Subclass of analytic functions defined by $q$-derivative operator associated with Pascal distribution series

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Abstract: The purpose of the present paper is to find the necessary and sufficient condition and inclusion relation for Pascal distribution series to be in the subclass $\mathcal{T}C_q(\lambda, \alpha)$ of analytic functions defined by $q$-derivative operator. Further, we consider an integral operator related to Pascal distribution series, and several corollaries and consequences of the main results are also considered.

Keywords: analytic functions; Hadamard product; q-starlike functions; q-convex functions; Pascal distribution series

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1. Introduction and definitions

Let $\mathcal{A}$ denote the class of the normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ consisting of functions of the form,

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in \mathbb{U}. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}(A, B), \tau \in \mathbb{C}\setminus\{0\}, -1 \leq B < A \leq 1$, if it satisfies the inequality

$$\left|\frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]}\right| < 1, \quad z \in \mathbb{U}.$$
This class was introduced by Dixit and Pal [13].

The theory of $q$-calculus operators are used in describing and solving various problems in applied science such as ordinary fractional calculus, optimal control, $q$-difference and $q$-integral equations, as well as geometric function theory of complex analysis. The application of $q$-calculus was initiated by Jackson [23]. Recently, many researchers studied $q$-calculus such as Srivastava et al. [52], Muhammad and Darus [31], Kanas and Răducanu [28], Aldweby and Darus [2–4] and Muhammad and Sokol [30]. For details on $q$-calculus one can refer [1, 5–7, 9, 20, 23, 25, 38, 39, 43, 44, 46, 48–51] and also the reference cited therein.

For $0 < q < 1$ the Jackson’s $q$-derivative of a function $f \in \mathcal{A}$ is, by definition, given as follows [23]

$$D_qf(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z} & \text{for } z \neq 0, \\ f'(0) & \text{for } z = 0, \end{cases} \quad (1.3)$$

and

$$D_q^2f(z) = D_q(D_qf(z)).$$

From (1.3), we have

$$D_qf(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \quad (1.4)$$

where

$$[n]_q = \frac{1-q^n}{1-q}, \quad (1.5)$$

is sometimes called the basic number $n$. If $q \to 1-$, $[n]_q \to n$.

For a function $h(z) = z^n$, we obtain

$$D_q h(z) = D_q z^n = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1},$$

and

$$\lim_{q \to 1-} D_q h(z) = \lim_{q \to 1-} ([n]_q z^{n-1}) = n z^{n-1} = h'(z),$$

where $h'$ is the ordinary derivative.

Using the above defined $q$-calculus, several subclasses belonging to the class $\mathcal{A}$ have already been investigated in geometric function theory. Ismail et al. [26] were the first who used the $q$-derivative operator $D_q$ to study the $q$-calculus analogous of the class $S^*$ of starlike functions in $\mathbb{U}$ (see Definition 1.1 below). However, a firm footing of the $q$-calculus in the context of geometric function theory was presented mainly and basic (or $q$-) hypergeometric functions were first used in geometric function theory in a book chapter by Srivastava (see, for details, [45], p.347 et seq.); see also [46]).

For $0 < q < 1$, we define the class $S^*_q(\alpha)$ of $q$- starlike functions and the class $C_q(\alpha)$ of $q$- convex functions of order $\alpha (0 \leq \alpha < 1)$ (see, [26, 40, 41]), as below:

**Definition 1.1.** A function $f \in \mathcal{A}$ is said to be in the class $S^*_q(\alpha)$ if it satisfies

$$\Re\left(\frac{z D_q f(z)}{f(z)}\right) > \alpha, \quad (z \in \mathbb{U}).$$
Definition 1.2. A function $f \in \mathcal{A}$ is said to be in the class $C_\gamma(\alpha)$ if it satisfies

$$\Re\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right) > \alpha, \ (z \in \mathbb{U}).$$

It is clear that $\lim_{q \to 1} S_q^\gamma(\alpha) = S^\gamma(\alpha)$ and $\lim_{q \to 1} C_q(\alpha) = C(\alpha)$, where $S^\gamma(\alpha)$ and $C(\alpha)$ are, respectively, well-known starlike and convex functions of order $\alpha$ in $\mathbb{U}$.

We now introduce a new subclass of analytic functions defined by $q$-derivative operator $D_q$.

Definition 1.3. A function $f \in \mathcal{A}$ is said to be in the class $C_q(\lambda, \alpha)$ if it satisfies

$$\Re\left(\frac{\lambda z^3(zD_qf(z))'' + (2\lambda + 1)z^2(zD_qf(z))'' + z(zD_qf(z))'}{\lambda z^2(zD_qf(z))'' + z(zD_qf(z))'}\right) > \alpha, \ (z \in \mathbb{U})$$

where $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1$.

We write

$$T C_q(\lambda, \alpha) = C_q(\lambda, \alpha) \cap T.$$ 

A variable $X$ is said to be Pascal distribution if it takes the values $0, 1, 2, 3, \ldots$ with probabilities

$$(1 - s)^m, \ \frac{s^1m(1 - s)^m}{1!}, \ \frac{s^2m(1 - s)^m}{2!}, \ \frac{s^3m(m + 1)(1 - s)^m}{3!}, \ldots,$$

respectively, where $s$ and $m$ are called the parameters, and thus

$$P(X = k) = \binom{k + m - 1}{m - 1} s^k(1 - s)^m, \ k = 0, 1, 2, 3, \ldots.$$ 

Very recently, El-Deeb et al. [15] (see also, [10, 34]) introduced a power series whose coefficients are probabilities of Pascal distribution, that is

$$\Psi^m_s(z) := z + \sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1} s^{n-1}(1 - s)^m z^n, \ z \in \mathbb{U},$$

where $m \geq 1$, $0 \leq s \leq 1$, and we note that, by ratio test the radius of convergence of above series is infinity. We also define the series

$$\Phi^m_s(z) := 2z - \Psi^m_s(z) = z - \sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1} s^{n-1}(1 - s)^m z^n, \ z \in \mathbb{U}.$$ (1.7)

Let consider the linear operator $I^m_s : \mathcal{A} \to \mathcal{A}$ defined by the convolution or Hadamard product

$$I^m_s f(z) := \Psi^m_s(z) * f(z) = z + \sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1} s^{n-1}(1 - s)^m a_n z^n, \ z \in \mathbb{U},$$

where $m \geq 1$ and $0 \leq s \leq 1$.

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions, using hypergeometric functions (see for example, [8, 11, 21, 29, 42, 47]),
Lemma 1.4. A function $f$ of the form (1.2) is in $\mathcal{T}C_q(\lambda, \alpha)$ if and only if it satisfies
\[
\sum_{n=2}^{\infty} |n| n(n - \alpha)(\lambda n - \lambda + 1)|a_n| \leq 1 - \alpha,
\]
where $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1$ and $z \in \mathbb{U}$.

Lemma 1.4 can be proved using the same technique as in [27].

Lemma 1.5. [13] If $f \in \mathcal{R}(A, B)$ is of the form (1.1), then
\[
|a_n| \leq (A - B)\frac{|\tau|}{n}, \quad n \in \mathbb{N}\setminus\{1\}.
\]
The result is sharp.

2. Necessary and sufficient condition for $\Phi^m_s \in \mathcal{T}C_q(\lambda, \alpha)$

For convenience throughout in the sequel, we use the following identities that hold for $m \geq 1$ and $0 \leq s < 1$:
\[
\sum_{n=0}^{\infty} \binom{n + m - 1}{m - 1} s^n = \frac{1}{(1 - s)^m}, \quad \sum_{n=0}^{\infty} \binom{n + m - 2}{m - 2} s^n = \frac{1}{(1 - s)^{m-1}},
\]
\[
\sum_{n=0}^{\infty} \binom{n + m}{m} s^n = \frac{1}{(1 - s)^{m+1}}, \quad \sum_{n=0}^{\infty} \binom{n + m + 1}{m + 1} s^n = \frac{1}{(1 - s)^{m+2}}.
\]

By simple calculations we derive the following relations:
\[
\sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1} s^{n-1} = \sum_{n=0}^{\infty} \binom{n + m - 1}{m - 1} s^n - 1 = \frac{1}{(1 - s)^m} - 1, \quad (2.1)
\]
\[
\sum_{n=2}^{\infty} (n - 1) \binom{n + m - 2}{m - 1} s^{n-1} = sm \sum_{n=0}^{\infty} \binom{n + m}{m} s^n = s \frac{(m-1)}{(1 - s)^{m+1}}, \quad (2.2)
\]
\[
\sum_{n=3}^{\infty} (n - 1)(n - 2) \binom{n + m - 2}{m - 1} s^{n-1} = 2s^2 \frac{(m+1)}{(1 - s)^{m+2}} \quad (2.3)
\]
\[
\sum_{n=4}^{\infty} (n - 1)(n - 2)(n - 3) \binom{n + m - 2}{m - 1} s^{n-1} = 6s^3 \frac{(m+2)}{(1 - s)^{m+3}} \quad (2.4)
\]
and
\[
\sum_{n=5}^{\infty} (n-1)(n-2)(n-3)(n-4) \binom{n+m-2}{m-1} s^{n-1} = 24 s^4 \binom{m+3}{m-1} \frac{(1-s)^{m+4}}{(1-s)^{m+4}}.
\] (2.5)

Unless otherwise mentioned, we shall assume in this paper that \(0 \leq \alpha < 1\) and \(0 \leq \lambda \leq 1\), \(0 < q < 1\) and \(0 \leq s < 1\).

Firstly, we obtain the necessary and sufficient conditions for \(\Phi^m_s\) to be in the class \(T_C q(\lambda, \alpha)\).

**Theorem 2.1.** Let \(m \geq 1\) and \(q \rightarrow 1\). Then \(\Phi^m_s \in T_C q(\lambda, \alpha)\) if and only if
\[
24 \lambda \binom{m+3}{m-1} s^4 \left(1-s\right)^{m+4} + 6(\lambda(9 - \alpha) + 1) \binom{m+2}{m-1} s^3 \left(1-s\right)^{m+3} + 2(4\lambda(2 - \alpha) + 7 - 3\alpha) \binom{m+1}{m-1} s^2 \left(1-s\right)^{m+2} \\
(4\lambda(2 - \alpha) + 7 - 3\alpha) \binom{m}{m-1} s \left(1-s\right)^{m+1} \\
\leq 1 - \alpha.
\] (2.6)

**Proof.** Since \(\Phi^m_s\) is defined by (1.7), in view of Lemma 1.4 it is sufficient to show that
\[
P_q := \sum_{n=2}^{\infty} [n]_q n(\lambda n - \lambda + 1) \binom{n+m-2}{m-1} s^{n-1} (1-s)^m \leq 1 - \alpha.
\]

Since \([n]_q \rightarrow n\), when \(q \rightarrow 1\), we get
\[
P_1 = \sum_{n=2}^{\infty} n^2(\lambda n - \lambda + 1) \binom{n+m-2}{m-1} s^{n-1} (1-s)^m \\
= \sum_{n=2}^{\infty} \left[\lambda n^4 + (1 - \lambda - \alpha \lambda) n^3 + \alpha(\lambda - 1)n^2\right] \binom{n+m-2}{m-1} s^{n-1} (1-s)^m.
\]

Writing
\[
n^2 = (n-1)(n-2) + 3(n-1) + 1, \quad (2.7)\\
n^3 = (n-1)(n-2)(n-3) + 6(n-1)(n-2) + 7(n-1) + 1, \quad (2.8)\\
n^4 = (n-1)(n-2)(n-3)(n-4) + 10(n-1)(n-2)(n-3) \\
+ 25(n-1)(n-2) + 15(n-1) + 1, \quad (2.9)
\]

and using (2.2)–(2.5), we have
Theorem 3.1. Let $m \geq 1$ and $q \to 1 - 1$ if $f \in \mathcal{R}(A, B)$ and the inequality

\[
(A - B)|\tau| \left[ 6 \left( \frac{m+2}{m-1} \right) s^3 - 2(\lambda(5 - \alpha) + 1) s^2 \left( \frac{m+1}{m-1} \right) \right] + (2\lambda(2 - \alpha) + 3 - \alpha) \left( \frac{m}{1-s} \right) s \leq 1 - \alpha.
\]

is satisfied then $I_s^m f \in \mathcal{T}_c(q, \alpha)$.

Proof. According to Lemma 1.4 it is sufficient to show that

\[
Q_q := \sum_{n=2}^{\infty} [n]_q n(n - \alpha)(\lambda n - \lambda + 1) \left( \frac{n+m-2}{m-1} \right) s^{n-1}(1 - s)^m |a_n| \leq 1 - \alpha.
\]
Since \( f \in \mathcal{R}(A, B) \), using Lemma 1.5 we have

\[
|a_n| \leq \frac{(A - B)|\tau|}{n}, \quad n \in \mathbb{N} \setminus \{1\},
\]

therefore

\[
Q_1 \leq (A - B)|\tau| \left[ \sum_{n=2}^{\infty} n(n - \alpha)(\lambda n - \lambda + 1) \left( \frac{n + m - 2}{m - 1} \right)^{n-1}(1 - s)^m \right] 
+ (\lambda(5 - \alpha) + 1) \sum_{n=3}^{\infty} (n - 1)(n - 2) \left( \frac{n + m - 2}{m - 1} \right)^{n-1}(1 - s)^m 
+ (2\lambda(2 - \alpha) + 3 - \alpha) \sum_{n=2}^{\infty} (n - 1) \left( \frac{n + m - 2}{m - 1} \right)^{n-1}(1 - s)^m 
+ (1 - \alpha) \sum_{n=2}^{\infty} \left( \frac{n + m - 2}{m - 1} \right)^{n-1}(1 - s)^m \right] 
= (A - B)|\tau| \left[ 6\lambda s^3 \left( \frac{m+2}{m-1} \right)^3 + 2(\lambda(5 - \alpha) + 1)s^2 \left( \frac{m+1}{m-1} \right)^2(1 - s)^2 
+ (2\lambda(2 - \alpha) + 3 - \alpha) \left( \frac{m}{m-1} \right)s + (1 - \alpha)(1 - (1 - s)^m) \right].
\]

but this last expression is upper bounded by \( 1 - \alpha \) if and only if (3.1) holds. \( \square \)

4. Integral operator

**Theorem 4.1.** Let \( m \geq 1 \) and \( \lambda \to 1 - \). If the integral operator \( \mathcal{G}^m_s \) is given by

\[
\mathcal{G}^m_s(z) := \int_0^z \frac{\Phi^m_s(t)}{t} dt, \quad z \in \mathbb{U},
\]

then \( \mathcal{G}^m_s \in \mathcal{TC}_q(\lambda, \alpha) \) if and only

\[
6\lambda s^3 \left( \frac{m+2}{m-1} \right)^3 + 2(\lambda(5 - \alpha) + 1)s^2 \left( \frac{m+1}{m-1} \right)^2(1 - s)^2 
+ (2\lambda(2 - \alpha) + 3 - \alpha) \left( \frac{m}{m-1} \right)s + (1 - \alpha)(1 - (1 - s)^m) \leq 1 - \alpha.
\]
Proof. According to (1.7) it follows that
\[
\mathcal{G}_s^m(z) = z - \sum_{n=2}^{\infty} \left( \frac{n+m-2}{m-1} \right) s^{n-1}(1-s)^n z^n, \quad z \in \mathbb{U}.
\]

Using Lemma 1.4, the function \(\mathcal{G}_s^m(z)\) belongs to \(\mathcal{TC}_q(\alpha, \alpha)\) if and only if
\[
R_q := \sum_{n=2}^{\infty} [n]_q n(n-\alpha)(\lambda n - \lambda + 1) \times \frac{1}{n} \left( \frac{n+m-2}{m-1} \right) s^{a-1}(1-s)^m \leq 1 - \alpha,
\]

Now,
\[
R_1 = \sum_{n=2}^{\infty} [\lambda n^3 + (1-\lambda-\alpha) n^2 + \alpha(\lambda - 1)n] \left( \frac{n+m-2}{m-1} \right) s^{a-1}(1-s)^m
\]

By a similar proof like those of Theorem 3.1 we get that \(\mathcal{G}_s^m f \in \mathcal{TC}_q(\alpha, \alpha)\) if and only if (4.2) holds.

\[\square\]

5. Corollaries and consequences

Corollary 5.1. Let \(m \geq 1\) and \(q \rightarrow 1 - \). Then \(\Phi_s^m \in \mathcal{TC}_q(0, \alpha)\), if and only if
\[
6 \left( \frac{m}{m-1} \right)^3 \frac{s^3}{(1-s)^{m+3}} + 2(7 - 3\alpha) \left( \frac{m+1}{m-1} \right)^3 \frac{s^2}{(1-s)^{m+2}} + (7 - 3\alpha) \left( \frac{m}{m-1} \right) \frac{s}{(1-s)^{m+1}} \leq 1 - \alpha.
\]

Corollary 5.2. Let \(m \geq 1\) and \(q \rightarrow 1 - \). If \(f \in \mathcal{R}(A, B)\) and the inequality
\[
(A - B)\left|\frac{1}{s} \left[ 2 \left( \frac{m+1}{m-1} \right) \frac{s^2}{(1-s)^2} + (3 - \alpha) \left( \frac{m}{m-1} \right) \frac{s}{1-s} + (1 - \alpha)(1 - (1-s)^m) \right] \right| \leq 1 - \alpha.
\]

is satisfied then \(I_s^m f \in \mathcal{TC}_q(0, \alpha)\).

Corollary 5.3. Let \(m \geq 1\) and \(q \rightarrow 1 - \). If the integral operator \(\mathcal{G}_s^m\) is given by (4.1), then \(\mathcal{G}_s^m \in \mathcal{TC}_q(0, \alpha)\) if and only
\[
2s^2 \left( \frac{m+1}{m-1} \right) \frac{s^2}{(1-s)^{m+2}} + (3 - \alpha) \left( \frac{m}{m-1} \right) \frac{s}{(1-s)^{m+1}} \leq 1 - \alpha.
\]

6. Conclusions

In this paper, we find the necessary and sufficient conditions and inclusion relations for Pascal distribution series to be in a subclass of analytic functions defined by \(q\)-derivative operator. Basic (or \(q\)-) series and basic (or \(q\)-) polynomials, especially the basic (or \(q\)-) hypergeometric functions and basic (or \(q\)-) hypergeometric polynomials, are applicable particularly in several diverse areas (see, for example, [45], pp.350–351) and [44], p.328]). Moreover, in this recently-published survey-cum-expository review article by Srivastava [44], the so-called \((p, q)\)-calculus was exposed to be a rather trivial and inconsequential variation of the classical \(q\)-calculus, the additional parameter \(p\) being redundant (see, for details, [44], p.340)). This observation by Srivastava [44] will indeed apply also to any attempt to produce the rather straightforward \((p, q)\)-variations of the results which we have presented in this paper.
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Conflicts of interest

The authors declare no conflict of interest.

References


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