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Research article

Common fixed point on the $b_v(s)$-metric space of function-valued mappings

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Abstract: In this paper, some common fixed point results are obtained in $b_v(s)$-metric spaces. $b_v(s)$-metric space generalizes not only b-metric space but also rectangular metric space, v-generalized metric space and rectangular b-metric space. Examples show how these results can be applied in concrete situations.

Keywords: fixed point; b-metric space; rectangular b-metric space; $b_v(s)$-metric space; contraction
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1. Introduction and Preliminaries

Let $(\mathcal{X}, d)$ be a metric space and T be a self mapping on $\mathcal{X}$. A point x is said to be a fixed point of T, if $Tx = x$. Fixed point theory is one of the most important theory in mathematics. It has many applications to very different type of problems arise in different branches. Also uniqueness and existence problems of fixed points are important. One of the fixed point theorems is Geraghty-type fixed point theorem. In 1973, Geraghty [17] proved a fixed point result, generalizing the Banach contraction principle. Several authors have proved later various results using Geraghty-type conditions. Fixed point results of this kind in b-metric spaces were obtained by Dukić et al. in [12].

It is well known that metric spaces are very important tool for all branches of mathematics. So mathematicians have been tried to generalize this space and transform their studies to more generalized metric spaces. As one of the most famous generalized metric spaces, in 1989, b-metric spaces was introduced by the following way.

Definition 1.1. [7] Let $\mathcal{X}$ be a nonempty set and $s \geq 1$ be a real number. A function $d : \mathcal{X}\times\mathcal{X} \to [0, \infty)$ is a b-metric if, for all $x, y, z \in \mathcal{X}$, the following conditions are satisfied

(b_1) $d(x,y) = 0$ iff $x = y$,
(b_2) $d(x,y) = d(y,x)$,
(b_3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.
The pair $(\mathcal{X}, d)$ is called a b-metric space.

It is important to notice that b-metric spaces are also not metrizable. In particular, a b-metric might not be a continuous function of its variables (see [18, Ex. 2]).

In fact a b-metric space for $s = 1$ is exactly a metric space. After this definition, many authors proved fixed point theorems for different type mappings in this space (see [3, 4, 5, 8, 9, 14, 15, 16, 19, 20, 22, 23, 29]) and the references cited therein.

Following b-metric spaces, some generalized version of this space such as extended b-metric space, dislocated b-metric space, rectangular b-metric space [13], partial b-metric space, partial ordered b-metric space, etc. were introduced. The latest generalized b-metric space was introduced by Mitrovic and Radenovic [24] in 2017 by the following way.

Definition 1.2. [24] Let $\mathcal{X}$ be a nonempty set, $d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ a function and $v \in \mathbb{N}$. d is called a $b_v(s)$-metric space if there exists a real number $s \geq 1$ such that for all $x, z \in \mathcal{X}$ and for all distinct points $y_1, y_2, \ldots, y_v \in \mathcal{X}$, each of them different from x and z the following conditions are satisfied
(b_1) $d(x,y) = 0$ iff $x = y$,
(b_2) $d(x,y) = d(y,x)$,
(b_3) $d(x,z) \leq s[d(x,y_1) + d(y_1,y_2) + \ldots + d(y_v,z)]$.
The pair $(\mathcal{X}, d)$ is called a $b_v(s)$-metric space.

Definition 1.3. [24] Let (X, d) be a $b_v(s)$-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) The sequence $\{x_n\}$ is called convergent in (X, d) and converges to x, if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) The sequence $\{x_n\}$ is called a Cauchy sequence in (X, d) if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \epsilon$ for all $n > n_0$, $p > 0$ or equivalently, if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$.

(iii) (X, d) is said to be a complete $b_v(s)$-metric space if every Cauchy sequence in X converges to some $x \in X$.

Example 1.4. [28] Consider the set $X = \{\frac{1}{n} \mid n \in \mathbb{N},\ n \geq 2\}$. Define $d : X \times X \to [0, \infty)$ by

$$d(\frac{1}{k}, \frac{1}{m}) = \begin{cases} |k-m|, & \text{if } |k-m| \neq 1 \\ \frac{1}{2}, & \text{if } |k-m| = 1. \end{cases}$$

Then (X, d) is a $b_3(3)$-metric space.

$b_v(s)$-metric space generalizes not only b-metric space but also rectangular metric space, v-generalized metric space and rectangular b-metric space. Also, Mitrovic and Radenovic prove Banach contraction principle and Reich fixed point theorem in this space. Aleksic et al. [2] prove the common fixed point theorem of Jungck in $b_v(s)$-metric spaces.

Abdullahi and Kumam [1] present the notion of partial $b_v(s)$-metric space. They obtain some topological properties and prove some fixed point theorems in this space with supporting examples.

Mitrovic et al. [25] prove Khan type and Dass-Gupta type fixed point theorems in $b_v(s)$-metric spaces. Aydi et al. [6] obtain some common fixed point theorems in partial $b_v(s)$-metric spaces. For other related studies, see [11, 21, 26, 27].

In this paper, using ideas from [10, 30] we prove some common fixed point theorems in $b_v(s)$-metric space. Moreover, we give some examples to support our new results. Thus, we obtain generalizations of several known fixed point results from the literature.

2. Main results

In this section, we give some common fixed point results. Following [12], for a real number $s > 1$, let $\mathcal{F}_s$ be the collection of all functions $\beta : [0, \infty) \to [0, \frac{1}{s})$ satisfying the following condition:

$$\limsup_{n\to\infty} \beta(t_n) = \frac{1}{s} \quad implies \quad \lim_{n\to\infty} t_n = 0.$$

As an example of a function in $\mathcal{F}_s$ may be given by $\beta(t) = e^{-2t}$, for $t > 0$ and $\beta(0) \in [0, \frac{1}{2})$. Let $(\mathcal{X}, \hat{d})$ be a complete $b_v(s)$-metric space and let $\mathcal{B}$ be the collection of all bounded functions f defined on $\mathcal{X}$ with metric $d(f, g) = \sup\{\hat{d}(f(x), g(y)) : x, y \in \mathcal{X}\}$. Recall that a function f on $\mathcal{X}$ is said to be bounded if $f(\mathcal{X})$ is a bounded subset in $\mathcal{X}$.

Remark 2.1. The metric d defined as above is a $b_v(s)$-metric on $\mathcal{B}$ because $d(f, g) = 0$ if and only if $\hat{d}(f(x), g(y)) = 0$, for all $x, y \in \mathcal{X}$, and so $f = g$. Also for all $f, g, h_1, \cdots, h_n \in \mathcal{B}$ and $x, y \in \mathcal{X}$, we have

$$\begin{aligned} \hat{d}\,(f(x), g(y)) \leq & s[\hat{d}\,(f(x), h_1(y)) + \hat{d}\,(h_1(y), h_2(y)) + \cdots + \hat{d}\,(h_n(y), g(y))\,] \\ \leq & s[\,\sup\{\hat{d}\,(f(x), h_1(y)) : x, y \in \mathcal{X}\} + \sup\{\hat{d}\,(h_1(y), h_2(y)) : x, y \in \mathcal{X}\} \\ & + \cdots + \sup\{\hat{d}\,(h_n(y), g(y) : x, y \in \mathcal{X})\}], \end{aligned}$$

and so

$$\begin{aligned} \sup\{\hat{d}\,(f(x), g(y)) : x, y \in \mathcal{X}\} \leq & s[\,\sup\{\hat{d}\,(f(x), h_1(y)) : x, y \in \mathcal{X}\} + \sup\{\hat{d}\,(h_1(y), h_2(y)) : x, y \in \mathcal{X}\} \\ & + \cdots + \sup\{\hat{d}\,(h_n(y), g(y) : x, y \in \mathcal{X})\}]. \end{aligned}$$

Therefore

$$d(f, g) \leq s[d(f, h_1) + d(h_1, h_2) + \cdots + d(h_n, g)].$$

Theorem 2.2. *Let $(\mathcal{X}, \hat{d})$ be a complete $b_v(s)$-metric space. Let $\mathcal{B}$ be the collection of all bounded functions f defined on $\mathcal{X}$ with metric*

$$d(f, g) = \sup\{\hat{d}(f(x), g(y)) : x, y \in \mathcal{X}\}.$$

Also, let T and I be the commuting mappings (i.e., $T(I(f)) = I(T(f))$ for all f in $\mathcal{B}$) defined on $\mathcal{B}$ in which $T(\mathcal{B}) \subseteq I(\mathcal{B})$, I be a continuous mapping satisfying the following contraction

$$d(Tf, Tg) \leq \beta(d(f, g))M(f, g), \tag{2.1}$$

where $\beta \in \mathcal{F}_s$ and

$$M(f, g) = \max\left\{d(If, Ig), \frac{d(If, Tf)d(Ig, Tg)}{1 + d(Tf, Tg)}, \frac{d(If, Tf)d(Ig, Tg)}{1 + d(If, Ig)}, \frac{d(If, Tf)d(Ig, Tg)}{1 + d(If, Tg) + d(Ig, Tf)}\right\},$$

for all $f, g \in \mathcal{B}$. Then T and I have a unique common fixed point.

Proof. Let $f_0 \in \mathcal{B}$ be arbitrary. Then Tf_0 and If_0 are well defined. Since $Tf_0 \in I(\mathcal{B})$, there exists a function $f_1 \in \mathcal{B}$ such that $If_1 = Tf_0$. In general, if f_n is chosen, then we can choose a point f_{n+1} in $\mathcal{B}$ such that $If_{n+1} = Tf_n$. If for some n, $If_n = If_{n+1} = Tf_n$, then $If_n = u$ is a function such that $Tu = Iu$, because we have

$$Tu = T(If_n) = (TIf_n) = (ITf_n) = I(Tf_n) = I(If_n) = Iu.$$

Now let $d(u, Tu) > 0$, then we get

$$\begin{aligned}
d(u, Tu) = d(Tf_n, Tu) \leq& \beta(d(f_n, u))M(f_n, u)\\
=& \beta(d(f_n, u)) \max\Big\{d(If_n, Iu), \frac{d(If_n, Tf_n)d(Iu, Tu)}{1 + d(Tf_n, Tu)},\\
& \frac{d(If_n, Tf_n)d(Iu, Tu)}{1 + d(If_n, Iu)}, \frac{d(If_n, Tf_n)d(Iu, Tu)}{1 + d(If_n, Tu) + d(Iu, Tf_n)}\Big\}\\
=& \beta(d(f_n, u)) \max\Big\{d(u, Iu), \frac{d(u, u)d(Iu, Tu)}{1 + d(u, Tu)},\\
& \frac{d(u, u)d(Iu, Tu)}{1 + d(u, Iu)}, \frac{d(u, u)d(Iu, Tu)}{1 + d(u, Tu) + d(Iu, u)}\Big\}\\
=& \beta(d(f_n, u))d(u, Iu)\\
=& \beta(d(f_n, u))d(u, Tu)\\
<& d(u, Tu),
\end{aligned}$$

but it is a contradiction. Hence $Tu = Iu = u$, and so the proof of this case is completed.

Now, let $If_n \neq If_{n+1}$, for all $n \geq 0$. We show that $\{If_n\}$ is a Cauchy sequence.

Step I. First we prove that $\lim_{n\to\infty} d(If_{n+1}, If_n) = l = 0$. From (2.1) for all $m, n \in \mathbb{N}$ we get

$$\begin{aligned}
d(If_{n+1}, If_n) =& d(Tf_n, Tf_{n-1})\\
\leq& \beta(d(f_n, f_{n-1}))M(f_n, f_{n-1})\\
<& M(f_n, f_{n-1})\\
=& \max\Big\{d(If_n, If_{n-1}), \frac{d(If_n, Tf_n)d(If_{n-1}, Tf_{n-1})}{1 + d(Tf_n, Tf_{n-1})},\\
& \frac{d(If_n, Tf_n)d(If_{n-1}, Tf_{n-1})}{1 + d(If_n, If_{n-1})}, \frac{d(If_n, Tf_n)d(If_n, Tf_{n-1})}{1 + d(If_n, Tf_{n-1}) + d(If_{n-1}, Tf_n)}\Big\}\\
=& \max\Big\{d(If_n, If_{n-1}), \frac{d(If_n, If_{n-1})d(Tf_n, Tf_{n-1})}{1 + d(Tf_n, Tf_{n-1})},\\
& \frac{d(If_n, If_{n+1})d(If_{n-1}, If_n)}{1 + d(If_n, If_{n-1})}, \frac{d(If_n, If_{n-1})d(If_n, If_{n-2})}{1 + d(If_n, If_{n-2}) + d(If_{n-1}, If_{n-1})}\Big\}\\
\leq& \max\Big\{d(If_n, If_{n-1}), d(If_n, If_{n-1}), d(If_n, If_{n+1}), d(If_n, If_{n-1})\Big\}\\
=& \max\Big\{d(If_n, If_{n-1}), d(If_n, If_{n+1})\Big\}.
\end{aligned}$$

If $\max\{d(If_n, If_{n-1})), d(If_n, If_{n+1})\} = d(If_n, If_{n+1})$, then

$$d(If_{n+1}, If_n) = d(Tf_n, Tf_{n-1}) \leq \beta(d(f_n, f_{n-1}))M(f_n, f_{n-1}) < d(If_n, If_{n+1}),$$

which is a contradiction. Hence $\max\{d(If_n, If_{n-1})), d(If_n, If_{n+1})\} = d(If_n, If_{n-1})$, so

$$d(If_{n+1}, If_n) \leq \beta(d(f_n, f_{n-1}))M(f_n, f_{n-1}) < d(If_n, If_{n-1}). \tag{2.2}$$

Therefore the sequence $\{d(If_{n+1}, If_n)\}$ is decreasing. So there exists $l \geq 0$ such that

$$\lim_{n\to\infty} d(If_{n+1}, If_n) = l.$$

If $l > 0$, then letting $n \to \infty$ in (2.1) we obtain $l \leq \beta(d(f_n, f_{n-1}))l$, which is impossible. Therefore

$$\lim_{n\to\infty} d(If_{n+1}, If_n) = 0. \tag{2.3}$$

Step II. Suppose that $If_n = If_m$, for some $n > m$. Hence $If_{n+1} = Tf_n = Tf_m = If_{m+1}$, and by continuing this process we get $If_{n+i} = If_{m+i}$, for each $i \in \mathbb{N}$. Then

$$\begin{aligned} d(If_m, If_{m+1}) &= d(If_n, If_{n+1}) \\ &= d(Tf_{n-1}, Tf_n) \\ &\leq \beta(d(f_{n-1}, f_n))M(f_{n-1}, f_n) \\ &\leq \max\{d(If_{n-1}, If_n), d(If_n, If_{n+1})\}. \end{aligned}$$

If $\max\{d(If_{n-1}, If_n), d(If_n, If_{n+1})\} = d(If_n, If_{n+1})$, then $d(If_n, If_{n+1}) < d(If_n, If_{n+1})$ which is a contradiction. If $\max\{d(If_{n-1}, If_n), d(If_n, If_{n+1})\} = d(If_{n-1}, If_n)$, then we have

$$\begin{aligned} d(If_m, If_{m+1}) &= d(If_n, If_{n+1}) \\ &= d(Tf_{n-1}, Tf_n) \\ &\leq \beta(d(f_{n-1}, f_n))M(f_{n-1}, f_n) < d(If_{n-1}, If_n) \\ &\leq \beta(d(f_{n-1}, f_n))M(f_{n-2}, f_{n-1}) \\ &\leq \max\{d(If_{n-2}, If_{n-1}), d(If_{n-1}, If_n)\} \\ &< d(If_{n-2}, If_{n-1}) \\ &\vdots \\ &< d(If_m, If_{m+1}), \end{aligned}$$

which is impossible. Therefore, we can assume that $If_n \neq If_m$ for $n \neq m$. Since d is a $b_v(s)$-metric space we have

$$\begin{aligned} d(If_n, If_m) &\leq s\left(d(If_n, If_{n+1}) + d(If_{n+1}, If_{m+1}) + d(If_{m+1}, If_m)\right) \\ &\leq s\left(d(If_n, If_{n+1}) + \beta(d(f_n, f_m))M(f_n, f_m) + d(If_{m+1}, If_m)\right). \end{aligned} \tag{2.4}$$

Moreover

$$\begin{aligned} d(If_n, If_m) \leq M(f_n, f_m) = \max\Big\{&d(If_n, If_m), \frac{d(If_n, Tf_n)d(If_m, Tf_m)}{1 + d(Tf_n, Tf_m)}, \\ &\frac{d(If_n, Tf_n)d(If_m, Tf_m)}{1 + d(If_n, If_m)}, \frac{d(If_n, Tf_n)d(If_m, Tf_m)}{1 + d(If_n, Tf_m) + d(If_m, Tf_n)}\Big\}. \end{aligned}$$

Taking the upper limit as $n, m \to \infty$ and using (2.3), we get

$$\limsup_{n,m\to\infty} M(f_n, f_m) = \limsup_{n,m\to\infty} d(If_n, If_m).$$

Using (2.4), we conclude that

$$\limsup_{n,m\to\infty} d(If_n, If_m) \leq s\beta(d(f_n, f_m)) \limsup_{n,m\to\infty} M(f_n, f_m) = s\beta(d(f_n, f_m)) \limsup_{n,m\to\infty} d(If_n, If_m),$$

since $\beta(d(f_n, f_m)) < \frac{1}{s}$, this is impossible elsewhere $\limsup_{n,m\to\infty} d(If_n, If_m) = 0$. Hence the sequence $\{If_n\}$ is a Cauchy sequence in $\mathcal{B}$. As $\mathcal{B}$ is the family of bounded functions defined on the complete metric space $(\mathcal{X}, d^*)$, so $(\mathcal{B}, d)$ is a complete metric space and thus the sequence $\{If_n\}$ is convergent to $f \in \mathcal{B}$, that is $\lim_{n\to\infty} If_n = \lim_{n\to\infty} Tf_{n-1} = f$.

Step III. In this step, we show that f is a common fixed point of T and I. Since T and I commute, we obtain

$$I(f) = I(\lim_{n\to\infty} Tf_n) = \lim_{n\to\infty} ITf_n = \lim_{n\to\infty} TIf_n = Tf.$$

Let $Tf = If = g$, then we get $Tg = TIf = ITf = Ig$. If $Tf \neq Tg$, then by (2.1), we obtain

$$\begin{aligned} d(Tf, Tg) \leq& \beta(d(f, g))M(f, g) \\ =& \beta(d(f, g)) \max\Big\{d(If, Ig), \frac{d(If, Tg)d(Ig, Tg)}{1 + d(Tf, Tg)}, \\ & \frac{d(If, Tf)d(Ig, Tg)}{1 + d(If, Ig)}, \frac{d(If, Tf)d(Ig, Tg)}{1 + d(If, Tg) + d(Ig, Tf)}\Big\} \\ <& d(If, Ig) = d(Tf, Tg), \end{aligned}$$

a contradiction. So we have $g = Tf = Tg$, and so $Tg = Ig = g$, i.e., g is a common fixed point for T and I. Condition (2.1) implies that g is the unique common fixed point. □

If in Theorem 2.2 we set $\beta(t) = \lambda < \frac{1}{s}$ for $t \geq 0$, then we get the following result.

Corollary 2.3. *Let $(\mathcal{X}, \hat{d})$ be a complete $b_v(s)$-metric space. Let $\mathcal{B}$ be the collection of all bounded functions f defined on $\mathcal{X}$ with metric $d(f, g) = \sup\{\hat{d}(f(x), g(y)) : x, y \in \mathcal{X}\}$. Also, let T and I be the commuting mappings defined on $\mathcal{B}$ in which $T(\mathcal{B}) \subseteq I(\mathcal{B})$, I be a continuous mapping satisfying the following contraction*

$$d(Tf, Tg) \leq \lambda M(f, g),$$

where $0 \leq \lambda < \frac{1}{s}$ and $M(f, g)$ is as in Theorem 2.2, for all $f, g \in \mathcal{B}$. Then T and I have a unique common fixed point.

Observe that

$$\begin{aligned} \alpha d(If, Ig) +& \gamma\frac{d(If, Tg)d(Ig, Tg)}{1 + d(Tf, Tg)} + \delta\frac{d(If, Tf)d(Ig, Tg)}{1 + d(If, Ig)} + \zeta\frac{d(If, Tf)d(Ig, Tg)}{1 + d(If, Tg) + d(Ig, Tf)} \\ \leq& (\alpha + \gamma + \delta + \zeta) \max\Big\{d(If, Ig), \frac{d(If, Tg)d(Ig, Tg)}{1 + d(Tf, Tg)}, \\ & \frac{d(If, Tf)d(Ig, Tg)}{1 + d(If, Ig)}, \frac{d(If, Tf)d(Ig, Tg)}{1 + d(If, Tg) + d(Ig, Tf)}\Big\} \\ =& (\alpha + \gamma + \delta + \zeta)M(f, g). \end{aligned}$$

Hence, putting $\beta(t) = \alpha + \gamma + \delta + \zeta$ in (2.1) we get the following corollary.

Corollary 2.4. *Let $(\mathcal{X}, \hat{d})$ be a complete $b_v(s)$-metric space. Let $\mathcal{B}$ be the collection of all bounded functions defined on $\mathcal{X}$ with metric $d(f,g) = \sup\{\hat{d}(f(x), g(y))) : x, y \in \mathcal{X}\}$. Also, let T and I be commuting mappings defined on $\mathcal{B}$ in which $T(\mathcal{B}) \subseteq I(\mathcal{B})$, I be a continuous mapping satisfying the following contraction*

$$\begin{aligned} d(Tf, Tg) \leq \alpha d(If, Ig) + \gamma \frac{d(If, Tg)d(Ig, Tg)}{1 + d(Tf, Tg)} + \delta \frac{d(If, Tf)d(Ig, Tg)}{1 + d(If, Ig)} \\ + \zeta \frac{d(If, Tf)d(Ig, Tg)}{1 + d(If, Tg) + d(Ig, Tf)}, \end{aligned}$$

for all $f, g \in \mathcal{B}$, where $\alpha, \gamma, \delta, \zeta \geq 0$ and $\alpha + \gamma + \delta + \zeta < \frac{1}{s}$. Then T and I have a unique common fixed point.

In following, we give two examples to support these results.

Example 2.5. Let $\mathcal{X} = [0, 1]$ and $\hat{d}$ be a $b_v(s)$-metric on $\mathcal{X}$ and $\mathcal{B} = C[0, 1]$, the set of all real-valued continuous functions on $[0, 1]$. Define the mappings T and I on $\mathcal{B}$ as $T(f)(x) = \frac{2}{3}f(x)$ and $I(f)(x) = f(x)$, for all $f \in C[0, 1]$ and $x \in [0, 1]$. Then for $\frac{2}{3} < \lambda < 1$ we have

$$\begin{aligned} d(Tf, Tg) &= d(\frac{2}{3}f, \frac{2}{3}g) \\ &= \sup\{\hat{d}(\frac{2}{3}f(x), \frac{2}{3}g(y)) : x, y \in \mathcal{X}\} \\ &= \frac{2}{3}\sup\{\hat{d}(f(x), g(y)) : x, y \in \mathcal{X}\} \\ &< \lambda d(f, g) \leq \lambda M(f, g), \end{aligned}$$

where

$$M(f, g) = \max\Big\{d(f, g), \frac{d(f, \frac{2}{3}f)d(g, \frac{2}{3}g)}{1 + d(\frac{2}{3}f, \frac{2}{3}g)}, \frac{d(f, \frac{2}{3}f)d(g, \frac{2}{3}g)}{1 + d(f, g)}, \frac{d(f, \frac{2}{3}f)d(g, \frac{2}{3}g)}{1 + d(f, \frac{2}{3}g) + d(g, \frac{2}{3}f)}\Big\}.$$

Since all the conditions required for Corollary 2.3 are satisfied, there exists a unique common fixed function of T and I. In fact, null function is a unique common fixed function.

Example 2.6. Let $\mathcal{X} = [0, 2]$ and $\hat{d}(x, y) = |x - y|$, and $\mathcal{B} = \{f, g\}$, where f, g are bounded functions on $[0, 2]$ defined as follows.

$$f(x) = \begin{cases} 1, & x \in [0, 1] \\ 0, & otherwise \end{cases}, \qquad g(x) = \begin{cases} -1, & x \in [0, 1] \\ 0, & otherwise \end{cases}$$

and

$$\beta(t) = \begin{cases} \frac{1}{2}, & t \in [0, 2] \\ 0, & otherwise. \end{cases}$$

Define the mappings T, I on $\mathcal{B}$ as $T(f) = f^2$ and $I(f) = f$, for all $f \in \mathcal{B}$ and let d be the metric defined on $\mathcal{B}$ as

$$d(f, g) = \int_0^2 |f(x) - g(x)| dx.$$

It is clear that $(\mathcal{X}, \hat{d})$ is a complete $b_1(s)$-metric space and T, I are continuous mappings. Also, f and g are bounded functions. T and I satisfy (2.1) because

$$d(Tf, Tg) = d(f^2, g^2) = \int_0^2 |f^2(x) - g^2(x)| dx = 0.$$

Moreover $M(f, g) = 2$ and so we have

$$d(Tf, Tg) = 0 < \beta(d(f, g))M(f, g) = 1.$$

Hence, all of conditions of Theorem 2.2 are fulfilled, and so there exists a common fixed function in $\mathcal{B}$. In fact, f is a common fixed function of T and I.

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Conflict of interest

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