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Research article

Global existence and blow-up of solutions for logarithmic Klein-Gordon equation

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Abstract: This arcitle concerns the initial-boundary value problem for a class of Klein-Gordon equation with logarithmic nonlinearity. By using Galerkin method and compactness criterion, we prove the existence of global solutions to this problem. Meanwhile, the blow-up of solutions in the unstable set is also obtained.

Keywords: Klein-Gordon equation; logarithmic nonlinearity; initial-boundary value problem; global solutions; blow-up
Mathematics Subject Classification: 35L05, 35L10, 35B40

1. Introduction

In this paper, we consider the following problem

$$u_{tt} - \Delta u + u = u \log |u|, \quad (x, t) \in \Omega \times R^+, \tag{1.1}$$

$$u(x, 0) = u_0(x), \; u_t(x, 0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$

$$u(x, t) = 0, \; (x, t) \in \partial\Omega \times R^+, \tag{1.3}$$

where $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$.

The model equation (1.1) arises in logarithmic quantum mechanics and is applied to nuclear physics, optics and geophysics [1–5]. P. Gorka [6] dealt with the equation

$$u_{tt} - u_{xx} = -u + \varepsilon u \log |u|^2, \quad (x, t) \in \mathcal{O} \times (0, T) \tag{1.4}$$

with initial datum

$$u(x, 0) = u_0(x), \; u_t(x, 0) = u_1(x), \quad x \in \mathcal{O} \tag{1.5}$$

and boundary value condition

$$u(x,t) = 0,\ (x,t) \in \partial O \times (0,T), \tag{1.6}$$

where $O = [a,b] \subset R^1$, $\varepsilon \in [0,1]$. By applying the Galerkin method, logarithmic Sobolev inequality and compactness theorem, he established the global weak solutions of the problem (1.4)–(1.6). K. Bartkowski and P. Korka [7] showed the classical solutions and weak solutions to the Cauchy problem of Eq (1.4) for $O = R^1$. In [8], T. Cazenave and A. Haraux investigated the local and global solutions for the Cauchy problem of the logarithmic wave equation $u_{tt} - \Delta u = u \log |u|$.

For the following nonlinear Klein-Gordon equation

$$u_{tt} - \Delta u + m^2 u = |u|^{p-2}u,\ x \in \Omega,\ t > 0, \tag{1.7}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \tag{1.8}$$

$$u(x,t) = 0,\ x \in \partial\Omega,\ t \geq 0. \tag{1.9}$$

For $n \geq 3$, P. Brenner [9] studied L_p-decay and scattering properties for the Cauchy problem of the Eq (1.7). As $n = 1$ and 2, K. Nakanishi [10] showed that the scattering operators for Eq (1.7) are well-defined in whole energy space in R^{1+n} with $p > 1 + \frac{4}{n}$. Under the condition of small energy data, such results were known for $n \geq 3$ [11–13].

As $m = 0$, for sufficiently large initial data, the blow-up results of the problem (1.7)–(1.9) in finite time was proved by H. A. Levine [14] and J. Ball [15]. Furthermore, Y. C. Liu [16], L. E. Payne and D. H. Sattinger [17] and D. H. Sattinger [18] obtained the results of the global existence and nonexistence of weak solutions for the problem (1.7)–(1.9) by establishing the method of potential wells. Also in [16, 19], the authors gave a threshold result of solutions and obtained the vacuum isolating of solutions.

At last we should mention that the logarithmic heat equation was studied by H. Chen and S. Y. Tian [20] and H. Chen, P. Luo and G. W. Liu [21]. Moreover, there were also many researches on the logarithmic Schrödinger equation [22–25].

In this paper, by applying Galerkin method and compactness criterion, we prove the global existence of the problem (1.1)–(1.3). Furthermore, in the sense of L^2 norm, the blow-up result for this problem is obtained by the concavity method.

2. Preliminaries

2.1. Some lemmas

For the applications through this paper, we list up some known lemmas.

Definition 2.1 If

$$u \in C([0,T], H_0^1(\Omega))\ ,u_t \in C([0,T], L^2(\Omega))\ ,u_{tt} \in C([0,T], H^{-1}(\Omega))$$

satisfies

$$\int_\Omega u_{tt}\varphi dx + \int_\Omega \nabla u \nabla \varphi dx + \int_\Omega u\varphi dx = \int_\Omega u \log |u| \varphi dx,\ \ \varphi \in H_0^1(\Omega).$$

Then the function u is called a weak solution of (1.1)–(1.3) on $[0,T]$.

Lemma 2.1 Assume that $2 \le r < +\infty,\ n \le 2$ and $2 \le r \le \frac{2n}{n-2},\ n > 2$. Then

$$\|u\|_r \le C\|\nabla u\|,\ \forall u \in H_0^1(\Omega),$$

where $C > 0$ is a constant depending on Ω and r.

Lemma 2.2 ([20, 21, 26]) If $u \in H_0^1(\Omega)$, then for each $a > 0$, one has the inequality

$$\int_\Omega |u|^2 \log |u| dx \le \|u\|^2 \log \|u\| + \frac{a^2}{2\pi}\|\nabla u\|^2 - \frac{n}{2}(1 + \log a)\|u\|^2.$$

Lemma 2.3 Let $u(t)$ be a solution of the problem (1.1)–(1.3), then the energy $\mathcal{E}(t)$ is conservation. Namely, $\mathcal{E}(t) = \mathcal{E}(0),\ \forall t > 0$, where

$$\mathcal{E}(t) = \frac{1}{2}(\|u_t\|^2 + \|\nabla u\|^2 - \int_\Omega u^2 \log |u| dx) + \frac{3}{4}\|u\|^2, \tag{2.1}$$

for $u \in H_0^1(\Omega),\ t \ge 0$ and

$$\mathcal{E}(0) = \frac{1}{2}(\|u_1\|^2 + \|\nabla u_0\|^2 - \int_\Omega u_0^2 \log |u_0| dx) + \frac{3}{4}\|u_0\|^2 \tag{2.2}$$

is the initial total energy.

Lemma 2.4[27] Let X be a Banach space, if $f \in L^p(0, T; X),\ \frac{\partial f}{\partial t} \in L^p(0, T; X)$, then f is a continuous injection from $[0, T]$ on to X when the value is transformed in the set of measure zero in $[0, T]$.

Lemma 2.5[28] Let $u_n(x)$ be a bounded sequence in $L^p(\Omega),\ 1 \le p < +\infty$ such that u_n almost everywhere converges to u. Then $u \in L^p(\Omega)$ and u_n weakly converges in $L^p(\Omega)$ to u, where $\Omega \subset R^n$ is a bounded domain.

The local existence result of the problem (1.1)–(1.3) is described as follows. For its detailed proof process, see references [31–33].

Theorem 2.1 (Local existence) Let $u_0 \in H_0^1(\Omega),\ u_1 \in L^2(\Omega)$. Then there exists $T > 0$ such that the problem (1.1)–(1.3) has a unique local solution $u(t)$ satisfying

$$u \in C([0, T];\ H_0^1(\Omega)),\ \ u_t \in C([0, T];\ L^2(\Omega)).$$

2.2. Potential wells

At first, we introduce some useful functionals

$$\mathcal{J}(u) = \frac{1}{2}(\|\nabla u\|^2 - \int_\Omega u^2 \log |u| dx) + \frac{3}{4}\|u\|^2, \tag{2.3}$$

$$\mathcal{K}(u) = \|\nabla u\|^2 + \|u\|^2 - \int_\Omega u^2 \log |u| dx. \tag{2.4}$$

By (2.1), (2.3) and (2.4), we have

$$\mathcal{J}(u) = \frac{1}{2}\mathcal{K}(u) + \frac{1}{4}\|u\|^2,\ \ \mathcal{E}(t) = \frac{1}{2}\|u_t\|^2 + \mathcal{J}(u), \tag{2.5}$$

for $u \in H_0^1(\Omega)$.

As in [17], the potential well depth is defined as

$$d = \inf\{\sup_{\lambda \geq 0} \mathcal{J}(\lambda u) : \ u \in H_0^1(\Omega)/\{0\}\}. \tag{2.6}$$

Now, we define the Nehari manifold ([29, 30]) by

$$\mathcal{N} = \{u \in H_0^1(\Omega)/\{0\}; \ \mathcal{K}(u) = 0\}.$$

The stable set $\mathcal{W}$ and the unstable set $\mathcal{U}$ can be defined respectively by

$$\mathcal{W} = \{u \in H_0^1(\Omega) : \ \mathcal{K}(u) > 0, \ \mathcal{J}(u) < d\} \cap \{0\},$$

and

$$\mathcal{U} = \{u \in H_0^1(\Omega) : \ \mathcal{K}(u) < 0, \ \mathcal{J}(u) < d\}.$$

It is to see that the potential well depth d may also be described as

$$d = \inf_{u \in \mathcal{N}} \mathcal{J}(u). \tag{2.7}$$

Lemma 2.6 Let $u \in H_0^1(\Omega)$ and $||u|| \neq 0$, then we have

$$(i) \ \lim_{\lambda \to 0^+} \mathcal{J}(\lambda u) = 0, \ \lim_{\lambda \to +\infty} \mathcal{J}(\lambda u) = -\infty;$$

$$(ii) \ \mathcal{K}(\lambda u) = \lambda \frac{d}{d\lambda} \mathcal{J}(\lambda u) \begin{cases} > 0, & 0 < \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda^* < \lambda < +\infty, \end{cases} \tag{2.8}$$

where

$$\lambda^* = \exp\Big(\frac{||\nabla u||^2 + ||u||^2 - \int_\Omega u^2 \log |u| dx}{||u||^2}\Big).$$

Proof. (i) By $\lim_{\lambda \to 0^+} \lambda^2 \log \lambda = 0, \ \lim_{\lambda \to +\infty} \log \lambda = +\infty$ and

$$\mathcal{J}(\lambda u) = \frac{\lambda^2}{2} ||\nabla u||^2 + \frac{3}{4} \lambda^2 ||u||^2 - \frac{1}{2} (\lambda^2 \log \lambda) ||u||^2 - \frac{1}{2} \lambda^2 \int_\Omega u^2 \log |u| dx,$$

we get

$$\lim_{\lambda \to 0^+} \mathcal{J}(\lambda u) = 0, \ \lim_{\lambda \to +\infty} \mathcal{J}(\lambda u) = -\infty.$$

(ii) By direct calculations, we obtain

$$\frac{d}{d\lambda} \mathcal{J}(\lambda u) = \lambda \Big[||\nabla u||^2 + ||u||^2 - \int_\Omega u^2 \log |u| dx \Big] - (\lambda \log \lambda) ||u||^2. \tag{2.9}$$

Let $\frac{d}{d\lambda} \mathcal{J}(\lambda u) = 0$, then we deduce that

$$\lambda^* = \exp\Big(\frac{||\nabla u||^2 + ||u||^2 - \int_\Omega u^2 \log |u| dx}{||u||^2}\Big).$$

From (3.2), we have

$$\mathcal{K}(\lambda u) = \lambda^2[\|\nabla u\|^2 + \|u\|^2 - \int_\Omega u^2 \log|u|dx] - (\lambda^2 \log\lambda)\|u\|^2. \tag{2.10}$$

By (2.9) and (2.10), the equality (2.8) is valid. □

Lemma 2.7 If $u \in H_0^1(\Omega)$, then

$$d = \frac{1}{4}(\sqrt{2\pi})^n e^{n+2}. \tag{2.11}$$

Proof. By Lemma 2.2, we have

$$\begin{aligned} \mathcal{K}(u) &= \|\nabla u\|^2 - \int_\Omega u^2 \log|u|dx + \|u\|^2 \\ &\geq \left(1 - \frac{a^2}{2\pi}\right)\|\nabla u\|^2 + [1 + \frac{n}{2}(1 + \log a) - \log\|u\|] \cdot \|u\|^2. \end{aligned} \tag{2.12}$$

By taking $a = \sqrt{2\pi}$, we obtain from (2.12) that

$$\mathcal{K}(u) \geq [1 + \frac{n}{2}(1 + \log a) - \log\|u\|] \cdot \|u\|^2. \tag{2.13}$$

Combining Lemma 2.6 and (2.5) yields that

$$\sup_{\lambda\geq 0} \mathcal{J}(\lambda u) = \mathcal{J}(\lambda^* u) = \frac{1}{2}\mathcal{K}(\lambda^* u) + \frac{1}{4}\|\lambda^* u\|^2. \tag{2.14}$$

We receive from (2.13) and Lemma 2.6 that

$$0 = \mathcal{K}(\lambda^* u) \geq [1 + \frac{n}{2}(1 + \log a) - \log\|\lambda^* u\|] \cdot \|\lambda^* u\|^2,$$

then

$$\|\lambda^* u\|^2 \geq a^n e^{n+2}. \tag{2.15}$$

It follows from(2.14) and (2.15) that

$$\sup_{\lambda\geq 0} \mathcal{J}(\lambda u) \geq \frac{1}{4} a^n e^{n+2}. \tag{2.16}$$

By (2.6) and (3.16), we have that $d = \frac{1}{4}(\sqrt{2\pi})^n e^{n+2}$. □

In order to further study the problem (1.1)–(1.3), for $0 < \varepsilon < 1$ and $u \in H_0^1(\Omega)$, we define some functionals as follows

$$\mathcal{J}_\varepsilon(u) = \frac{\varepsilon}{2}\|\nabla u\|^2 - \frac{1}{2}\int_\Omega u^2 \log|u|dx + \frac{3}{4}\|u\|^2, \tag{2.17}$$

$$\mathcal{K}_\varepsilon(u) = \varepsilon\|\nabla u\|^2 + \|u\|^2 - \int_\Omega u^2 \log|u|dx. \tag{2.18}$$

Let

$$\mathcal{N}_\varepsilon(u) = \{u \in H_0^1(\Omega);\ J_\varepsilon(u) = 0,\ \|\nabla u\| \neq 0\}, \tag{2.19}$$

then we define $d(\varepsilon)$ as

$$d(\varepsilon) = \inf_{u\in\mathcal{N}_\varepsilon(u)} \mathcal{J}(u). \tag{2.20}$$

Proposition 2.1 If $d(\varepsilon)$ is defined by (2.20), then

$$d(\varepsilon) = 2\lambda_1 de(1-\varepsilon)\varepsilon^{\frac{n}{2}}, \tag{2.21}$$

where λ_1 is the first eigenvalue of the following boundary value problem

$$\begin{cases} -\Delta u = \lambda u, \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega. \end{cases} \tag{2.22}$$

Proof. By $u \in \mathcal{N}_\varepsilon(u)$, we get

$$\mathcal{J}(u) = \frac{1-\varepsilon}{2}\|\nabla u\|^2 + \mathcal{J}_\varepsilon(u) = \frac{1-\varepsilon}{2}\|\nabla u\|^2. \tag{2.23}$$

From (2.17) and Lemma 2.2 that

$$\begin{aligned} \varepsilon\|\nabla u\|^2 &= \int_\Omega u^2 \log|u| dx - \frac{3}{2}\|u\|^2 \\ &\le \frac{a^2}{2\pi}\|\nabla u\|^2 + \|u\|^2\log\|u\| - \frac{n}{2}(1+\log a)\|u\|^2 - \frac{3}{2}\|u\|^2. \end{aligned} \tag{2.24}$$

By taking $a^2 = 2\pi\varepsilon$ in (2.24), we obtain

$$[\log\|u\|^2 - n(1+\log a) - 3]\|u\|^2 \ge 0,$$

which implies that

$$\|u\|^2 \ge a^n e^{n+3}.$$

Since the eigenvalue λ_1 satisfies the problem (2.22), so we gain

$$\|\nabla u\|^2 \ge \lambda_1 a^n e^{n+3} = 4\lambda_1 de\varepsilon^{\frac{n}{2}}. \tag{2.25}$$

It follows from (2.23) and (2.25) that

$$\mathcal{J}(u) \ge 2\lambda_1 de(1-\varepsilon)\varepsilon^{\frac{n}{2}}.$$

Thus, by (2.20), we have

$$d(\varepsilon) = 2\lambda_1 de(1-\varepsilon)\varepsilon^{\frac{n}{2}}.$$

□

Proposition 2.2 As a function of ε, $d(\varepsilon)$ have the following properties for $\varepsilon \in [0,1]$:

(a) $d(0) = d(1) = 0$.

(b) $d(\varepsilon)$ is increasing on $[0,\varepsilon_0]$ and decreasing on $[\varepsilon_0, 1]$. Thus, $d(\varepsilon)$ gets the maximum at $\varepsilon_0 = \frac{n}{n+2}$, and $d(\varepsilon_0) = \frac{4\lambda_1 de}{n+2}(\frac{n}{n+2})^{\frac{n}{2}}$.

(c) For $\forall h \in (0, d(\varepsilon_0))$, the equation $d(\varepsilon) = h$ has two roots ε_1 and ε_2 in the interval $(0,\varepsilon_0)$ and $(\varepsilon_0, 1)$, respectively.

Proof. (a) is easy to be proved. Here we omit the proof of it.

(b) By calculation, we have

$$d'(\varepsilon) = 2\lambda_1 de\Big[-\varepsilon^{\frac{n}{2}} + \frac{n}{2}(1-\varepsilon)\varepsilon^{\frac{n}{2}-1}\Big] = \lambda_1 de\varepsilon^{\frac{n}{2}-1}[n-(n+2)\varepsilon]. \tag{2.26}$$

From $d'(\varepsilon) = 0$, we conclude that $\varepsilon_0 = \frac{n}{n+2}$. In addition, by (2.26), we get $d'(\varepsilon) > 0$ on $(0, \varepsilon_0)$ and $d'(\varepsilon) < 0$ on $(\varepsilon_0, 1)$. Therefore, $d(\varepsilon)$ takes the maximum value at $\varepsilon_0 = \frac{n}{n+2}$, and

$$d(\varepsilon_0) = 2\lambda_1 de(1-\varepsilon_0)\varepsilon_0^{\frac{n}{2}} = \frac{4\lambda_1 de}{n+2}(\frac{n}{n+2})^{\frac{n}{2}}.$$

(c) Let $f(\varepsilon) = d(\varepsilon) - h$, then $f(0) = -h < 0,\ f(\varepsilon_0) = d(\varepsilon_0) - h > 0,\ f(1) = -h < 0$. According to the continuity of function $f(\varepsilon)$ on interval $[0, 1]$, the equation $f(\varepsilon) = 0$ i.e. $d(\varepsilon) = h$ has two roots $\varepsilon_1 \in (0, \delta_0)$ and $\varepsilon_2 \in (\delta_0, 1)$. □

Proposition 2.3 Let $r(\varepsilon) = 4\lambda_1 de\varepsilon^{\frac{n}{2}}$, then

(1) If $\mathcal{J}(u) \le d(\varepsilon)$, then $0 < \|\nabla u\|^2 \le r(\varepsilon)$ if only and if $J_\varepsilon(u) \ge 0$.

(2) If $\mathcal{J}_\varepsilon(u) < 0$, then $\|\nabla u\|^2 > r(\varepsilon)$.

Proof. (1) For $a^2 = 2\pi\varepsilon$, we have

$$\begin{aligned}
\mathcal{J}_\varepsilon(u) &= \frac{\varepsilon}{2}\|\nabla u\|^2 - \frac{1}{2}\int_\Omega u^2 \log|u| dx + \frac{3}{4}\|u\|^2 \\
&\ge \Big(\frac{\varepsilon}{2} - \frac{a^2}{4\pi}\Big)\|\nabla u\|^2 - \frac{1}{2}\|u\|^2 \log\|u\| + \frac{n}{4}(1+\log a)\|u\|^2 + \frac{3}{4}\|u\|^2 \\
&= \frac{1}{4}\|u\|^2[-2\log\|u\| + n(1+\log a) + 3] \\
&= \frac{1}{4}\|u\|^2 \log\frac{\lambda_1 a^n e^{n+3}}{\|\nabla u\|^2} = \frac{1}{4}\|u\|^2 \log\frac{4\lambda_1 de\varepsilon^{\frac{n}{2}}}{\|\nabla u\|^2}.
\end{aligned} \tag{2.27}$$

By $0 < \|\nabla u\|^2 \le r(\varepsilon)$, we see that $\log\frac{4\lambda_1 de\varepsilon^{\frac{n}{2}}}{\|\nabla u\|^2} \ge 0$. Therefore, we conclude from (2.27) that $\mathcal{J}_\varepsilon(u) \ge 0$.

If $\mathcal{J}_\varepsilon(u) \ge 0$, then from (2.20) and

$$\mathcal{J}(u) = \frac{1-\varepsilon}{2}\|\nabla u\|^2 + \mathcal{J}_\varepsilon(u) \le d(\varepsilon), \tag{2.28}$$

we have

$$\frac{1-\varepsilon}{2}\|\nabla u\|^2 \le 2\lambda_1 de(1-\varepsilon)\varepsilon^{\frac{n}{2}},$$

which implies that $\|\nabla u\|^2 \le r(\varepsilon)$.

(b) By (2.24) and $\mathcal{J}_\varepsilon(u) < 0$, we have $\log\frac{4\lambda_1 de\varepsilon^{\frac{n}{2}}}{\|\nabla u\|^2} < 0$, which implies that $\|\nabla u\|^2 > r(\varepsilon)$. □

On the basis of Proposition 2.1 and Proposition 2.2, we define a family of potential wells by

$$\mathcal{W}_\varepsilon = \{u \in H_0^1(\Omega) :\ \mathcal{J}_\varepsilon(u) > 0,\ \mathcal{J}(u) < d(\varepsilon)\} \cap \{0\},$$

and

$$\mathcal{U}_\varepsilon = \{u \in H_0^1(\Omega) :\ \mathcal{J}_\varepsilon(u) < 0,\ \mathcal{J}(u) < d(\varepsilon)\},$$

for $\varepsilon \in (0, 1)$.

Remark From $\mathcal{J}_\varepsilon(u) > 0$ and

$$\mathcal{J}(u) = \frac{1-\varepsilon}{2}\|\nabla u\|^2 + \mathcal{J}_\varepsilon(u),$$

we have $\mathcal{J}(u) > 0$.

3. Global existence of solutions

In this section, by applying Galerkin method and the compactness principle, we study the global solutions of the problem (1.1)–(1.3).

Theorem 3.1 If $u_0 \in \mathcal{W},\ u_1 \in L^2(\Omega)$ satisfy $0 < \mathcal{E}(0) < d$, then the problem (1.1)–(1.3) admits a global solution $u(x,t)$ such that $u(x,t) \in L^\infty([0,+\infty); H_0^1(\Omega)),\ u_t(x,t) \in L^\infty([0,+\infty); L^2(\Omega))$.

Proof. Let $\{\omega_j\}_{j=1}^\infty$ be a basis for $H_0^1(\Omega)$. We are going to find out the approximate solution $u_m(t)$ in the form $u_m(t) = \sum_{j=1}^m g_{jm}(t)\omega_j$ with $g_{jm}(t) \in C^2[0,T],\ \forall T > 0$, where the unknown functions $g_{jm}(t)$ are determined by the following ordinary differential equation

$$(u_{mtt}(t), \omega_j) + (\nabla u_m(t), \nabla\omega_j) + (u_m(t), \omega_j) = (u_m(t)\log|u_m(t)|, \omega_j),\ j = 1, 2, \cdots, m \tag{3.1}$$

with initial data

$$u_m(0) = u_{0m},\ u_{mt}(0) = u_{1m}. \tag{3.2}$$

By the density of $H_0^1(\Omega)$ in $L^2(\Omega)$, there exist α_{jm} and β_{jm}, $j = 1, 2, \cdots, m$ such that

$$u_{0m} = \sum_{j=1}^m \alpha_{jm}\omega_j \to u_0(x) \textit{ strongly in } H_0^1(\Omega),\ m \to \infty, \tag{3.3}$$

$$u_{1m} = \sum_{j=1}^m \beta_{jm}\omega_j \to u_1(x) \textit{ strongly in } L^2(\Omega),\ m \to \infty. \tag{3.4}$$

By a Picard's iteration method, there exists solution $g_{jm}(t)$ of the problem (3.1) and (3.2) in interval $[0, t_m^1)$ for some $t_m^1 \le T$. From the uniformly boundedness of function $g_{jm}(t)$ and the extension theorem, we can extend this solution to the whole interval $[0,T]$ for any given $T > 0$ by making use of the a priori estimates below.

Multiplying both sides of (3.1) by $g'_{jm}(t)$ and summing with respect to j from 1 to m, and integrating over $[0,t]$, we have from (2.1) and (2.3) that

$$\mathcal{E}_m(t) = \frac{1}{2}\|u_{mt}(t)\|^2 + \mathcal{J}(u_m(t)) = \frac{1}{2}\|u_{mt}(0)\|^2 + \mathcal{J}(u_m(0)) = \mathcal{E}_m(0) < d. \tag{3.5}$$

By (3.5), we can verify

$$u_m(t) \in \mathcal{W},\ \forall t \in [0,T]. \tag{3.6}$$

In fact, suppose that (3.6) is false and let τ be the smallest time for that $u_m(\tau) \notin \mathcal{W}$. Then in virtue of the continuity of $u_m(t)$, we see $u_m(\tau) \in \partial\mathcal{W}$. From the continuity $\mathcal{J}(u(t))$ and $\mathcal{K}(u(t))$ with respect to t, we have either $\mathcal{J}(u_m(\tau)) = d$ or $\mathcal{K}(u_m(\tau)) = 0$. By (3.5), we get $\mathcal{J}(u_m(\tau)) < d$. So, the former case is impossible. Assume that $\mathcal{K}(u_m(\tau)) = 0$ is valid, then $u_m(\tau) \in \mathcal{N}$. From (2.7), we obtain $\mathcal{J}(u_m(\tau)) \geq d$ which is contradictive with (3.5). Therefore, the latter case is impossible as well.

We deduce from (2.5), (3.5) and (3.6) that

$$d > J(u_m(t)) = \frac{1}{4}\|u_m(t)\|^2 + \frac{1}{2}K(u_m(t)) > \frac{1}{4}\|u_m(t)\|^2, \tag{3.7}$$

which implies that

$$\|u_m(t)\|^2 < 4d. \tag{3.8}$$

From (2.1), (3.5) and Lemma 2.2, we obtain

$$\begin{aligned} &\|u_{mt}(t)\|^2 + \|\nabla u_m(t)\|^2 + \frac{3}{2}\|u_m(t)\|^2 \leq 2d + \int_\Omega u_m^2(t)\log|u_m(t)|dx \\ &\leq 2d + \|u_m\|^2\log\|u_m(t)\| + \frac{a^2}{2\pi}\|\nabla u_m(t)\|^2 - \frac{n}{2}(1+\log a)\|u_m(t)\|^2. \end{aligned} \tag{3.9}$$

Let $a = \sqrt{\pi}$, then we have from (3.8) and (3.9)

$$\begin{aligned} 2\|u_{mt}(t)\|^2 + \|\nabla u_m(t)\|^2 &\leq 4d + (\log\|u_m(t)\|^2 - \log(\sqrt{\pi}e)^n - 3)\|u_m(t)\|^2 \\ &\leq 4d[1 + \log 4d - \log(\sqrt{\pi})^n e^{n+3}] = 2nd\log 2, \end{aligned} \tag{3.10}$$

which implies

$$\|u_{mt}(t)\| < \sqrt{nd\log 2}, \quad \|\nabla u_m(t)\| \leq \sqrt{2nd\log 2}. \tag{3.11}$$

We know that $u_{mtt}(t)$ is uniformly bounded in $L^\infty(0,T;H^{-1}(\Omega))$ by a standard discussion. Then, there exists a function $u(t)$ and a convergent subsequence of $\{u_\mu\}$, still denoted by $\{u_m\}$. As $m \to \infty$, we obtain

$$u_m \to u \textit{ weakly star in } L^\infty(0,T;H_0^1(\Omega)), \tag{3.12}$$

$$u_{mt} \to u_t \textit{ weakly star in } L^\infty(0,T;L^2(\Omega)), \tag{3.13}$$

$$u_{mtt} \to u_{tt} \textit{ weakly star in } L^\infty(0,T;H^{-1}(\Omega)). \tag{3.14}$$

From (3.12)–(3.14) and Aubin-Lions lemma, we have

$$u_m \to u \textit{ strongly in } L^2(0,T;L^2(\Omega)), \tag{3.15}$$

which implies

$$u_m \to u \textit{ a.e. in } (0,T)\times\Omega. \tag{3.16}$$

By (3.16), we can infer that

$$u_m\log|u_m| \to u\log|u| \textit{ a.e. in } (0,T)\times\Omega. \tag{3.17}$$

Let $\Omega_1 = \{x \in \Omega;\ |u_m(x)| \le 1\}$ and $\Omega_2 = \{x \in \Omega;\ |u_m(x)| > 1\}$, then by direct calculation, we get from (3.11)

$$\begin{aligned} \int_\Omega |u_m(t)\log|u_m(t)||^2 dx &= \int_{\Omega_1} |u_m(t)|^2(\log|u_m(t)|)^2 dx \\ &\quad + \int_{\Omega_2} |u_m(t)|^2(\log|u_m(t)|)^2 dx \\ &\le e^{-2}|\Omega| + \left(\frac{n-2}{2}\right)^2 \int_{\Omega_2} |u_m(t)|^{\frac{2n}{n-2}} dx \\ &\le e^{-2}|\Omega| + \left(\frac{n-2}{2}\right)^2 C^{\frac{2n}{n-2}} \|\nabla u_m(t)\|^{\frac{2n}{n-2}} \\ &\le e^{-2}|\Omega| + \left(\frac{n-2}{2}\right)^2 (2ndC^2\log 2)^{\frac{n}{n-2}}. \end{aligned} \tag{3.18}$$

The estimate (3.18) indicates that $u_m\log|u_m|$ is uniformly bounded in $L^\infty(0,T;L^2(\Omega))$. Thus there exists a function χ such that

$$u_m\log|u_m| \to \chi \textit{ weakly star in } L^\infty(0,T;L^2(\Omega)). \tag{3.19}$$

From (3.17), (3.18) and Lemma 2.5, we have

$$u_m\log|u_m| \to u\log|u| \textit{ weakly in } L^\infty(0,T;L^2(\Omega)). \tag{3.20}$$

It follows from (3.19) and (3.20) that

$$\chi = u\log|u|. \tag{3.21}$$

Let $m\to\infty$ in (3.1), by using (3.12), (3.14), (3.19) and (3.20), we obtain

$$(u_{tt},\omega_j) + (\nabla u,\nabla\omega_j) + (u,\omega_j) = (u\log|u|,\omega_j),\ \ \forall j.$$

By the density of the system $\{\omega_j\}_{j=1}^\infty$ in $H_0^1(\Omega)$, we deduce that

$$(u_{tt},\varphi) - (\Delta u,\varphi) + (u,v) = (u\log|u|,\varphi)$$

for $\forall\varphi\in H_0^1(\Omega)$. That is to say u satisfies the Eq (1.1) in the weak sense.

Next, we prove that $u(0) = u_0, u_t(0) = u_1$ are held. It follows from (3.12), (3.13) and Lemma 2.4 that $u(t) : [0,T] \to L^2(\Omega)$ is continuous. Hence, we gain that $u(0)$ is valid and $u_m(0)\to u(0)$ weakly in $L^2(\Omega)$. By (3.3), we obtain $u(0) = u_0$.

To prove $u_t(0) = u_1$, we note that

$$\int_0^T (u_{mtt},\xi\omega_j)dt = -\int_0^T (u_{mt},\xi_t\omega_j)dt - (u_{mt}(0),\omega_j),$$

where $\xi(t)$ is a smooth function with $\xi(0) = 1, \xi(T) = 0$.

For given j, as $m\to\infty$, in the distribution sense, we have

$$\int_0^T (u_{tt},\xi\omega_j)dt = -\int_0^T (u_t,\xi_t\omega_j)dt - (u_t(0),\omega_j) \tag{3.22}$$

in $\mathcal{D}'([0,T])$. On the other hand, by (3.1), we get

$$\int_0^T (u_{mtt}, \xi\omega_j)dt = \int_0^T [(\Delta u_m, \xi\omega_j) - (u_m, \xi\omega_j) + (u_m \log|u_m|, \xi\omega_j)]dt. \tag{3.23}$$

Taking the limitation on both sides of (3.23) as $m \to \infty$, we obtain

$$\int_0^T (u_{tt}, \xi\omega_j)dt = \int_0^T [(\Delta u, \xi\omega_j) - (u, \xi\omega_j) + (u \log|u|, \xi\omega_j)]dt.$$

Therefore,

$$\int_0^T (u_{tt}, \xi\omega_j)dt = -\int_0^T (u_t, \xi_t\omega_j)dt - (u_1, \omega_j). \tag{3.24}$$

It follows from (3.22) and (3.24) that $(u_t(0), \omega_j) = (u_1, \omega_j)$. By the density of $\{\omega_j\}_{j=1}^m$ in $L^2(\Omega)$, we get $u_t(0) = u_1$.

The proof of Theorem 3.1 is completed. □

For the case $\mathcal{K}(u_0) \geq 0$ and $\mathcal{E}(0) = d$, the global existence result of the problem (1.1)–(1.3) reads as follows:

Theorem 3.2 Given that $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. If $\mathcal{E}(0) = d$ and $\mathcal{K}(u_0) \geq 0$, then there exists a global weak solution $u(x,t)$ for the problem (1.1)–(1.3) such that $u(x,t) \in L^\infty([0,+\infty); H_0^1(\Omega))$, $u_t(x,t) \in L^\infty([0,+\infty); L^2(\Omega))$.

Proof. Let $\rho_k = 1 - \frac{1}{k}$ and $u_{0k} = \rho_k u_0$ for $k \geq 2$. We consider the following problem

$$\begin{cases} u_{tt} + \Delta u + u = u \log|u|, \ \ (x,t) \in \Omega \times R^+, \\ u(x,0) = u_{0k}(x), \ u_t(x,0) = u_1(x), \ \ x \in \Omega, \\ u(x,t) = 0, \ (x,t) \in \partial\Omega \times R^+. \end{cases} \tag{3.25}$$

By $K(u_0) \geq 0$ and Lemma 2.6, we have $\lambda^* = \lambda^*(u_0) \geq 1$. Therefore, we conclude that $\mathcal{K}(u_{0k}) > 0$. Thus, we have

$$\mathcal{J}(u_{0k}) = \frac{1}{4}\|u_{0k}\|^2 + \frac{1}{2}\mathcal{K}(u_{0k}) > 0$$

and $\mathcal{J}(u_{0k}) = \mathcal{J}(\mu_k u_0) < \mathcal{J}(u_0)$. Therefore,

$$0 < \mathcal{E}_k(0) = \frac{1}{2}\|u_1\|^2 + J(u_{0k}) < \frac{1}{2}\|u_1\|^2 + \mathcal{J}(u_0) = \mathcal{E}(0) = d.$$

So, we obtain $u_{0k} \in \mathcal{W}$. For each k, by Theorem 3.1, the problem (3.25) admits a global weak solution $u_k(t)$ which satisfies that $u_k(t) \in L^\infty([0,+\infty); H_0^1(\Omega))$, $u_{kt}(t) \in L^\infty([0,+\infty); L^2(\Omega))$ and

$$(u_{kt}, \varphi) + \int_0^t [(\Delta u_k, \varphi) + (u_k, \varphi)]ds = (u_1, \varphi) + \int_0^t (u_k \log|u_k|, \varphi)ds \tag{3.26}$$

for any $\varphi \in H_0^1(\Omega)$. In addition,

$$\mathcal{E}_k(t) = \frac{1}{2}\|u_{kt}\|^2 + \mathcal{J}(u_k) = \frac{1}{2}\|u_1\|^2 + \mathcal{J}(u_{0k}) = \mathcal{E}_k(0) < d. \tag{3.27}$$

By using the formula (3.27) and the same argument as (3.6), we may verify $u_k(t) \in \mathcal{W}$.

The remainder proof for Theorem 3.2 is the same process as Theorem 3.1. Here, we omit it. □

Next, we study the global existence of solution to the problem (1.1)–(1.3) in a family of potential wells $\mathcal{W}_\varepsilon$. For this purpose, we need the following lemmas

Lemma 3.1 Suppose that $u_0 \in H_0^1(\Omega),\ u_1 \in L^2(\Omega)$ and $0 < \mathcal{E}(0) < d(\varepsilon_0)$. $\varepsilon_1, \varepsilon_2$ are the two roots of the equation $d(\varepsilon) = \mathcal{E}(0)$, $u(x,t)$ is a solution of the problem (1.1)–(1.3). Then

(i) If $\mathcal{J}_{\varepsilon_0}(u_0) > 0$, then $u(t) \in \mathcal{W}_\varepsilon$ for $\forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$.

(ii) If $\mathcal{J}_{\varepsilon_0}(u_0) < 0$, then $u(t) \in \mathcal{U}_\varepsilon$ for $\forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$.

Proof. Firstly, under the conditions in Lemma 4.1, we prove the sign of $\mathcal{J}_\varepsilon(u)$ is invariant on the interval $(\varepsilon_1, \varepsilon_2)$.

Multiplying both sides of the Eq (1.1) by u_t, then we get from integrating over $\Omega \times [0, t]$ that

$$\mathcal{E}(t) = \frac{1}{2}\|u_t\|^2 + \mathcal{J}(u) = \frac{1}{2}\|u_1\|^2 + \mathcal{J}(u_0) = \mathcal{E}(0) = d(\varepsilon). \tag{3.28}$$

By (3.28) and $0 < \mathcal{E}(0) < d(\varepsilon_0)$, it is easy to see that $0 < \mathcal{J}(u) < d(\varepsilon_0)$. Namely, $\|\nabla u\| \neq 0$.

By contradiction, we suppose that the sign of $\mathcal{J}_\varepsilon(u)$ is variable on $(\varepsilon_1, \varepsilon_2)$, then there exists $\varepsilon' \in (\varepsilon_1, \varepsilon_2)$ such that $\mathcal{J}_{\varepsilon'}(u) = 0$. From (3.28), (2.20) and Proposition 2.2, we gain that $\mathcal{E}(0) \geq \mathcal{J}(u) \geq d(\varepsilon') > d(\varepsilon_1) = d(\varepsilon_2)$, which is contradictive with $\mathcal{E}(0) = d(\varepsilon_1) = d(\varepsilon_2)$.

(i) Because $\mathcal{J}_{\varepsilon_0}(u_0) > 0$ and the sign of $\mathcal{J}_\varepsilon(u)$ is not changed for $(\varepsilon_1, \varepsilon_2)$, we have $\|\nabla u_0\| \neq 0$ and $\mathcal{J}_\varepsilon(u_0) > 0,\ \forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$. From (3.28), we get $\mathcal{J}(u_0) \leq \mathcal{E}(0) < d(\varepsilon)$. Thus, we obtain $u_0 \in \mathcal{W}_\varepsilon, \forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$.

Next we prove $u(t) \in \mathcal{W}_\varepsilon$ for $\forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$ and $0 < t < T$, where T is the existence time of $u(t)$. Assume that there exists a number $t_1 \in (0, T)$ such that $u(t_1) \notin \mathcal{W}_\varepsilon$. Then, in virtue of the continuity of $u(t)$, we see $u(t_1) \in \partial\mathcal{W}_\varepsilon, \forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$. From the definition of $\mathcal{W}_\varepsilon$ and the continuity of $\mathcal{J}(u(t))$ and $\mathcal{J}_\varepsilon(u(t))$ with respect to t, we have

$$\mathcal{J}_\varepsilon(u(t_1)) = 0,\ \|\nabla u(t_1)\| \neq 0, \tag{3.29}$$

or

$$\mathcal{J}(u(t_1)) = d(\varepsilon). \tag{3.30}$$

It follows from (3.28) that

$$\mathcal{J}(u(t)) < \mathcal{E}(0) = d(\varepsilon),\ t \in (0, T). \tag{3.31}$$

Thus, the case (3.30) is impossible. If (3.29) holds, then, by (3.17), we have $\mathcal{J}(u(t_1)) \geq d(\varepsilon)$ which is contradictive with (3.31). Consequently, the case (3.29) is also impossible. Thus, we conclude that $u(t) \in \mathcal{W}_\varepsilon,\ \forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$.

(ii) Since the sign of $\mathcal{J}_\varepsilon(u)$ is not changed for $(\varepsilon_1, \varepsilon_2)$, by $\mathcal{J}_{\varepsilon_0}(u_0) < 0$, we get $\mathcal{J}_\varepsilon(u_0) < 0$ for $\forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$. Thus, we have $u_0 \in \mathcal{U}_\varepsilon$ from $\mathcal{J}(u_0) < \mathcal{E}(0) = d(\varepsilon)$. Now we prove $u(t) \in \mathcal{U}_\varepsilon$ for $\forall \varepsilon \in (\varepsilon_1, \varepsilon_2),\ 0 < t < T$. If it is not right, then there exists $t_2 \in (0, T)$ with $u(t_2) \in \partial\mathcal{U}_\varepsilon$ for $\forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$, i.e. either $\mathcal{J}_\varepsilon(u(t_2)) = 0$ or $\mathcal{J}(u(t_2)) = d(\varepsilon)$. By (3.31), $\mathcal{J}(u(t_2)) = d(\varepsilon)$ is impossible. Moreover, let t_2 be the first time such that $\mathcal{J}_\varepsilon(u(t_2) = 0$, then $\mathcal{J}_\varepsilon(u(t)) < 0$ for $0 \leq t < t_2$. Combining (3.28) and Proposition 2.3, we get $\|\nabla u(t)\| > r(\varepsilon)$ for $t \in [0, t_2)$. Hence, we obtain $\|\nabla u(t_2)\| \geq r(\varepsilon)$. From (3.17), it follows that $\mathcal{J}(u(t_2)) \geq d(\varepsilon)$ which is contradictive with (3.31). This implies that $\mathcal{J}_\varepsilon(u(t_2)) = 0$ is also impossible. Therefore, we have $u(t) \in \mathcal{U}_\varepsilon,\ \forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$. □

Lemma 3.2 Assume that $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$ and $0 < \mathcal{E}(0) \le h < d(\varepsilon_0)$. $\varepsilon_1, \varepsilon_2$ are the two roots of the equation $d(\varepsilon) = h$, $u(x,t)$ are solutions of the problem (1.1)–(1.3). Then

(i) If $\mathcal{J}_{\varepsilon_0}(u_0) > 0$, then $u(t) \in \mathcal{W}_\varepsilon$ for $\forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$.

(ii) If $\mathcal{J}_{\varepsilon_0}(u_0) < 0$, then $u(t) \in U_\varepsilon$ for $\forall \varepsilon \in (\varepsilon_1, \varepsilon_2)$.

We can prove Lemma 3.2 by means of the similar method shown in the proof of Lemma 3.1. Here, we omit it.

Theorem 3.3 Suppose that $\varepsilon_1, \varepsilon_2$ are the two roots of the equation $d(\varepsilon) = \mathcal{E}(0)$ and $\mathcal{J}_{\varepsilon_2}(u_0) > 0$. If $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $0 < \mathcal{E}(0) < d(\varepsilon)$, then the problem (1.1)–(1.3) admits a global weak solution $u(x,t)$ such that

$$u(x,t) \in L^\infty(0,T; H_0^1(\Omega)), \;\; u_t(x,t) \in L^\infty(0,T; L^2(\Omega)),$$

for any $T > 0$.

Proof. By using the similar argument as Theorem 3.1, we are going to prove Theorem 3.3. Under the conditions in Theorem 3.3, by Lemma 3.1, we have $u_0 \in \mathcal{W}_\varepsilon$ for $\varepsilon \in (\varepsilon_1, \varepsilon_2)$. For any given $\varepsilon_1 < \varepsilon < \varepsilon_2$, we derive $\mathcal{J}_\varepsilon(u_m(0)) > 0$ and $\mathcal{E}_m(0) < d(\varepsilon)$, which implies that $u_m(0) \in \mathcal{W}_\varepsilon$. Once again, we get $u_m(t) \in \mathcal{W}_\varepsilon$ by Lemma 3.1. Here, the approximate solutions $u_m(t)$ are given in the proof of Theorem 3.1.

Multiplying both sides of (3.1) by $g'_{jm}(t)$, summing over j from 1 to m and integrating with respect to t, we obtain

$$\mathcal{E}_m(t) = \frac{1}{2}\|u_{mt}(t)\|^2 + \mathcal{J}(u_m(t)) = \frac{1}{2}\|u_{mt}(0)\|^2 + \mathcal{J}(u_m(0)) = \mathcal{E}_m(0) < d(\varepsilon). \tag{3.32}$$

From (2.3) and (2.17), we deduce

$$\mathcal{J}(u_m(t)) = \frac{1-\varepsilon}{2}\|\nabla u_m(t)\|^2 + \mathcal{J}_\varepsilon(u_m(t)). \tag{3.33}$$

Combining (2.21), (3.32), (3.33), by $u_m(t) \in \mathcal{W}_\varepsilon$, we get $\mathcal{J}(u_m(t)) > 0$ and the following estimates

$$\|\nabla u_m(t)\| < 2\sqrt{\lambda_1 de}\, \varepsilon^{\frac{n}{4}}. \tag{3.34}$$

From (2.21) and (3.32), we find that

$$\|u_{mt}(t)\| < 2\sqrt{\lambda_1 de(1-\varepsilon)}\, \varepsilon^{\frac{n}{4}}. \tag{3.35}$$

By means of the same procedure as the estimates (3.18), we obtain

$$\int_\Omega |u_m(t)\log|u_m(t)\|^2 dx \le e^{-2}|\Omega| + \left(\frac{n-2}{2}\right)^2 (4\lambda_1 deC^2\varepsilon^{\frac{n}{2}})^{\frac{n}{n-2}}. \tag{3.36}$$

The remainder of the proof for Theorem 3.3 is the same as those of Theorem 3.1. Here, we omit them. □

4. Blow-up of solution

In this section, we establish the blow-up property of solution for the problem (1.1)–(1.3).

Lemma 4.1 Let $u(t)$ be a solution of (1.1)–(1.3). If $u_0 \in \mathcal{U}$ and $\mathcal{E}(0) < d$, then $u(t) \in \mathcal{U}$ and $\mathcal{E}(t) < d$, for all $t \geq 0$.

Proof. It follows from Lemma 2.3 that

$$\mathcal{E}(t) = \mathcal{E}(0) < d, \ \forall t \geq 0.$$

By (2.5), we obtain

$$\mathcal{J}(u) \leq \mathcal{E}(t) < d, \ \forall t \geq 0. \tag{4.1}$$

By contradiction, we assume that there exists $t^* \in [0, +\infty)$ such that $u(t^*) \notin \mathcal{U}$, then, from the continuity of $\mathcal{K}(u(t))$ on t, we have $\mathcal{K}(u(t^*)) = 0$. This implies that $u(t^*) \in \mathcal{N}$. We get from (2.7) that $\mathcal{J}(u(t^*)) \geq d$, which is contradiction with (4.1). Consequently, Lemma 4.1 is valid. □

Lemma 4.2 Suppose that $u \in \mathcal{U}$, then $\mathcal{K}(u(t)) < 2[\mathcal{J}(u(t)) - d]$.

Proof. If $u \in \mathcal{U}$, then it follows from Lemma 2.6 that there exists a λ^* such that $0 < \lambda^* < 1$ and $\mathcal{K}(\lambda^* u) = 0$. By the definition of d in (2.6), we get

$$d < \mathcal{J}(\lambda^* u) = \frac{1}{2}\mathcal{K}(\lambda^* u) + \frac{1}{4}\|\lambda^* u\|^2 = \frac{1}{4}\|\lambda^* u\|^2 < \frac{1}{4}\|u\|^2.$$

We have from (2.5) that $d < \mathcal{J}(u(t)) - \frac{1}{2}\mathcal{K}(u(t))$, which implies that $\mathcal{K}(u(t)) < 2[\mathcal{J}(u(t)) - d]$. □

Theorem 4.1 If the initial datum $u_0 \in \mathcal{U},\ u_1 \in L^2(\Omega)$ satisfy that $\mathcal{E}(0) < d$ and $\int_\Omega u_0 u_1 dx > 0$, then the solution $u(t)$ in Theorem 2.1 of the problem (1.1)–(1.3) blows up as time t goes to infinity, which means that

$$\lim_{t \to +\infty} \|u(t)\|^2 = +\infty.$$

Proof. Let $\mathcal{P}(t) = \|u(t)\|^2$, then $\mathcal{P}(t) > 0,\ t \geq 0$. Direct computations show that

$$\mathcal{P}'(t) = 2(u, u_t). \tag{4.2}$$

From (1.1) and (2.4), we get

$$\begin{aligned} \mathcal{P}''(t) &= 2\|u_t\|^2 + 2\int_\Omega u u_{tt} dx \\ &= 2\|u_t\|^2 - 2\Big(\|\nabla u\|^2 + \|u\|^2 - \int_\Omega u^2 \log|u| dx\Big) \\ &= 2\|u_t\|^2 - 2\mathcal{K}(u). \end{aligned} \tag{4.3}$$

It follows from Cauchy-Schwarz inequality and (4.2) that

$$|\mathcal{P}'(t)|^2 \leq 4\mathcal{P}(t)\|u_t\|^2,\ t \geq 0. \tag{4.4}$$

Then we have from (2.5) and Lemma 2.3 that

$$\begin{aligned}\mathcal{P}''(t)\mathcal{P}(t)-[\mathcal{P}'(t)]^2 &\geq 2\mathcal{P}(t)[\|u_t\|^2-\mathcal{K}(u(t))]-4\mathcal{P}(t)\|u_t\|^2\\ &=-2\mathcal{P}(t)[\|u_t\|^2+\mathcal{K}(u(t))]\\ &\geq -2\mathcal{P}(t)[2\mathcal{E}(t)-2\mathcal{J}(u(t))+\mathcal{K}(u(t))].\end{aligned} \tag{4.5}$$

From $u_0\in\mathcal{U}$, $\mathcal{E}(0)<d$ and Lemma 4.1, we have $u\in\mathcal{U}$, $\mathcal{E}(t)<d$. Hence by Lemma 4.2, we obtain that

$$2\mathcal{E}(t)-2\mathcal{J}(u(t))+\mathcal{K}(u(t))<2d-2\mathcal{J}(u(t))+2(\mathcal{J}(u(t))-d)=0. \tag{4.6}$$

We conclude from (4.5) and (4.6) that

$$\mathcal{P}''(t)\mathcal{P}(t)-[\mathcal{P}'(t)]^2>0. \tag{4.7}$$

Furthermore, by direct calculation, it is easy to see that

$$(\log|\mathcal{P}(t)|)'=\frac{\mathcal{P}'(t)}{\mathcal{P}(t)}. \tag{4.8}$$

$$(\log|\mathcal{P}(t)|)''=\left(\frac{\mathcal{P}'(t)}{\mathcal{P}(t)}\right)'=\frac{\mathcal{P}''(t)\mathcal{P}(t)-[\mathcal{P}'(t)]^2}{\mathcal{P}^2(t)}>0. \tag{4.9}$$

We know from (4.9) that the function $(\log|\mathcal{P}(t)|)'=\frac{\mathcal{P}'(t)}{\mathcal{P}(t)}$ is increasing on time t. By integrating both sides of (4.8) from 0 to t, we get

$$\log|\mathcal{P}(t)|-\log|\mathcal{P}(0)|=\int_0^t(\log|\mathcal{P}(s)|)'ds=\int_0^t\frac{\mathcal{P}'(s)}{\mathcal{P}(s)}ds\geq\frac{\mathcal{P}'(0)}{\mathcal{P}(0)}t,$$

for $t>0$. Therefore,

$$\mathcal{P}(t)\geq\mathcal{P}(t_0)\exp\left(\frac{\mathcal{P}'(t_0)}{\mathcal{P}(t_0)}(t-t_0)\right). \tag{4.10}$$

From the definition of $\mathcal{P}(t)$, (4.10) means that

$$\lim_{t\to+\infty}\|u(t)\|^2=+\infty.$$

This finishes the proof of Theorem 4.1. □

5. Conclusions

By applying logarithmic Sobolev inequality, the Galerkin method and compactness theorem, we prove the global existence results of the problem (1.1)–(1.3) under the conditions that the initial values $u_0\in\mathcal{W}$, $u_1\in L^2(\Omega)$ satisfy (i) $0<\mathcal{E}(0)<d$ or (ii) $\mathcal{K}(u_0)\geq 0$ and $\mathcal{E}(0)=d$. Meanwhile, under the condition of positive initial energy, by using the concavity analysis method, we establish the finite time blow-up result of solutions in the sense of L^2 norm. On the other hand, the global existence of solution for this problem is also obtained in a family of potential wells $\mathcal{W}_\varepsilon$. Our result implies that the polynomial nonlinearity is important for the solutions of such kinds of Klein-Gordon equation to be blow-up in finite time.

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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