Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(3): 2778–2795.
DOI: 10.3934/math.2021170
Received: 25 October 2020
Accepted: 22 December 2020
Published: 06 January 2021

Research article

Robust passivity analysis of mixed delayed neural networks with interval nondifferentiable time-varying delay based on multiple integral approach

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Abstract: New results on robust passivity analysis of neural networks with interval nondifferentiable and distributed time-varying delays are investigated. It is assumed that the parameter uncertainties are norm-bounded. By construction an appropriate Lyapunov-Krasovskii containing single, double, triple and quadruple integrals, which fully utilize information of the neuron activation function and use refined Jensen's inequality for checking the passivity of the addressed neural networks are established in linear matrix inequalities (LMIs). This result is less conservative than the existing results in literature. It can be checked numerically using the effective LMI toolbox in MATLAB. Three numerical examples are provided to demonstrate the effectiveness and the merits of the proposed methods.

Keywords: passivity analysis; neural networks; uncertainties; nondifferentiable delay; time-varying delays
Mathematics Subject Classification: 34K25, 93D05, 93D09

1. Introduction

Recently, neural networks (NNs) have drawn considerable attention in many fields of science and engineering applications for example associative memories, fixed-point computation, control, static image processing and combinatorial optimization Ref. [1–3]. However, time-delay is common in various biological and physical phenomena, which is demonstrated by applying of mathematical modelling with time-delay in a wide range of applications for instance mechanical transmission, fluid transmission, metallurgical processes and networked control systems which is frequently a source of chaos, instability and poor control performance. These applications are extensively dependent upon the stability of the equilibrium of NNs. So, stability is much importance in dynamical properties of NNs when NNs are designed. As research results, the stability problem and the performance of the NNs with time-delay have been improved in Ref. [4–12]. However, most results were discussed only on the discrete delay in NNs. In contrast, the distributed delay should be associated with a model of a system that there exists a distribution of propagation delays over a period of time in some cases as discussed in Ref. [13, 14]. Therefore, there has been an increasing interest in the delayed NNs, and a great number of results on these topics have been reported in the literature Ref. [15–23] as well.

On the other hand, the passivity is interesting problem and is closely related to the circuit analysis method. The properties of the passivity are that the system can keep the system internally stable Ref. [24, 25]. Especially, the passive system employs the product of input and output as the energy provision and embodies the energy attenuation character. A passive system only burns energy without energy production and passivity represents the property of energy consumption Ref. [26]. The issue of passivity performance analysis has been used in various areas such as fuzzy control, signal processing, networked control and sliding mode control Ref. [27]. Due to these features, the passivity problems have been an active area of research in recently decades with NNs.

In the same way, Ref. [28–30] also studied the passivity analysis of neural network with discrete and distributed delays. In addition, many uncertain factors such as uncertain parameters, disturbance and environmental noise are regularly encountered in many practical and engineering systems, and these make it difficult to develop an exact mathematical model. Therefore, the parameter uncertainties are very important and unavoidable while modelling NNs in both theoretical and practical cases. Meanwhile, improved the delay-dependent approach to passivity analysis for uncertain NNs with discrete interval and distributed time-varying delays has also discussed in Ref. [31, 32]. However, there are few results for studying this problem with uncertainties to the best of the authors's knowledge, we study delay-dependent passivity criteria for uncertain NNs with discrete interval and distributed time-varying delays.

Recently, several approaches to reduce the conservatism for the system with time delay have been reported in the literature, namely an appropriate Lyapunov-Krasovskii functional method by utilizing information of the neuron activation function and some techniques to evaluate the bounds on some cross-terms product arising in the analysis of the delay-dependent stability problem such as integral inequality, refined Jensen's inequality, and free weighting matrices approach etc. These approaches will give better maximum allowable upper bound for time-varying delay over some existing ones Ref. [33–40]. However, these previous works still study on delay-derivative-dependent stability criteria. Practically time delays can occur in an irregular fashion such as sometimes the time-varying delays are not differentiable.

Therefore in this paper, we have followed robust passivity analysis of NNs with interval nondifferentiable and distributed time-varying delays to obtain a better maximum bound value and to relax the derivative condition of delay. Moreover, system is assumed that the parameter uncertainties are norm-bounded for checking the passivity of the addressed NNs in LMIs, which can be checked numerically using the effective LMI toolbox in MATLAB. This is the first time that we apply the methods to study the networks model to reduce the condition of delays being non-differentiable delays. Moreover, the system can be turned into the delayed NNs proposed in Ref. [36, 39, 40] which means that this work is more general than them. Furthermore, the main ideas of this work are given as follows:

- The challenge of this paper is studying the new result on robust passivity analysis of NNs with non-differentiable mixed time-varying delays which mean that this work can be used for various systems with fast time-varying delays compared with previous works considered on differentiable delay ($\dot{r}(t) \leq \mu$).
- The new Lyapunov Krasovskii functional establishes more relationships among different vectors, avoids the extra conservatism arising from estimating the time-varying delays and utilizes more information about the upper and lower bounds of the time delays existing in the systems.
- The new sufficient conditions based on refined Jensen-based inequalities proposed in Ref. [41], are less conservative than the others proposed Ref. [33–40] which are shown in the comparison examples.

The rest of paper is organized as follows: Section 2 provides some mathematical preliminaries and network model. Section 3 presents the passivity analysis of uncertain NNs with interval and distributed time-varying delays. Numerical examples are given in Section 4. Finally, the conclusion is provided in Section 5.

2. Network model and mathematics preliminaries

Notations: $\mathbb{R}^n$ is the n-dimensional Euclidean space; $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ real matrices; I_n represents the n-dimensional identity matrix. Let $\mathbb{S}_n^+$ denotes the set of symmetric positive definite matrices in $\mathbb{R}^{n\times n}$. We also denoted by $\mathbb{D}_n^+$ the set of positive diagonal matrices. A matrix $D = \text{diag}\{d_1, d_2, ..., d_n\} \in \mathbb{D}_n^+$ if $d_i > 0$ ($i = 1, 2, ..., n$). The notation $X \geq 0$ (respectively, $X > 0$) means that X is positive semi-definite (respectively, positive definite); diag(...) denotes a block diagonal matrix; $\begin{bmatrix} X & Y \\ \star & Z \end{bmatrix}$ stands for $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix}$; Matrix dimensions, if not explicitly stated, are assumed to be compatible for algebraic operations.

Consider the following of NNs with nondifferentiable interval and distributed time-varying delays in the form:

$$\begin{cases} \dot{p}(t) = -Dp(t) + Ag(p(t)) + A_1 g(p(t - r(t))) + A_2 \int_{t-d(t)}^{t} g(p(s))\, ds + u(t), \\ q(t) = C_1 g(p(t)) + C_2 g(p(t - r(t))) + C_3 \int_{t-d(t)}^{t} g(p(s))\, ds + C_4 u(t), \\ p(t) = \phi(t), \ \ t \in [-\theta, 0], \ \ \theta = \max\{r_2, d\}, \end{cases} \tag{2.1}$$

where n denotes the number of neurons in the network, $p(t) = [p_1(t),\ p_2(t), ..., p_n(t)]^T \in \mathbb{R}^n$ is the neurons state vector, $q(t) \in \mathbb{R}^n$ is the output vector and $u(t)$ is the external input of the network, $D = \text{diag}\{d_1, d_2, ..., d_n\}$ is a positive diagonal matrix, A, A_1, A_2 are interconnection weight matrices, C_1, C_2, C_3, C_4 are real matrices, $g(p(t)) = [g_1(p_1(t)),\ g_2(p_2(t)), ..., g_n(p_n(t))]^T \in \mathbb{R}^n$ denotes the activation function, $g(p(t - r(t))) = [g_1(p_1(t - r(t))),\ g_2(p_2(t - r(t))), ...,\ g_n(p_n(t - r(t)))]^T \in \mathbb{R}^n$. and $\phi(t) \in \mathbb{R}^n$ is the initial function.

The variables $r(t)$ and $d(t)$ represent the mixed delays of the model in (2.1) and satisfy

$$0 \leq r_1 \leq r(t) \leq r_2 \quad \text{and} \quad 0 \leq d(t) \leq d, \ \ \forall t \geq 0, \tag{2.2}$$

where r_1, r_2, and d are constants.

The neural activation functions $g_i(p_i(t))$ are continuous $g_i(0) = 0$ and there exist constants l_i^-, l_i^+ $(i = 1, 2, ..., n)$ such that

$$l_i^- \leq \frac{g_i(p) - g_i(q)}{p - q} \leq l_i^+, \quad \forall p, q \in \mathbb{R},\ p \neq q. \tag{2.3}$$

Definition 2.1. *[5] The neural network (2.1) is said to be passive if there exists a scalar $\gamma > 0$ such that for all $t_f \geq 0$*

$$2\int_0^{t_f} q^T(s)u(s)\,ds \ \geq\ -\gamma\int_0^{t_f} u^T(s)u(s)\,ds, \tag{2.4}$$

under the zero initial condition.

Lemma 2.2. *[41] For a given matrix $Q \in \mathbb{S}_n^+$ and a function $e : [u, v] \to \mathbb{R}^n$ whose derivative $\dot{e} \in C([u, v], \mathbb{R}^n)$, the following inequalities hold:*

$$\int_u^v \dot{e}^T(s)Q\dot{e}(s)\,ds \quad \geq \quad \frac{1}{v-u}\hat{\varepsilon}^T\bar{Q}\,\hat{\varepsilon}, \tag{2.5}$$

$$\int_u^v\int_s^v \dot{e}^T(\alpha)Q\dot{e}(\alpha)\,d\alpha\,ds \quad \geq \quad 2\hat{\Gamma}^T\hat{Q}\,\hat{\Gamma}, \tag{2.6}$$

where $\bar{Q} = \mathrm{diag}\{Q, 3Q, 5Q\}$, $\hat{Q} = \mathrm{diag}\{Q, 2Q\}$, $\hat{\varepsilon} = [\varepsilon_1^T, \varepsilon_2^T, \varepsilon_3^T]^T$, $\hat{\Gamma} = [\Gamma_1^T, \Gamma_2^T]^T$ *and*

$$\begin{aligned}
\varepsilon_1 &= e(v) - e(u),\\
\varepsilon_2 &= e(v) + e(u) - \frac{2}{v-u}\int_u^v e(s)\,ds,\\
\varepsilon_3 &= e(v) - e(u) + \frac{6}{v-u}\int_u^v e(s)\,ds - \frac{12}{(v-u)^2}\int_u^v\int_s^v e(\alpha)\,d\alpha\,ds,\\
\Gamma_1 &= e(v) - \frac{1}{v-u}\int_u^v e(s)\,ds,\\
\Gamma_2 &= e(v) + \frac{2}{v-u}\int_u^v e(s)\,ds - \frac{6}{(v-u)^2}\int_u^v\int_s^v e(\alpha)\,d\alpha\,ds.
\end{aligned}$$

Lemma 2.3. *[6] For a positive definite matrix $P > 0$, and an integral function $\{e(\alpha)|\alpha \in [u, v]\}$, then the following inequalities hold:*

$$\int_u^v e^T(\alpha)Pe(\alpha)\,d\alpha \ \geq\ \frac{1}{v-u}\left(\int_u^v e(\alpha)\,d\alpha\right)^T P\left(\int_u^v e(\alpha)\,d\alpha\right), \tag{2.7}$$

$$\begin{aligned}
&\int_u^v\int_\beta^v\int_s^v e^T(\alpha)Pe(\alpha)\,d\alpha\,d\beta\,ds\\
&\quad\geq\ \frac{6}{(v-u)^3}\left(\int_u^v\int_\beta^v\int_s^v e(\alpha)\,d\alpha\,d\beta\,ds\right)^T P\left(\int_u^v\int_\beta^v\int_s^v e(\alpha)\,d\alpha\,d\beta\,ds\right).
\end{aligned} \tag{2.8}$$

Lemma 2.4. *[6] Let M, N and $F(t)$ be real matrices of appropriate dimensions with $F(t)$ satisfying $F^T(t)F(t) \leq I$. Then for any scalar $\epsilon > 0$.*

$$MF(t)N + (MF(t)N)^T \ \leq\ \epsilon^{-1}MM^T + \epsilon N^TN. \tag{2.9}$$

Lemma 2.5. *[6] Given constant symmetric matrices P, Q, R with appropriate dimensions satisfying $P = P^T$, $Q = Q^T > 0$. Then $P + R^TQ^{-1}R < 0$ if and only if*

$$\begin{bmatrix} P & R^T \\ R & -Q \end{bmatrix} < 0 \quad or \quad \begin{bmatrix} -Q & R \\ R^T & P \end{bmatrix} < 0. \tag{2.10}$$

3. Main results

In this section, the new result on robust passivity analysis for NNs with interval nondifferentiable and distributed time-varying delays will be established. Let us set

$L_1 = \text{diag}\{l_1^-, l_2^-, ..., l_n^-\}$, $L_2 = \text{diag}\{l_1^+, l_2^+, ..., l_n^+\}$, $r_{12} = r_2 - r_1$,

$\bar{S}_k = \text{diag}\{S_k, 3S_k, 5S_k\}, k = 1, 2,$ $T = \begin{bmatrix} I_n & -I_n & 0 & 0 \\ I_n & I_n & -2I_n & 0 \\ I_n & -I_n & 6I_n & -6I_n \end{bmatrix}$,

$\Sigma_1(r) = [\varphi_1^T \ \ r_1\varphi_9^T \ \ (r - r_1)\varphi_{10}^T + (r_2 - r)\varphi_{11}^T \ \ \frac{r_1^2}{2}\varphi_{12}^T]^T$,

$\Sigma_2 = [\mathcal{A}^T \ \ (\varphi_1 - \varphi_2)^T \ \ (\varphi_2 - \varphi_4)^T \ \ r_1(\varphi_1 - \varphi_9)^T]^T$,

$\Sigma_3 = [\varphi_2^T \ \ \varphi_8^T]^T$, $\Sigma_4 = [\varphi_4^T \ \ \varphi_7^T]^T$, $\Sigma_5 = [\varphi_1^T \ \ \varphi_5^T]^T$,

$\Sigma_6 = [\varphi_1^T \ \ \varphi_2^T \ \ \varphi_9^T \ \ \varphi_{12}^T]^T$, $\Sigma_7 = [\varphi_3^T \ \ \varphi_4^T \ \ \varphi_{11}^T \ \ \varphi_{14}^T]^T$, $\Sigma_8 = [\varphi_2^T \ \ \varphi_3^T \ \ \varphi_{10}^T \ \ \varphi_{13}^T]^T$,

$\Sigma_9(r) = [((r - r_1)\varphi_{10} + (r_2 - r)\varphi_{11})^T \ \ (\varphi_{16} + \varphi_{17})^T]^T$,

$\Sigma_{10} = \frac{r_1^2}{2}\varphi_1 - \frac{r_2^2}{2}\varphi_{12}$, $\Sigma_{11}(r) = \frac{r_{12}^2}{2}\varphi_2 - \frac{(r-r_1)^2}{2}\varphi_{13} - \frac{(r_2-r)^2}{2}\varphi_{14}$,

$\Sigma_{12} = \varphi_5 - L_1\varphi_1$, $\Sigma_{13} = L_2\varphi_1 - \varphi_5$,

$\Sigma_{14} = \varphi_6 - L_1\varphi_3$, $\Sigma_{15} = L_2\varphi_3 - \varphi_6$,

$\Sigma_{16} = \varphi_5 - \varphi_6 - L_1(\varphi_1 - \varphi_3)$, $\Sigma_{17} = L_2(\varphi_1 - \varphi_3) - \varphi_5 + \varphi_6$,

$\Sigma_{18} = \varphi_5 - \varphi_8 - L_1(\varphi_1 - \varphi_2)$, $\Sigma_{19} = L_2(\varphi_1 - \varphi_2) - \varphi_5 + \varphi_8$,

$\Sigma_{20} = \varphi_5 - \varphi_7 - L_1(\varphi_1 - \varphi_4)$, $\Sigma_{21} = L_2(\varphi_1 - \varphi_4) - \varphi_5 + \varphi_7$,

$\Sigma_{22} = \varphi_7 - \varphi_8 - L_1(\varphi_4 - \varphi_2)$, $\Sigma_{23} = L_2(\varphi_4 - \varphi_2) - \varphi_7 + \varphi_8$,

$\Sigma_{24} = \varphi_6 - \varphi_8 - L_1(\varphi_3 - \varphi_2)$, $\Sigma_{25} = L_2(\varphi_3 - \varphi_2) - \varphi_6 + \varphi_8$,

$\Sigma_{26} = \varphi_7 - \varphi_6 - L_1(\varphi_4 - \varphi_3)$, $\Sigma_{27} = L_2(\varphi_4 - \varphi_3) - \varphi_7 + \varphi_6$,

$\Gamma_0(r) = \Sigma_1^T(r)P\Sigma_2 + \varphi_1^T(L_2W_2 - L_1W_1)\mathcal{A} - \varphi_5^T(W_2 - W_1)\mathcal{A}$,

$\Gamma_1 = \varphi_1^T(Q_2 + Q_3)\varphi_1 - \varphi_2^TQ_2\varphi_2 - \varphi_4^TQ_3\varphi_4 + \Sigma_3^TQ_1\Sigma_3 - \Sigma_4^TQ_1\Sigma_4 + r_{12}^2\Sigma_5^TS_3\Sigma_5 + d^2\varphi_5^TS_4\varphi_5$,

$\Gamma_2 = \mathcal{A}^T[r_1^2S_1 + r_{12}^2S_2 + \frac{r_1^2}{2}R_1 + \frac{r_{12}^2}{2}R_2 + \frac{r_1^6}{36}Z_1 + \frac{(r_2^3-r_1^3)(r_2-r_1)^3}{36}Z_2]\mathcal{A}$,

$\Gamma_3 = \varphi_{15}^T(\gamma I_n + 2C_4)\varphi_{15} + \text{Sym}(\varphi_{15}^T(C_1\varphi_5 + C_2\varphi_6 + C_3\varphi_{18}))$,

$\Omega_1(r) = \text{Sym}(\Gamma_0(r)) + \Gamma_1 + \Gamma_2 - \Gamma_3$,

$\Omega_2 = \Sigma_6^TT^T\bar{S}_1T\Sigma_6$, $\Omega_3 = \Sigma_7^TT^T\bar{S}_2T\Sigma_7$, $\Omega_4 = \Sigma_8^TT^T\bar{S}_2T\Sigma_8$,

$\Omega_5 = 2(\varphi_1 - \varphi_9)^TR_1(\varphi_1 - \varphi_9) + 4(\varphi_1 + 2\varphi_9 - 3\varphi_{12})^TR_1(\varphi_1 + 2\varphi_9 - 3\varphi_{12})$,

$\Omega_6 = 2(\varphi_2 - \varphi_{10})^TR_2(\varphi_2 - \varphi_{10}) + 4(\varphi_2 + 2\varphi_{10} - 3\varphi_{13})^TR_2(\varphi_2 + 2\varphi_{10} - 3\varphi_{13}) + 2(\varphi_3 - \varphi_{11})^TR_2(\varphi_3 - \varphi_{11}) + 4(\varphi_3 + 2\varphi_{11} - 3\varphi_{14})^TR_2(\varphi_3 + 2\varphi_{11} - 3\varphi_{14})$,

$\Pi(r) = \Sigma_9^T(r)S_3\Sigma_9(r) + \Sigma_{10}^TZ_1\Sigma_{10} + \Sigma_{11}^T(r)Z_2\Sigma_{11}(r)$,

$\Delta_0 = \Sigma_{12}^T\Delta_1\Sigma_{13} + \Sigma_{14}^T\Delta_2\Sigma_{15} + \Sigma_{16}^T\Delta_3\Sigma_{17} + \Sigma_{18}^T\Delta_4\Sigma_{19} + \Sigma_{20}^T\Delta_5\Sigma_{21} + \Sigma_{22}^T\Delta_6\Sigma_{23} + \Sigma_{24}^T\Delta_7\Sigma_{25} + \Sigma_{26}^T\Delta_8\Sigma_{27}$,
$\mathcal{A} = D\varphi_1 + A\varphi_5 + A_1\varphi_6 + A_2\varphi_{18} + \varphi_{15}$,
and $\varphi_i = [0_{n\times(i-1)n}\ \ I_n\ \ 0_{n\times(18-i)n}]\ \ (i = 1, 2, ..., 18)$.

Based on the Lyapunov–Krasovskii functional approach, we present our new theorem for passivity of NNs (2.1).

Theorem 3.1. *The delayed neural network in (2.1) is passive in the sense of definition 2.1 for any delays $r(t)$ and $d(t)$ satisfying $0 \le r_1 \le r(t) \le r_2$ and $0 \le d(t) \le d$ if there exist matrices $P \in \mathbb{S}_{4n}^+; Q_1, S_3 \in \mathbb{S}_{2n}^+; Q_2, Q_3, S_1, S_2, S_4, R_1, R_2, Z_1, Z_2 \in \mathbb{S}_n^+; \Delta_k, W_\sigma \in \mathbb{D}_n^+$, $(k = 1, 2, ..., 8; \sigma = 1, 2)$, and a scalar $\gamma > 0$ satisfy the following LMI:*

$$\Phi(r) = \mathrm{Sym}(\Delta_0) - \Pi(r) - \Omega_1(r) - \sum_{k=2}^{6}\Omega_k < 0. \tag{3.1}$$

Proof. Consider the following Lyapunov-Krasovskii functional:

$$V(t, p_t) = \sum_{i=1}^{5} V_i(t, p_t),$$

where,

$$\begin{aligned}
V_1(t, p_t) &= \eta_1^T(t)P\eta_1(t) + 2\sum_{i=1}^{n} w_{1i}\int_0^{p_i(t)} (g_i(s) - l_i^- s)\,ds \\
&\quad +2\sum_{i=1}^{n} w_{2i}\int_0^{p_i(t)} (l_i^+ s - g_i(s))\,ds, \\
V_2(t, p_t) &= \int_{t-r_2}^{t-r_1} \eta_2^T(s)Q_1\eta_2(s)\,ds + \int_{t-r_1}^{t} p^T(s)Q_2p(s)\,ds + \int_{t-r_2}^{t} p^T(s)Q_3p(s)\,ds, \\
V_3(t, p_t) &= r_1\int_{-r_1}^{0}\int_{t+s}^{t} \dot{p}^T(u)S_1\dot{p}(u)\,du\,ds + r_{12}\int_{-r_2}^{-r_1}\int_{t+s}^{t} \dot{p}^T(u)S_2\dot{p}(u)\,du\,ds \\
&\quad +r_{12}\int_{-r_2}^{-r_1}\int_{t+s}^{t} \eta_2^T(u)S_3\eta_2(u)\,du\,ds + d\int_{-d}^{0}\int_{t+s}^{t} g^T(p(u))S_4g(p(u))\,du\,ds, \\
V_4(t, p_t) &= \int_{t-r_1}^{t}\int_{s}^{t}\int_{u}^{t} \dot{p}^T(\lambda)R_1\dot{p}(\lambda)\,d\lambda\,du\,ds + \int_{-r_2}^{-r_1}\int_{s}^{-r_1}\int_{t+u}^{t} \dot{p}^T(\lambda)R_2\dot{p}(\lambda)\,d\lambda\,du\,ds, \\
V_5(t, p_t) &= \frac{r_1^3}{6}\int_{t-r_1}^{t}\int_{s}^{t}\int_{\lambda}^{t}\int_{u}^{t} \dot{p}^T(\theta)Z_1\dot{p}(\theta)\,d\theta\,du\,d\lambda\,ds \\
&\quad +\frac{(r_2^3 - r_1^3)}{6}\int_{-r_2}^{-r_1}\int_{s}^{-r_1}\int_{\lambda}^{-r_1}\int_{t+u}^{t} \dot{p}^T(\theta)Z_2\dot{p}(\theta)\,d\theta\,du\,d\lambda\,ds.
\end{aligned}$$

Let $\Delta_k = \mathrm{diag}\{\lambda_{k1}, \lambda_{k2}, ..., \lambda_{kn}\}$ $(k = 1, 2, ..., 8)$, $W_\sigma = \mathrm{diag}\{w_{\sigma1}, w_{\sigma2}, ..., w_{\sigma n}\}$ $(j = 1, 2)$, and

$$\begin{aligned}
\eta_1(t) &= \left[p^T(t)\ \ \int_{t-r_1}^{t} p^T(s)\,ds\ \ \int_{t-r_2}^{t-r_1} p^T(s)\,ds\ \ \int_{t-r_1}^{t}\int_{s}^{t} p^T(u)\,du\,ds\right]^T, \\
\eta_2(t) &= \left[p^T(t)\ \ g^T(p(t))\right]^T,
\end{aligned}$$

$$
\begin{aligned}
\xi(t) \;=\;& \Big[p^T(t), p^T(t-r_1), p^T(t-r(t)), p^T(t-r_2), g^T(p(t)), g^T(p(t-r(t))), g^T(p(t-r_2)),\\
& g^T(p(t-r_1)), \frac{1}{r_1}\int_{t-r_1}^{t} p^T(s)\,ds, \frac{1}{r(t)-r_1}\int_{t-r(t)}^{t-r_1} p^T(s)\,ds, \frac{1}{r_2-r(t)}\int_{t-r_2}^{t-r(t)} p^T(s)\,ds,\\
& \frac{2}{r_1^2}\int_{t-r_1}^{t}\int_{s}^{t} p^T(u)\,du\,ds, \frac{2}{(r(t)-r_1)^2}\int_{t-r(t)}^{t-r_1}\int_{s}^{t-r_1} p^T(u)\,du\,ds,\\
& \frac{2}{(r_2-r(t))^2}\int_{t-r_2}^{t-r(t)}\int_{s}^{t-r(t)} p^T(u)\,du\,ds, u^T(t), \int_{t-r(t)}^{t-r_1} g^T(p(s))\,ds, \int_{t-r_2}^{t-r(t)} g^T(p(s))\,ds,\\
& \int_{t-d(t)}^{t} g^T(p(s))\,ds\Big]^T.
\end{aligned}
$$

The derivative of $V(t, p_t)$ along the solution of system (2.1) as follows:

$$
\begin{aligned}
\dot{V}_1(t, p_t) \;=\;& 2\eta_1^T(t)P\dot{\eta}_1(t) + 2\sum_{i=1}^{n} w_{1i}\Big[g_i(p_i(t))\dot{p}_i(t) - l_i^-(p_i(t))\dot{p}_i(t)\Big]\\
& +2\sum_{i=1}^{n} w_{2i}\Big[l_i^+(p_i(t))\dot{p}_i(t) - g_i(p_i(t))\dot{p}_i(t)\Big]\\
=\;& 2\eta_1^T(t)P\dot{\eta}_1(t) + 2\Big[p^T(t)(L_2W_2 - L_1W_1) - g^T(p(t))(W_2 - W_1)\Big]\dot{p}(t),\\
\dot{V}_2(t, p_t) \;=\;& \eta_2^T(t-r_1)Q_1\eta_2(t-r_1) - \eta_2^T(t-r_2)Q_1\eta_2(t-r_2) + p^T(t)(Q_2+Q_3)p(t)\\
& -p^T(t-r_1)Q_2p(t-r_1) - p^T(t-r_2)Q_3p(t-r_2),\\
\dot{V}_3(t, p_t) \;=\;& r_1^2\dot{p}^T(t)S_1\dot{p}(t) + r_{12}^2\dot{p}^T(t)S_2\dot{p}(t) + r_{12}^2\eta_2^T(t)S_3\eta_2(t)\\
& -r_1\int_{t-r_1}^{t}\dot{p}^T(s)S_1\dot{p}(s)\,ds - r_{12}\int_{t-r_2}^{t-r_1}\dot{p}^T(s)S_2\dot{p}(s)\,ds\\
& +d^2g^T(p(t))S_4g(p(t)) - r_{12}\int_{t-r_2}^{t-r_1}\eta_2^T(s)S_3\eta_2(s)\,ds\\
& -d\int_{t-d}^{t} g^T(p(s))S_4g(p(s))\,ds,\\
\dot{V}_4(t, p_t) \;=\;& \frac{r_1^2}{2}\dot{p}^T(t)R_1\dot{p}(t) + \frac{r_{12}^2}{2}\dot{p}^T(t)R_2\dot{p}(t) - \int_{t-r_1}^{t}\int_{s}^{t}\dot{p}^T(u)R_1\dot{p}(u)\,du\,ds\\
& -\int_{-r_2}^{-r_1}\int_{t+s}^{t-r_1}\dot{p}^T(u)R_2\dot{p}(u)\,du\,ds,\\
\dot{V}_5(t, p_t) \;=\;& \frac{r_1^6}{36}\dot{p}^T(t)Z_1\dot{p}(t) + \frac{(r_2^3-r_1^3)(r_2-r_1)^3}{36}\dot{p}^T(t)Z_2\dot{p}(t)\\
& -\frac{r_1^3}{6}\int_{t-r_1}^{t}\int_{s}^{t}\int_{\lambda}^{t}\dot{p}^T(u)Z_1\dot{p}(u)\,du\,d\lambda\,ds\\
& -\frac{(r_2^3-r_1^3)}{6}\int_{-r_2}^{-r_1}\int_{s}^{-r_1}\int_{t+\lambda}^{t-r_1}\dot{p}^T(u)Z_2\dot{p}(u)\,du\,d\lambda\,ds.
\end{aligned}
$$

We conclude that,

$$
\dot{V}(t, p_t) \;\leq\; \xi^T(t)\Big(\mathrm{Sym}(\Gamma_0(r)) + \Gamma_1 + \Gamma_2\Big)\xi(t)
$$

$$
\begin{aligned}
&-r_1\int_{t-r_1}^{t}\dot{p}^T(s)S_1\dot{p}(s)\,ds - r_{12}\int_{t-r_2}^{t-r_1}\dot{p}^T(s)S_2\dot{p}(s)\,ds\\
&-r_{12}\int_{t-r_2}^{t-r_1}\eta_2^T(s)S_3\eta_2(s)\,ds - d(t)\int_{t-d(t)}^{t}g^T(p(s))S_4g(p(s))\,ds\\
&-\int_{t-r_1}^{t}\int_{s}^{t}\dot{p}^T(u)R_1\dot{p}(u)\,du\,ds - \int_{t-r_2}^{t-r_1}\int_{s}^{t-r_1}\dot{p}^T(u)R_2\dot{p}(u)\,du\,ds\\
&-\frac{r_1^3}{6}\int_{t-r_1}^{t}\int_{s}^{t}\int_{\lambda}^{t}\dot{p}^T(u)Z_1\dot{p}(u)\,du\,d\lambda\,ds\\
&-\frac{(r_2^3-r_1^3)}{6}\int_{-r_2}^{-r_1}\int_{s}^{-r_1}\int_{t+\lambda}^{t-r_1}\dot{p}^T(u)Z_2\dot{p}(u)\,du\,d\lambda\,ds. \qquad (3.2)
\end{aligned}
$$

According to Lemma 2.2, we have

$$
-r_1\int_{t-r_1}^{t}\dot{p}^T(s)S_1\dot{p}(s)\,ds \;\le\; -\xi^T(t)\Sigma_6^T T^T\bar{S}_1T\Sigma_6\xi(t) = -\xi^T(t)\Omega_2\xi(t). \qquad (3.3)
$$

By splitting, we have

$$
-r_{12}\int_{t-r_2}^{t-r_1}\dot{p}^T(s)S_2\dot{p}(s)\,ds \;=\; -r_{12}\int_{t-r_2}^{t-r(t)}\dot{p}^T(s)S_2\dot{p}(s)\,ds - r_{12}\int_{t-r(t)}^{t-r_1}\dot{p}^T(s)S_2\dot{p}(s)\,ds. \qquad (3.4)
$$

Applying Lemma 2.2 yields

$$
-r_{12}\int_{t-r_2}^{t-r(t)}\dot{p}^T(s)S_2\dot{p}(s)\,ds \;\le\; -\xi^T(t)\Sigma_7^T T^T\bar{S}_2T\Sigma_7\xi(t) \;=\; -\xi^T(t)\Omega_3\xi(t). \qquad (3.5)
$$

$$
-r_{12}\int_{t-r(t)}^{t-r_1}\dot{p}^T(s)S_2\dot{p}(s)\,ds \;\le\; -\xi^T(t)\Sigma_8^T T^T\bar{S}_2T\Sigma_8\xi(t) \;=\; -\xi^T(t)\Omega_4\xi(t). \qquad (3.6)
$$

$$
-\int_{t-r_1}^{t}\int_{s}^{t}\dot{p}^T(u)R_1\dot{p}(u)\,du\,ds \;\le\; -\xi^T(t)\Omega_5\xi(t). \qquad (3.7)
$$

$$
-\int_{t-r_2}^{t-r_1}\int_{s}^{t-r_1}\dot{p}^T(u)R_2\dot{p}(u)\,du\,ds \;\le\; -\xi^T(t)\Omega_6\xi(t). \qquad (3.8)
$$

In the same way, applying the inequalities (2.7) and (2.8), then we obtain

$$
\begin{aligned}
-r_{12}\int_{t-r_2}^{t-r_1}\eta_2^T(s)S_3\eta_2(s)\,ds &\le -\Big(\int_{t-r_2}^{t-r_1}\eta_2(s)\,ds\Big)^T S_3\Big(\int_{t-r_2}^{t-r_1}\eta_2(s)\,ds\Big)\\
&\le -\xi^T(t)\Sigma_9^T(r)S_3\Sigma_9(r)\xi(t), \qquad (3.9)
\end{aligned}
$$

$$
-d(t)\int_{t-d(t)}^{t}g(p(s))^T(s)\,S_4\,g(p(s))\,ds \;\le\; -\Big(\int_{t-d(t)}^{t}g(p(s))\,ds\Big)^T S_4\Big(\int_{t-d(t)}^{t}g(p(s))\,ds\Big)
$$

$$= -\xi^T(t)\varphi_{18}^T S_4\varphi_{18}\xi(t), \tag{3.10}$$

$$\begin{aligned} -\frac{r_1^3}{6}\int_{t-r_1}^{t}\int_{s}^{t}\int_{\lambda}^{t}\dot{p}^T(u)Z_1\dot{p}(u)\,du\,d\lambda\,ds &\le -\xi^T(t)\Big(\frac{r_1^2}{2}\varphi_1-\frac{r_1^2}{2}\varphi_{12}\Big)^T Z_1\Big(\frac{r_1^2}{2}\varphi_1-\frac{r_1^2}{2}\varphi_{12}\Big)\xi(t)\\ &= -\xi^T(t)\Sigma_{10}^T Z_1\Sigma_{10}\xi(t), \end{aligned}\tag{3.11}$$

$$-\frac{(r_2^3-r_1^3)}{6}\int_{-r_2}^{-r_1}\int_{s}^{-r_1}\int_{t+\lambda}^{t-r_1}\dot{p}^T(u)Z_2\dot{p}(u)\,du\,d\lambda\,ds \le -\xi^T(t)\Sigma_{11}^T(r)Z_2\Sigma_{11}(r)\xi(t). \tag{3.12}$$

For $\lambda_{1i}>0,\ i=1,2,\dots,n$, it can be deduced from (2.3) that

$$2(g_i(p_i(t))-l_i^-p_i(t))\lambda_{1i}(l_i^+p_i(t)-g_i(p_i(t))) \ge 0,$$

and thus

$$\xi^T(t)\mathrm{Sym}\Big(\Sigma_{12}^T\Delta_1\Sigma_{13}\Big)\xi(t) \ge 0. \tag{3.13}$$

From (3.13), we have

$$\xi^T(t)\mathrm{Sym}(\Delta_0)\xi(t) \ge 0. \tag{3.14}$$

Then, to show that NNs (2.1) is passive, we define $J(t_f)=\int_0^{t_f}\Big[-\gamma u^T(t)u(t)-2q^T(t)u(t)\Big]dt$ where $t_f\ge 0$. Consider the zero initial condition and we have

$$\begin{aligned} J(t_f) &= \int_0^{t_f}\Big[\dot{V}(p_t)-\gamma u^T(t)u(t)-2q^T(t)u(t)\Big]dt - V(p_{t_f})\\ &\le \int_0^{t_f}\Big[\dot{V}(p_t)-\gamma u^T(t)u(t)-2q^T(t)u(t)\,dt\Big]. \end{aligned}$$

From (3.2) to (3.14), it can be deduced that

$$\dot{V}(p_t)-\gamma u^T(t)u(t)-2q^T(t)u(t) \le \xi^T(t)\Phi(r)\xi(t).$$

where $\Phi(r)$ is an affine function in r, for $\Phi(r)<0$, $r\in[r_1,r_2]$ if and only if $\Phi(r_1)<0$ and $\Phi(r_2)<0$. if (3.1) holds for $r=r_1$ and $r=r_2$ and we have $\Phi(r)<0$, then

$$\dot{V}(p_t)-\gamma u^T(t)u(t)-2q^T(t)u(t) \le 0.$$

Considering, we have $J(t_f)<0$ for any $t_f\ge 0$ if condition (2.3) is satisfied. Thus, the system of NNs (2.1) is passive. The proof is completed. □

Remark 1. *We can see that the time delay in this work is a continuous function which belongs to a given interval. It means that the lower and upper bounds of the time-varying delay are available. Moreover, there is no need to be differentiable for the delay function. Therefore, the delays considering in this brief are more general than those studied in [29, 33, 34, 38].*

Remark 2. *The activation function in inequality (2.3) studied by [40] is more general than [28, 33, 36, 39] because the constants l_i^- and l_i^+ can be positive, zero or negative. We can see that the activation function under (2.3) can be unbounded, non-monotonic, non-differentiable. Hence, the passivity condition is considered in this work is less conservative than Ref. [28, 33, 36, 39].*

Remark 3. *The proof of theorem 3.1 shows estimating of integral terms by lemma 2.2 applying equations (3.3), (3.4), (3.5), (3.6), (3.7) and (3.8), which obtained a tighter upper bound than Jensen's inequality used in Ref. [25, 29, 33, 36].*

Based on the presented passivity condition in theorem 3.1, we will develop passivity analysis of uncertain NNs established as follows. Consider

$$\begin{cases} \dot{p}(t) = -(D + \Delta D(t))p(t) + (A + \Delta A(t))g(p(t)) + (A_1 + \Delta A_1(t))g(p(t - r(t))) \\ \qquad\qquad +(A_2 + \Delta A_2(t)) \int_{t-d(t)}^{t} g(p(s))\, ds + u(t), \\ q(t) = C_1 g(p(t)) + C_2 g(p(t - r(t))) + C_3 \int_{t-d(t)}^{t} g(p(s))\, ds + u(t), \\ p(t) = \phi(t), \quad t \in [-\theta, 0], \quad \theta = \max\{r_2, d\}, \end{cases} \tag{3.15}$$

where $\Delta D(t)$, $\Delta A(t)$, $\Delta A_1(t)$, $\Delta A_2(t)$ are the time-varying parameter uncertainties, which are assumed to be of the form

$$[\Delta D(t)\ \ \Delta A(t)\ \ \Delta A_1(t)\ \ \Delta A_2(t)] \ = \ MF(t)[N_1\ \ N_2\ \ N_3\ \ N_4], \tag{3.16}$$

where M, N_1, N_2, N_3 and N_4 are known real constant matrices, and $F(\cdot)$ is an unknown time-varying matrix function satisfying $F^T(t)F(t) \leq I$ then we have the following result.

Theorem 3.2. *The delayed neural network in (3.15) is passive in the sense of definition 2.1 for any delays $\tau(t)$ and $d(t)$ satisfying $0 \leq r_1 \leq r(t) \leq r_2$ and $0 \leq d(t) \leq d$ if there exist matrices $P \in \mathbb{S}_{4n}^+; Q_1, S_3 \in \mathbb{S}_{2n}^+; Q_2, Q_3, S_1, S_2, S_4, R_1, R_2, Z_1, Z_2 \in \mathbb{S}_n^+; \Delta_k, W_\sigma \in \mathbb{D}_n^+$, $(k = 1, 2, ..., 8; \sigma = 1, 2)$, and a scalar $\gamma > 0$ satisfy the following LMI:*

$$\begin{bmatrix} \Phi(r) + \epsilon \Xi_2^T \Xi_2 & \Xi_1^T \\ \Xi_1 & -\epsilon I \end{bmatrix} \ < \ 0, \tag{3.17}$$

where

$$\begin{aligned} \Xi_1 &= [P_1 M\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0], \\ \Xi_2 &= [-N_1\ \ 0\ \ 0\ \ 0\ \ N_2\ \ N_3\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ N_4\ \ 0], \end{aligned}$$

and $\Phi(r)$ is defined in theorem 3.1.

Proof. Replacing $D, A, A_1,$ and A_2 in theorem 3.1 with $D+MF(t)N_1, A+MF(t)N_2, A_1+MF(t)N_3, A_2+MF(t)N_4$ respectively, so we have

$$\Phi(r) + \Xi_1^T F(t)\Xi_2 + \Xi_2^T F(t)\Xi_1 \ < \ 0. \tag{3.18}$$

Applying lemma 2.4, it can be deduced that for $\epsilon > 0$

$$\Phi(r) + \epsilon^{-1}\Xi_1^T \Xi_1 + \epsilon \Xi_2^T \Xi_2 \ < \ 0. \tag{3.19}$$

From lemma 2.5 shows that (3.19) is equivalent to (3.17), therefore the proof is completed. □

Remark 4. *If delayed NNs (2.1) are setting as $C_2 = 0, C_3 = 0$ and $C_4 = 0$, the networks model turns into the delayed NNs proposed in [28, 33, 38, 40]:*

$$\begin{cases} \dot{p}(t) = -Dp(t) + Ag(p(t)) + A_1 g(p(t - r(t))) + A_2 \int_{t-d(t)}^{t} g(p(s))\, ds + u(t), \\ q(t) = C_1 g(p(t)), \\ p(t) = \phi(t), \quad t \in [-d, 0]. \end{cases} \tag{3.20}$$

Hence, our network model (2.1) includes previous network model, which can be regarded as a special case of neural network (2.1).

Remark 5. *To illuminate how to solve the upper bound of r_2 for system (2.1) satisfying time-varying delays (2.2) and neural activation functions (2.3), the following steps are performed.*

Step 1: Given positive diagonal matrix D, real matrices $A, A_1, A_2, C_1, C_2, C_3$ and positive constants r_1, d.

Step 2: Select a positive constant γ.

Step 3: Define variable matrices with appropriate dimensions $P, Q_1, S_3, Q_2, Q_3, S_1, S_2, S_4, R_1, R_2, Z_1, Z_2, \Delta_k, W_\sigma$ $(k = 1, 2, ..., 8; \sigma = 1, 2)$.

Step 4: Use matlab software to compute the value of the variable.

Step 5: Calculate the value of LMIs, in (3.1).

4. Numerical examples

In this section, three numerical examples are given to illustrate the merits of the proposed robust passivity results.

Example 4.1. *Consider a neural network (2.1) with the following parameters:*

$$D = \begin{bmatrix} 2.2 & 0 \\ 0 & 1.8 \end{bmatrix}, \; A = \begin{bmatrix} 1.2 & 1 \\ -0.2 & 0.3 \end{bmatrix}, \; A_1 = \begin{bmatrix} 0.8 & 0.4 \\ -0.2 & 0.1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$L_1 = \text{diag}\{0, 0\}$, $L_2 = \text{diag}\{1, 1\}$, $C_1 = I$, *and* $C_2 = C_3 = C_4 = 0$. *The neural activation functions are assumed to be* $g_i(p_i) = \frac{1}{2}(|p_i + 1| - |p_i - 1|)$ $(i = 1, 2)$. *It is easy to check that the neural activation functions are satisfied (2.3) with $l_i^- = 0$ and $l_i^+ = 0$ $(i = 1, 2)$. Using Matlab LMI Toolbox, we can conclude that the upper bound of r_2 without non differentiable μ which is shown in Table 1 is feasibility of the LMI in theorem 3.1. In addition, the results from [36–40] without distributed delay are listed in Table 1. As shown in this table, the criterion of this paper is less conservative than those results obtained in [36–40]. According to Figure 1, it can be confirmed that neural network (2.1) under zero input and the initial condition $[p_1(t), p_2(t)]^T = [-1, 1]^T$ is stable.*

Table 1. Upper bound of r_2 for Example 4.1.

μ	$\mu = 0.5$	unknown
[36]	0.5227	-
[39]	1.3752	-
[40]	3.0430	-
[38]	3.0835	-
[37]	3.6566	-
Theorem 3.1	-	4.1010

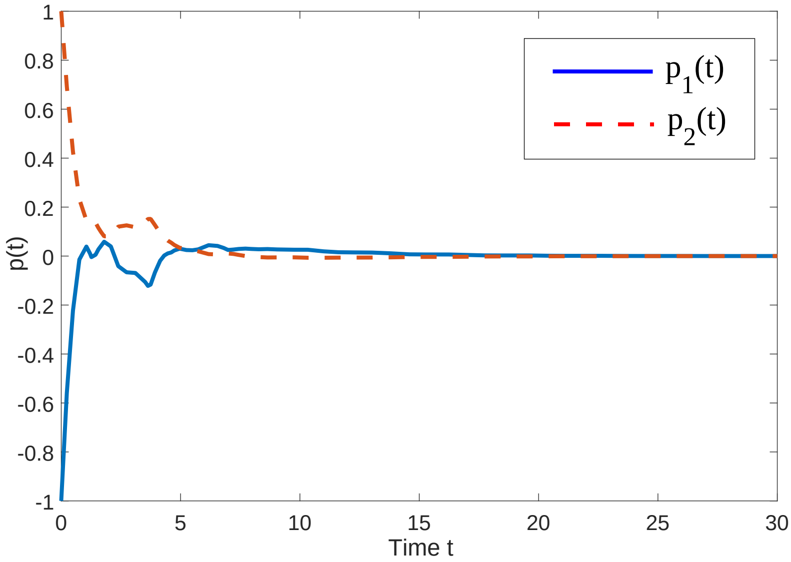

Figure 1. State trajectory of neural network in Example 4.1.

Example 4.2. *Consider a uncertain neural network (3.15) with the following parameters:*

$$D = \begin{bmatrix} 2.3 & 0 \\ 0 & 2.5 \end{bmatrix},\ A = \begin{bmatrix} 0.3 & 0.2 \\ 0.4 & 0.1 \end{bmatrix},\ A_1 = \begin{bmatrix} 0.5 & 0.7 \\ 0.7 & 0.4 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.5 & -0.3 \\ 0.2 & 1.2 \end{bmatrix},$$

$M = \mathrm{diag}\{0.1, 0.1\}$, $N_1 = N_2 = N_3 = N_4 = \mathrm{diag}\{1, 1\}$. *With these parameters, we can conclude that the upper bound of* r_2 *are shown in Table 2 is feasibility of the LMI in theorem 3.2. Moreover, the results from [33, 35, 40] are listed in Table 2. As shown in this table, the criteria of this paper is less conservative than those results obtained in [33, 35, 40]. We have activation functions as above and set* $\Delta D(t) = \Delta A(t) = \Delta A_1(t) = \Delta A_2(t) = \begin{bmatrix} 0.1\sin(t) & 0 \\ 0 & 0.1\sin(t) \end{bmatrix}$ *shown in Figure 2. From Figure 2, it can be confirmed that the neural network (3.15) without input* $u(t)$ *is robustly stable, which the initial condition* $[p_1(t), p_2(t)]^T = [1.5, -1.5]^T$.

Table 2. Upper bound of r_2 for Example 4.2.

	μ=0.1	unknown μ
[33]	0.5005	0.4269
[40]	0.5504	-
[35]	0.6621	-
Theorem 3.2	-	3.0420

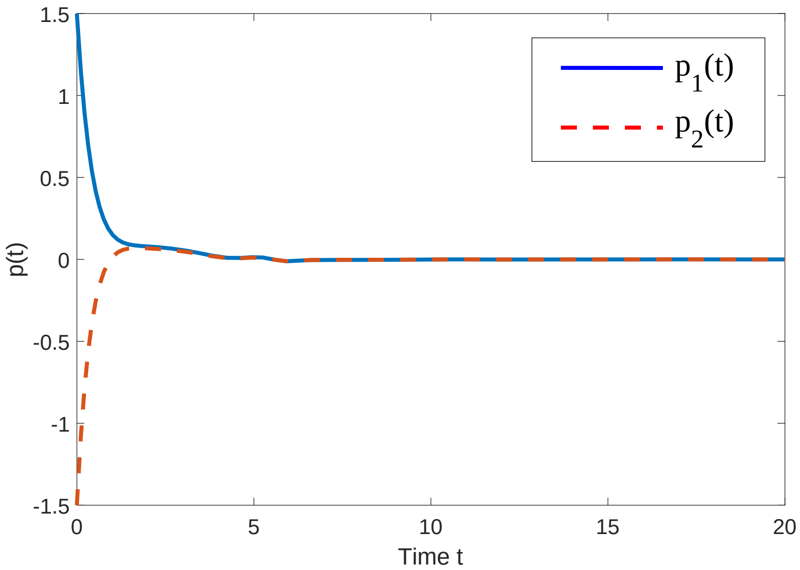

Figure 2. State trajectory of neural network in Example 4.2.

Example 4.3. *Consider a uncertain neural network (3.15) with the following parameters:*

$$D = \begin{bmatrix} 2.2 & 0 \\ 0 & 1.5 \end{bmatrix}, \; A = \begin{bmatrix} 1 & 0.6 \\ 0.1 & 0.3 \end{bmatrix}, \; A_1 = \begin{bmatrix} 1 & -0.1 \\ 0.1 & 0.2 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$M = \text{diag}\{0.1, 0.1\}$, $N_1 = 0.1, N_2 = 0.2I$, $N_3 = 0.3I$, $N_4 = \text{diag}\{0, 0\}$ *and* $C_1 = C_2 = C_3 = C_4 = I$. *In this example, we can conclude that the upper bounds of* r_2 *are shown in Table 3 is feasibility of the LMI in theorem 3.2. Moreover, the results from [33, 34, 40] without distributed delay are listed in Table 3. As shown in this table, the criterion of this paper is less conservative than those results obtained in [33, 34, 40]. We have activation functions as above and set* $\Delta D(t) = \begin{bmatrix} 0.01\sin(t) & 0 \\ 0 & 0.01\sin(t) \end{bmatrix}$, $\Delta A(t) = \begin{bmatrix} 0.02\sin(t) & 0 \\ 0 & 0.02\sin(t) \end{bmatrix}$, $\Delta A_1(t) = \begin{bmatrix} 0.03\sin(t) & 0 \\ 0 & 0.03\sin(t) \end{bmatrix}$ *shown in Figure 3. From Figure 3, it can be confirmed that the neural network (3.15) without input* $u(t)$ *is robustly stable, which the initial condition* $[p_1(t), p_2(t)]^T = [2, -1]^T$.

Table 3. Upper bound of r_2 for Example 4.3.

	μ=0.3	unknown μ
[33]	0.4197	0.3994
[40]	1.9091	-
[34]	2.1350	-
Theorem 3.2	-	2.3220

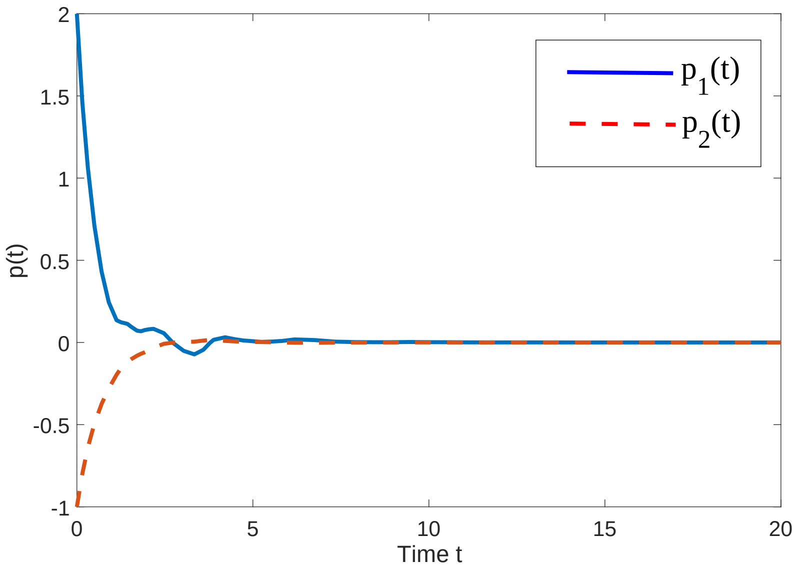

Figure 3. State trajectory of neural network in Example 4.3.

Remark 6. *An important property in linear circuit and system theory is passivity which is applicable to the analysis of properties of immittance or hybrid matrices of various classes of neural networks, inverse problem of linear optimal control, Popov criterion, circle criterion and spectral factorization by algebra [42]. In the recent years, passivity properties have also been related to the neural networks [36–40]. It should be pointed out that the aforementioned results have the restrictions on the derivative time-varying delays which mean that the delayed conditions in this work are more applicable in the real-world system by establishing Lyapunov-Krasovskii functional fully of the information of the delays* r_1, r_2 *and* d*. On the other hand, in this work, we use the refined Jensen's inequality to estimate single and double integrals. By applying the aforementioned techniques, we obtain the less conservative results than the others [25, 29, 33, 36].*

Example 4.4. *Consider a uncertain neural network (3.15) with the following parameters:*

$$D = \begin{bmatrix} 2.0 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 2.3 \end{bmatrix}, \qquad A = \begin{bmatrix} -2.7 & 0.6 & 0.34 \\ 0.5 & 1.0 & 0.16 \\ 0.8 & 2.0 & -1.0 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.8 & 0.15 & 0.16 \\ 0.5 & 0.1 & 0.25 \\ 0.5 & 0.25 & 1.0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0.37 & 0.9 & -1.38 \\ 0.18 & 0.1 & -0.13 \\ 0.5 & 0.7 & -0.5 \end{bmatrix},$$

$M = \text{diag}\{0.1, 0.1, 0.1\}$, $N_1 = 0.1I$, $N_2 = 0.2I$, $N_3 = 0.2I$, $N_4 = 0.1I$, $C_1 = C_2 = C_3 = C_4 = I$ *and*

$$\Delta D(t) = \begin{bmatrix} 0.2\sin(t) & 0 & 0 \\ 0 & 0.2\sin(t) & 0 \\ 0 & 0 & 0.3\sin(t) \end{bmatrix}, \quad \Delta A(t) = \begin{bmatrix} 0.15\sin(t) & 0 & 0 \\ 0 & 0.05\sin(t) & 0 \\ 0 & 0 & 0.1\sin(t) \end{bmatrix},$$

$$\Delta A_1(t) = \begin{bmatrix} 0.15\sin(t) & 0 & 0 \\ 0 & 0.1\sin(t) & 0 \\ 0 & 0 & 0.05\sin(t) \end{bmatrix}, \ \Delta A_2(t) = \begin{bmatrix} 0.1\sin(t) & 0 & 0 \\ 0 & 0.2\sin(t) & 0 \\ 0 & 0 & 0.1\sin(t) \end{bmatrix}.$$

In Example 4.4, the state trajectory of neural network (3.15) for $r_1 = 1, r_2 = 2, d = 0.5$ *and* $g_1(s) = g_2(s) = g_3(s) = tanh(s)$ *without input* $u(t)$ *has been analyzed. The result is robustly stable shown in Figure 4 with the initial condition* $[p_1(t), p_2(t), p_3(t)]^T = [2, 1, -2,]^T$.

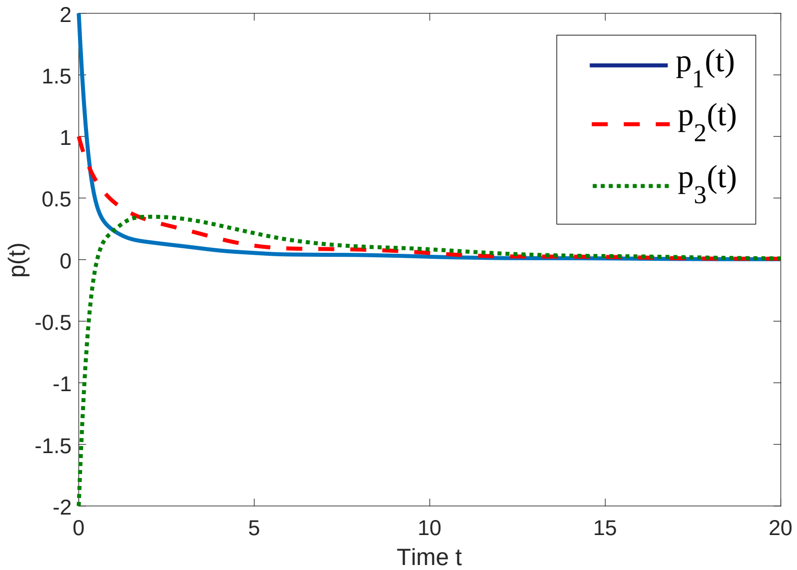

Figure 4. State trajectory of neural network in Example 4.4.

5. Conclusions

In this research, we focused on new results for robust passivity analysis of NNs with interval nondifferentiable and distributed time-varying delays. Using refined Jensen's inequalities, and applying the Lyapunov-Krasovskii functional containing single, double, triple and quadruple integrals, the new conditions were obtained in terms of LMI which can be checked by using LMI toolbox in MATLAB. Moreover, These results are less conservative than the existing ones and can be an effective method. Compared with existing ones, the obtained criteria are more effective because of the application of refined Jensen-based inequality technique comprising single and double inequalities evaluating. Three numerical examples have been proposed to show the effectiveness of the methods. For further research, we can use these methods to consider the dynamic networks with Markovian jumping delayed complex networks or stochastic delayed complex networks.

Acknowledgments

The first author was financially supported by the Thailand Research Fund (TRF), the Office of the Higher Education Commission (OHEC) (grant number : MRG6280149) and Khon Kaen University. The fourth author was financial supported by University of Pha Yao. The fifth author was supported by Rajamangala University of Technology Isan (RMUTI) and Thailand Science Research and Innovation (TSRI). Contract No.Contract No. FRB630010/0174-P6-03.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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