Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(2): 1101–1109.
DOI: 10.3934/math.2021066
Received: 10 September 2020
Accepted: 28 October 2020
Published: 10 November 2020

Research article

On the Davenport constant of a two-dimensional box $[\![-1,1]\!] \times [\![-m,n]\!]$

Guixin Deng*and Shuxin Wang

College of Mathematics and Statistics, Nanning Normal University, Nanning, 530000, China

* **Correspondence:** Email: dengguixin@live.com.

Abstract: Let G be an abelian group and X be a nonempty subset of G. A sequence S over X is called zero-sum if the sum of all terms of S is zero. A nonempty zero-sum sequence S is called minimal zero-sum if all nonempty proper subsequences of S are not zero-sum. The Davenport constant of X, denoted by $\mathsf{D}(X)$, is defined to be the supremum of lengths of all minimal zero-sum sequences over X. In this paper, we study the minimal zero-sum sequences over $X = [\![-1,1]\!] \times [\![-m,n]\!] \subset \mathbb{Z}^2$. We completely determine the structure of minimal zero-sum sequences of maximal length over X and obtain that $\mathsf{D}(X) = 2(n+m)$.

Keywords: zero-sum; minimal zero-sum; Davenport constant
Mathematics Subject Classification: 11B30, 11P70

1. Introduction

Zero-sum theory mainly study the problem relative to minimal zero-sum sequences of abelian groups. This theory has applications in groups theory, graph theory and factorization theory, see the survey article [6], and the monographs [7, 9]. The study of zero-sum problems in finite abelian groups have a long history, see for example [4, 5, 10, 12, 13, 16, 18]. In 1960s, Davenport found that the principle ideal generated by an irreducible element in an algebraic number field F is the product of at most n prime ideals, where n is exactly the Davenport constant of the class group of F. In general, let H be a Krull monoid with class group G and let $X \subset G$ be the set of classes containing prime divisors. The factorization properties of H have a strong connection with zero-sum sequences over X, see [9].

The study of zero-sum problems in infinite abelian groups G mainly focus in the case that G is the free abelian groups $\mathbb{Z}^n$, see for instance [2, 8]. It was difficult to compute the exact value of Davenport constant for a general subset of an abelian group. In particular, it was suggested in [1] that we could study the Davenport constant of subsets with simple geometric structure (e.g., the product of integral interval). For two real numbers a, b, let $[\![a,b]\!] = \{n \in \mathbb{Z} : a \le n \le b\}$. For a sequence $S = x_1, \ldots, x_n$, we use $\|S\|$ to denote the number of terms appeared in S. Lambert [11] showed that

$\mathsf{D}([\![-n,n]\!]) = \max\{2, 2n-1\}$. In [15], it was shown that if S is a minimal zero-sum sequence over $\mathbb{Z}$, then $\|S^-\| \le \max\{S\}$ and $\|S^+\| \le -\min\{S\}$, where S^+, (S^-, resp) is the subsequence of S consisting of positive (negative, resp) elements. As an immediate consequence, we have

$$\mathsf{D}(X) \le \operatorname{diam}(X) = \sup_{x,y\in X} |x-y|,$$

where X is a finite subset of $\mathbb{Z}$ containing both positive and negative integers. With the same notation, Sissokho [17] showed that $\|S^+\| \cdot \|S^-\|$ is no more than the sum of all terms of S^+.

In [14], the authors used a simple method to prove that

$$\sup_{x,y\in X, x<0, y>0} \frac{|x-y|}{\gcd(x,y)} \le \mathsf{D}(X) \le \operatorname{diam}(X) = \sup_{x,y\in X} |x-y|.$$

The lower bound above comes from the example of minimal zero-sum sequence $x,\dots,x,y,\dots,y$, where x appears $\frac{y}{\gcd(x,y)}$ times and y appears $\frac{-x}{\gcd(x,y)}$ times. The structure of minimal zero-sum sequences over $[\![-m,n]\!]$ whose length are close to $n+m$ was investigated in detail in [3] and [19], and it was proved that the above lower bound is the exact value of the Davenport constant of the interval $[\![-m,n]\!]$ for all but only finitely pairs of $n,m>0$.

The exact value of Davenport constant are widely open for high dimensions. In particular, [14] showed that $\mathsf{D}([\![-1,1]\!]^2) = 4$ and $(2m-1)^2 \le \mathsf{D}([\![-m,m]\!]^2) \le (2m+1)(4m+1)$. In this paper, we determine the structure of minimal zero-sum sequences of maximal length over $[\![-1,1]\!] \times [\![-m,n]\!]$ and also obtain that $\mathsf{D}([\![-1,1]\!] \times [\![-m,n]\!]) = 2(m+n)$ for any positive integers m,n.

2. Preliminaries

Our notation and terminology are consistent with [6] and [9]. Let $\mathbb{Z}$ denote the set of integers. Let G be an abelian group (written additively) and let X be a nonempty subset of G. A sequence over X is an unordered finite sequence of terms from X for which repetition of terms is allowed. We always view sequences over X as elements of the free abelian monoid $\mathcal{F}(X)$. A sequence $S \in \mathcal{F}(X)$ is written in the form

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{i=1}^{l} g_i = \prod_{g\in X} g^{[\mathsf{v}_g(S)]},$$

where $\mathsf{v}_g(S)$ is the times of g appeared in S, which is called the multiplicity of g in S.

We call

- $\|S\| = l = \sum_{g\in X} \mathsf{v}_g(S)$ the length of S;
- $\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g\in X} \mathsf{v}_g(S) g$ the sum of S.

A sequence T is called a subsequence of S if $\mathsf{v}_g(T) \le \mathsf{v}_g(S)$ for all g. For any subset Y of G, let $S|_Y = \prod_{g\in Y} g^{[\mathsf{v}_g(S)]}$ be the subsequence of S consisting of terms of S from Y, and let $\mathsf{v}_Y(S) = \|S|_Y\| = \sum_{g\in Y} \mathsf{v}_g(S)$ be the length of $S|_Y$. In particular, if S is a sequence over $\mathbb{Z}$, let S^+ and S^- denote the subsequence consisting of all positive (resp. negative) terms of S.

A nonempty sequence S is called

- zero-sum if $\sigma(S) = 0$;
- minimal zero-sum if $\sigma(S) = 0$ and $\sigma(T) \ne 0$ for any nonempty proper subsequence T of S.

The Davenport constant of X, denoted by $\mathsf{D}(X)$, is defined as the supremum of lengths of minimal zero-sum sequences over X. We recall two basic results concerning minimal zero-sum sequences over integers.

Lemma 1. ([14, Theorem 2]) *Let S be a minimal zero-sum sequence over $[\![-m,n]\!]$. Then $\|S^+\| \le m$ and $\|S^-\| \le n$. In particular, if $\|S\| = n+m$, then $S = (-m)^{[n]} \cdot n^{[m]}$ and* $\gcd(m,n) = 1$.

Lemma 2. ([3, Lemma 3.2]) *Let S be a minimal zero-sum sequence over $[\![-m,n]\!]$ such that $\|S^+\| = u$ and $\|S^-\| = v$. Then*

$$\mathsf{v}_{[\![v,n]\!]}(S) \ge \|S\| - n, \quad and \quad \mathsf{v}_{[\![-m,-u]\!]}(S) \ge \|S\| - m.$$

3. Main results

We present and prove our main result as follows. Let $\pi_x, \pi_y : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ be the reflections such that

$$\pi_x((a,b)) = (a,-b), \quad \pi_y((a,b)) = (-a,b).$$

Then the following are equivalent.

(1) $S = \prod_{i=1}^{l} g_i$ is a minimal zero-sum sequence over $[\![-1,1]\!] \times [\![-m,n]\!]$;
(2) $\pi_y(S) = \prod_{i=1}^{l} \pi_y(g_i)$ is a minimal zero-sum sequence over $[\![-1,1]\!] \times [\![-m,n]\!]$;
(3) $\pi_x(S) = \prod_{i=1}^{l} \pi_x(g_i)$ is a minimal zero-sum sequence over $[\![-1,1]\!] \times [\![-n,m]\!]$.

Hence we have $\mathsf{D}([\![-1,1]\!] \times [\![-m,n]\!]) = \mathsf{D}([\![-1,1]\!] \times [\![-n,m]\!])$ and it suffices to deal with the case that $n \ge m$.

Theorem 1. *Let S be a minimal zero-sum sequence over $[\![-1,1]\!] \times [\![-m,n]\!]$ of length $\|S\| \ge 2(n+m)$ and $n \ge m$. Then S or $\pi_y(S)$ is one of the following:*
(1) $S = (-1,-m)^{[n]} \cdot (1,-m)^{[n]} \cdot (0,n)^{[2m]}$, $\gcd(2m,n) = 1$;
(2) $S = (-1,n)^{[m]} \cdot (1,n)^{[m]} \cdot (0,-m)^{[2n]}$, $\gcd(m,2n) = 1$;
(3) $S = (-1,n)^{[m-b]} \cdot (-1,-m)^{[n+b]} \cdot (1,b)^{[n+m]}$, $-m \le b < m$, $\gcd(n+m, m-b) = 1$;
(4) $S = (-1,n)^{[k]} \cdot (-1,-m)^{[n+m-k]} \cdot (1,n)^{[2m-k]} \cdot (1,-m)^{[n-m+k]}, n > m, 0 \le k \le 2m$, $\gcd(n+m, 2m) = 1$.

Since S is zero-sum, we may write

$$S = \prod_{i=1}^{l}(-1,a_i) \cdot \prod_{i=1}^{l}(1,b_i) \cdot \prod_{j=1}^{t}(0,c_j),$$

where $\|S\| = 2l + t$. We see that

$$S_\theta = \prod_{i=1}^{l}(a_i + b_{\theta(i)}) \cdot \prod_{j=1}^{t} c_j$$

is also a minimal zero-sum sequence over $\mathbb{Z}$ for any permutation θ of $\{1,\ldots,l\}$. The key of the proof of the main theorem is to study the structure of S_θ for some special permutations.

Lemma 3. *Suppose that θ satisfies that $\sigma(S_\theta^+) \le \sigma(S_\eta^+)$ for any permutation η. Then*

(1) *If $a_i + b_{\theta(i)} > 0$ and $a_j + b_{\theta(j)} < 0$, then $a_i \le a_j$ or $b_{\theta(i)} \le b_{\theta(j)}$;*
(2) *Suppose that $a_i + b_{\theta(i)} = \max_{1\le j\le l}\{a_j + b_{\theta(j)}\}$. Then*

$$a_k + b_{\theta(k)} \ge -m + \min\{a_i, b_{\theta(i)}\}, \ k = 1,\ldots,l.$$

Proof. After a permutation of indices if necessary, we may assume that θ is the identity map, $a_1 + b_1 = \max_{1\le i\le l}\{a_i + b_i\}$. We need the following two claims.

Claim 1. If $a_i + b_i > 0$ and $a_j + b_j < 0$, then one of $\{a_i + b_j, a_j + b_i\}$ is positive and the other one is negative.

Let η be the permutation which exchanges i and j and fixes other indices. If both $a_i + b_j$ and $a_j + b_i$ are positive, then $\sigma(S^+_\eta) - \sigma(S^+_\theta) = (a_i + b_j) + (a_j + b_i) - (a_i + b_i) = a_j + b_j < 0$. If both $a_i + b_j$ and $a_j + b_i$ are negative, then $\sigma(S^+_\eta) - \sigma(S^+_\theta) = -a_i - b_i < 0$. This contradicts to the choice of θ. Claim 1 is true.

If $a_i + b_j > 0$ and $a_j + b_i < 0$, then $\sigma(S^+_\eta) - \sigma(S^+_\theta) = a_i + b_j - (a_i + b_i) = b_j - b_i \ge 0$. If $a_i + b_j < 0$ and $a_j + b_i > 0$, then $\sigma(S^+_\eta) - \sigma(S^+_\theta) = a_j + b_i - (a_i + b_i) = a_j - a_i \ge 0$. This proves statement (1).

Claim 2.

- If each $a_i + b_i$ is positive, then $a_i = b_i = n$ for $i = 1, \ldots, l$;
- If each $a_i + b_i$ is negative, then $a_i = b_i = -m$ for $i = 1, \ldots, l$.

We only deal with the former situation. If each $a_i + b_i$ is positive, then $S_\theta = \prod_{i=1}^{l}(a_i + b_i) \cdot \prod_{j=1}^{t} c_j$ is a minimal zero-sum sequence over $[\![-m, 2n]\!]$. By Lemma 1, we have $l \le \|S^+_\theta\| \le m$. Combing with $\|S\| = 2l + t \ge 2(n + m)$, one has $\|S_\theta\| = l + t \ge 2n + m$. By Lemma 1 again,

$$S_\theta = (-m)^{[2n]} \cdot (2n)^{[m]}, \ \ \gcd(m, 2n) = 1.$$

Hence, $a_i + b_i = 2n$ and $a_i = b_i = n$. This finishes the proof of Claim 2.

The proof of statement (2) was divided into three cases.

If $a_1 + b_1 < 0$, then each $a_k + b_k$ is negative. By Claim 2, we obtain $a_i = b_i = -m$ and $a_k + b_k = -2m = -m + \min\{a_1, b_1\}$.

If $a_1 + b_1 > 0$ and $a_k + b_k < 0$, by statement (1) we have $a_k \ge a_1 \ge \min\{a_1, b_1\}$ or $b_k \ge b_1 \ge \min\{a_1, b_1\}$. Hence,

$$a_k + b_k = \min\{a_k, b_k\} + \max\{a_k, b_k\} \ge -m + \min\{a_1, b_1\}.$$

If $a_1 + b_1 > 0$ and $0 < a_k + b_k < -m + \min\{a_1, b_1\}$, then $\min\{a_1, b_1\} > m$. It follows that both $a_1 + b_j$ and $a_j + b_1$ are positive for any $j = 1, \ldots, l$. By Claim 1, each $a_j + b_j$ is positive. So $a_k = b_k = n$ by Claim 2. This contradicts to that $a_k + b_k < -m + \min\{a_1, b_1\}$. □

Proof of Theorem 1 After a permutation of indices, we may assume that $a_1 + b_1 \ge a_2 + b_2 \ge \cdots \ge a_l + b_l$ and $T = \prod_{i=1}^{l}(a_i + b_i) \cdot \prod_{j=1}^{t} c_j$ satisfies the condition in Lemma 3, that is, $\sigma(T^+) \le \sigma(S^+_\eta)$. We also assume that $a_1 \ge b_1$. (If $a_1 < b_1$, we could replace S by $\pi_y(S)$.)

Let $u = \|T^+\|$ and $v = \|T^-\|$. We divide the proof into two major cases.

Case 1: Suppose $t > 0$. Since $\|S\| = 2l + t \ge 2(n + m)$, one has

$$\|T\| = u + v = l + t \ge n + m + 1.$$

Subcase 1.1: Suppose $b_1 \ge 0$. By Lemma 3, $a_i + b_i \ge -m + b_1 \ge -m$ and T is a minimal zero-sum sequence over $[\![-m, \max\{T^+\}]\!]$. By Lemma 1, we obtain that $u \le m$, and thus $\max\{T^+\} \ge v \ge n + 1$. Hence, $\max\{T^+\} = \max_{1\le i\le l}\{a_i + b_i\} = a_1 + b_1 > n$. We have

$$\begin{aligned} u = \|T^+\| &\ge \mathsf{v}_{[\![v, a_1+b_1]\!]}(T) \\ &\ge \|T\| - a_1 - b_1 \qquad \text{(By Lemma 2)} \end{aligned}$$

$$\begin{aligned} &\geq (n-a_1)+(m-b_1)+1 \quad (\text{Because } \|T\| \geq n+m+1)\\ &\geq m-b_1+1. \end{aligned}$$

Noting that $a_1+b_1,\dots,a_l+b_l$ are contained in $[\![-m+b_1,a_1+b_1]\!]$. Since T is a minimal zero-sum sequence over $[\![-m,a_1+b_1]\!]$, by Lemma 2

$$\mathsf{v}_{[\![-m,-u]\!]}(T) \geq \|T\|-m.$$

Since $u \geq m-b_1+1$, each term of $T|_{[\![-m,-u]\!]}$ comes from $c_1,\dots,c_t$ and

$$t \geq \mathsf{v}_{[\![-m,-u]\!]}(T) \geq \|T\|-m = l+t-m,$$

so $l \leq m$. Combing with $2l+t \geq 2(n+m)$ and $l+t \leq a_1+b_1+m \leq 2n+m$, one has

$$2n+2(m-l) \leq t \leq 2n+(m-l).$$

Hence, $l=m$, $t=2n$ and $a_1=b_1=n$. We obtain that T is a minimal zero-sum sequence over $[\![-m,2n]\!]$ of length $2n+m$. By Lemma 1,

$$T=(-m)^{[2n]}\cdot(2n)^{[m]}, \quad \gcd(2n,m)=1.$$

So

$$S=(-1,n)^{[m]}\cdot(1,n)^{[m]}\cdot(0,-m)^{[2n]}, \quad \gcd(m,2n)=1.$$

Subcase 1.2: Suppose $b_1 \leq 0$. Then $\max\{T\} \leq n$ and we have $v \leq n$ by Lemma 1. Since

$$\|T\| = u+v \geq n+m+1,$$

one has $-m+b_1 \leq \min\{T\} = \min_{1\leq i\leq l}\{a_i+b_i\} < -m$. Then $u \geq n+m+1-v \geq m+1$. We have

$$\begin{aligned} v=\|T^-\| &\geq \mathsf{v}_{[\![-m+b_1,-u]\!]}(T)\\ &\geq \|T\|-m+b_1 \quad (\text{By Lemma 2})\\ &\geq n+b_1+1 \quad (\text{Because } \|T\| \geq n+m+1)\\ &\geq a_1+b_1+1. \quad (\text{Because } n \geq a_1) \end{aligned}$$

By Lemma 2,

$$\mathsf{v}_{[\![v,n]\!]}(T) \geq \|T\|-n = l+t-n.$$

Noting that the terms of $T|_{[\![a_1+b_1+1,n]\!]}$ come from $\prod_{j=1}^{t} c_j$. We have

$$t \geq \mathsf{v}_{[\![v,n]\!]}(T) \geq \|T\|-n = l+t-n.$$

So $l \leq n$. Combing with $2l+t \geq 2(n+m)$, one has $t \geq 2m$.

Since $u \geq m+1$ and all terms of $T|_{[\![-m+b_1,-m-1]\!]}$ come from $\prod_{i=1}^{l}(a_i+b_i)$, one has

$$l \geq \mathsf{v}_{[\![-m+b_1,-u]\!]}(T) \geq \|T\|-m+b_1 = l+t-m+b_1.$$

So $t \leq m - b_1 \leq 2m$. We obtain that $t = 2m, b_1 = -m$, $l = n$, and T is a minimal zero-sum sequence over $[\![-2m, n]\!]$ with length $2m + n$. By Lemma 1,

$$T = (-2m)^{[n]} \cdot n^{[2m]}, \quad \gcd(2m, n) = 1.$$

Hence,

$$S = (-1, -m)^{[n]} \cdot (1, -m)^{[n]} \cdot (0, n)^{[2m]}, \quad \gcd(2m, n) = 1.$$

Case 2: Suppose $t = 0$. By Lemma 3, T is a minimal zero-sum sequence over $[\![-m + b_1, a_1 + b_1]\!]$ of length

$$\|T\| = l \geq n + m \geq a_1 + m.$$

By Lemma 1, we have $a_1 = n$ and

$$T = (-m + b_1)^{[n+b_1]} \cdot (n + b_1)^{[m-b_1]}, \quad \gcd(m - b_1, n + b_1) = 1.$$

Thus, $a_i + b_i \in \{-m + b_1, n + b_1\}$ for any i.

We claim that: at least one of $\{a_i, b_i\}$ is b_1 for $i = 1, \ldots, n + m$.

If $a_j + b_j = -m + b_1 < 0$, by Lemma 3 we have $a_j \geq a_1 \geq n$ or $b_j \geq b_1$. If $a_j \geq n$, then $a_j + b_j \geq n - m \geq 0$, contradiction. So $b_j \geq b_1$. Combing with $a_j \geq -m$ and $a_j + b_j = -m + b_1$, we obtain that $a_j = -m$ and $b_j = b_1$.

If $a_j + b_j = a_1 + b_1 = n + b_1 > 0$ and $\max\{a_j, b_j\} < n$, then $\min\{a_j, b_j\} > b_1$. By Lemma 3 again, for any $k = 1, 2, \ldots, n + m$

$$a_k + b_k < 0 \Rightarrow a_k + b_k \geq -m + \min\{a_j, b_j\} > -m + b_1.$$

This contradicts to that $a_k + b_k = -m + b_1$.

Let $b = b_1$. We divide the remains of the proof into two subcases.

Subcase 2.1: Suppose $b > -m$. In this situation, we will show that

$$b_1 = b_2 = \cdots = b_{m+n} = b.$$

Choose $a_j + b_j = -m + b$ and $a_i + b_i = n + b$, then one of $\{a_j, b_j\}$ ($\{a_i, b_i\}$, resp) is b and the other one is $-m$ (n, resp). By Lemma 3,

$$a_1 = n \leq a_j, \text{ or } b_1 = b \leq b_j.$$

Since $n \geq m$, the situation $a_j \geq n$ cannot happen, so $b \leq b_j$. Since $b > -m$, one has $b_j = b$ and $a_j = -m$. We obtain that $b_j = b, a_j = -m$ if $a_j + b_j < 0$.

Since $a_i + b_i = n + b > 0$, by Lemma 3 again we have

$$a_i \leq a_j = -m, \text{ or } b_i \leq b_j = b.$$

If $a_i \leq -m$, then $a_i = -m$ and $b_i = b$ by the claim and the hypothesis $b \neq -m$. It follows that $a_i + b_i = -m + b < 0$, which contradicts to that $a_i + b_i > 0$. So $b_i \leq b$ and $a_i = n$. We see that in this subcase

$$a_i \in \{n, -m\}, \quad b_1 = b_2 = \cdots = b_{m+n} = b.$$

Hence,

$$S = (-1, n)^{[m-b]} \cdot (-1, -m)^{[n+b]} \cdot (1, b)^{[n+m]}, \quad \gcd(n + m, m - b) = 1.$$

Subcase 2.2: Suppose $b = -m$. If $a_j + b_j = -m + b = -2m$, then $a_j = b_j = -m$. If $a_i + b_i = n - m > 0$, then one of $\{a_i, b_i\}$ is $-m$ and the other one is n. So $a_i, b_i \in \{-m, n\}$ and $n \neq m$ in this subcase. Writing

$$S = (-1, n)^{[k]} \cdot (-1, -m)^{[n+m-k]} \cdot (1, n)^{[r]} \cdot (1, -m)^{[n+m-r]}.$$

Since S is zero-sum, one has $(k + r)n = m(2n + 2m - k - r)$ and thus $r = 2m - k$.

We show that S or $\pi_y(S)$ is one of the form (1) – (4). It is straight to verify that these sequences are all minimal zero-sum sequences. Here we only prove that (4) is a minimal zero-sum sequence. Let

$$R = (-1, n)^{[x_1]} \cdot (-1, -m)^{[x_2]} \cdot (1, n)^{[x_3]} \cdot (1, -m)^{[x_4]}$$

be a nonempty zero-sum subsequence of (4). Then

$$\begin{cases} x_1 + x_2 = x_3 + x_4 \\ n(x_1 + x_3) = m(x_2 + x_4). \end{cases}$$

Since $\gcd(n, m) = 1$, we have

$$\begin{cases} x_1 + x_3 = m \\ x_2 + x_4 = n, \end{cases} \quad \text{or} \quad \begin{cases} x_1 + x_3 = 2m \\ x_2 + x_4 = 2n. \end{cases}$$

If $x_1 + x_3 = m$ and $x_2 + x_4 = n$, then $2(x_1 + x_2) = n + m$, which contradicts to that $\gcd(n + m, 2m) = 1$. If $x_1 + x_3 = 2m$ and $x_2 + x_4 = 2n$, then $R = S$. This shows that (4) is minimal. The proof is complete.

Corollary 1. $\mathsf{D}([\![-1, 1]\!] \times [\![-m, n]\!]) = 2(n + m)$ *for any positive integers n and m.*

Proof. It follows immediately from Theorem 1 that either

$$(-1, n) \cdot (-1, -m)^{[n+m-1]} \cdot (1, m - 1)^{[n+m]}$$

or

$$(-1, -m) \cdot (-1, n)^{[n+m-1]} \cdot (1, -n + 1)^{[n+m]}$$

is a minimal zero-sum sequence over $[\![-1, 1]\!] \times [\![-m, n]\!]$ of length $2(n + m)$. □

4. Discussions and conclusions

The computation of the exact value of the Davenport constant of a general high-dimensional box seems to be very difficult. Plagne and Tringali [14] constructed minimal zero-sum sequences recursively of length $(2m - 1)^d$ over the d-dimensional box $[\![-m, m]\!]^d$. In fact, using their method one can show that there exist minimal zero-sum sequences of length $(n + m)^d$ over $[\![-m, n]\!]^d$ when $\gcd(m, n) = 1$. In particular, they showed that

$$(2m - 1)^2 \leq \mathsf{D}([\![-m, m]\!]^2) \leq (2m + 1)(4m + 1), \ m \geq 2.$$

The following example shows that the above lower bound is not sharp.

Example 1. *Let*

$$S_p = (-m,-m)^{[m^2-pm+p]} \cdot (m,-m+1)^{[m^2+pm]} \cdot (-p,m)^{[2m^2-m]},\ p \in [\![-m,m]\!],$$

be a zero-sum sequence of length $4m^2 - m + p$ *over* $[\![-m,m]\!]$. *It is easy to verify that* S *is minimal zero-sum if and only if* $\gcd(2m-1, m+p) = \gcd(m,p) = 1$. *By the Betrand hypothesis, there exists a prime* P *such that* $m < P < 2m$. *Hence,* $p_0 = P - m$ *satisfies the conditions, and we obtain that*

$$\mathsf{D}([\![-m,m]\!]^2) > 4m^2 - m.$$

In particular, if m *is odd, then*

$$(-m,-m)^{[3m-2]} \cdot (m,-m+1)^{[2m^2-2m]} \cdot (-m+2,m)^{[2m^2-m]},$$

is a minimal zero-sum sequence of length $4m^2 - 2$ *over* $[\![-m,m]\!]^2$.

Another interesting problem is to study the asymptotic behavior of the Davenport constant of $\prod_{i=1}^{d}[\![-m_i,n_i]\!]$ when m_i, n_i are growing. In [14], it was shown that for fixed $d > 0$, the quantity $\mathsf{D}([\![-m,m]\!]^d)$ grows like m^d. But it is not sure that a constant a_d exists such that

$$\mathsf{D}([\![-m,m]\!]^d) \sim a_d m^d,\ \ as\ \ m \longrightarrow \infty.$$

In the two-dimension case, to the best of our knowledge, we believe the following is true.

Conjecture 1. *Let* m_i, n_i *be positive integers,* $i = 1,2$. *Then*

$$\mathsf{D}([\![-m_1,n_1]\!] \times [\![-m_2,n_2]\!]) \le (m_1+n_1)(m_2+n_2),$$

and

$$\mathsf{D}([\![-m_1,n_1]\!] \times [\![-m_2,n_2]\!]) \sim (m_1+n_1)(m_2+n_2),$$

as $\min\{m_1,m_2,n_1,n_2\} \longrightarrow \infty$.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (11801104 and 11961050).

Conflict of interest

The author declares there is no conflicts of interest in this paper.

References

1. N. R. Baeth, A. Geroldinger, D. J. Grynkiewicz, D. Smertnig, A semigroup-theoretical view of direct-sum decompositions and associated combinatorial problems, *J. Algebra Appl.,* **14** (2015), 1–42.

2. P. Baginski, S. T. Chapman, R. Rodriguez, G. J. Schaeffer, Y. She, On the delta set and catenary degree of Krull monoids with infinite cyclic divisor class group, *J. Pure Appl. Algebra.,* **214** (2010), 1334–1339.

3. G. Deng, X. Zeng, Long minimal zero-sum sequences over a finite set of $\mathbb{Z}$, *European J. Combin.,* **67** (2018), 78–86.

4. P. van Emde Boas, A combinatorial problem on finite abelian groups II, *Math. Centrum Amsterdam ZW*, **7** (1969), 1–60.

5. P. van Emde Boas, D. Kruyswijk, A combinatorial problem on finite abelian groups III, *Math. Centrum Amsterdam ZW*, **8** (1969), 1–34.

6. W. Gao, A. Geroldinger, Zero-sum problems in abelian groups: A survey, *Expo. Math.,* **24** (2006), 337–369.

7. A. Geroldinger, *Additive group theory and non-unique factorizations, Combinatorial Number Theory and Additive Group Theory*, Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, 2009.

8. A. Geroldinger, D. J. Grynkiewicz, G. J. Schaeffer, W. A. Schmid, On the arithmetic of Krull monoids with infinite cyclic class groups, *J. Pure Appl. Algebra,* **214** (2010), 2219–2250.

9. A. Geroldinger, F. Halter-Koch, *Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory*, Pure and Applied Mathematics, Chapman & Hall/CRC, 2006.

10. A. Geroldinger, R. Schneider, On Davenportis constant, *J. Combin. Theory Ser. A*, **61** (1992), 147–152.

11. J. Lambert, Une borne pour les générateurs des solutions entières positives d'une équation diophantienne linéaire, *C. R. Acad. Sci. Paris Ser. I Math.,* **305** (1987), 39–40.

12. J. E. Olson, A combinatorial problem on finite Abelian groups I, *J. Number Theory*, **1** (1969), 8–10.

13. J. E. Olson, A combinatorial problem on finite Abelian groups II, *J. Number Theory*, **1** (1969), 195–199.

14. A. Plagne, S. Tringali, The Davenport constant of a box, *Acta Arith.,* **171** (2015), 197–219.

15. M. L. Sahs, P. A. Sissokho, J. N. Torf, A zero-sum theorem over $\mathbb{Z}$, *Integers*, **13** (2013), 1–11.

16. S. Savchev, F. Chen, Long zero-free sequences in finite cyclic groups, *Discrete Math.*, **307** (2007), 2671–2679.

17. P. A. Sissokho, A note on minimal zero-sum sequences over $\mathbb{Z}$, *Acta Arith.,* **166** (2014), 279–288.

18. P. Yuan, On the index of minimal zero-sum sequences over finite cyclic groups, *J. Combin. Theory Ser. A,* **114** (2007), 1545–1551.

19. X. Zeng, G. Deng, Minimal zero-sum sequences over $[\![-m, n]\!]$, *J. Number Theory*, **203** (2019), 230–241.

AIMS Press