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Research article

The Hermite-Hadamard inequality for s-Convex functions in the third sense

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Abstract: In this paper, the Hermite-Hadamard inequality for s-convex functions in the third sense is provided. In addition, some integral inequalities for them are presented. Also, the new functions based on the integral and double integral of s-convex functions in the third sense are defined and under certain conditions, the third sense s-convexity of these functions are shown and some inequality relations for these are expressed.

Keywords: Hermite-Hadamard inequality; convex set; convex function; s-convex function in the third sense
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1. Introduction

Among the integral inequalities, Hermite-Hadamard inequality is one of the elegant inequalities involving convex functions, which discloses the relation among the integral mean of a convex function, both the arithmetic mean of the images and the image of the arithmetic mean of the integral limits. Its origin goes back to the studies of Hermite that is published in Mathesis 3 (1883, p.82) and, ten years later, by Hadamard. After it has been called Hadamard's inequality for a long time, the Hermite-Hadamard inequality is commonly used. This inequality asserts that for a convex function defined on the interval $[a, b]$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This inequality implies that the mean value of the integral of the convex function f on $[a, b]$ interpolates between the image of the arithmetic mean of the endpoints of $[a, b]$ and the arithmetic mean of the images of the endpoints of $[a, b]$.

This inequality attracts special interests of many researchers. They presented various refinements, extensions, generalizations, extensions for differents function types (see [1, 2, 8, 12, 13, 15, 16, 22, 24, 27–31] and the references therein). Especially, many of the extension studies relates with the new type of convexities. s-convexity, which originates from the studies on modular spaces [6], is one of them and has some applications, especially, in fractal theory [21]. Let us recall the definition of classical convex function for a clear understanding of s-convexity.

Let U be a convex set in a vector space X and $f : U \to \mathbb{R}$. f is said to be convex function, if the inequality

$$f(\lambda x + \mu y) \le \lambda f(x) + \mu f(y) \tag{1.2}$$

holds for all $x, y \in U$ and the positive numbers λ, μ with $\lambda + \mu = 1$.

s-Convexity is obtained either by replacing λ, μ with λ^s, μ^s in the condition $\lambda + \mu = 1$ or by replacing λ, μ with λ^s, μ^s or $\lambda^{\frac{1}{s}}, \mu^{\frac{1}{s}}$ in right handside of (1.2) or by making both properly. According to these replacements, different types of s-convexity are defined. The first sense, the second sense are well-known ones and given as follows:

Let $U \subseteq \mathbb{R}^n$ and $0 < s \le 1$. If for each $x, y \in U$, $\lambda, \mu \ge 0$ such that $\lambda^s + \mu^s = 1$, $\lambda x + \mu y \in U$, then U is called an s-convex set in $\mathbb{R}^n$. This definition is the same as the definition of p-convex set given in article [5]. Therefore, when s-convex set is mentioned in this article, p-convex set will be understood.

Let $U \subseteq \mathbb{R}^n$ be an s-convex set such that $s \in (0, 1]$. A function $f : U \to \mathbb{R}$ is said to be s-convex in the first sense if the inequality

$$f(\lambda x + \mu y) \le \lambda^s f(x) + \mu^s f(y) \tag{1.3}$$

holds for all $x, y \in U$ and $\lambda, \mu \ge 0$ with $\lambda^s + \mu^s = 1$.

Let $U \subseteq \mathbb{R}^n$ be a convex set and $s \in (0, 1]$. A function $f : U \to \mathbb{R}$ is said to be s-convex in the second sense if the inequality

$$f(\lambda x + \mu y) \le \lambda^s f(x) + \mu^s f(y) \tag{1.4}$$

holds for all $x, y \in U$ and $\lambda, \mu \ge 0$ with $\lambda + \mu = 1$.

Also the third sense s-convexity is introduced [26]. In the literature, there are quite a number of studies on Hermite-Hadamard inequality and extensions for s-convex functions in the first and especially, second sense [3, 7, 9, 10, 14, 18–20, 23, 25].

In this study, we present the Hermite-Hadamard inequality for the s-convex functions in the third sense. Also some integral inequalities are given.

2. Preliminaries

Throughout this paper, $\mathbb{R}_+$ and $\mathbb{Z}_{++}$ denote the set of nonnegative real numbers and the set of positive integers, respectively. Let us express a proposition that will be needed hereafter:

Proposition 1. *[4] For $a > 0$, $[0, a)$ is an s-convex set.*

Definition 2. *[17] Let $s \in (0, 1]$, $U \subseteq \mathbb{R}^n$ be an s-convex set and $f : U \to \mathbb{R}$. If for all $x, y \in U$ and $t, v \ge 0$ such that $t^s + v^s = 1$,*

$$f(tx + vy) \le t^{\frac{1}{s}} f(x) + v^{\frac{1}{s}} f(y), \tag{2.1}$$

then f is said to be s-convex function in the third sense. The class of these functions are denoted by K_s^3.

It is clear that, we obtain classical convexity of a function for $s = 1$ in (2.1).

Inequality (2.1) can be expressed in terms of one parameter in two ways: Firstly, since $t^s + v^s = 1$, hence, $v = (1 - t^s)^{\frac{1}{s}}$, we can write

$$f(tx + (1 - t^s)^{\frac{1}{s}} y) \leq t^{\frac{1}{s}} f(x) + (1 - t^s)^{\frac{1}{s^2}} f(y).$$

Secondly, replacing the condition $t^s + v^s = 1$ with $\left(t^{\frac{1}{s}}\right)^s + \left(v^{\frac{1}{s}}\right)^s = 1$, we have

$$f(t^{\frac{1}{s}} x + (1 - t)^{\frac{1}{s}} y) \leq t^{\frac{1}{s^2}} f(x) + (1 - t)^{\frac{1}{s^2}} f(y).$$

Gamma, beta functions and some relations used in some of the results are below. These functions are defined as follows, respectively: For $x, y > 0$,

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \ \text{ and } \ B(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} dt.$$

Beta and gamma functions have the following properties:

For $x, y > 0$ and $n \in \mathbb{Z}_{++}$,

$$B(x, y) = B(y, x)\ ,\ B(x + 1, y) = \frac{x}{x + y} B(x, y)\ ,\ B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}\ ,\ \Gamma(n) = (n - 1)!.$$

Now, let us give an inequality on beta function we will use later:

Theorem 3. *[11] Let $p, q > 0$. Then*

$$\left| B(p + 1, q + 1) - \frac{1}{(1 + p)(1 + q)} \right| \leq \frac{1}{4}. \tag{2.2}$$

3. Main results

Throughout this paper, the set U will be taken as an s-convex subset of $\mathbb{R}$.

Theorem 4. *Let $U \subseteq \mathbb{R}_+$ and $f : U \to \mathbb{R}_+$ be an integrable s-convex function in the third sense. For $a, b \in U$ with $a < b$, the following inequality holds*

$$2^{\frac{1}{s^2}-1} f(\frac{a + b}{2^{\frac{1}{s}}})(b - a) \leqslant \int_a^b f(x)dx \leqslant \frac{1}{s + 1}\left(\frac{1}{s}\left[f(a)b + f(b)a\right] B(\frac{1}{s^2}, \frac{1}{s}) + s\left[f(b)b + f(a)a\right]\right).$$

Proof. To show the right part of the inequality, we change variable $x = t^{\frac{1}{s}} b + (1 - t)^{\frac{1}{s}} a$, then we have

$$\int_a^b f(x)dx = \frac{1}{s} \int_0^1 f(t^{\frac{1}{s}} b + (1 - t)^{\frac{1}{s}} a)(bt^{\frac{1}{s}-1} - a(1 - t)^{\frac{1}{s}-1})dt.$$

Using the s-convexity of f, we can write

$$\begin{aligned}
\int_a^b f(x)dx &\leq \frac{1}{s}\int_0^1 \left(\left[t^{\frac{1}{s^2}} f(b) + (1-t)^{\frac{1}{s^2}} f(a)\right]\left[bt^{\frac{1}{s}-1} + a(1-t)^{\frac{1}{s}-1}\right]\right) dt \\
&= \frac{1}{s}\int_0^1 \left(f(a)\left[bt^{\frac{1}{s}-1}(1-t)^{\frac{1}{s^2}} + a(1-t)^{\frac{1}{s^2}+\frac{1}{s}-1}\right] + f(b)\left[at^{\frac{1}{s^2}}(1-t)^{\frac{1}{s}-1} + bt^{\frac{1}{s^2}+\frac{1}{s}-1}\right]\right) dt \\
&= \frac{1}{s}\left(f(a)\int_0^1 \left[bt^{\frac{1}{s}-1}(1-t)^{\frac{1}{s^2}} + a(1-t)^{\frac{1}{s^2}+\frac{1}{s}-1}\right] dt + f(b)\int_0^1 \left[at^{\frac{1}{s^2}}(1-t)^{\frac{1}{s}-1} + bt^{\frac{1}{s^2}+\frac{1}{s}-1}\right] dt\right) \\
&= \frac{1}{s}\left(f(a)b\frac{\Gamma\left(1+\frac{1}{s^2}\right)\Gamma\left(\frac{1}{s}\right)}{\Gamma\left(1+\frac{1}{s}+\frac{1}{s^2}\right)} + f(a)a\frac{s^2}{s+1} + f(b)a\frac{\Gamma\left(1+\frac{1}{s^2}\right)\Gamma\left(\frac{1}{s}\right)}{\Gamma\left(1+\frac{1}{s}+\frac{1}{s^2}\right)} + f(b)b\frac{s^2}{s+1}\right) \\
&= \frac{1}{s}\left[f(a)b + f(b)a\right]\frac{\Gamma\left(1+\frac{1}{s^2}\right)\Gamma\left(\frac{1}{s}\right)}{\Gamma\left(1+\frac{1}{s}+\frac{1}{s^2}\right)} + \left[f(b)b + f(a)a\right]\frac{s}{s+1} \\
&= \frac{1}{s(s+1)}\left[f(a)b + f(b)a\right]B\left(\frac{1}{s^2}, \frac{1}{s}\right) + \left[f(b)b + f(a)a\right]\frac{s}{s+1} \\
&= \frac{1}{s+1}\left(\frac{1}{s}\left[f(a)b + f(b)a\right]B\left(\frac{1}{s^2}, \frac{1}{s}\right) + s\left[f(b)b + f(a)a\right]\right).
\end{aligned}$$

For the first part of the inequality, since $f \in K_s^3$, we have

$$f\left(\frac{x+y}{2^{\frac{1}{s}}}\right) \leqslant \frac{f(x)+f(y)}{2^{\frac{1}{s^2}}}$$

for all $x, y > 0$. Let $x = ta + (1-t)b$ and $y = tb + (1-t)a$ for $t \in (0, 1]$. Then

$$f\left(\frac{a+b}{2^{\frac{1}{s}}}\right) \leqslant \frac{f(ta + (1-t)b) + f((1-t)a + tb)}{2^{\frac{1}{s^2}}}.$$

Integrating both side and using the fact that

$$\int_0^1 f(ta + (1-t)b)\,dt = \int_0^1 f(tb + (1-t)a)\,dt,$$

one can have

$$2^{\frac{1}{s^2}-1} f\left(\frac{a+b}{2^{\frac{1}{s}}}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx.$$

□

Corollary 5. *It is obtained a Hermite-Hadamard type inequality for the convex functions in case $s = 1$ in Theorem 4.*

Theorem 6. *Let $f : U \to \mathbb{R}$ be an integrable s-convex function in the third sense. For $a, b \in U$ with $a < b$, the following inequality holds*

$$\int_0^1 f\left((1-t^s)^{\frac{1}{s}}\, a + tb\right)\left[1 + (1-t^s)^{\frac{1}{s}-1} t^{s-1}\right] dt \le \frac{1}{s+1}\left(s + \frac{1}{s}\, B(\frac{1}{s^2}, \frac{1}{s})\right)[f(a) + f(b)]. \tag{3.1}$$

Proof. Using the s-convexity in the third sense of f, for $t \in [0,1]$ and $a, b \in U$ with $a < b$, we have

$$f\left(ta + (1-t^s)^{\frac{1}{s}}\, b\right) \le t^{\frac{1}{s}} f(a) + (1-t^s)^{\frac{1}{s^2}}\, f(b)$$

and

$$f\left((1-t^s)^{\frac{1}{s}}\, a + tb\right) \le (1-t^s)^{\frac{1}{s^2}}\, f(a) + t^{\frac{1}{s}} f(b).$$

By summing these inequalities side by side, it is derived that

$$f\left(ta + (1-t^s)^{\frac{1}{s}}\, b\right) + f\left((1-t^s)^{\frac{1}{s}}\, a + tb\right) \le [f(a) + f(b)]\left[(1-t^s)^{\frac{1}{s^2}} + t^{\frac{1}{s}}\right].$$

Using the definite integration on $[0,1]$ yields

$$\int_0^1 f\left(ta + (1-t^s)^{\frac{1}{s}}\, b\right) dt + \int_0^1 f\left((1-t^s)^{\frac{1}{s}}\, a + tb\right) dt \le [f(a) + f(b)] \int_0^1 \left[(1-t^s)^{\frac{1}{s^2}} + t^{\frac{1}{s}}\right] dt. \tag{3.2}$$

By changing variable $v = (1-t^s)^{\frac{1}{s}}$ at first integral of left side of the inequality above, $t^s = 1 - v^s$, $v \in [0,1]$ and $t^{s-1}dt = -v^{s-1}dv$, hence

$$\begin{aligned}\int_0^1 f\left(ta + (1-t^s)^{\frac{1}{s}}\, b\right) dt &= -\int_1^0 f\left((1-v^s)^{\frac{1}{s}}\, a + vb\right)\left[\frac{v}{(1-v^s)^{\frac{1}{s}}}\right]^{s-1} dv \\ &= \int_0^1 f\left((1-t^s)^{\frac{1}{s}}\, a + tb\right)(1-t^s)^{\frac{1}{s}-1} t^{s-1} dt.\end{aligned}$$

Using this equality,

$$\int_0^1 f\left(ta + (1-t^s)^{\frac{1}{s}}\, b\right) dt + \int_0^1 f\left((1-t^s)^{\frac{1}{s}}\, a + tb\right) dt = \int_0^1 f\left((1-t^s)^{\frac{1}{s}}\, a + tb\right)\left[1 + (1-t^s)^{\frac{1}{s}-1} t^{s-1}\right] dt.$$

Using the equality $\int_0^1 (1-t^s)^{\frac{1}{s^2}}\, dt = \frac{1}{s(s+1)} B(\frac{1}{s^2}, \frac{1}{s})$ related on beta function, we have

$$\int_0^1 \left[(1-t^s)^{\frac{1}{s^2}} + t^{\frac{1}{s}}\right] dt = \frac{1}{s+1}\left(s + \frac{1}{s}\, B(\frac{1}{s^2}, \frac{1}{s})\right).$$

From the inequality (3.2), we obtain

$$\int_0^1 f\left((1-t^s)^{\frac{1}{s}}\, a + tb\right)\left[1 + (1-t^s)^{\frac{1}{s}-1} t^{s-1}\right] dt \le \frac{1}{s+1}\left(s + \frac{1}{s}\, B(\frac{1}{s^2}, \frac{1}{s})\right)[f(a) + f(b)].$$

□

The upper bound for the integral at the left side of (3.1) is given in terms of beta function, which is also another integral. We can find the upper bound without beta function.

Theorem 7. *Let $f : U \to \mathbb{R}$ be an integrable s-convex function in the third sense. For $a, b \in U$ with $a < b$, the following inequality holds*

$$\int_0^1 f\left((1-t^s)^{\frac{1}{s}}\, a + t\, b\right)\left[1 + (1-t^s)^{\frac{1}{s}-1} t^{s-1}\right] dt \le \left(s + \frac{1}{4s}\right)[f(a) + f(b)].$$

Proof. From (2.2), we obtain

$$B\left(\frac{1}{s^2}, \frac{1}{s}\right) \le \frac{1}{4} + s^3.$$

Using this inequality in Theorem 6, we can write

$$\begin{aligned}\int_0^1 f\left((1-t^s)^{\frac{1}{s}}\, a + t\, b\right)\left[1 + (1-t^s)^{\frac{1}{s}-1} t^{s-1}\right] dt &\le \frac{1}{s+1}\left(s + s^2 + \frac{1}{4s}\right)[f(a) + f(b)] \\ &\le \left(s + \frac{1}{4s(s+1)}\right)[f(a) + f(b)] \\ &\le \left(s + \frac{1}{4s}\right)[f(a) + f(b)].\end{aligned}$$

□

Theorem 8. *Let $U \subseteq \mathbb{R}_+$ and $f : U \to \mathbb{R}$ be a nondecreasing s-convex function in the third sense. For $a, b \in U$ with $a < b$, the following inequality holds:*

$$f\left(\tfrac{a+b}{2^{\frac{2}{s}-1}}\right) \le \int_0^1 f\left(\tfrac{a+b}{2^{\frac{1}{s}}}\left[t + (1-t^s)^{\frac{1}{s}}\right]\right) dt \le \tfrac{1}{2^{\frac{1}{s^2}}} \int_0^1 f\left((1-t^s)^{\frac{1}{s}}\, a + tb\right)\left[1 + (1-t^s)^{\frac{1}{s}-1} t^{s-1}\right] dt. \tag{3.3}$$

Proof. Since $h(x) = x^{\frac{1}{s}}$ is a convex function on $[0, \infty)$ for $0 < s < 1$, we know that

$$h\left(\frac{x+y}{2}\right) \le \frac{h(x) + h(y)}{2}.$$

Writing $x = t^s$ and $y = 1 - t^s$, we have

$$\left(\frac{t^s + 1 - t^s}{2}\right)^{\frac{1}{s}} = \frac{1}{2^{\frac{1}{s}}} \le \frac{(t^s)^{\frac{1}{s}} + (1-t^s)^{\frac{1}{s}}}{2}$$

$$\frac{a+b}{2^{\frac{1}{s}}} \cdot \frac{1}{2^{\frac{1}{s}}} \le \frac{a+b}{2^{\frac{1}{s}}} \cdot \frac{t + (1-t^s)^{\frac{1}{s}}}{2}$$

hence,

$$\frac{a+b}{2^{\frac{2}{s}-1}} \le \frac{a+b}{2^{\frac{1}{s}}} \cdot \left[t + (1-t^s)^{\frac{1}{s}}\right].$$

Using the fact that f is nondecreasing on $(0, \infty)$, we obtain

$$f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \le f\left(\frac{a+b}{2^{\frac{1}{s}}}\left[t + (1-t^s)^{\frac{1}{s}}\right]\right) \text{ for all } t \in [0,1].$$

Integration on $[0, 1]$ yields to the upper inequality in (3.3). From f being s-convex, it is known that for all $x, y \in [0, \infty)$,

$$f\left(\frac{x+y}{2^{\frac{1}{s}}}\right) \leq \frac{f(x)+f(y)}{2^{\frac{1}{s^2}}} \tag{3.4}$$

Putting $x = ta + (1-t^s)^{\frac{1}{s}}\, b$ and $y = (1-t^s)^{\frac{1}{s}}\, a + tb$ with $t \in [0, 1]$ above, we deduce

$$f\left(\frac{a+b}{2^{\frac{1}{s}}}\left[t + (1-t^s)^{\frac{1}{s}}\right]\right) \leq \frac{1}{2^{\frac{1}{s^2}}}\left[f\left(ta + (1-t^s)^{\frac{1}{s}}\, b\right) + f\left((1-t^s)^{\frac{1}{s}}\, a + tb\right)\right].$$

If each side of inequality is integrated on [0,1] over t, then

$$\int_0^1 f\left(\frac{a+b}{2^{\frac{1}{s}}}\left[t + (1-t^s)^{\frac{1}{s}}\right]\right) dt \leq \frac{1}{2^{\frac{1}{s^2}}}\left[\int_0^1 f\left(ta + (1-t^s)^{\frac{1}{s}}\, b\right) dt + \int_0^1 f\left((1-t^s)^{\frac{1}{s}}\, a + tb\right) dt\right]. \tag{3.5}$$

By changing variable $v = (1-t^s)^{\frac{1}{s}}$ as indicated in the proof of Theorem 6 we have,

$$\frac{1}{2^{\frac{1}{s^2}}}\left[\int_0^1 f\left(ta + (1-t^s)^{\frac{1}{s}}\, b\right) dt + \int_0^1 f\left((1-t^s)^{\frac{1}{s}}\, a + tb\right) dt\right] = \frac{1}{2^{\frac{1}{s^2}}}\int_0^1 f\left((1-t^s)^{\frac{1}{s}}\, a + tb\right)\left[1 + (1-t^s)^{\frac{1}{s}-1} t^{s-1}\right] dt.$$

Thus

$$\int_0^1 f\left(\frac{a+b}{2^{\frac{1}{s}}}\left[t + (1-t^s)^{\frac{1}{s}}\right]\right) dt \leq \frac{1}{2^{\frac{1}{s^2}}}\int_0^1 f\left((1-t^s)^{\frac{1}{s}}\, a + tb\right)\left[1 + (1-t^s)^{\frac{1}{s}-1} t^{s-1}\right] dt$$

is obtained. □

Combining Theorem 6 with Theorem 8 and Theorem 7, we conclude the following result.

Corollary 9. *Let $U \subseteq \mathbb{R}_+$ and $f : U \to \mathbb{R}$ be a nondecreasing s-convex function in the third sense. For $a, b \in U$ with $a < b$, the following inequality holds:*

$$2^{\frac{1}{s^2}} f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \leq \frac{1}{s+1}\left(s + \frac{1}{s}\, B(\frac{1}{s^2}, \frac{1}{s})\right)[f(a) + f(b)] \leq \left(s + \frac{1}{4s}\right)[f(a) + f(b)].$$

Theorem 10. *Let $f : U \to \mathbb{R}_+$ be an s-convex function in the third sense. If $a, b \in U$ with $a < b$, then*

$$\int_0^1 f(t^{\frac{1}{s}} a + (1-t)^{\frac{1}{s}} b) dt \leq \frac{f(a) + f(b)}{2}.$$

Proof. Using s-convexity in the third sense of f, for all $x, y \in U$ and $t \in [0, 1]$,

$$f(t^{\frac{1}{s}} a + (1-t)^{\frac{1}{s}} b) \leq t^{\frac{1}{s^2}} f(a) + (1-t)^{\frac{1}{s^2}} f(b).$$

Since $t^{\frac{1}{s^2}} \leq t$ and $(1-t)^{\frac{1}{s^2}} \leq 1-t$, we have

$$f(t^{\frac{1}{s}} a + (1-t)^{\frac{1}{s}} b) \leq t f(a) + (1-t) f(b).$$

Integration on t gives

$$\int_0^1 f(t^{\frac{1}{s}}a + (1-t)^{\frac{1}{s}}b)dt \leq \frac{f(a)+f(b)}{2}.$$

□

Theorem 11. *Let $U \subseteq \mathbb{R}_+$ and $f : U \to \mathbb{R}$ be an increasing s-convex function in the third sense. If $a, b \in U$ with $a < b$, then the following inequalities holds:*

$$f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \leq \int_0^1 f\left(\frac{a+b}{2^{\frac{1}{s}}}\left[t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right]\right)dt \leq 2^{1-\frac{1}{s^2}} \int_0^1 f\left(at^{\frac{1}{s}} + b(1-t)^{\frac{1}{s}}\right)dt \leq \frac{2s^2}{2^{\frac{1}{s^2}}(s^2+1)}[f(a)+f(b)].$$

Proof. The fact that $h(x) = x^{\frac{1}{s}}$ is a convex function for $s \in (0, 1]$ on $[0, \infty)$ implies

$$\frac{1}{2^{\frac{1}{s}}} \leq \frac{t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}}{2}$$

for all $t \in [0, 1]$. From the monotonicity of f, one can get

$$f\left(\frac{a+b}{2^{\frac{1}{s}}} \cdot \frac{2}{2^{\frac{1}{s}}}\right) = f\left(\frac{a+b}{2^{\frac{2}{s}-1}}\right) \leq f\left(\frac{a+b}{2^{\frac{1}{s}}}\left[t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right]\right)$$

for all $t \in [0, 1]$. Thus the first inequality (from left to right) is obtained.

By using the s-convexity of f, we have

$$f\left(\frac{a+b}{2^{\frac{1}{s}}}\left[t^{\frac{1}{s}} + (1-t)^{\frac{1}{s}}\right]\right) \leq \frac{1}{2^{\frac{1}{s^2}}}\left[f\left(at^{\frac{1}{s}} + b(1-t)^{\frac{1}{s}}\right) + f\left(a(1-t)^{\frac{1}{s}} + bt^{\frac{1}{s}}\right)\right]. \tag{3.6}$$

From the changing of variables, it is clear that

$$\int_0^1 f\left(a(1-t)^{\frac{1}{s}} + bt^{\frac{1}{s}}\right)dt = \int_0^1 f\left(at^{\frac{1}{s}} + b(1-t)^{\frac{1}{s}}\right)dt.$$

So (3.6) yields to the second inequality (from left to right). On the other hand, the s-convexity of f on $[0, \infty)$ implies

$$f\left(t^{\frac{1}{s}}a + (1-t)^{\frac{1}{s}}b\right) \leq t^{\frac{1}{s^2}} f(a) + (1-t)^{\frac{1}{s^2}} f(b)$$

for all $t \in [0, 1]$. By integrating this inequality over t in $[0, 1]$, one can have that

$$\int_0^1 f\left(t^{\frac{1}{s}}a + (1-t)^{\frac{1}{s}}b\right)dt \leq \frac{s^2}{s^2+1}[f(a) + f(b)].$$

By multiplying each side with $2^{1-\frac{1}{s^2}}$, we get the last part of the inequality. □

Theorem 12. *Let $U \subseteq \mathbb{R}_+$ and $f : U \to \mathbb{R}_+$ be an integrable s-convex function in the third sense and let $0 < a < b$. If*

$$\int_a^\infty x^{\frac{s+1}{s(s-1)}} f(x)dx$$

is finite, then

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{s}{1-s}\left[a^{\frac{1+s^2}{s(1-s)}}\int_a^\infty x^{\frac{s+1}{s(s-1)}}f(x)dx + b^{\frac{1+s^2}{s(1-s)}}\int_b^\infty x^{\frac{s+1}{s(s-1)}}f(x)dx\right].$$

Proof. Using the s-convexity of f on U for any $z, y \in U$ and $u \in [0,1]$, we have

$$f\left(u^{\frac{1}{s}}z + (1-u)^{\frac{1}{s}}y\right) \leq u^{\frac{1}{s^2}}f(z) + (1-u)^{\frac{1}{s^2}}f(y).$$

By making substitutions $z = u^{1-\frac{1}{s}}a, u \in (0,1]$ and $y = (1-u)^{1-\frac{1}{s}}b, u \in [0,1)$, the following is obtained:

$$f(ua + (1-u)b) \leq u^{\frac{1}{s^2}}f\left(u^{1-\frac{1}{s}}a\right) + (1-u)^{\frac{1}{s^2}}f\left((1-u)^{1-\frac{1}{s}}b\right) \tag{3.7}$$

for all $u \in (0,1)$. To show the validity of the inequality above, first, let us testify that $\int_0^1 u^{\frac{1}{s^2}}f\left(u^{1-\frac{1}{s}}a\right)du$ is finite. The change of the variable $x = u^{1-\frac{1}{s}}a, u \in (0,1]$ yields

$$u = \left(\frac{x}{a}\right)^{\frac{1}{1-\frac{1}{s}}} = \left(\frac{x}{a}\right)^{\frac{s}{s-1}} = \frac{x^{\frac{s}{s-1}}}{a^{\frac{s}{s-1}}}$$

and

$$du = \frac{s}{s-1}\cdot\frac{1}{a^{\frac{s}{s-1}}}x^{\frac{s}{s-1}-1}dx = \frac{s}{s-1}\cdot\frac{1}{a^{\frac{s}{s-1}}}x^{\frac{1}{s-1}}dx.$$

Thus,

$$\int_0^1 u^{\frac{1}{s^2}}f\left(u^{1-\frac{1}{s}}a\right)du = \int_\infty^a\left[\frac{x^{\frac{1}{s(s-1)}}}{a^{\frac{1}{s(s-1)}}}\cdot\frac{s}{s-1}\cdot\frac{x^{\frac{1}{s-1}}}{a^{\frac{s}{s-1}}}f(x)\right]dx = \frac{s}{1-s}\cdot a^{\frac{1+s^2}{s(1-s)}}\int_a^\infty x^{\frac{s+1}{s(s-1)}}f(x)dx < \infty.$$

For the integral

$$\int_0^1 (1-u)^{\frac{1}{s^2}}f\left((1-u)^{1-\frac{1}{s}}b\right)du,$$

the change of variable $t = 1-u$, $u \in [0,1)$ yields to

$$\int_0^1 t^{\frac{1}{s^2}}f\left(t^{1-\frac{1}{s}}b\right)dt.$$

In a similar way above, by the substitution $x = t^{1-\frac{1}{s}}b$, $t \in (0,1]$,

$$\int_0^1 t^{\frac{1}{s^2}}f\left(t^{1-\frac{1}{s}}b\right)dt = \frac{s}{1-s}\cdot b^{\frac{1+s^2}{s(1-s)}}\int_b^\infty x^{\frac{s+1}{s(s-1)}}f(x)dx < \infty$$

is deduced. Integrating the inequality (3.7) on (0,1) over u, taking into account that

$$\int_0^1 f(ua + (1-u)b)du = \frac{1}{b-a}\int_a^b f(x)dx$$

and

$$\int_0^1 u^{\frac{1}{s^2}} f\left(u^{1-\frac{1}{s}} a\right) du = \frac{s}{1-s} \cdot a^{\frac{1+s^2}{s(1-s)}} \int_a^\infty x^{\frac{s+1}{s(s-1)}} f(x) dx,$$

$$\int_0^1 (1-u)^{\frac{1}{s^2}} f\left((1-u)^{1-\frac{1}{s}} b\right) du = \frac{s}{1-s} \cdot b^{\frac{1+s^2}{s(1-s)}} \int_b^\infty x^{\frac{s+1}{s(s-1)}} f(x) dx$$

respectively, one can obtain the desired inequality. □

Theorem 13. *Let $U \subseteq \mathbb{R}_+$ and $f : U \to \mathbb{R}_+$ be an s-convex function in the third sense and let $0 < a < b$. Then*

(i) Let $g : [0,1] \to \mathbb{R}$ be a function defined as follows

$$g(t) = \frac{1}{b-a} \int_a^b f(tx + (1-t)\frac{a+b}{2}) dx.$$

If f is decreasing (or noninreasing) function then $g \in K_s^3$.

(ii) If f is integrable on $[a,b]$, then the following inequality holds for $t \in (0,1]$, $g(t) \geq 2^{\frac{1}{s^2}-1} f\left(\frac{a+b}{2^{\frac{1}{s}}}\right)$.

Proof. (i) Let $x, y \in [0,1]$ and $\lambda, \mu \geq 0$ with $\lambda^s + \mu^s = 1$. Using $\lambda + \mu \leq 1$, monotonicity and s-convexity of f, we can write

$$\begin{aligned}
g(\lambda t_1 + \mu t_2) &= \frac{1}{b-a} \int_a^b f\left((\lambda t_1 + \mu t_2) x + [1 - (\lambda t_1 + \mu t_2)] \frac{a+b}{2}\right) dx \\
&= \frac{1}{b-a} \int_a^b f\left(\lambda t_1 x - \lambda t_1 \frac{a+b}{2} + \mu t_2 x - \mu t_2 \frac{a+b}{2} + \frac{a+b}{2}\right) dx \\
&\leq \frac{1}{b-a} \int_a^b f\left(\lambda t_1 x - \lambda t_1 \frac{a+b}{2} + \mu t_2 x - \mu t_2 \frac{a+b}{2} + (\lambda + \mu) \frac{a+b}{2}\right) dx \\
&= \frac{1}{b-a} \int_a^b f\left(\lambda \left(t_1 x + (1 - t_1) \frac{a+b}{2}\right) + \mu \left(t_2 x + (1 - t_2) \frac{a+b}{2}\right)\right) dx \\
&\leq \frac{1}{b-a} \int_a^b \left[\lambda^{\frac{1}{s}} f\left(t_1 x + (1 - t_1) \frac{a+b}{2}\right) + \mu^{\frac{1}{s}} f\left(t_2 x + (1 - t_2) \frac{a+b}{2}\right)\right] dx \\
&= \lambda^{\frac{1}{s}} g(t_1) + \mu^{\frac{1}{s}} g(t_2).
\end{aligned}$$

(ii) Assume that $t \in (0,1]$. By changing variable $u = tx + (1-t)\frac{a+b}{2}$, we have

$$g(t) = \frac{1}{t(b-a)} \int_{ta+(1-t)\frac{a+b}{2}}^{tb+(1-t)\frac{a+b}{2}} f(u) du = \frac{1}{m-n} \int_n^m f(u) du$$

where $m = tb + (1-t)\frac{a+b}{2}$ and $n = ta + (1-t)\frac{a+b}{2}$. The left part of the Hermite-Hadamard inequality gives

$$\frac{1}{m-n}\int_n^m f(u)du \geq 2^{\frac{1}{s^2}-1} f\left(\frac{m+n}{2^{\frac{1}{s}}}\right) = 2^{\frac{1}{s^2}-1} f\left(\frac{a+b}{2^{\frac{1}{s}}}\right)$$

hence, we get the required inequality. □

Theorem 14. *Let $f : U \to \mathbb{R}_+$ be an s-convex function in the third sense. Let $a, b \in U$ with $a < b$. Consider the function*

$$h(t) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx + (1-t)y)dxdy,\ t \in [0,1].$$

(i) *If f is decreasing function, then h is also s-convex function in the third sense on* $[0, 1]$.
(ii) *If f is integrable on* $[a, b]$, *then the following inequality holds:*

$$2^{1-\frac{1}{s^2}} h(t) \geq \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2^{\frac{1}{s}}}\right)dxdy,\ t \in [0,1].$$

Proof. (i) Let $x, y \in [0, 1]$ and $\lambda, \mu \geq 0$ with $\lambda^s + \mu^s = 1$. Taking into account that $\lambda^{s-1} - t_1 \geq 1 - t_1$ for $t_1, \lambda \in [0, 1]$ and $\mu^{s-1} - t_2 \geq 1 - t_2$ for $t_2, \mu \in [0, 1]$, then using the monotonicity and s-convexity of f, respectively, we have

$$\begin{aligned}
h(\lambda t_1 + \mu t_2) &= \frac{1}{(b-a)^2}\int_a^b\int_a^b f((\lambda t_1 + \mu t_2)x + (1 - (\lambda t_1 + \mu t_2))y)dxdy \\
&= \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\lambda\left[t_1 x + \left(\lambda^{s-1} - t_1\right)y\right] + \mu\left[t_2 x + \left(\mu^{s-1} - t_2\right)y\right]\right)dxdy \\
&\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\lambda\left[t_1 x + (1-t_1)y\right] + \mu\left[t_2 x + (1-t_2)y\right]\right)dxdy \\
&\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b \{\lambda^{\frac{1}{s}} f\left(\left[t_1 x + (1-t_1)y\right]\right) + \mu^{\frac{1}{s}} f(\left[t_2 x + (1-t_2)y\right])\}dxdy \\
&= \lambda^{\frac{1}{s}} h(t_1) + \mu^{\frac{1}{s}} h(t_2)
\end{aligned}$$

(ii) From (3.4), we can write

$$f\left(\frac{x+y}{2^{\frac{1}{s}}}\right) \leq \frac{f(tx + (1-t)y) + f(ty + (1-t)x)}{2^{\frac{1}{s^2}}}$$

for all $t \in [0, 1]$ and $x, y \in [a, b]$. Integrating this inequality on $[a, b]^2$ we get

$$\frac{1}{2^{\frac{1}{s^2}}}\left[\int_a^b\int_a^b f(tx + (1-t)y)dxdy + \int_a^b\int_a^b f((1-t)x + ty)dxdy\right] \geq \int_a^b\int_a^b f\left(\frac{x+y}{2^{\frac{1}{s}}}\right)dxdy$$

since

$$\int_a^b \int_a^b f(tx + (1-t)y)dxdy = \int_a^b \int_a^b f((1-t)x + ty)dxdy$$

the above inequality gives us the desired result. □

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Conflict of interest

The authors declare no conflict of interest.

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