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Research article

Oscillation theorems for higher order dynamic equations with superlinear neutral term

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Abstract: In this paper, several oscillation criteria for a class of higher order dynamic equations with superlinear neutral term are established. The proposed results provide a unified platform that adequately covers both discrete and continuous equations and further sufficiently comments on oscillatory behavior of more general class of equations than the ones reported in the literature. We conclude the paper by demonstrating illustrative examples.

Keywords: oscillation criteria; superlinear neutral term; higher order dynamic equations
Mathematics Subject Classification: 34N05, 39A10

1. Introduction

In this paper, we establish new oscillation criteria for a class of higher order dynamic equations with superlinear neutral term of the form

$$(a(\mu)y^{\Delta^{n-1}}(\mu))^{\Delta} + q(\mu)x^{\beta}(\tau(\mu)) = 0, \tag{1.1}$$

where $y(\mu) := x(\mu) + p(\mu)x^{\alpha}(\delta(\mu))$ on an arbitrary time scale $\mathbb{T} \subseteq \mathbb{R}$ and under the assumptions:

(i) a, p and $q \in C_{rd}([\mu_0, \infty)_{\mathbb{T}}, (0, \infty))$ and $a^{\Delta}(\mu) \geq 0$ for $\mu \in [t_0, \infty)_{\mathbb{T}}$,

(ii) α, β are the ratios of positive odd integers with $\beta \leq \alpha$ and $\alpha \geq 1$,

(iii) $\tau, \delta \in C_{rd}(\mathbb{T}_0, \mathbb{T})$ such that $x(\mu) = \delta^{-1}(\tau(\mu)) \leq \mu$, $\delta(\mu) \geq \mu$, δ is nondecreasing and invertible and x is nondecreasing with $\lim_{\mu\to\infty} x(\mu) = \infty$.

Assume that

$$A(v, u) = \int_u^v \frac{1}{a(s)} \Delta s, \;\; v \geq u \geq \mu_0,$$

and we also assume that

$$\lim_{\mu\to\infty} A(\mu,\mu_0) < \infty. \tag{1.2}$$

Define the time scale interval to be $[\mu_0,\infty)_{\mathbb{T}} := [\mu_0,\infty)\bigcap\mathbb{T}$. Since we are interested in the oscillatory behavior of solutions of (1.1), we assume that $\sup\mathbb{T}=\infty$. Recall that a solution of (1.1) is a nontrivial real-valued function x satisfying Eq (1.1) for $\mu \geq \mu_0$. We exclude from our consideration all solutions vanishing in some neighborhood of infinity. A solution x of (1.1) is called *oscillatory* if it is neither eventually positive nor eventually negative, otherwise it is called *nonoscillatory*.

The theory of dynamic equations has been introduced to unify difference and differential equations; see, for instance [1]. Meanwhile, different theoretical aspects of this theory have been discussed in the last years. Particularly, the oscillation of dynamic equations has been the target of many researchers who succeeded in reporting relevant results. Exploring the literature, however, one can observe that most of the obtained results for the oscillation of dynamic equations have been carried out using the comparison, integral averaging, and Riccati transformation techniques [2–12]. Based on authors' observation, never the less, there are no known results regarding the oscillation of higher order dynamic equations with nonlinear (superlinear/sublinear) neutral term and advanced argument. The reader can consult relevant results for difference and differential equations in [13, 14], and the references cited therein. More precisely, the existing literature does not provide any criteria for the oscillation of Eq (1.1). Motivated by this inspiration, we consider new sufficient conditions that ensure that all solutions of (1.1) are either oscillatory or converge to zero. Eq (1.1) is commonly used in a variety of applied problems. We state, in particular, the use of Eq (1.1) in the study of non-Newtonian fluid theory and the turbulent flow of a polytrophic gas in a porous medium; see the papers [15, 16] for further details.

2. Main results

As our results will be based on Taylor monomials, we give the following definition.

Definition 2.1. [1] Taylor monomials are the functions $h_n : \mathbb{T}\times\mathbb{T}\to\mathbb{R}$, $n\in\mathbb{N}_0=\{0,1,2,\ldots\}$ which are recursively defined as

$$h_0(\mu,s)\equiv 1,\quad \mu,s\in\mathbb{T},$$

and for $n\in\mathbb{N}_0$

$$h_{n+1}(\mu,s)=\int_s^{\mu} h_n(\tau,s)\triangle\tau,\quad \mu,s\in\mathbb{T}.$$

It follows that $h_1(\mu,s)=\mu-s$ on any time scale.

One should observe that finding h_n for $n\geq 2$ is not an easy task in general. For a particular time scale such as $\mathbb{T}=\mathbb{R}$ or $\mathbb{T}=\mathbb{Z}$, we can easily find the functions h_n. Indeed, we have

$$h_n(\mu,s)=\frac{(\mu-s)^n}{n!},\quad \mu,s\in\mathbb{R}\quad\text{and}\quad h_n(\mu,s)=\frac{(\mu-s)^{\overline{n}}}{n!},\quad \mu,s\in\mathbb{Z}, \tag{2.1}$$

where $\mu^{\overline{n}} := \mu(\mu+1)\ldots(\mu+n-1)$.

Lemma 2.2. (Kneser's Theorem) [1, Theorem 5] *Let* $\sup\mathbb{T}=\infty$, $n\in\mathbb{N}$ *and* $x\in C^n_{rd}(\mathbb{T},\mathbb{R}^+)$. *Suppose that* $x^{\triangle^n}(\mu)\neq 0$ *is either nonnegative or nonpositive on* $\mathbb{T}$. *Then there exists* $m\in[0,n)_{\mathbb{Z}}$ *such that* $(-1)^{n-m}x^{\triangle^n}(\mu)\geq 0$ *holds for all sufficiently large* μ. *Moreover, both of the following conditions hold:*

(i) $0 \le k < m$ *implies* $x^{\triangle^k}(\mu) > 0$, *for all sufficiently large* μ.
(ii) $m \le k < n$ *implies* $(-1)^{m+k} x^{\triangle^k}(\mu) > 0$, *for all sufficiently large* μ.

In what follows, we provide the following lemma which plays a crucial role in the sequel.

Lemma 2.3. [1] *Let* $\sup \mathbb{T} = \infty$ *and* $x \in C^n_{rd}(\mathbb{T}, \mathbb{R}^+)$, $n \ge 2$. *Moreover, suppose that Kneser's Theorem holds with* $m \in [0, n)_{\mathbb{Z}}$ *and* $x^{\triangle^n}(\mu) \le 0$ *on* $\mathbb{T}$. *Then, there exists a sufficiently large* $\mu_1 \in \mathbb{T}$ *such that*

$$x^{\triangle}(\mu) \ge h_{m-1}(\mu, \mu_1) x^{\triangle^m}(\mu), \quad \textit{for all} \ \ \mu \in [\mu_1, \infty)_{\mathbb{T}}.$$

If $m = n - 1$ *in the above inequality, then upon integration it becomes*

$$x(\mu) \ge h_{n-1}(\mu, \mu_1) x^{\triangle^{n-1}}(\mu), \quad \textit{for all} \ \ \mu \in [\mu_1, \infty)_{\mathbb{T}}.$$

For convenience, we let

$$B(\mu) = \frac{1}{p(\delta^{-1}(\mu))}\left(1 - \frac{c}{p^{\frac{1}{\alpha}}(\delta^{-1}(\delta^{-1}(\mu)))}\right) \ge 0,$$

for any constant $c > 0$ and $\mu \ge \mu_1$, $\mu_1 \in [\mu_0, \infty)_{\mathbb{T}}$. We introduce the notations

$$Q_1(\mu) = q(\mu)\Big(h^{\frac{\beta}{\alpha}}_{n-1}(x(\mu), \mu_1) B^{\frac{\beta}{\alpha}}(\tau(\mu)) a^{-\frac{\beta}{\alpha}}(x(\mu))\Big) \tag{2.2}$$

and

$$Q_2(\mu) = \frac{1}{a(\mu)} \int_{\mu_1}^{\mu} \Big(q(s) h^{\frac{\beta}{\alpha}}_{n-2}(x(s), \mu_1) B^{\frac{\beta}{\alpha}}(\tau(s))\Big) \triangle s \tag{2.3}$$

for $\mu \ge \mu_1$, $\mu_1 \in [\mu_0, \infty)_{\mathbb{T}}$.

Theorem 2.4. *Assume that conditions (i)–(iii) and (1.2) hold. If*

$$\limsup_{\mu \to \infty} \int_{x(\mu)}^{\mu} Q_1(s) \triangle s = \infty \tag{2.4}$$

and there exists $\mu_1 \in [\mu_0, \infty)_{\mathbb{T}}$ *such that*

$$\limsup_{\mu \to \infty} \int_{x(\mu)}^{\mu} Q_2(s) \triangle s = \infty, \tag{2.5}$$

then every solution of Eq (1.1) is either oscillatory or converges to zero.

Proof. Let x be a nonoscillatory solution of Eq (1.1), say $x(\mu) > 0$ with $\lim_{\mu \to \infty} x(\mu) \ne 0$, $x(\tau(\mu)) > 0$ and $x(\delta(\mu)) > 0$ for $\mu \ge \mu_1$ and $\mu_1 \in [\mu_0, \infty)_{\mathbb{T}}$. Then Eq (1.1) implies that

$$(a(\mu) y^{\triangle^{n-1}}(\mu))^{\triangle} = -q(\mu) x^{\beta}(\tau(\mu)) < 0. \tag{2.6}$$

Hence $a(\mu) y^{\triangle^{n-1}}(\mu)$ is nonincreasing and of one sign. That is, there exists $\mu_2 \ge \mu_1$ such that

(I) $y^{\triangle^{n-1}}(\mu) > 0$, and $(a(\mu) y^{\triangle^{n-1}}(\mu))^{\triangle} < 0$, for $\mu \ge \mu_2$,
or

(II) $y^{\Delta^{n-1}}(\mu) < 0, \quad$ and $(a(\mu)y^{\Delta^{n-1}}(\mu))^{\Delta} < 0, \quad$ for $\mu \geq \mu_2$.

We observe that the above cases hold for all n, while the subcases including $y^{\Delta}(\mu) \leq 0$ on $[\mu_1, \infty)$ are excluded due to the condition $\lim_{\mu\to\infty} x(\mu) > 0$.

For case I: Here, we have $y^{\Delta}(\mu) > 0$ for $\mu \geq \mu_2$. There exist positive constants c_1 and $\mu_3 \geq \mu_2$ such that $y(\delta^{-1}(\mu)) \geq c_1$ and $0 < y^{\frac{1}{\alpha}-1}(\delta^{-1}(\mu)) \leq c_1^{\frac{1}{\alpha}-1} := c$.

Clearly, we get $y(\mu) \geq x(\mu)$ and

$$x^{\alpha}(\delta(\mu)) = \frac{1}{p(\mu)}(y(\mu) - x(\mu)) \leq \frac{y(\mu)}{p(\mu)}.$$

It is easy to see that

$$x^{\alpha}(\delta^{-1}(\mu)) \leq \frac{1}{p^{\frac{1}{\alpha}}(\delta^{-1}(\delta^{-1}(\mu)))} y^{\frac{1}{\alpha}}(\delta^{-1}(\delta^{-1}(\mu)))$$

and

$$\begin{aligned} x^{\alpha}(\mu) &= \frac{1}{p(\delta^{-1}(\mu))}\Big(y(\delta^{-1}(\mu)) - x(\delta^{-1}(\mu))\Big) \\ &\geq \frac{1}{p(\delta^{-1}(\mu))}\Big(y(\delta^{-1}(\mu)) - \frac{1}{p^{\frac{1}{\alpha}}(\delta^{-1}(\delta^{-1}(\mu)))} y^{\frac{1}{\alpha}}(\delta^{-1}(\delta^{-1}(\mu)))\Big). \end{aligned}$$

Using the fact that y is a nondecreasing, $\delta(\mu) \geq \mu$ and the fact that $\delta^{-1}(\delta^{-1}(\mu)) \leq \delta^{-1}(\mu)$, we have $y^{\frac{1}{\alpha}}(\delta^{-1}(\delta^{-1}(\mu))) \leq y^{\frac{1}{\alpha}}(\delta^{-1}(\mu))$. Therefore, we get

$$\begin{aligned} x^{\alpha}(\mu) &\geq \frac{1}{p(\delta^{-1}(\mu))}\Big(1 - \frac{y^{\frac{1}{\alpha}-1}(\delta^{-1}(\mu))}{p^{\frac{1}{\alpha}}(\delta^{-1}(\delta^{-1}(\mu)))}\Big) y(\delta^{-1}(\mu)) \\ &\geq \frac{1}{p(\delta^{-1}(\mu))}\Big(1 - \frac{c}{p^{\frac{1}{\alpha}}(\delta^{-1}(\delta^{-1}(\mu)))}\Big) y(\delta^{-1}(\mu)) \\ &\geq B(\mu) y(\delta^{-1}(\mu)), \end{aligned}$$

or

$$x^{\alpha}(\tau(\mu)) \geq B(\tau(\mu)) y(\delta^{-1}(\tau(\mu))) = B(\tau(\mu)) y(x(\mu)). \tag{2.7}$$

Using (2.7) in (2.6), we get

$$(a(\mu)y^{\Delta^{n-1}}(\mu))^{\Delta} \leq -q(\mu) B^{\frac{\beta}{\alpha}}(\tau(\mu)) y^{\frac{\beta}{\alpha}}(x(\mu)). \tag{2.8}$$

By Lemma 2.3, we find

$$y(\mu) \geq h_{n-1}(\mu, \mu_1) y^{\Delta^{n-1}}(\mu), \quad \text{for } \mu \geq \mu_1. \tag{2.9}$$

Using (2.9) in (2.8), we get

$$(a(\mu)y^{\Delta^{n-1}}(\mu))^{\Delta} \leq -q(\mu) h_{n-1}^{\frac{\beta}{\alpha}}(x(\mu), \mu_1) B^{\frac{\beta}{\alpha}}(\tau(\mu)) \Big(y^{\Delta^{n-1}}(x(\mu))\Big)^{\frac{\beta}{\alpha}}, \tag{2.10}$$

or

$$Y^{\Delta}(\mu) + Q_1(\mu) Y^{\frac{\beta}{\alpha}}(x(\mu)) \leq 0, \quad \text{for } \mu \geq \mu_3, \tag{2.11}$$

where $Y(\mu) = a(\mu)y^{\Delta^{n-1}}(\mu)$ and Q_1 is defined in (2.2). Integrating (2.11) from $x(\mu)$ to μ, and using that $x(\mu)$, $a(\mu)$ and $y^{\Delta^{n-1}}(\mu)$ are nondecreasing, we have

$$Y(x(\mu)) \geq -Y(\mu) + Y(x(\mu)) \geq \Big[\int_{x(\mu)}^{\mu} Q_1(s)\Delta s\Big] Y^{\frac{\beta}{\alpha}}(x(\mu)),$$

or

$$Y^{1-\frac{\beta}{\alpha}}(x(\mu)) \geq \int_{x(\mu)}^{\mu} Q_1(s)\Delta s, \quad for \ \ \mu \geq \mu_3.$$

Taking limsup of both sides as $\mu \to \infty$, we arrive at the desired contradiction.

For case II: Here, we see that

$$y(\mu) > 0,\ y^{\Delta}(\mu) > 0, \dots, y^{\Delta^{n-2}}(\mu) > 0,\ y^{\Delta^{n-1}}(\mu) < 0 \text{ and } \Big(a(\mu)y^{\Delta^{n-1}}(\mu)\Big)^{\Delta} < 0,\ \text{for } \mu \geq \mu_2.$$

Proceeding as in the above case, we obtain (2.8) and by Lemma 2.3 with $m = n-2$, we see that

$$y(\mu) \geq h_{n-2}(\mu,\mu_1)y^{\Delta^{n-2}}(\mu),\ \text{for } \mu \geq \mu_3. \tag{2.12}$$

Using (2.12) in (2.8), we have

$$(a(\mu)y^{\Delta^{n-1}}(\mu))^{\Delta} \leq -q(\mu)h_{n-2}^{\frac{\beta}{\alpha}}(x(\mu),\mu_1)B^{\frac{\beta}{\alpha}}(\tau(\mu))\Big(y^{\Delta^{n-2}}(x(\mu))\Big)^{\frac{\beta}{\alpha}},\ \text{ for } \ \mu \geq \mu_3. \tag{2.13}$$

Integrating this inequality from μ_3 to μ, we have

$$-(a(\mu)y^{\Delta^{n-1}}(\mu)) \geq \int_{\mu_3}^{\mu} \Big(q(s)h_{n-2}^{\frac{\beta}{\alpha}}(x(s),\mu_1)B^{\frac{\beta}{\alpha}}(\tau(s))\Big)\Big(y^{\Delta^{n-2}}(x(s))\Big)^{\frac{\beta}{\alpha}}\Delta s, \quad for\ \mu \geq \mu_3,$$

or

$$-y^{\Delta^{n-1}}(\mu) \geq Q_2(\mu)\Big(y^{\Delta^{n-2}}(x(\mu))\Big)^{\frac{\beta}{\alpha}}, \quad \mu \geq \mu_3,$$

where Q_2 is defined in (2.3). It follows that

$$Z^{\Delta}(\mu) + Q_2(\mu)Z^{\frac{\beta}{\alpha}}(x(\mu)) \leq 0,$$

where $Z(\mu) = y^{\Delta^{n-2}}(\mu)$. The remaining part of the proof is similar to that of the above case and hence is omitted. □

Define

$$Q(\mu) = \min\{Q_1(\mu), Q_2(\mu)\}.$$

By virtue of the above theorem, we conclude the following immediate consequence.

Corollary 2.5. *Assume that conditions (i)–(iii) and (1.2) are satisfied. If*

$$\limsup_{\mu\to\infty} \int_{x(\mu)}^{\mu} Q(s)\Delta s = \infty, \tag{2.14}$$

then every solution of Eq (1.1) is either oscillatory or converges to zero.

Remark 2.6. For $\mathbb{T}=\mathbb{R}$, conditions (2.4) and (2.5) become:

$$\limsup_{\mu\to\infty}\int_{x(\mu)}^{\mu} q(s)\Big(x^{(n-1)\frac{\beta}{\alpha}}(s)B^{\frac{\beta}{\alpha}}(\tau(s))a^{-\frac{\beta}{\alpha}}(x(s))\Big)ds=\infty, \tag{2.15}$$

and

$$\limsup_{\mu\to\infty}\int_{x(\mu)}^{\mu}\frac{1}{a(u)}\int_{\mu_1}^{u}\Big(q(s)x^{(n-2)\frac{\beta}{\alpha}}(s)B^{\frac{\beta}{\alpha}}(\tau(s))\Big)dsdu=\infty. \tag{2.16}$$

For the case when $\mathbb{T}=\mathbb{Z}$, on the other hand, Conditions (2.4) and (2.5) reduce to

$$\limsup_{\mu\to\infty}\sum_{s=x(\mu)}^{\mu-1} q(s)\Big(x^{(n-1)\frac{\beta}{\alpha}}(s)B^{\frac{\beta}{\alpha}}(\tau(s))a^{-\frac{\beta}{\alpha}}(x(s))\Big)ds=\infty, \tag{2.17}$$

and

$$\limsup_{\mu\to\infty}\sum_{u=x(\mu)}^{\mu-1}\frac{1}{a(u)}\sum_{s=\mu_1}^{u-1}q(s)x^{(n-2)\frac{\beta}{\alpha}}(s)B^{\frac{\beta}{\alpha}}(\tau(s))=\infty, \tag{2.18}$$

for $\mu\geq\mu_1,\ \ \mu_1\in[\mu_0,\infty)_{\mathbb{T}}$..

The following example is illustrative.

Example 2.7. Consider the following differential equations

$$\Big(\mu^6 y^{(n-1)}(\mu)\Big)'+\mu^{-n-7}x^3(\tau(\mu))=0 \tag{2.19}$$

and

$$\Big(e^{\mu}y^{(n-1)}(\mu)\Big)'+\frac{e^{\mu}}{\mu^{n-3}}x^3(\tau(\mu))=0, \tag{2.20}$$

where $y(\mu):=x(\mu)+\mu x^3(2\mu)$. We consider the cases $\tau(\mu)=\frac{3}{2}\mu$, $\tau(\mu)=\mu$ or $\tau(\mu)=\frac{2}{3}\mu$. Since $\delta(\mu)=2\mu$, we have $x(\mu)=\frac{3}{4}\mu$, $x(\mu)=\frac{1}{2}\mu$ or $x(\mu)=\frac{1}{3}\mu$ respectively. Clearly, $B(\mu)=\frac{2(\mu^{\frac{1}{3}}-4^{\frac{1}{3}})}{\mu^{\frac{4}{3}}}\geq 0$ for $\mu\geq 4$. One can easily prove that conditions (2.15) and (2.16) are satisfied and hence we conclude that every solution of Eq (2.19) or (2.20) is either oscillatory or tends to zero.

Remark 2.8. Results reported in the literature cannot comment on the oscillatory behavior of Eqs (2.19) and (2.20).

Remark 2.9. One may observe that the selection of different values of the argument $\tau(\mu)$ implies that the current results work for all types of equations, that is, for equations with delay, ordinary or advanced arguments.

For the case

$$\lim_{\mu\to\infty}A(\mu,\mu_0)=\infty \tag{2.21}$$

we can derive the following immediate result.

Theorem 2.10. *Let $n\geq 2$ and condition (2.21) be satisfied. Then, the following statements hold true:*

(i) If condition (2.4) holds, then every solution of even order Eq (1.1) is oscillatory.

(ii) If condition (2.4) holds, then every solution of odd order Eq (1.1) is either oscillatory or converges to zero.

Theorem 2.11. *Let $n \geq 2$ and condition (2.21) hold. If $x(\mu) \geq \mu$ and*

$$\limsup_{\mu\to\infty}\left(\frac{h_{n-1}(\mu,\mu_1)}{a(\mu)}\Big(\int_{\mu}^{\infty} q(s)B^{\frac{\beta}{\alpha}}(\tau(s))\triangle s\Big)\right) = \infty, \tag{2.22}$$

for $\mu_1 \in [\mu_0,\infty)_{\mathbb{T}}$, then the conclusions of Theorem 2.10 hold.

Proof. Let x be a nonoscillatory solution of Eq (1.1), say $x(\mu) > 0$ with $\lim_{\mu\to\infty} x(\mu) \neq 0$, $x(\tau(\mu)) > 0$ and $x(\delta(\mu)) > 0$ for $\mu \geq \mu_1$ for $\mu_1 \in [\mu_0,\infty)_{\mathbb{T}}$. Proceeding as in the proof of Theorem 2.4 until we obtain (2.8). Integrating (2.8) from μ to u and letting $u \to \infty$, we have

$$y^{\triangle^{n-1}}(\mu) \geq \frac{1}{a(\mu)}\left(\int_{\mu}^{\infty} q(s)B^{\frac{\beta}{\alpha}}(\tau(s))\triangle s\right)y^{\frac{\beta}{\alpha}}(\mu). \tag{2.23}$$

By virtue of Lemma 2.3, we find

$$y(\mu) \geq h_{n-1}(\mu,\mu_1)y^{\triangle^{n-1}}(\mu), \quad \text{for } \mu \geq \mu_1. \tag{2.24}$$

Using (2.24) in (2.23), we get

$$y^{1-\frac{\beta}{\alpha}}(\mu) \geq h_{n-1}(\mu,\mu_1)y^{\triangle^{n-1}}(\mu) \geq h_{n-1}(\mu,\mu_1)\left(\frac{1}{a(\mu)}\Big(\int_{\mu}^{\infty} q(s)B^{\frac{\beta}{\alpha}}(\tau(s))\triangle s\Big)\right).$$

Taking limsup of both sides of this inequality as $\mu \to \infty$, we obtain a contradiction to (2.22). □

Example 2.12. Consider the dynamic equation

$$y^{\triangle^{n}}(\mu) + \frac{1}{\mu^n}x^3\left(\frac{\mu}{2}\right) = 0, \tag{2.25}$$

where $y(\mu) := x(\mu) + x^3(2\mu)$. It is easy to see that conditions of Theorem 2.11 are satisfied and hence we conclude that every solution of Eq (2.25) is either oscillatory or tends to zero.

Remark 2.13. The oscillatory behavior of Eq (2.25) cannot be addressed via results existing in the literature.

3. Conclusions

Unlike previous techniques, a new approach is employed to establish easily verifiable sufficient conditions for the oscillation of higher order dynamic equations with superlinear neutral term and advanced argument. The obtained results improve and complement some previous theorems in the literature. Examples are provided to support theoretical findings. The results in this paper are presented in an essentially new form and of a high degree of generality. For future consideration, it will be of great importance to study the oscillation of Eq (1.1) when $\beta > \alpha$.

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Conflict of interest

The authors declare that there is no competing interest concerning this work.

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