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*Research article*

# Existence and multiplicity of solutions for generalized asymptotically linear Schrödinger-Kirchhoff equations

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**Abstract:** In this paper, we investigate the nonlinear Schrödinger-Kirchhoff equations on the whole space. By using the Morse index of the reduced Schrödinger operator, we show the existence and multiplicity of solutions for this problem with asymptotically linear nonlinearity via variational methods.

**Keywords:** Schrödinger-Kirchhoff equation; Morse index; variational method
**Mathematics Subject Classification:** 35B33, 35B35

## 1. Introduction

In this paper, we study the existence and multiplicity of solutions for the following Schrödinger-Kirchhoff equation

$$-(1+b\int_{\mathbb{R}^N}|\nabla u|^2dx)\Delta u+V(x)u=f(x,u),\ \text{in } \mathbb{R}^N, \tag{1.1}$$

where $b>0$ is a parameter. Eq (1.1) arises in an interesting physical context. When $V\equiv 0$ and $\mathbb{R}^N$ is replaced by a bounded domain $\Omega\subset\mathbb{R}^N$, Eq (1.1) reduces to the following Dirichlet problem:

$$-(1+b\int_{\mathbb{R}^N}|\nabla u|^2dx)\Delta u=f(x,u),\ \text{in } \Omega,\ \ u=0 \text{ on } \Omega, \tag{1.2}$$

which is related to the stationary analogue of the equation

$$\rho\frac{\partial^2 u}{\partial t^2}-(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx)\frac{\partial^2 u}{\partial x^2}=0,$$

proposed by Kirchhoff [9] as an extension of classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. There is a large literature on existence and multiplicity results. In 1978, Lions [13] proposed an abstract framework for the Eq (1.2). Since then, Eq (1.2) have been investigated by many authors, see for instance [5, 6, 10, 17–19, 22, 26, 27] and the reference therein. Recently, a lot of attentions have been focused on the study of solutions of (1.1) on the whole space $\mathbb{R}^N$ (see [7, 8, 11, 12, 15, 25, 28] and references therein).

Motivated by [7, 8, 11, 12, 15, 25, 28], we consider the asymptotical linear Kirchhoff type Eq (1.1) on the whole space $\mathbb{R}^N$ and assume the potential $V$ satisfies the following condition:

$(V_1)$ $V \in L^q_{loc}(\mathbb{R}^N)$ is real-valued, and $V^- := \min\{V, 0\} \in L^\infty(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ for some $q \in [2, +\infty) \cap (\frac{N}{2}, +\infty)$.

It is known that the assumption $(V_1)$ ensures that the reduced schrödinger operator $A = -\Delta + V$ is self-adjoint and semi-bounded on $L^2(\mathbb{R}^N)$ (see Theorem A.2.7 in Simon [23]). We denote by $\sigma(A)$ $\sigma_d(A)$ and $\sigma_{ess}(A)$ the spectrum, the discrete spectrum (eigenvalue with finite multiplicities), the essential spectrum of $A$, respectively.

In this paper, we assume the following general spectrum assumption

$(V_2)$ $a := \inf \sigma(A)$, $M := \inf \sigma_{ess}(A)$, $-\infty < a < M$, $M > 0$.

Condition $(V_2)$ implies that the potential $V$ is not periodic.

**Remark 1.1.** *The following potentials satisfy* $(V_2)$*:*

*Ex 1.* $V_\lambda = \lambda g(x) + 1$, *for some* $g \in C(\mathbb{R}^N, \mathbb{R})$, $g \geq 0$, *where* $\Omega := int(g^{-1}(0))$ *is not empty and is a bounded domain (see [2]).*

*Ex 2.* $V \in C(\mathbb{R}^N)$ *is bounded from below and there exists* $M > 0$ *such that* $\Lambda = \{x \in \mathbb{R}^N | V(x) \leq M\}$ *has finite Lebesgue measure.*

Due to the presence of essential spectrum, the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ is not compact and the problem becomes more difficult. To overcome the loss of compactness, we need to control the interplay between the nonlinear term $f$ and the essential spectrum part $\sigma_{ess}(A)$. Inspired by Liu, Su, and Weth [14], we assume

$(f_1)$ $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $\frac{f(x,u)}{u}$ is bounded on $\mathbb{R}^N \times (\mathbb{R} \setminus \{0\})$.

$(f_2)$ $f^* = \limsup_{|x|\to\infty} \sup_{u\neq 0} \frac{f(x,u)}{u} < M$.

Condition $(f_2)$ allows that the nonlinearity $f$ locally intersects with the essential spectrum. Motivated by $(f_2)$, we define

$$\Upsilon = \{B | B \text{ is a bounded, continuous real-valued function with} \limsup_{|x|\to\infty} B(x) < M\}.$$

The nonlinear function $f$ is assumed to be asymptotically linear at infinity and at origin in the following sense: There exist $B_0(x), B_\infty(x) \in \Upsilon$, such that

$(f_0)$ $f(x, u) = B_0(x)u + o(|u|)$ as $|u| \to 0$, uniformly in $x \in \mathbb{R}^N$,

$(f_\infty)$ $f(x, u) = B_\infty(x)u + o(|u|)$ as $|u| \to \infty$, uniformly in $x \in \mathbb{R}^N$.

A quantitative way to measure the twisting between $(f_0)$ and $(f_\infty)$ is the index theory which were widely used to investigate the periodic solutions of Hamiltonian systems (see Ekland [4], Long [16] and the references therein). However, the index theories constructed in [4, 16] depends on the compactness of Sobolev embedding or equivalently the spectral property $\sigma(A) = \sigma_d(A)$. Thus, the classical index theories can not work here. We will introduce a classification theory for the reduced linear Schrödinger equation with $(V_2)$ (see [21]). More precisely, for $B \in \Upsilon$ we classify the following reduced linear Schrödinger equation

$$-\Delta u + V(x)u - B(x)u = 0.$$

This classification gives a pair of numbers $(i(B), \nu(B))$, where $i(B)$ is the number of the negative eigenvalues of $A - B$ and $\nu(B)$ =dim ker$(A - B)$. We will briefly recall the definitions and some useful properties of the Morse index function in Section 2. Our main result is the following: Assume that $F(x, u) = \int_0^u f(x, s)ds$.

**Theorem 1.1.** *Let $(V_1)$, $(V_2)$, $(f_1)$ and $(f_2)$ be satisfied. Moreover, $F(x, u) \leq 0$, $(f_0)$ and $(f_\infty)$ hold with $i(B_0) > i(B_\infty) + \nu(B_\infty)$. Then (1.1) has at least $i(B_0) - i(B_\infty) - \nu(B_\infty)$ pairs of solutions provided $f(x, u)$ is odd in $u$.*

The aim of this paper is to extend the index theory to the study of Schrödinger-Kirchhoff equations. By using the Morse index of the reduced linear Schrödinger equation, we show how the behavior of the nonlinearity at the origin and at the infinity affects the number of solutions. In our setting, the main obstacle is the lack of compactness due to the presence of essential spectrum and the nonlocal part. Benefitting from some of the techniques used in [12, 14, 15, 28], we regain the compactness.

**Remark 1.2.** *(1) Inspired by works of Ding and L. Jeanjean [3], we add the sign condition $F \leq 0$ in order to control the compactness.*

*(2) Usually, if $\nu(B_\infty) = 0$, we write $f$ is non-resonance at infinity. When $b = 0$, non-resonance condition plays a very important role in the verification of Palais-Smale condition, Fortunately, because of the nonlocal part of the system (1.1), in this paper, we can consider system (1.1) without the non-resonance condition.*

*(3) Let the eigenvalues of $A$ be denoted by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \inf \sigma_{ess}(A)$, counting their multiplicities. Assume that all the assumptions of Theorem 1.1 hold. If we assume that*

$$\lambda_k < B_0(x) < \lambda_{k+1}, \ \ \lambda_l < B_\infty(x) \leq \lambda_{l+1}, \ \ \forall x \in \mathbb{R}^N,$$

*then we have $i(B_0) = k$, $i(B_\infty) = l$ and (1.1) has at least $k - l$ pairs of solutions; if we assume*

$$\lambda_k < B_0(x) < \lambda_{k+1}, \ \ B_\infty(x) \leq \lambda_1, \ \ \forall x \in \mathbb{R}^N,$$

*then we have $i(B_0) = k$, $i(B_\infty) = 0$ and (1.1) has at least $k$ pairs of solutions.*

This paper is organized as follows. In Section 2, we first present variational framework to deal with problem (1.1). We also recall some propositions and lemmas about the classification of the reduced linear Schrödinger equation which will be used to prove our main results. In Section 3, we give the proof of the main results.

We use the following notations:

- inf denotes the infimum of a set or a function in given domain.
- sup denotes the supremum of a set or a function in given domain.
- int denotes the interior of a set.
- dim denotes the dimension of a subspace.
- codim denotes the codimension of a subspace.
- $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = \infty$ for $N = 1$ and $N = 2$.
- $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*} : \nabla u \in L^2(\mathbb{R}^N)\}$.

## 2. Preliminaries

### 2.1. Variational settings

In what follows by $|\cdot|_q$ we denote the usual $L^q-$norm, and by $(\cdot,\cdot)_2$ the usual $L^2$-inner product. Define $A = -\Delta + V$. By $(V_1)$, $A$ is self-adjoint and semi-bounded on $L^2(\mathbb{R}^N)$ with domain $D(A) \subseteq H^2(\mathbb{R}^N)$. Note that 0 is at most an eigenvalue of finite multiplicites of $A$. Without loss of generality, throughout this paper, we assume $0 \notin \sigma(A)$. Thus, condition $(V_2)$ introduces an orthogonal decomposition

$$L^2 = L^+ \oplus L^-, \ \ u = u^+ + u^-,$$

corresponding to the spectrum of $A$ such that $A$ is negative definite on $L^-$ and positive definite on $L^+$. Denoting the absolute value of $A$ by $|A|$, let $E = D(|A|^{\frac{1}{2}})$ be the Hilbert space with the inner product

$$(u, v) = (|A|^{\frac{1}{2}}u, |A|^{\frac{1}{2}}v)_2,$$

and norm $\|u\| = (u, u)^{\frac{1}{2}}$. We have a decomposition

$$E = E^+ \oplus E^-, \text{ where } E^\pm = E \cap L^\pm,$$

which are orthogonal to each other with respect to the inner product $(\cdot,\cdot)$ and $(\cdot,\cdot)_2$.

**Lemma 2.1.** *The space E embeds continuously into $H^1(\mathbb{R}^N)$, and hence, E embeds continuously into $L^p(\mathbb{R}^N)$ for $p \in [2, 2^*]$ and compactly into $L^p_{loc}(\mathbb{R}^N)$ for $p \in [2, 2^*)$.*

Let us define the functional $I_b(u) : E \to \mathbb{R}$ by

$$I_b(u) = \|u^+\|^2 - \|u^-\|^2 + \frac{b}{4}(\int_{\mathbb{R}^N} |\nabla u|^2 dx)^2 - \int_{\mathbb{R}^N} F(x, u)dx. \tag{2.1}$$

Our hypotheses on $f$ imply that $I_b(u) \in C^1(E, \mathbb{R})$ and the critical points of $I_b$ are the weak solutions for problem (1.1).

### 2.2. Classification for reduced linear Schrödinger equation

Recall that we define

$$\Upsilon = \{B | B \text{ is a bounded, continuous real-valued function with} \limsup_{|x|\to\infty} B(x) < b\}.$$

For any $B(x) \in \Upsilon$, we also denote $B$ as the operator multiplication by $B(x)$ in $L^2(\mathbb{R}^N)$ without causing any confusing. By Lemma 2.1 of [21],

**Lemma 2.2.** *For any $B \in \Upsilon$, the essential spectrum $\sigma_{ess}(A - B)$ of $A - B$ is contained in $(0, +\infty)$.*

Let us recall the standard definitions and results on Rayleigh-Ritz quotients (see e.g. [20] ). Let $T$ be a self-adjoint operator in a Hilbert space $X$, with domain $D(T)$ and form-domain $F(T)$. If $T$ is bounded from below, we may define a sequence of min-max levels

$$\lambda_k(T) = \inf_{Y \text{ subspace of } F(T), \ \dim Y = k} \ \sup_{x \in Y \setminus \{\theta\}} \frac{(x, Tx)}{\|x\|_X^2}.$$

To each $k$ we also associate the (possible infinite) multiplicity number

$$m_k(T) = \text{card}\{k' \geq 1,\ \ \lambda_{k'}(T) = \lambda_k(T)\} \geq 1.$$

Then $\lambda_k(T) \leq \inf \sigma_{ess}(T)$. In the case $\lambda_k(T) < \inf \sigma_{ess}(T)$, $\lambda_k$ is an eigenvalue of $T$ with multiplicity $m_k(T)$.

**Definition 2.1.** *For any $B \in \Upsilon$, we define $i(B) = \sharp\{i|\lambda_i(A - B) < 0\}$, $\nu(B) = \sharp\{i|\lambda_i(A - B) = 0\}$.*

We define the following quadratic form:

$$q_B(u, v) = \frac{1}{2}(u^+, v^+) - \frac{1}{2}(u^-, v^-) - \frac{1}{2}(Bu, v)_2,\ \ \forall u, v \in E. \tag{2.2}$$

The following proposition lists some properties concerning the index function $(i(B), \nu(B))$ and the bilinear form $q_B$. For the details of the proofs, we refer to [21].

**Proposition 2.1.** *(i) The $E$ can be divided into three subspaces*

$$E = E^+(B) \oplus E^0(B) \oplus E^-(B),$$

*such that $q_B$ is positive definite, zero and negative definite on $E^+(B)$, $E^0(B)$ and $E^-(B)$, respectively. Furthermore, $E^0(B)$ and $E^-(B)$ are finite dimensional subspaces. Moreover, $i(B)$ =dim $E^-(B)$, $\nu(B)$ =dim $E^0(B)$.*

*(ii) $i(B) = \Sigma_{\lambda<0}\nu(B + \lambda)$ and $i(B)$ is the Morse index of $q_B$ on $E$; $\nu(B)$ =dim ker$(A - B)$.*

*(iii) For any $B_1$, $B_2 \in \Upsilon$ with $B_1 < B_2$, we have*

$$i(B_2) - i(B_1) = \sum_{\lambda\in[0,1)} \nu(B_1 + \lambda(B_2 - B_1)).$$

*(iv) There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, we have*

$$\begin{aligned}
&\nu(B + \epsilon) = 0 = \nu(B - \epsilon),\\
&i(B - \epsilon) = i(B),\\
&i(B + \epsilon) = i(B) + \nu(B).
\end{aligned}$$

*(v) $(-q_B(u, u))^{\frac{1}{2}}$ is an equivalent norm on $E^-(B)$ and there exists $c > 0$ such that $(q_B(u, u))^{\frac{1}{2}} \geq c\|u\|^2$, $\forall u \in E^+(B)$.*

## 3. Proof of main result

In order to prove Theorem 1.1, we use the symmetric Mountain-Pass Theorem (see [1,24]). Recall that $(u_n) \in E$ is a Palais-Smale ((PS) for short) sequence of $\Phi$ if $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$. $\Phi$ is said to satisfy the (PS)-condition if any such sequence contains a convergent subsequence.

**Theorem 3.1.** *Let $\Phi \in C^1(E, \mathbb{R})$ be an even functional on a Banach space $E$. Assume $\Phi(0) = 0$ and $\Phi$ satisfies the (PS)-condition. Suppose that*

*($\Phi_1$) there exists $E_1 \subset X$, $dimE_1 = k_1$ and $\rho > 0$ such that*

$$\sup_{u \in E_1 \cap S_\rho} \Phi(u) < 0;$$

*($\Phi_2$) there exists $E_2 \subset X$, $codimE_2 = k_2 < k_1$ such that*

$$\inf_{u \in E_2} \Phi(u) > -\infty.$$

*Then $\Phi$ has at least $k_1 - k_2$ pairs of critical points with negative critical values.*

**Lemma 3.1.** *Any (PS)-sequence of the functional $I_b$ defined as in (2.1) is bounded.*

*Proof.* Let $(u_n) \in E$ be such that $I_b(u_n) \to c$ and $I_b'(u_n) \to 0$. To prove $\{u_n\}$ is bounded, we develop a contradiction argument. We assume that, up to a subsequence, $\|u_n\| \to \infty$, and set $w_n = \frac{u_n}{\|u_n\|}$. Assume that

$$\begin{aligned} &w_n \rightharpoonup w, \text{ in } E, \;\; w_n(x) \to w(x) \text{ a.e. in } \mathbb{R}^N \\ &w_n^+ \rightharpoonup w^+, \text{ in } E^+, \;\; w_n^- \to w^-, \text{ in } E^-. \end{aligned}$$

We first claim that $w \neq 0$. Assume on the contrary that $w = 0$. It follows that

$$\begin{aligned} o(1) \quad &= \frac{(I_b'(u_n), u_n)}{\|u_n\|^2} \\ &= \|w_n^+\|^2 - \|w_n^-\|^2 - \int_{\mathbb{R}^N} \frac{f(x,u_n)}{u_n} w_n^2 dx + \frac{b(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx)^2}{\|u_n\|^2} \\ &\geq o(1) + \|w_n^+\|^2 - \int_{\mathbb{R}^N} \frac{f(x,u_n)}{u_n} w_n^2 dx. \end{aligned} \tag{3.1}$$

Here we use the fact that $(\int_{\mathbb{R}^N} |\nabla u|^2 dx)^2 \geq 0$. Let $\{F_\lambda\}$ denote the spectral family of $A$. We define the following projections:

$$P_1 = \int_{M-\epsilon}^{+\infty} dF_\lambda, \;\; P_2 = \int_{-\infty}^{M-\epsilon} dF_\lambda \tag{3.2}$$

with $\epsilon < M - f^*$, where $f^*$ is defined as in ($f_2$) and $M$ is defined as in ($V_2$). We have $P_1 w_n^+ = P_1 w_n$ and $P_2 E^+ = \int_0^{M-\epsilon} dF_\lambda E$. In particular, $P_2 E$ and $P_2 E^+$ are finite dimensional subspaces, and

$$\|w_n^+\| = \|P_1 w_n\| + o(1), \;\; \|P_1 w_n\|_2^2 \leq \frac{1}{M-\epsilon} \|P_1 w_n\|^2. \tag{3.3}$$

Thus, by (3.1) and (3.3),

$$\begin{aligned} o(1) \quad &\geq o(1) + \|P_1 w_n\|^2 - f^* \int_{\mathbb{R}^N} w_n^2 dx \\ &\geq o(1) + (1 - \frac{f^*}{M-\epsilon}) \|P_1 w_n\|^2. \end{aligned}$$

The above inequality implies that $\|P_1 w_n\| \to 0$ in $E$. Hence $1 = \|w_n\|^2 = \|P_1 w_n\|^2 + \|P_2 w_n\|^2 \to 0$, which is a contradiction.

On the other hand, if $w \neq 0$,

$$\begin{aligned} o(1) &= \frac{(I_b'(u_n), u_n)}{\|u_n\|^4} \\ &= \frac{\|u_n^+\|^2 - \|u_n^-\|^2}{\|u_n\|^4} - \frac{\int_{\mathbb{R}^N} \frac{f(x,u_n)}{u_n} w_n^2 dx}{\|u_n\|^2} + \frac{b(\int_{\mathbb{R}^N} |\nabla w_n|^2 dx)^2}{\|u_n\|^2}. \end{aligned}$$

Recall that $\frac{f(x,z)}{z}$ is bounded for all $x$ and $z$, one has

$$\frac{\int_{\mathbb{R}^N} \frac{f(x,u_n)}{u_n} w_n^2 dx}{\|u_n\|^2} = o(1).$$

Thus,

$$o(1) = b(\int_{\mathbb{R}^N} |\nabla w_n|^2 dx)^2.$$

Set $g(z) = \int_{\mathbb{R}^N} |\nabla z|^2 dx$. It is easy to see that $g(z)$ is lower semi-continuous. Consequently, $\liminf_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla w_n|^2 dx)^2 \geq (\int_{\mathbb{R}^N} |\nabla w|^2 dx)^2$, and $o(1) \geq (\int_{\mathbb{R}^N} |\nabla w|^2 dx)^2$. Thus,

$$\int_{\mathbb{R}^N} |\nabla w|^2 dx = 0.$$

Since $E$ embeds continuously into $D^{1,2}(\mathbb{R}^N)$, we have $w \in D^{1,2}(\mathbb{R}^N)$ and $w = 0$. This is a contradiction. Hence, $\{u_n\}$ is bounded. □

**Lemma 3.2.** *Any (PS)-sequence has a convergent subsequence.*

*Proof.* Assume that $u_n \rightharpoonup u$ in $E$ and let $v_n = u_n - u$. Then, up to a subsequence, $v_n \rightharpoonup 0$ in $E$. Then $v_n^+ \rightharpoonup 0$, $v_n^- \to 0$ in $E$ and

$$\begin{aligned} o(1) &= (I_b'(u_n), v_n) \\ &= (u_n^+, v_n^+) - (u_n^-, v_n^-) - \int_{\mathbb{R}^N} f(x, u_n) v_n dx + b \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \int_{\mathbb{R}^N} \nabla u_n \nabla v_n dx \\ &\geq o(1) + \|v_n^+\|^2 - f^* \int_{\mathbb{R}^N} v_n^2 dx + b \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \int_{\mathbb{R}^N} \nabla u \nabla v_n dx. \end{aligned}$$

Since $E$ embeds continuously into $D^{1,2}(\mathbb{R}^N)$, we have $h_u(v) = \int_{\mathbb{R}^N} \nabla u \nabla v dx$ is continuous on $v \in E$. Moreover, since $v_n \rightharpoonup 0$, we have $h_u(v_n) \to 0$. Thus,

$$o(1) \geq \|v_n^+\|^2 - f^* \int_{\mathbb{R}^N} v_n^2 dx.$$

Then the lemma follows from the procedure as in Lemma 3.1. More precisely, using the definitions and properties of $P_1$, $P_2$ as in (3.2), we deduce that

$$o(1) \geq o(1) + (1 - \frac{f^*}{M - \epsilon}) \|P_1 v_n\|^2, \tag{3.4}$$

which implies $P_1 v_n \to 0$ in $E$ and $v_n = P_1 v_n + P_2 v_n \to 0$ in $E$. This completes the proof. □

Set $f(x,u) = B_0(x)u + f_1(x,u)$. By condition $(f_0)$, we have $f_1(x,u) = o(u)$ as $|u| \to 0$ uniformly in $x$. Set $F_1(x,u) = \int_0^1 f_1(x,\theta u)d\theta u$. Fix any $2 < p < 2^*$. For any $\epsilon > 0$, there is a $C_\epsilon > 0$ such that

$$|f_1(x,u)| \leq \epsilon|u| + C_\epsilon|u|^{p-1},$$

which implies that

$$|F_1(x,u)| \leq \frac{\epsilon}{2}|u|^2 + \frac{C_\epsilon}{p}|u|^p,$$

and therefore

$$|\int_{\mathbb{R}^N} F_1(x,u)dx| \leq \frac{\epsilon}{2}\|u\|_2^2 + \frac{C_\epsilon}{p}\|u\|_p^p.$$

By the continuity of the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$, there exists a positive constant $C_\epsilon^*$ which depends on $N$ and $p$, such that

$$|\int_{\mathbb{R}^N} F_1(x,u)dx| \leq \frac{\epsilon}{2}\|u\|_2^2 + C_\epsilon^*\|u\|^p.$$

**Lemma 3.3.** *For any $b > 0$, there exists a $\rho > 0$ and $E_1$ with $dimE_1 = i(B_0)$ such that*

$$\sup_{u\in E_1\cap S_\rho} I_b(u) < 0.$$

*Proof.* Since $E$ embeds into $D^{1,2}(\mathbb{R}^N)$ continuously, there exists $C_1 > 0$ such that $\|\nabla u\|_2^2 \leq C_1\|u\|^2$ and

$$\begin{aligned} I_b(u) \quad & \leq \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \frac{1}{2}(B_0(x)u,u)_2 + \frac{\epsilon}{2}\|u\|_2^2 + C_\epsilon^*\|u\|^p \\ & +\frac{b}{4}(\int_{\mathbb{R}^N} |\nabla u|^2dx)^2 \\ & \leq \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \frac{1}{2}((B_0-\epsilon)u,u)_{L^2} + C_\epsilon^*\|u\|^p + \frac{b}{4}C_1^2\|u\|^4 \\ & = q_{B_0+\epsilon}(u,u) + C_\epsilon^*\|u\|^p + \frac{b}{4}C_1^2\|u\|^4. \end{aligned} \tag{3.5}$$

Pick $E_1 = E^-(B_0-\epsilon)$. For any $u \in E_1$, $\sqrt{-q_{B_0-\epsilon}(u,u)}$ is an equivalence norm on $E_1$, and thus there exists a constant $c_2$ such that

$$q_{B_0-\epsilon}(u,u) \leq -c_2\|u\|^2, \quad \forall u \in E_1. \tag{3.6}$$

Hence, we have

$$\begin{aligned} I_b(u) \quad & \leq -c_2\|u\|^2 + C_\epsilon^*\|u\|^p + \frac{b}{4}C_1^2\|u\|^4 \\ & = (-c_2 + C_\epsilon^*\|u\|^{p-2} + \frac{b}{4}C_1^2\|u\|^2)\|u\|^2. \end{aligned}$$

Moreover, for $\epsilon$ small enough, we have $\dim E_1 = i(B_0-\epsilon) = i(B_0)$. Thus, this lemma follows by $\rho < (\frac{c_2}{C_\epsilon^* + \frac{b}{4}C_1^2})^{\frac{1}{\max(p-2,2)}}$. □

**Lemma 3.4.** *Assume $E_2 = E^+(B_\infty)$. There exists a $R > 0$, such that for any $b > 0$,*

$$\inf_{u\in E_2,\ \|u\|\geq R} I_b(u) > 0.$$

*Proof.* It is sufficient to show that for any $b$, $I_b(u) \to \infty$ as $\|u\| \to \infty$. Arguing indirecting, we assume that for some sequence $(x_n) \in E_2$, with $\|x_n\| \to \infty$, there is $\gamma > 0$ such that $I_b(x_n) \leq \gamma$ for all $b > 0$. Setting $y_n = \frac{x_n}{\|x_n\|}$, we have $\|y_n\| = 1$, $y_n \rightharpoonup y$ in $E_2$, $y_n^+ \rightharpoonup y^+$ in $E_2$, $y_n^- \to y^-$ in $E_2$, and

$$\begin{aligned}\frac{\gamma}{\|x_n\|^2} \geq \frac{I_b(x_n)}{\|x_n\|^2} \quad &= \|y_n^+\|^2 - \|y_n^-\|^2 - \int_{\mathbb{R}^N} \frac{F(x,x_n)}{\|x_n\|^2}dx \\ &+\frac{b(\int_{\mathbb{R}^N} |\nabla x_n|^2dx)^2}{\|x_n\|^2} \\ &\geq \|y_n^+\|^2 - \|y_n^-\|^2 - \int_{\mathbb{R}^N} \frac{F(x,x_n)}{\|x_n\|^2}dx. \end{aligned} \tag{3.7}$$

Here we use the fact that $(\int_{\mathbb{R}^N} |\nabla x_n|^2dx)^2 \geq 0$.

We claim that $y^- \neq 0$. In fact, if not we assume $y^- = 0$. Since $E^-$ is a finite dimensional subspace, we obtain $\|y_n^-\|^2 = o(1)$. Since $F \leq 0$, from (3.7), we deduce that $y_n^+ \to 0$, and $1 = \|y_n\|^2 = \|y_n^+\|^2 + \|y_n^-\|^2 \to 0$ in $E$. This is a contradiction and implies that $y^- \neq 0$. Moreover, $y \neq 0$ and

$$q_{B_\infty}(y,y) > 0.$$

There exists $r > 0$ such that

$$\|y^+\|^2 - \|y^-\|^2 - \int_{B_r(0)} B_\infty(x)y^2dx > 0, \tag{3.8}$$

where $B_r(0) = \{x \in \mathbb{R}^N : |x| \leq r\}$. Assume that $f_2(x,u) = f(x,u) - B_\infty(x)u$. Then $f_2(x,u) = o(|u|)$ as $|u| \to \infty$ uniformly in $x$. Set $F_2(x,u) = \int_0^1 f_2(x,\theta u)ud\theta$. Note that $y_n \to y$ in $L^2(B_r(0))$. It follows that

$$\begin{aligned}|\int_{B_r(0)} \frac{F_2(x,x_n)}{\|x_n\|^2}dx| \quad &\leq \int_{B_r(0)} \frac{|F_2(x,x_n)|}{x_n^2}y_n^2dx \\ &\leq \int_{B_r(0)} \frac{|F_2(x,x_n)|}{x_n^2}|y_n - y|^2dx + \int_{B_r(0)} \frac{|F_2(x,x_n)|}{x_n^2}y^2dx \\ &= o(1). \end{aligned} \tag{3.9}$$

Thus, from (3.7)–(3.9),

$$\begin{aligned} o(1) \quad &\geq \lim_{n\to\infty}(\|y_n^+\|^2 - \|y_n^-\|^2 - \int_{B_r(0)} \frac{F(x,x_n)}{\|x_n\|^2}dx) \\ &\geq \|y^+\|^2 - \|y^-\|^2 - \int_{B_r(0)} B_\infty(x)y^2dx > 0. \end{aligned}$$

This is a contradiction. We complete the proof. □

**Proof of Theorem 1.1.** $I_b$ is even provided $f(x,u)$ is odd in $u$. With $E_1 = E^-(B_0 - \epsilon)$ and $E_2 = E^+(B_\infty)$ the condition $(\Phi_1)$ of Theorem 3.1 holds by Lemma 3.3 and $(\Phi_2)$ of Theorem 3.1 holds by Lemma 3.4. Moreover, $\dim E_1 = i(B_0)$ and $\operatorname{codim} E_2 = i(B_\infty) + \nu(B_\infty)$. Lemma 3.1 and Lemma 3.2 imply $I_b$ satisfies the (PS)-condition. Therefore, $I_b$ has at least $i(B_0) - i(B_\infty) - \nu(B_\infty)$ pairs of nontrivial critical points by Theorem 3.1.

## 4. Conclusions

This manuscript has employed the minimax method to study the existence and multiplicity of solutions of Schrödinger-Kirchhoff equations with asymptotically linear nonlinearities. By using the Morse index of the reduced linear Schrödinger equation, we show how the behavior of the nonlinearity at origin and at infinity affects the number of solutions.

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## Conflict of interest

There is no conflict of interest.

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