{e_1, ...e_n\}$ is a local orthonormal frame on M.

Let ψ^T be the tangential component of the immersion $\psi : M \to \mathbf{E}^{m+1}$ and $\sigma = \langle\psi, \xi\rangle$ be the support function of the hypersurface M. Then, we have $\psi = \psi^T + \sigma\xi$ and using Eq (2.1), we get

$$\nabla_X \psi^T = X + \sigma AX, \quad \text{grad}\,\sigma = -A\psi^T, \quad X \in \mathfrak{X}(M). \tag{2.7}$$

Using above equation, we have

$$\text{div}\,\psi^T = m(1 + \sigma\alpha). \tag{2.8}$$

Thus, for a compact hypersurface M of the Euclidean space $\mathbf{E}^m$, on integrating the above equation, we have the following Minkowski's formula

$$\int_M (1 + \sigma\alpha) = 0. \tag{2.9}$$

On a compact Riemannian manifold (M, g), the Laplace operator Δ acting on a smooth function $h : M \to \mathbf{R}$ is defined by $\Delta h = \text{div}\;(\text{grad}\,h)$ and the Hessian operator H_h for the smooth function h is a symmetric operator defined by

$$H_h(X) = \nabla_X \text{grad}\,h, \quad X \in \mathfrak{X}(M).$$

On a compact Riemannian manifold (M, g), we have the following formula known as Bochner's formula

$$\int_M Ric\,(\text{grad}\,h, \text{grad}\,h) = \int_M \left((\Delta h)^2 - \|H_h\|^2\right). \tag{2.10}$$

On the other hand, given a Riemannian manifold (M, g) and a vector field $X \in \mathfrak{X}(M)$,, we let θ_X denote the dual one form of X (that is, defined by $\theta_X(Y) = g(X, Y)$) and A_X be the $(1, 1)$-tensor (viewed as an endomorphism) defined by

$$A_X(Y) = \nabla_Y X$$

Write as usual

$$L_X g(Y, Z) + d\theta_X(Y, Z) = 2g(A_X(Y), Z),$$

for all $Y, Z \in \mathfrak{X}(M)$.

Let B and ϕ be the symmetric and anti-symmetric parts of A_X. In other words, we have

$$L_X g(Y, Z) = 2g(B(Y), Z)$$
$$d\theta_X(Y, Z) = 2g(\phi(Y), Z)$$

Now, formula $\nabla_X \psi^T = X + \sigma AX$ in Eq (2.7) is nothing but $A_{\psi^T} = I + \sigma A$. It follows that $B = I + \sigma A$ and $\phi = 0$, and this implies that ψ^T is gradient.

3. Characterizations of spheres

Let M be an orientable compact immersed hypersurface of the Euclidean space $\mathbf{E}^{m+1}$ with immersion $\psi : M \to \mathbf{E}^{m+1}$ and unit normal ξ, shape operator A. In this section, we answer the questions raised in the introduction and find two new characterizations of the Euclidean spheres.

Theorem 1. *Let $\psi : M \to \mathbf{E}^{m+1}$ be an immersion of a compact simply connected hypersurface with ψ^T a non-trivial geodesic vector field, $m \geq 2$. Then the Ricci curvature satisfies*

$$Ric\left(\psi^T, \psi^T\right) \geq \frac{m-1}{m}\left(\operatorname{div}\psi^T\right)^2,$$

if and only if, the mean curvature α is a constant, ψ^T is a non-homothetic conformal vector field, and M is isometric to the sphere $\mathbf{S}^m\left(\alpha^2\right)$.

Proof. Suppose ψ^T is a geodesic vector field and the Ricci curvature of the hypersurface M satisfies

$$Ric\left(\psi^T, \psi^T\right) \geq \frac{m-1}{m}\left(\operatorname{div}\psi^T\right)^2. \tag{3.1}$$

Then, using Eqs (1.3) and (2.7), we have $\sigma A\psi^T = -\psi^T$. Taking covariant derivative with respect to $X \in \mathfrak{X}(M)$ in this equation and using Eq (2.7), we get

$$X(\sigma) A\psi^T + \sigma(\nabla A)(X, \psi^T) + \sigma A(X + \sigma AX) = -X - \sigma AX,$$

that is

$$\sigma(\nabla A)(X, \psi^T) = -X(\sigma) A\psi^T - X - 2\sigma AX - \sigma^2 A^2 X, \quad X \in \mathfrak{X}(M).$$

Now, for a local orthonormal frame $\{e_1, ..., e_m\}$ on M, choosing $X = e_i$ in above equation and taking the inner product with e_i in above equation and summing the resulting equation, we conclude

$$m\sigma\psi^T(\alpha) = -g\left(A\psi^T, \operatorname{grad}\sigma\right) - m - 2m\sigma\alpha - \sigma^2\|A\|^2,$$

where we used symmetry of the shape operator A and Eq (2.6). Now, using Eq (2.7) in above equation, we have

$$m\sigma\psi^T(\alpha) = \left\|A\psi^T\right\|^2 - m - 2m\sigma\alpha - \sigma^2\|A\|^2. \tag{3.2}$$

Note that $\operatorname{div}\left(\alpha\left(\sigma\psi^T\right)\right) = \sigma\psi^T(\alpha) + \alpha\operatorname{div}\left(\sigma\psi^T\right)$ and using Eqs (2.7), (2.8), we get

$$\sigma\psi^T(\alpha) = \operatorname{div}\left(\alpha\left(\sigma\psi^T\right)\right) + \alpha g\left(A\psi^T, \psi^T\right) - m\sigma\alpha(1 + \sigma\alpha). \tag{3.3}$$

Inserting the above equation in Eq (3.2) and using Eq (2.3), we conclude

$$Ric\left(\psi^T, \psi^T\right) - m^2\sigma\alpha(1 + \sigma\alpha) + \operatorname{div}\left(\alpha\left(\sigma\psi^T\right)\right) = -m - 2m\sigma\alpha - \sigma^2\|A\|^2.$$

Integrating the above equation while using Minkowski's formula (2.9), we have

$$\int_M \left(Ric\left(\psi^T, \psi^T\right) + m(m-1) - m^2\sigma^2\alpha^2 + \sigma^2\|A\|^2\right) = 0,$$

that is,

$$\int_M \left(Ric\left(\psi^T, \psi^T\right) - m(m-1)\left(\sigma^2\alpha^2 - 1\right)\right) = \int_M \sigma^2\left(m\alpha^2 - \|A\|^2\right). \tag{3.4}$$

Now, we use div $\psi^T = m(1 + \sigma\alpha)$ and Eq (2.9), to arrive at

$$\int_M \left(\operatorname{div} \psi^T\right)^2 = m^2 \int_M \left(1 + 2\sigma\alpha + \sigma^2\alpha^2\right) = m^2 \int_M \left(\sigma^2\alpha^2 - 1\right)$$

and inserting the above equation in Eq (3.4), we have

$$\int_M \left(Ric\left(\psi^T, \psi^T\right) - \frac{m-1}{m}\left(\operatorname{div} \psi^T\right)^2\right) = \int_M \sigma^2\left(m\alpha^2 - \|A\|^2\right). \tag{3.5}$$

Using inequality (3.1) in Eq (3.5), we get

$$\int_M \sigma^2\left(m\alpha^2 - \|A\|^2\right) \geq 0$$

and above inequality in view of the Schwart's inequality $\|A\|^2 \geq m\alpha^2$ implies

$$\sigma^2\left(m\alpha^2 - \|A\|^2\right) = 0.$$

If $\sigma = 0$, then by Minkowski's formula (2.9), we get a contradiction. Thus, we have $\|A\|^2 = m\alpha^2$ and this equality holds, if and only if, $A = \alpha I$. In other words, M is shown to be totally umbilical. Moreover, using $A = \alpha I$, we have

$$(\nabla A)(X, Y) = X(\alpha)\, Y,$$

and we get

$$\sum_{i=1}^{m} (\nabla A)(e_i, e_i) = \operatorname{grad} \alpha.$$

Using Eq (2.6), we get $(m-1)\operatorname{grad}\alpha = 0$ and with restriction on dimension m, we conclude α is a constant. Moreover, this constant $\alpha \neq 0$ due to the fact that the Euclidean space does not have a compact minimal hypersurface. Inserting $A = \alpha I$ in Eq (2.2), we see that the simply connected hypersurface M has constant curvature α^2 and M being compact, it is complete. Hence, M is complete simply connected hypersurface of constant positive curvature α^2 and is therefore isometric to the sphere $\mathbf{S}^m(\alpha^2)$. Since $B = (1 + \sigma\alpha)\, I$, we get $L_{\psi^T} g = 2(1 + \sigma\alpha)\, g$. In other words, ψ^T is a conformal vector field which is non-homothetic, given that the function $1 + \sigma\alpha$ is not constant as ψ^T is supposed to be non-trivial. The converse is trivial as ψ^T for the natural embedding $\psi : \mathbf{S}^m(\alpha^2) \to \mathbf{E}^{m+1}$ has $\psi^T = 0$, which satisfies the hypothesis of the Theorem. □

Theorem 2. *Let $\psi : M \to \mathbf{E}^{m+1}$ be an immersion of a compact simply connected hypersurface with $\Box\psi^T = 0$, $m \geq 2$. Then the mean curvature α and support function σ satisfies*

$$|\sigma\alpha| \leq 1,$$

if and only if, the mean curvature α is a constant, ψ^T is a parallel vector field (i.e., the covariant derivative of ψ^T vanishes), and M is isometric to the sphere $\mathbf{S}^m\left(\alpha^2\right)$.

Proof. Let M be a compact simply connected hypersurface of the Euclidean space $\mathbf{E}^{m+1}$ with $\square\psi^T = 0$, and

$$|\sigma\alpha| \leq 1. \tag{3.6}$$

We use Eqs (2.6) and (2.7) in computing $\Delta\psi^T$, where Δ is rough Laplace operator, and obtain

$$\Delta\psi^T = A(\text{grad}\,\sigma) + m\sigma\text{grad}\,\alpha.$$

Using Eq (2.7), we get

$$\Delta\psi^T = -A^2\left(\psi^T\right) + m\sigma\text{grad}\,\alpha. \tag{3.7}$$

Also, in view of Eq (2.4), we have

$$S\left(\psi^T\right) = m\alpha A\left(\psi^T\right) - A^2\left(\psi^T\right). \tag{3.8}$$

Now, using Eqs (1.2), (3.7) and (3.8), we conclude

$$\square\psi^T = -2A^2\left(\psi^T\right) + m\alpha A\left(\psi^T\right) + m\sigma\text{grad}\,\alpha$$

and taking the inner product in above equation with ψ^T and using $\square\psi^T = 0$, we arrive at

$$m\alpha g\left(A\psi^T, \psi^T\right) - 2g\left(A\psi^T, A\psi^T\right) + m\sigma\psi^T(\alpha) = 0.$$

Using Eq (3.3), we conclude

$$2m\alpha g\left(A\psi^T, \psi^T\right) - 2\left\|A\psi^T\right\|^2 + m\text{div}\,\left(\alpha\left(\sigma\psi^T\right)\right) - m^2\sigma\alpha(1 + \sigma\alpha) = 0$$

and in view of Eq (2.3), we get

$$2Ric\left(\psi^T, \psi^T\right) + m\text{div}\,\left(\alpha\left(\sigma\psi^T\right)\right) - m^2\sigma\alpha(1 + \sigma\alpha) = 0.$$

Integrating the above equation, while using Eq (2.9), we get

$$\int_M \left(2Ric\left(\psi^T, \psi^T\right) + m^2 - m^2\sigma^2\alpha^2\right) = 0. \tag{3.9}$$

Define a smooth function h on the hypersurface M, by $h = \frac{1}{2}\|\psi\|^2$, which has gradient grad $h = \psi^T$ and $\Delta h = m(1 + \sigma\alpha)$. Also using Eq (2.7), the Hessian H_h is given by

$$H_h = I + \sigma A. \tag{3.10}$$

Thus, using Bochner's formula (2.10), we have

$$\begin{aligned}\int_M Ric\left(\psi^T, \psi^T\right) &= \int_M \left((\Delta h)^2 - \|H_h\|^2\right)\\ &= \int_M \left(\frac{1}{m}(\Delta h)^2 - \|H_h\|^2 + \frac{m-1}{m}(\Delta h)^2\right).\end{aligned}$$

Inserting $\Delta h = m(1 + \sigma\alpha)$ in the last term of the right hand side of above equation, we have

$$\int_M \left(\frac{1}{m}(\Delta h)^2 - \|H_h\|^2\right) = \int_M \left(Ric\left(\psi^T, \psi^T\right) - m(m-1)(1+\sigma\alpha)^2\right)$$

and using Eq (2.9), we get

$$2\int_M \left(\frac{1}{m}(\Delta h)^2 - \|H_h\|^2\right) = \int_M \left(2Ric\left(\psi^T, \psi^T\right) - 2m(m-1)(\sigma^2\alpha^2 - 1)\right). \tag{3.11}$$

Combining Eqs (3.9) and (3.11), we arrive at

$$2\int_M \left(\frac{1}{m}(\Delta h)^2 - \|H_h\|^2\right) = m(m-2)\int_M \left(1 - \sigma^2\alpha^2\right).$$

Using Schwarz' inequality $\|H_h\|^2 \geq \frac{1}{m}(\Delta h)^2$ and inequality (3.6) with $m \geq 2$, in above equation, we conclude the equality $\|H_h\|^2 = \frac{1}{m}(\Delta h)^2$ and this equality holds, if and only if

$$H_h = \frac{\Delta h}{m} I.$$

Now, using Eq (3.10) and $\Delta h = m(1 + \sigma\alpha)$ in above equation, we get

$$\sigma(A - \alpha I) = 0.$$

If $\sigma = 0$, we get a contradiction by Eq (2.9). Thus, we get $A = \alpha I$ and following the proof of Theorem 3.1, we get M is isometric to the sphere $\mathbf{S}^n(\alpha^2)$. We also deduce that $L_{\psi^T} g = 2(1 + \sigma\alpha) g$, with both σ and α constants. It follows that $1 + \sigma\alpha = 0$, since otherwise ψ^T becomes a homothetic vector field and consequently M is isometric to the Euclidean space, a contradiction. Thus, $\nabla X = A_{\psi^T} = 0$, that is ψ^T is parallel. The converse is trivial as on $\mathbf{S}^m(\alpha^2)$ as hypersurface of the Euclidean space $\mathbf{E}^{m+1}$, we have $\psi^T = 0$ and $\sigma = -\frac{1}{\alpha}$, and $|\sigma\alpha| = 1$. □

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Conflict of interest

The authors declare that there is no conflict of interest.

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