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Research article

Miscellaneous reverse order laws and their equivalent facts for generalized inverses of a triple matrix product

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Abstract: Reverse order laws for generalized inverses of products of matrices are a class of algebraic matrix equalities that are composed of matrices and their generalized inverses, which can be used to describe the links between products of matrix and their generalized inverses and have been widely used to deal with various computational and applied problems in matrix analysis and applications. ROLs have been proposed and studied since 1950s and have thrown up many interesting but challenging problems concerning the establishment and characterization of various algebraic equalities in the theory of generalized inverses of matrices and the setting of non-commutative algebras. The aim of this paper is to provide a family of carefully thought-out research problems regarding reverse order laws for generalized inverses of a triple matrix product ABC of appropriate sizes, including the preparation of lots of useful formulas and facts on generalized inverses of matrices, presentation of known groups of results concerning nested reverse order laws for generalized inverses of the product AB, and the derivation of several groups of equivalent facts regarding various nested reverse order laws and matrix equalities. The main results of the paper and their proofs are established by means of the matrix rank method, the matrix range method, and the block matrix method, so that they are easy to understand within the scope of traditional matrix algebra and can be taken as prototypes of various complicated reverse order laws for generalized inverses of products of multiple matrices.

Keywords: matrix product; Moore–Penrose generalized inverse; reverse order law; orthogonal projector; rank; range
Mathematics Subject Classification: 15A09, 15A24

1. Introduction

Throughout this paper, let $\mathbb{C}^{m\times n}$ denote the collection of all $m\times n$ matrices over the field of complex numbers; A^* denote the conjugate transpose; $r(A)$ denote the rank of A, i.e., the maximum order of the invertible submatrix of A; $\mathscr{R}(A) = \{Ax \,|\, x \in \mathbb{C}^n\}$ denote the range (column space) of a matrix $A \in \mathbb{C}^{m\times n}$.

An $A \in \mathbb{C}^{m\times m}$ is said to be idempotent if $A^2 = A$; to be Hermitian if $A = A^*$; to be an orthogonal projector if $A^2 = A = A^*$; to be EP (range-Hermitian) if $\mathscr{R}(A) = \mathscr{R}(A^*)$; to be normal if $AA^* = A^*A$. The Moore–Penrose generalized inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the four Penrose equations

$$(1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (3)\ (AX)^* = AX,\ \ (4)\ (XA)^* = XA. \tag{1.1}$$

A matrix X is called a $\{i, \dots, j\}$-generalized inverse of A, denoted by $A^{(i,\dots,j)}$, if it satisfies the ith, ..., jth equations in (1.1). The collection of all $\{i, \dots, j\}$-generalized inverses of A is denoted by $\{A^{(i,\dots,j)}\}$. There can be found many different kinds of definitions of generalized inverses for a matrix. In comparison, generalized inverses that are defined by the four Penrose equations are most popular and well developed. It can be seen that there are all 15 types of $\{i, \dots, j\}$-generalized inverses of A by the above definition. All these generalized inverses have rich structures and occur in various theoretical and applied problems, yet $A^\dagger$, $A^{(1,3,4)}$, $A^{(1,2,4)}$, $A^{(1,2,3)}$, $A^{(1,4)}$, $A^{(1,3)}$, $A^{(1,2)}$, and $A^{(1)}$ are usually called the eight commonly-used types of generalized inverses of A in the literature. In particular, the Moore–Penrose generalized inverse of a matrix A was specially studied and recognized because $AA^\dagger$, $A^\dagger A$, $I_m - AA^\dagger$, and $I_n - A^\dagger A$ are four orthogonal projectors onto the ranges and kernels of A and A^*, respectively. Hence, it optimizes a number of interesting properties, and thus has extraordinary values in theoretical and computational mathematics with applications. For brief, we denote $P_A = AA^\dagger$ and $E_A = I_m - AA^\dagger$ in the sequel. It can also be used to represent other generalized inverses by means of certain algebraic operations of A and $A^\dagger$. The four matrix equations in (1.1) were proposed by Sir Penrose in his seminal paper [17]. It is obvious that generalized inverses of a matrix are fully determined by the four equations in (1.1), which are direct extensions to singular matrix of the four fundamental inverse operation properties $AA^{-1}A = A$, $A^{-1}AA^{-1} = A^{-1}$, $(AA^{-1})^* = AA^{-1}$, and $(A^{-1}A)^* = A^{-1}A$ for a nonsingular matrix A, and thus the algebraic connotations and characteristics of generalized inverses of a matrix are consistent with these equalities and their variations when computing and utilizing generalized inverses of matrices under various situations. In fact, generalized inverses, as an important branch of current matrix algebra, have been already developed as an independent theoretical system and analysis method in mathematics and applications. It, however, would still be of great practical and theoretical significance to deepen the study of generalized inverses from the perspective of many conceptual and fundamental problems. We refer the reader to the traditional reference books [1, 2] for more expositions on generalized inverses of matrices with a century's history.

Notice that generalized inverses of a matrix are defined to be certain common solutions of the four Penrose matrix equations. Hence it is natural to conduct various operations of matrices and generalized inverses and to establish various equalities for matrices and their generalized inverses from theoretical and applied points of view. As usual, the theory of generalized inverses processes its own lists of (more or less celebrated) problems and open questions, sometimes they are hard to appreciate or just to understand if one does not work in this field. We begin with a simple case to illustrate. Assume that $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $C \in \mathbb{C}^{p\times q}$. Then the pair of matrix products AB and ABC are defined accordingly. If they are all square matrices of the same size and assume that they are invertible matrices, then the following two best-known matrix equalities for the ordinary inverses of the pair of matrix products

$$(AB)^{-1} = B^{-1}A^{-1},\ \ (ABC)^{-1} = C^{-1}B^{-1}A^{-1} \tag{1.2}$$

hold, which are usually called the reverse order laws (ROLs) for the ordinary inverses of the matrix products AB and ABC, respectively. As fundamental properties of inverse operations of matrices, they can be used to simplify various matrix expressions that involve the inverses of products of matrices. If these matrices and their products are singular, then $A^{\dagger}$, $B^{\dagger}$, $C^{\dagger}$, $(AB)^{\dagger}$ and $(ABC)^{\dagger}$ do exist. In these cases, it is also necessary to describe the relations among these matrices. In particular, it is easy to propose the following two ROLs for the Moore–Penrose generalized inverses of AB and ABC:

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger}, \quad (ABC)^{\dagger} = C^{\dagger}B^{\dagger}A^{\dagger} \tag{1.3}$$

as natural extensions of the ROLs for the ordinary inverses of matrix products in (1.2). It is not until in 1960s that mathematicians paid attention to the study of ROLs for generalized inverses of matrix products due to the non-commutativity of matrix algebra and the lack of methodologies for dealing with complicated matrix expressions and matrix equalities that involve generalized inverses. Since then, there has been a long-term interest in the research of the two ROLs in (1.3), and there are many classic and profound results that have been obtained on the two ROLs and their variations; see e.g., some earlier and recent work [5,6,8–10,12,18,20,25–27]. In addition to these standard ROLs for the Moore–Penrose generalized inverses, it may be useful to examine further other kinds of reasonable expressions composed of generalized inverses of matrix products according to the conventional algebraic operations of matrices. Surprisingly, there exist many kinds of matrix equalities that are composed of nested products of matrices and their generalized inverses. Here we mention a group of well-known nested ROLs:

$$(AB)^{\dagger} = (A^{\dagger}AB)^{\dagger}A^{\dagger}, \tag{1.4}$$
$$(AB)^{\dagger} = B^{\dagger}(ABB^{\dagger})^{\dagger}, \tag{1.5}$$
$$(AB)^{\dagger} = B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}, \tag{1.6}$$
$$(AB)^{\dagger} = (A^{\dagger}AB)^{\dagger}(ABB^{\dagger})^{\dagger}, \tag{1.7}$$
$$(AB)^{\dagger} = (A^{\dagger}AB)^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}(ABB^{\dagger})^{\dagger}, \tag{1.8}$$
$$(AB)^{\dagger} = B^{\dagger}(ABB^{\dagger})^{\dagger}AB(A^{\dagger}AB)^{\dagger}A^{\dagger} \tag{1.9}$$

for the Moore–Penrose generalized inverse of AB, and two groups of nested ROLs:

$$(ABC)^{\dagger} = (A^{\dagger}ABC)^{\dagger}A^{\dagger}, \tag{1.10}$$
$$(ABC)^{\dagger} = C^{\dagger}(ABCC^{\dagger})^{\dagger}, \tag{1.11}$$
$$(ABC)^{\dagger} = C^{\dagger}(A^{\dagger}ABCC^{\dagger})^{\dagger}A^{\dagger}, \tag{1.12}$$
$$(ABC)^{\dagger} = (A^{\dagger}ABC)^{\dagger}B(ABCC^{\dagger})^{\dagger}, \tag{1.13}$$
$$(ABC)^{\dagger} = (A^{\dagger}ABC)^{\dagger}B(A^{\dagger}ABCC^{\dagger})^{\dagger}B(ABCC^{\dagger})^{\dagger}, \tag{1.14}$$
$$(ABC)^{\dagger} = C^{\dagger}(ABCC^{\dagger})^{\dagger}ABC(A^{\dagger}ABC)^{\dagger}A^{\dagger}, \tag{1.15}$$

and

$$(ABC)^{\dagger} = ((AB)^{\dagger}ABC)^{\dagger}(AB)^{\dagger}, \tag{1.16}$$
$$(ABC)^{\dagger} = (BC)^{\dagger}(ABC(BC)^{\dagger})^{\dagger}, \tag{1.17}$$
$$(ABC)^{\dagger} = (BC)^{\dagger}((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger}(AB)^{\dagger}, \tag{1.18}$$
$$(ABC)^{\dagger} = ((AB)^{\dagger}ABC)^{\dagger}B^{\dagger}(ABC(BC)^{\dagger})^{\dagger}, \tag{1.19}$$

$$(ABC)^{\dagger} = ((AB)^{\dagger}ABC)^{\dagger}B^{\dagger}((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger}B^{\dagger}(ABC(BC)^{\dagger})^{\dagger}, \tag{1.20}$$

$$(ABC)^{\dagger} = (BC)^{\dagger}(ABC(BC)^{\dagger})^{\dagger}ABC((AB)^{\dagger}ABC)^{\dagger}(AB)^{\dagger} \tag{1.21}$$

for the Moore–Penrose generalized inverse of the triple matrix product ABC. It should be pointed out that the common feature of the matrix equalities in (1.4)–(1.21) is: they involve the orthogonal projectors $A^{\dagger}A$, $BB^{\dagger}$, $CC^{\dagger}$, $(AB)^{\dagger}AB$, and $BC(BC)^{\dagger}$ in different places in the nested products of matrices on the right-hand sides, respectively, rather than the straightforward reverse order products $B^{\dagger}A^{\dagger}$ and $C^{\dagger}B^{\dagger}A^{\dagger}$. Therefore, they provide a mixture of multiple expressions of the Moore–Penrose generalized inverses of AB and ABC from the standpoint of orthogonal projectors. On the other hand, they all agree with (1.2) when A, B, and C are invertible matrices. To understand the motivation and reasonability of (1.4)–(1.21), it is instructive to rewrite AB and ABC as the following multiple matrix products:

$$AB = A(A^{\dagger}AB) = (ABB^{\dagger})B = A(A^{\dagger}ABB^{\dagger})B, \tag{1.22}$$

$$ABC = A(A^{\dagger}ABC) = (ABCC^{\dagger})C = A(A^{\dagger}ABCC^{\dagger})C \tag{1.23}$$

according to the definition of the Moore–Penrose generalized inverse of matrix. In these situations, applying (1.3) to the nested products in (1.22) and (1.23), respectively, we obtain (1.4)–(1.6) and (1.10)–(1.12). Equations (1.13)–(1.15) and (1.19)–(1.21) result from the products of the right-hand sides of (1.4), (1.5), (1.10), and (1.11) with AB and ABC, respectively, and simplification. Moreover, it is worth noting that the triple matrix product ABC can be written as

$$ABC = AB(AB)^{\dagger}ABB^{\dagger}BC(BC)^{\dagger}BC = AB((AB)^{\dagger}ABC(BC)^{\dagger})BC, \tag{1.24}$$

which is in fact a special situation of (1.23) by replacing A, B, and C by AB, $B^{\dagger}$, and BC, respectively. Applying (1.10)–(1.15) to (1.24) leads to (1.16)–(1.21), respectively. So that there is reasonable discourse for suggesting the ROLs in (1.4)–(1.21). This kind of consideration was first given in [5], see also [7, 11, 21–23] for some similar work. It should be pointed out that nested ROLs in (1.4)–(1.21) are different from the two ROLs in (1.3) and also they are not necessarily equivalent to the two ROLs in (1.3). However, there is solid reason for deriving various necessary and sufficient conditions for all these matrix equalities to hold by means of various usual algebraic operations of matrices. In fact, people approached the ROLs in (1.4)–(1.21) respectively for matrices, as well as elements in rings, Hilbert spaces, and C^{*}-algebras in the literature, such as, parts of (1.4)–(1.9) were formulated and approached in [3, 7, 11, 14, 15, 21, 22] and parts of (1.10)–(1.21) were considered in [4, 12, 23, 26, 28, 29] among others. It has been realized that ROLs may have millions of reasonable forms and thus they have become one of the fruitful research fields in the matrix algebra.

Because of important roles of generalized inverses in dealing with singular matrices and the reasonability of construction of ROLs, people have paid great attention to the research of ROLs under various general assumptions since 1960s. In fact, ROLs have been being conceptually amongst the most pleasant research topics concerning algebraic equalities composed of generalized inverses of matrices for people to work with. Recall that equalities and equations of matrices can be constructed arbitrarily through various conventional algebraic operations of matrices. But it is not easy to adequately address the challenges identified in determining consistency conditions of a given matrix equation and finding general solutions of the equation under general assumptions. It can be seen from (1.3)–(1.21) that reverse order laws problems are mainly concerned with establishing various reasonable equalities for products of singular matrices and their generalized inverses. Because of the non-commutativity of matrix algebra, (1.3)–(1.21) do not necessarily hold. On the other hand, observe that the right-hand sides of (1.4)–(1.9), and (1.10)–(1.21) have different structures of matrix operations. Hence the matrix equalities in (1.4)–(1.9), and (1.10)–(1.21) are not necessarily equivalent to the two ROLs in (1.3). Thus it is natural to seek conditions under which (1.4)–(1.21) hold, respectively. In the past several decades, people noticed many fundamental facts regarding the equivalences of equalities composed of matrices and their generalized inverses, some of them were discovered during the investigation of reverse order laws for generalized inverses of matrix products. During the formulation and characterization of matrix equalities, people have realized that the rank of matrix (one of the basic concepts in linear algebra, which can be calculated by counting the number of leading 1s in any row-echelon matrix to which a matrix can be carried by row operations) and various traditional algebraic and symbolical methods for calculating ranks of matrices (called the matrix rank method), can well be used to deal with various simple and complicated matrix equalities and matrix set inclusions. Now it is possible to derive identifying conditions for various ROLs for generalized inverses of matrix products to hold by means of various ordinary and effective matrix analysis tools, including the matrix rank method, the matrix range method, and the block matrix method.

A main focus of this paper is to present a full approach to the nested ROLs in (1.10)–(1.21) and their variations. The work contains the preparations of lots of useful formulas and facts on generalized inverses of matrices, and derivations of many necessary and sufficient conditions for the nested ROLs to hold through use of the matrix rank method and the block matrix method. The results obtained show essential equivalences among various matrix equalities that are composed of products of matrices and their generalized inverses. The paper is organized as follows. In Section 2, we present some preliminary formulas and results regarding operations of generalized inverses, ranks and ranges of matrices, as well as two groups of known results on ROLs of generalized inverses of products of two matrices and their variations. In Sections 3 and 4, we present miscellaneous equalities that are composed of the triple matrix products and their generalized inverses, and give detailed derivations of the main results. Conclusions and some open problems on ROLs for generalized inverses of multiple matrix products are given in Section 5.

2. Some preliminaries

We begin with presenting an assortment of necessary preliminary results, which will be used as tools in the derivations of the main results in the sequel. The formulas and facts in the following eight lemmas are well recognized and scattered in the literatures [1, 2, 13, 16, 19, 24] or easy to prove by the definitions of ranks, ranges, and generalized inverses of matrices.

Lemma 2.1. *Let $A \in \mathbb{C}^{m\times n}$. Then the following results hold*

$$(A^{\dagger})^{*} = (A^{*})^{\dagger}, \quad (A^{\dagger})^{\dagger} = A, \tag{2.1}$$

$$A^{\dagger} = A^{*}(AA^{*})^{\dagger} = (A^{*}A)^{\dagger}A^{*} = A^{*}(A^{*}AA^{*})^{\dagger}A^{*}, \tag{2.2}$$

$$(A^{*})^{\dagger}A^{*} = (AA^{\dagger})^{*} = AA^{\dagger}, \;\; A^{*}(A^{*})^{\dagger} = (A^{\dagger}A)^{*} = A^{\dagger}A, \tag{2.3}$$

$$(AA^{*})^{\dagger} = (A^{\dagger})^{*}A^{\dagger}, \;\; (A^{*}A)^{\dagger} = A^{\dagger}(A^{\dagger})^{*}, \;\; (AA^{*}A)^{\dagger} = A^{\dagger}(A^{\dagger})^{*}A^{\dagger}, \tag{2.4}$$

$$\mathscr{R}(A) = \mathscr{R}(AA^{*}) = \mathscr{R}(AA^{*}A) = \mathscr{R}(AA^{\dagger}) = \mathscr{R}((A^{\dagger})^{*}), \tag{2.5}$$

$$\mathscr{R}(A^*) = \mathscr{R}(A^*A) = \mathscr{R}(A^*AA^*) = \mathscr{R}(A^\dagger) = \mathscr{R}(A^\dagger A), \tag{2.6}$$
$$r(A) = r(A^*) = r(AA^*) = r(A^*A) = r(AA^*A) = r(A^*AA^*) = r(A^\dagger) = r(AA^\dagger) = r(A^\dagger A). \tag{2.7}$$

Lemma 2.2. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times k}$. Then the following results*

$$r[\,A,\ B\,] = r(A) + r(E_AB) = r(B) + r(E_BA), \tag{2.8}$$
$$r[\,A,\ B\,] = r(A) + r(B) - r(P_AP_B) - r(P_BP_A) + r[P_AP_B,\ P_BP_A], \tag{2.9}$$
$$2r[A,\ B] = r(A) + r(B) + r(P_A - P_B) \tag{2.10}$$

hold. In particular, the following results hold.

(a) $r[\,A,\ B\,] = r(A) \Leftrightarrow \mathscr{R}(B) \subseteq \mathscr{R}(A) \Leftrightarrow AA^\dagger B = B \Leftrightarrow E_AB = 0 \Leftrightarrow r(E_BA) = r(A) - r(B) \Leftrightarrow r(P_A - P_B) = r(A) - r(B)$.

(b) $\mathscr{R}(A) = \mathscr{R}(B) \Leftrightarrow r[A,\ B] = r(A) = r(B) \Leftrightarrow 2r[A,\ B] = r(A) + r(B) \Leftrightarrow 2r[P_AP_B,\ P_BP_A] = 2r(P_AP_B) + 2r(P_BP_A) - r(A) - r(B) \Leftrightarrow AA^\dagger = BB^\dagger$.

Lemma 2.3. *Let $M \in \mathbb{C}^{m\times n}$ and assume that $A,\ B \in \{M^{(2)}\}$. Then the following rank equality*

$$r(A - B) = r\begin{bmatrix} A \\ B \end{bmatrix} + r[A,\ B] - r(A) - r(B) \tag{2.11}$$

holds. In consequence,

$$A = B \Leftrightarrow r\begin{bmatrix} A \\ B \end{bmatrix} + r[A,\ B] = r(A) + r(B) \Leftrightarrow \mathscr{R}(A) = \mathscr{R}(B) \text{ and } \mathscr{R}(A^*) = \mathscr{R}(B^*). \tag{2.12}$$

Lemma 2.4. *Let $A_1 \in \mathbb{C}^{m\times n_1}$, $A_2 \in \mathbb{C}^{m\times n_2}$, $B_1 \in \mathbb{C}^{m\times p_1}$, and $B_2 \in \mathbb{C}^{m\times p_2}$, and assume that $\mathscr{R}(A_1) = \mathscr{R}(B_1)$ and $\mathscr{R}(A_2) = \mathscr{R}(B_2)$. Then*

$$\mathscr{R}[A_1,\ A_2] = \mathscr{R}[B_1,\ B_2] \text{ and } r[A_1,\ A_2] = r[B_1,\ B_2]. \tag{2.13}$$

Lemma 2.5. *Let $A \in \mathbb{C}^{m\times n}$ and $G \in \mathbb{C}^{n\times m}$. Then*

$$G \in \{A^{(1)}\} \Leftrightarrow AGA = A, \tag{2.14}$$
$$G \in \{A^{(1,2)}\} \Leftrightarrow AGA = A \text{ and } r(G) = r(A), \tag{2.15}$$
$$G \in \{A^{(1,3)}\} \Leftrightarrow AG = AA^\dagger \Leftrightarrow A^*AG = A^*, \tag{2.16}$$
$$G \in \{A^{(1,4)}\} \Leftrightarrow GA = A^\dagger A \Leftrightarrow GAA^* = A^*, \tag{2.17}$$
$$\begin{aligned} G \in \{A^{(1,2,3)}\} &\Leftrightarrow A^*AG = A^* \text{ and } r(G) = r(A) \\ &\Leftrightarrow A^*AG = A^* \text{ and } GE_A = 0, \end{aligned} \tag{2.18}$$
$$\begin{aligned} G \in \{A^{(1,2,4)}\} &\Leftrightarrow GAA^* = A^* \text{ and } r(G) = r(A) \\ &\Leftrightarrow GAA^* = A^* \text{ and } F_AG = 0, \end{aligned} \tag{2.19}$$
$$G \in \{A^{(1,3,4)}\} \Leftrightarrow A^*AG = A^* \text{ and } GAA^* = A^*, \tag{2.20}$$
$$\begin{aligned} G = A^\dagger &\Leftrightarrow G \in \{A^{(1,3)}\},\ G \in \{A^{(1,4)}\},\ \text{and } r(G) = r(A) \\ &\Leftrightarrow AG = AA^\dagger,\ GA = A^\dagger A,\ \text{and } r(G) = r(A) \\ &\Leftrightarrow A^*AG = A^*,\ GAA^* = A^*,\ \text{and } r(G) = r(A) \\ &\Leftrightarrow AG = AA^\dagger,\ GA = A^\dagger A,\ GE_A = 0,\ \text{and } F_AG = 0. \end{aligned} \tag{2.21}$$

Lemma 2.6. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$, $P \in \mathbb{C}^{p\times m}$, and $Q \in \mathbb{C}^{q\times n}$. Then the following results hold*

$$\mathscr{R}(A) \subseteq \mathscr{R}(B) \text{ and } r(A) = r(B) \Rightarrow \mathscr{R}(A) = \mathscr{R}(B), \tag{2.22}$$
$$\mathscr{R}(A) \subseteq \mathscr{R}(B) \Rightarrow \mathscr{R}(PA) \subseteq \mathscr{R}(PB), \tag{2.23}$$
$$\mathscr{R}(A) = \mathscr{R}(B) \Rightarrow \mathscr{R}(PA) = \mathscr{R}(PB), \tag{2.24}$$
$$\mathscr{R}(AQ^{\dagger}Q) = \mathscr{R}(AQ^{\dagger}) = \mathscr{R}(AQ^{*}Q) = \mathscr{R}(AQ^{*}). \tag{2.25}$$

Lemma 2.7. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, $M \in \mathbb{C}^{n\times n}$. Then the following range equalities*

$$\mathscr{R}(ABB^{*}A^{*}) = \mathscr{R}(ABB^{*}) = \mathscr{R}(AB), \tag{2.26}$$
$$\mathscr{R}(B^{*}A^{*}AB) = \mathscr{R}(B^{*}A^{*}A) = \mathscr{R}(B^{*}A^{*}), \tag{2.27}$$
$$\mathscr{R}(ABB^{\dagger}A^{\dagger}AB) = \mathscr{R}(ABB^{\dagger}A^{\dagger}) = \mathscr{R}(AB), \tag{2.28}$$
$$\mathscr{R}(B^{\dagger}A^{\dagger}ABB^{\dagger}A^{\dagger}) = \mathscr{R}(B^{\dagger}A^{\dagger}AB) = \mathscr{R}(B^{\dagger}A^{\dagger}) = \mathscr{R}(B^{\dagger}A^{*}) \tag{2.29}$$

hold, and the following rank equalities hold

$$r(AA^{*}ABB^{*}B) = r(A^{*}ABB^{*}) = r(ABB^{*}A^{*}) = r(B^{*}A^{*}AB) = r(AB), \tag{2.30}$$
$$r(B^{\dagger}A^{\dagger}) = r(B^{*}A^{\dagger}) = r(B^{\dagger}A^{*}) = r(AB), \tag{2.31}$$
$$r((A^{\dagger})^{*}(B^{\dagger})^{*}) = r((A^{\dagger})^{*}B) = r(A(B^{\dagger})^{*}) = r(AB), \tag{2.32}$$
$$r(BB^{\dagger}A^{\dagger}A) = r(BB^{\dagger}A^{*}A) = r(BB^{*}A^{\dagger}A) = r(AB), \tag{2.33}$$
$$r(A^{\dagger}ABB^{\dagger}) = r(A^{\dagger}ABB^{*}) = r(A^{*}ABB^{\dagger}) = r(AB), \tag{2.34}$$
$$r(ABB^{\dagger}A^{\dagger}) = r(ABB^{\dagger}A^{*}) = r(ABB^{*}A^{\dagger}) = r(AB), \tag{2.35}$$
$$r(B^{\dagger}A^{\dagger}AB) = r(B^{\dagger}A^{*}AB) = r(B^{*}A^{\dagger}AB) = r(AB), \tag{2.36}$$
$$r(ABB^{\dagger}A^{\dagger}AB) = r(ABB^{\dagger}A^{*}AB) = r(ABB^{*}A^{\dagger}AB) = r(AB), \tag{2.37}$$
$$r((BB^{*})^{\dagger}(A^{*}A)^{\dagger})) = r((BB^{*})^{\dagger}(A^{*}A)) = r(BB^{*})(A^{*}A)^{\dagger})) = r(AB), \tag{2.38}$$
$$r(B^{\dagger}(A^{*}A)^{\dagger}) = r(B^{\dagger}A^{*}A) = r((B^{*}(A^{*}A)^{\dagger})) = r(AB), \tag{2.39}$$
$$r((BB^{*})^{\dagger}A^{\dagger}) = r((BB^{*})^{\dagger}A^{*}) = r((BB^{*}A^{\dagger}) = r(AB), \tag{2.40}$$
$$r(B^{\dagger}MA^{\dagger}) = r(B^{*}MA^{\dagger}) = r(B^{\dagger}MA^{*}) = r(B^{*}MA^{*}), \tag{2.41}$$
$$r(BB^{\dagger}MA^{\dagger}A) = r(BB^{*}MA^{\dagger}) = r(BB^{\dagger}MA^{*}A) = r(BB^{*}MA^{*}A) = r(B^{*}MA^{*}). \tag{2.42}$$

Lemma 2.8. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times n}$, $C \in \mathbb{C}^{m\times q}$, and P, $Q \in \mathbb{C}^{s\times t}$. Then the following results hold.*

(a) $AB^{*} = 0 \Leftrightarrow BA^{*} = 0 \Leftrightarrow AB^{\dagger} = 0 \Leftrightarrow BA^{\dagger} = 0$.

(b) $A^{*}C = 0 \Leftrightarrow C^{*}A = 0 \Leftrightarrow A^{\dagger}C = 0 \Leftrightarrow C^{\dagger}A = 0$.

(c) $PQ^{*} = 0$ *and* $Q^{*}P = 0 \Rightarrow (P \pm Q)^{\dagger} = P^{\dagger} \pm Q^{\dagger}$.

Lemma 2.9. *Let $M \in \mathbb{C}^{m\times n}$ and assume that $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ are two orthogonal projectors, and denote $\widehat{P} = I_m - P$ and $\widehat{Q} = I_n - Q$. Then the following matrix equalities hold*

$$(PM\widehat{Q})^{\dagger} + (\widehat{P}MQ)^{\dagger} = (PM + MQ - 2PMQ)^{\dagger}, \tag{2.43}$$
$$(PM\widehat{Q})^{\dagger} - (\widehat{P}MQ)^{\dagger} = (PM - MQ)^{\dagger}, \tag{2.44}$$

$$(\widehat{P}M\widehat{Q})^{\dagger} + (PMQ)^{\dagger} = (M - PM - MQ + 2PMQ)^{\dagger}, \tag{2.45}$$
$$(\widehat{P}M\widehat{Q})^{\dagger} - (PMQ)^{\dagger} = (M - PM - MQ)^{\dagger}, \tag{2.46}$$

and the following matrix equalities hold

$$Q(PM + MQ - 2PMQ)^{\dagger}P = 0, \tag{2.47}$$
$$Q(PM - MQ)^{\dagger}P = 0, \tag{2.48}$$
$$(PMQ)^{\dagger} = Q(M - PM - MQ + 2PMQ)^{\dagger}P, \tag{2.49}$$
$$(PMQ)^{\dagger} = Q(PM + MQ - M)^{\dagger}P. \tag{2.50}$$

Proof. It is easy to verify that

$$PM\widehat{Q} + \widehat{P}MQ = PM + MQ - 2PMQ, \tag{2.51}$$
$$PM\widehat{Q} - \widehat{P}MQ = PM - MQ, \tag{2.52}$$
$$\widehat{P}M\widehat{Q} + PMQ = M - PM - MQ + 2PMQ, \tag{2.53}$$
$$\widehat{P}M\widehat{Q} - PMQ = M - PM - MQ \tag{2.54}$$

hold by expanding the left-hand sides of (2.51)–(2.54). In these cases, applying Lemma 2.8 (c) to both sides of (2.51)–(2.54) yields (2.43)–(2.46). Pre- and post-multiplying both sides of (2.43)–(2.46) with Q and P, respectively, and simplifying lead to (2.47)–(2.50). □

Finally, we give a summary of known solutions (with some modifications) established in [27] to the two nested ROLs in (1.4) and (1.5), which will directly be used to derive the main results in Section 3.

Lemma 2.10. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then the following 62 statements are equivalent*:

⟨1⟩ $\{(AB)^{(1,2,3)}\} \ni (A^{\dagger}AB)^{\dagger}A^{\dagger}$.

⟨2⟩ $\{(AB)^{(1,2,3)}\} \ni (A^{*}AB)^{\dagger}A^{*}$.

⟨3⟩ $\{(A^{\dagger}AB)^{(1,2,3)}\} \ni (AB)^{\dagger}A$.

⟨4⟩ $\{(A^{*}AB)^{(1,2,3)}\} \ni (AB)^{\dagger}(A^{\dagger})^{*}$.

⟨5⟩ $\{(AB)^{(1,2,3)}\} \ni B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}$.

⟨6⟩ $\{(AB)^{(1,2,3)}\} \ni B^{*}(A^{*}ABB^{*})^{\dagger}A^{*}$.

⟨7⟩ $\{(A^{\dagger}ABB^{\dagger})^{(1,2,3)}\} \ni B(AB)^{\dagger}A$.

⟨8⟩ $\{(A^{*}ABB^{*})^{(1,2,3)}\} \ni (B^{\dagger})^{*}(AB)^{\dagger}(A^{\dagger})^{*}$.

⟨9⟩ $(AB)^{\dagger} = (A^{\dagger}AB)^{\dagger}A^{\dagger}$.

⟨10⟩ $AB = ((A^{\dagger}AB)^{\dagger}A^{\dagger})^{\dagger}$.

⟨11⟩ $AB = A((AB)^{\dagger}A)^{\dagger}$.

⟨12⟩ $(AB)^{\dagger}A = (A^{\dagger}AB)^{\dagger}$.

⟨13⟩ $B(AB)^{\dagger}A = B(A^{\dagger}AB)^{\dagger}$.

⟨14⟩ $AB(AB)^{\dagger} = AB(A^{\dagger}AB)^{\dagger}A^{\dagger}$.

⟨15⟩ $AB(AB)^{\dagger}A = AB(A^{\dagger}AB)^{\dagger}$.

⟨16⟩ $B^{\dagger}A^{\dagger}AB(AB)^{\dagger} = B^{\dagger}A^{\dagger}$.

⟨17⟩ $(B^\dagger A^\dagger)^\dagger B^\dagger A^\dagger AB = AB$.

⟨18⟩ $AB(AB)^\dagger = (B^\dagger A^\dagger)^\dagger B^\dagger A^\dagger$.

⟨19⟩ $(AB)^\dagger = (A^*AB)^\dagger A^*$.

⟨20⟩ $AB = ((A^*AB)^\dagger A^*)^\dagger$.

⟨21⟩ $AB = (A^\dagger)^*((AB)^\dagger (A^\dagger)^*)^\dagger$.

⟨22⟩ $(AB)^\dagger (A^\dagger)^* = (A^*AB)^\dagger$.

⟨23⟩ $B(AB)^\dagger (A^\dagger)^* = B(A^*AB)^\dagger$.

⟨24⟩ $AB(AB)^\dagger = AB(A^*AB)^\dagger A^*$.

⟨25⟩ $AB(AB)^\dagger (A^\dagger)^* = AB(A^*AB)^\dagger$.

⟨26⟩ $(ABB^\dagger A^\dagger)^\dagger = (B^\dagger A^\dagger)^\dagger (AB)^\dagger$.

⟨27⟩ $AB(A^\dagger AB)^\dagger A^\dagger$ *is an orthogonal projector.*

⟨28⟩ $AB(A^*AB)^\dagger A^*$ *is an orthogonal projector.*

⟨29⟩ $A^\dagger (AB)(AB)^\dagger A$ *is an orthogonal projector.*

⟨30⟩ $A(A^\dagger ABB^\dagger)^\dagger A^\dagger$ *is an orthogonal projector.*

⟨31⟩ $ABB^*(A^*ABB^*)^\dagger A^*$ *is an orthogonal projector.*

⟨32⟩ $AB(AB)^\dagger$ *and* AA^* *commute.*

⟨33⟩ $AB(A^\dagger AB)^\dagger A^\dagger$ *and* AA^* *commute.*

⟨34⟩ $AB(A^*AB)^\dagger A^*$ *and* AA^* *commute.*

⟨35⟩ $A^\dagger (AB)(AB)^\dagger A$ *and* A^*A *commute.*

⟨36⟩ $ABB^\dagger A^\dagger$ *is EP.*

⟨37⟩ $\{((A^\dagger)^*B)^{(1,2,3)}\} \ni (A^\dagger AB)^\dagger A^*$.

⟨38⟩ $\{((A^\dagger)^*B)^{(1,2,3)}\} \ni ((A^*A)^\dagger B)^\dagger A^\dagger$.

⟨39⟩ $\{(A^\dagger AB)^{(1,2,3)}\} \ni ((A^\dagger)^*B)^\dagger (A^\dagger)^*$.

⟨40⟩ $\{((A^*A)^\dagger B)^{(1,2,3)}\} \ni ((A^\dagger)^*B)^\dagger A$.

⟨41⟩ $\{((A^\dagger)^*B)^{(1,2,3)}\} \ni B^\dagger (A^\dagger ABB^\dagger)^\dagger A^*$.

⟨42⟩ $\{((A^\dagger)^*B)^{(1,2,3)}\} \ni B^*((A^*A)^\dagger BB^*)^\dagger A^\dagger$.

⟨43⟩ $\{(A^\dagger ABB^\dagger)^{(1,2,3)}\} \ni B((A^\dagger)^*B)^\dagger (A^\dagger)^*$.

⟨44⟩ $\{((A^*A)^\dagger BB^*)^{(1,2,3)}\} \ni (B^\dagger)^*((A^\dagger)^*B)^\dagger A$.

⟨45⟩ $((A^\dagger)^*B)^\dagger = (A^\dagger AB)^\dagger A^*$.

⟨46⟩ $((A^\dagger)^*B)^\dagger = ((A^*A)^\dagger B)^\dagger A^\dagger$.

⟨47⟩ $(A^\dagger AB)^\dagger = ((A^\dagger)^*B)^\dagger (A^\dagger)^*$.

⟨48⟩ $((A^*A)^\dagger B)^\dagger = ((A^\dagger)^*B)^\dagger A$.

⟨49⟩ $A(B^\dagger A^\dagger A)^\dagger B^\dagger A^\dagger$ *is an orthogonal projector.*

⟨50⟩ $(A^\dagger)^*(B^\dagger (A^*A)^\dagger)^\dagger B^\dagger A^\dagger$ *is an orthogonal projector.*

⟨51⟩ $A^\dagger (B^\dagger A^\dagger)^\dagger (B^\dagger A^\dagger)A$ *is an orthogonal projector.*

⟨52⟩ $A(BB^\dagger A^\dagger A)^\dagger A^\dagger$ *is an orthogonal projector.*

⟨53⟩ $(A^\dagger)^*(BB^*)^\dagger ((A^*A)^\dagger (BB^*)^\dagger)^\dagger A^\dagger$ *is an orthogonal projector.*

$\langle 54\rangle$ $(B^\dagger A^\dagger)^\dagger(B^\dagger A^\dagger)$ *and* $(AA^*)^\dagger$ *commute*.

$\langle 55\rangle$ $A(B^\dagger A^\dagger A)^\dagger B^\dagger A^\dagger$ *and* $(AA^*)^\dagger$ *commute*.

$\langle 56\rangle$ $(A^\dagger)^*(B^\dagger(A^*A)^\dagger)^\dagger B^\dagger A^\dagger$ *and* $(AA^*)^\dagger$ *commute*.

$\langle 57\rangle$ $A^\dagger(B^\dagger A^\dagger)^\dagger(B^\dagger A^\dagger)A$ *and* $(A^*A)^\dagger$ *commute*.

$\langle 58\rangle$ $\mathscr{R}(AA^*AB) = \mathscr{R}(AB)$.

$\langle 59\rangle$ $\mathscr{R}((A^\dagger)^*(B^\dagger)^*) = \mathscr{R}(AB)$.

$\langle 60\rangle$ $r[AA^*AB,\ AB] = r(AB)$.

$\langle 61\rangle$ $r[(A^\dagger)^*(B^\dagger)^*,\ AB] = r(AB)$.

$\langle 62\rangle$ $r[(A^*AA^*)^\dagger B,\ (A^\dagger)^*B] = r((A^\dagger)^*B)$.

Lemma 2.11. *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$. *Then the following 62 statements are equivalent*:

$\langle 1\rangle$ $\{(AB)^{(1,2,4)}\} \ni B^\dagger(ABB^\dagger)^\dagger$.

$\langle 2\rangle$ $\{(AB)^{(1,2,4)}\} \ni B^*(ABB^*)^\dagger$.

$\langle 3\rangle$ $\{(ABB^\dagger)^{(1,2,4)}\} \ni B(AB)^\dagger$.

$\langle 4\rangle$ $\{(ABB^*)^{(1,2,4)}\} \ni (B^\dagger)^*(AB)^\dagger$.

$\langle 5\rangle$ $\{(AB)^{(1,2,4)}\} \ni B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$.

$\langle 6\rangle$ $\{(AB)^{(1,2,4)}\} \ni B^*(A^*ABB^*)^\dagger A^*$.

$\langle 7\rangle$ $\{(A^\dagger ABB^\dagger)^{(1,2,4)}\} \ni B(AB)^\dagger A$.

$\langle 8\rangle$ $\{(A^*ABB^*)^{(1,2,4)}\} \ni (B^\dagger)^*(AB)^\dagger(A^\dagger)^*$.

$\langle 9\rangle$ $(AB)^\dagger = B^\dagger(ABB^\dagger)^\dagger$.

$\langle 10\rangle$ $AB = (B^\dagger(ABB^\dagger)^\dagger)^\dagger$.

$\langle 11\rangle$ $AB = (B(AB)^\dagger)^\dagger B$.

$\langle 12\rangle$ $B(AB)^\dagger = (ABB^\dagger)^\dagger$.

$\langle 13\rangle$ $B(AB)^\dagger A = (ABB^\dagger)^\dagger A$.

$\langle 14\rangle$ $(AB)^\dagger AB = B^\dagger(ABB^\dagger)^\dagger AB$.

$\langle 15\rangle$ $B(AB)^\dagger AB = (ABB^\dagger)^\dagger AB$.

$\langle 16\rangle$ $(AB)^\dagger ABB^\dagger A^\dagger = B^\dagger A^\dagger$.

$\langle 17\rangle$ $ABB^\dagger A^\dagger(B^\dagger A^\dagger)^\dagger = AB$.

$\langle 18\rangle$ $(AB)^\dagger AB = B^\dagger A^\dagger(B^\dagger A^\dagger)^\dagger$.

$\langle 19\rangle$ $(AB)^\dagger = B^*(ABB^*)^\dagger$.

$\langle 20\rangle$ $AB = (B^*(ABB^*)^\dagger)^\dagger$.

$\langle 21\rangle$ $AB = ((B^\dagger)^*(AB)^\dagger)^\dagger(B^\dagger)^*$.

$\langle 22\rangle$ $(B^\dagger)^*(AB)^\dagger = (ABB^*)^\dagger$.

$\langle 23\rangle$ $(B^\dagger)^*(AB)^\dagger A = (ABB^*)^\dagger A$.

$\langle 24\rangle$ $(AB)^\dagger AB = B^*(ABB^*)^\dagger AB$.

$\langle 25\rangle$ $(B^\dagger)^*(AB)^\dagger AB = (ABB^*)^\dagger AB$.

$\langle 26\rangle$ $(B^\dagger A^\dagger AB)^\dagger = (AB)^\dagger(B^\dagger A^\dagger)^\dagger$.

⟨27⟩ $B^{\dagger}(ABB^{\dagger})^{\dagger}AB$ *is an orthogonal projector.*

⟨28⟩ $B^{*}(ABB^{*})^{\dagger}AB$ *is an orthogonal projector.*

⟨29⟩ $B(AB)^{\dagger}(AB)B^{\dagger}$ *is an orthogonal projector.*

⟨30⟩ $B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}B$ *is an orthogonal projector.*

⟨31⟩ $B^{*}(A^{*}ABB^{*})^{\dagger}A^{*}AB$ *is an orthogonal projector.*

⟨32⟩ $(AB)^{\dagger}AB$ *and* $B^{*}B$ *commute.*

⟨33⟩ $B^{\dagger}(ABB^{\dagger})^{\dagger}AB$ *and* $B^{*}B$ *commute.*

⟨34⟩ $B^{*}(ABB^{*})^{\dagger}AB$ *and* $B^{*}B$ *commute.*

⟨35⟩ $B(AB)^{\dagger}(AB)B^{\dagger}$ *and* BB^{*} *commute.*

⟨36⟩ $B^{\dagger}A^{\dagger}AB$ *is EP.*

⟨37⟩ $\{(A(B^{\dagger})^{*})^{(1,2,4)}\} \ni B^{*}(ABB^{\dagger})^{\dagger}$.

⟨38⟩ $(A(B^{\dagger})^{*})^{(1,2,4)} \ni B^{\dagger}(A(BB^{*})^{\dagger})^{\dagger}$.

⟨39⟩ $\{(ABB^{\dagger})^{(1,2,4)}\} \ni (B^{\dagger})^{*}(A(B^{\dagger})^{*})^{\dagger}$.

⟨40⟩ $\{(A(BB^{*})^{\dagger})^{(1,2,4)}\} \ni B(A(B^{\dagger})^{*})^{\dagger}$.

⟨41⟩ $\{(A(B^{\dagger})^{*})^{(1,2,4)}\} \ni B^{*}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger}$.

⟨42⟩ $\{(A(B^{\dagger})^{*})^{(1,2,4)}\} \ni B^{\dagger}(A^{*}A(BB^{*})^{\dagger})^{\dagger}A^{*}$.

⟨43⟩ $\{(A^{\dagger}ABB^{\dagger})^{(1,2,4)}\} \ni (B^{\dagger})^{*}(A(B^{\dagger})^{*})^{\dagger}A$.

⟨44⟩ $\{(A^{*}A(BB^{*})^{\dagger})^{(1,2,4)}\} \ni B(A(B^{\dagger})^{*})^{\dagger}(A^{\dagger})^{*}$.

⟨45⟩ $(A(B^{\dagger})^{*})^{\dagger} = B^{*}(ABB^{\dagger})^{\dagger}$.

⟨46⟩ $(A(B^{\dagger})^{*}) = B^{\dagger}(A(BB^{*})^{\dagger})^{\dagger}$.

⟨47⟩ $(ABB^{\dagger})^{\dagger} = (B^{\dagger})^{*}(A(B^{\dagger})^{*})^{\dagger}$.

⟨48⟩ $(A(B^{\dagger})^{*})^{\dagger} = B(A(BB^{*})^{\dagger})^{\dagger}$.

⟨49⟩ $B^{\dagger}A^{\dagger}(BB^{\dagger}A^{\dagger})^{\dagger}B$ *is an orthogonal projector.*

⟨50⟩ $B^{\dagger}A^{\dagger}((BB^{*})^{\dagger}A^{\dagger})^{\dagger}(B^{\dagger})^{*}$ *is an orthogonal projector.*

⟨51⟩ $B(B^{\dagger}A^{\dagger})(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger}$ *is orthogonal projector.*

⟨52⟩ $B^{\dagger}(BB^{\dagger}A^{\dagger}A)^{\dagger}B$ *is an orthogonal projector.*

⟨53⟩ $B^{\dagger}((A^{*}A)^{\dagger}(BB^{*})^{\dagger})^{\dagger}(A^{*}A)^{\dagger}(B^{\dagger})^{*}$ *is an orthogonal projector.*

⟨54⟩ $(B^{\dagger}A^{\dagger})(B^{\dagger}A^{\dagger})^{\dagger}$ *and* $(B^{*}B)^{\dagger}$ *commute.*

⟨55⟩ $B^{\dagger}A^{\dagger}(BB^{\dagger}A^{\dagger})^{\dagger}B$ *and* $(B^{*}B)^{\dagger}$ *commute.*

⟨56⟩ $B^{\dagger}A^{\dagger}((BB^{*})^{\dagger}A^{\dagger})^{\dagger}(B^{\dagger})^{*}$ *and* $(B^{*}B)^{\dagger}$ *commute.*

⟨57⟩ $B(B^{\dagger}A^{\dagger})(B^{\dagger}A^{\dagger})^{\dagger}B^{\dagger}$ *and* $(BB^{*})^{\dagger}$ *commute.*

⟨58⟩ $\mathscr{R}(B^{*}BB^{*}A^{*}) = \mathscr{R}(B^{*}A^{*})$.

⟨59⟩ $\mathscr{R}(B^{\dagger}A^{\dagger}) = \mathscr{R}(B^{*}A^{*})$.

⟨60⟩ $r[B^{*}BB^{*}A^{*},\ B^{*}A^{*}] = r(AB)$.

⟨61⟩ $r[B^{\dagger}A^{\dagger},\ B^{*}A^{*}] = r(AB)$.

⟨62⟩ $r[(BB^{*}B)^{\dagger}A^{\dagger},\ B^{\dagger}A^{\dagger}] = r(B^{\dagger}A^{\dagger})$.

3. Main results I

In this section, we are primarily concerned with the nested ROLs in (1.10)–(1.15). To begin, we present a group of known results in [26] on ROLs for $\{1\}$- and $\{1,2\}$-generalized inverses of matrices associated with (1.10)–(1.15).

Lemma 3.1. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, *and* $C \in \mathbb{C}^{p\times q}$.

⟨1⟩ *The following 3 matrix set inclusions always hold*

$$\{(ABC)^{(1)}\} \supseteq \{(A^{(1)}ABC)^{(1)}A^{(1)}\}, \tag{3.1}$$
$$\{(ABC)^{(1)}\} \supseteq \{C^{(1)}(ABCC^{(1)})^{(1)}\}, \tag{3.2}$$
$$\{(ABC)^{(1)}\} \supseteq \{C^{(1)}(A^{(1)}ABCC^{(1)})^{(1)}A^{(1)}\}. \tag{3.3}$$

⟨2⟩ *The following 3 matrix set inclusions always hold*

$$\{(ABC)^{(1,2)}\} \supseteq \{(A^{(1,2)}ABC)^{(1,2)}A^{(1,2)}\}, \tag{3.4}$$
$$\{(ABC)^{(1,2)}\} \supseteq \{C^{(1,2)}(ABCC^{(1,2)})^{(1,2)}\}, \tag{3.5}$$
$$\{(ABC)^{(1,2)}\} \supseteq \{C^{(1,2)}(A^{(1,2)}ABCC^{(1,2)})^{(1,2)}A^{(1,2)}\}. \tag{3.6}$$

⟨3⟩ *The following 4 results always hold*

$$\{(ABC)^{(1,2)}\} \ni (A^{\dagger}ABC)^{\dagger}A^{\dagger}, \tag{3.7}$$
$$\{(ABC)^{(1,2)}\} \ni C^{\dagger}(ABCC^{\dagger})^{\dagger}, \tag{3.8}$$
$$\{(ABC)^{(1,2)}\} \ni C^{\dagger}(A^{\dagger}ABCC^{\dagger})^{\dagger}A^{\dagger}, \tag{3.9}$$
$$\{(ABC)^{(1,2)}\} \ni C^{\dagger}(ABCC^{\dagger})^{\dagger}ABC(A^{\dagger}ABC)^{\dagger}A^{\dagger}. \tag{3.10}$$

⟨4⟩ *The two matrix equalities in* (1.13) *and* (1.14) *always hold. In particular, if* $A^{\dagger}ABC = BC$ *and* $ABCC^{\dagger} = AB$, *then* $(ABC)^{\dagger} = (BC)^{\dagger}B(AB)^{\dagger}$.

The correctness of (3.1)–(3.10) can directly be verified by the definitions of the generalized inverses and ordinary operations of the given matrices. Given Lemmas 2.10 and 2.11, we are now ready to establish a wide coverage of necessary and sufficient conditions for (1.10) and (1.11) to hold, respectively.

Theorem 3.2. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, *and* $C \in \mathbb{C}^{p\times q}$. *Then the following 62 statements are equivalent*:

⟨1⟩ $\{(ABC)^{(1,2,3)}\} \ni (A^{\dagger}ABC)^{\dagger}A^{\dagger}$.

⟨2⟩ $\{(ABC)^{(1,2,3)}\} \ni (A^{*}ABC)^{\dagger}A^{*}$.

⟨3⟩ $\{(A^{\dagger}ABC)^{(1,2,3)}\} \ni (ABC)^{\dagger}A$.

⟨4⟩ $\{(A^{*}ABC)^{(1,2,3)}\} \ni (ABC)^{\dagger}(A^{\dagger})^{*}$.

⟨5⟩ $\{(ABC)^{(1,2,3)}\} \ni (BC)^{\dagger}(A^{\dagger}ABC(BC)^{\dagger})^{\dagger}A^{\dagger}$.

⟨6⟩ $\{(ABC)^{(1,2,3)}\} \ni (BC)^{*}(A^{*}ABC(BC)^{*})^{\dagger}A^{*}$.

⟨7⟩ $\{(A^{\dagger}ABC(BC)^{\dagger})^{(1,2,3)}\} \ni BC(ABC)^{\dagger}A$.

⟨8⟩ $\{(A^{*}ABC(BC)^{*})^{(1,2,3)}\} \ni ((BC)^{\dagger})^{*}(ABC)^{\dagger}(A^{\dagger})^{*}$.

⟨9⟩ $(ABC)^\dagger = (A^\dagger ABC)^\dagger A^\dagger$.

⟨10⟩ $ABC = ((A^\dagger ABC)^\dagger A^\dagger)^\dagger$.

⟨11⟩ $ABC = A((ABC)^\dagger A)^\dagger$.

⟨12⟩ $(ABC)^\dagger A = (A^\dagger ABC)^\dagger$.

⟨13⟩ $BC(ABC)^\dagger A = BC(A^\dagger ABC)^\dagger$.

⟨14⟩ $ABC(ABC)^\dagger = ABC(A^\dagger ABC)^\dagger A^\dagger$.

⟨15⟩ $ABC(ABC)^\dagger A = ABC(A^\dagger ABC)^\dagger$.

⟨16⟩ $(BC)^\dagger A^\dagger ABC(ABC)^\dagger = (BC)^\dagger A^\dagger$.

⟨17⟩ $((BC)^\dagger A^\dagger)^\dagger (BC)^\dagger A^\dagger ABC = ABC$.

⟨18⟩ $ABC(ABC)^\dagger = ((BC)^\dagger A^\dagger)^\dagger (BC)^\dagger A^\dagger$.

⟨19⟩ $(ABC)^\dagger = (A^* ABC)^\dagger A^*$.

⟨20⟩ $ABC = ((A^* ABC)^\dagger A^*)^\dagger$.

⟨21⟩ $ABC = (A^\dagger)^*((ABC)^\dagger (A^\dagger)^*)^\dagger$.

⟨22⟩ $(ABC)^\dagger (A^\dagger)^* = (A^* ABC)^\dagger$.

⟨23⟩ $BC(ABC)^\dagger (A^\dagger)^* = BC(A^* ABC)^\dagger$.

⟨24⟩ $ABC(ABC)^\dagger = ABC(A^* ABC)^\dagger A^*$.

⟨25⟩ $ABC(ABC)^\dagger (A^\dagger)^* = ABC(A^* ABC)^\dagger$.

⟨26⟩ $(ABC(BC)^\dagger A^\dagger)^\dagger = ((BC)^\dagger A^\dagger)^\dagger (ABC)^\dagger$.

⟨27⟩ $ABC(A^\dagger ABC)^\dagger A^\dagger$ *is an orthogonal projector*.

⟨28⟩ $ABC(A^* ABC)^\dagger A^*$ *is an orthogonal projector*.

⟨29⟩ $A^\dagger(ABC)(ABC)^\dagger A$ *is an orthogonal projector*.

⟨30⟩ $A(A^\dagger ABC(BC)^\dagger)^\dagger A^\dagger$ *is an orthogonal projector*.

⟨31⟩ $ABC(BC)^*(A^* ABC(BC)^*)^\dagger A^*$ *is an orthogonal projector*.

⟨32⟩ $ABC(ABC)^\dagger$ *and* AA^* *commute*.

⟨33⟩ $ABC(A^\dagger ABC)^\dagger A^\dagger$ *and* AA^* *commute*.

⟨34⟩ $ABC(A^* ABC)^\dagger A^*$ *and* AA^* *commute*.

⟨35⟩ $A^\dagger(ABC)(ABC)^\dagger A$ *and* $A^* A$ *commute*.

⟨36⟩ $ABC(BC)^\dagger A^\dagger$ *is EP*.

⟨37⟩ $\{((A^\dagger)^* BC)^{(1,2,3)}\} \ni (A^\dagger ABC)^\dagger A^*$.

⟨38⟩ $\{((A^\dagger)^* BC)^{(1,2,3)}\} \ni ((A^* A)^\dagger BC)^\dagger A^\dagger$.

⟨39⟩ $\{(A^\dagger ABC)^{(1,2,3)}\} \ni ((A^\dagger)^* BC)^\dagger (A^\dagger)^*$.

⟨40⟩ $\{((A^* A)^\dagger BC)^{(1,2,3)}\} \ni ((A^\dagger)^* BC)^\dagger A$.

⟨41⟩ $\{(((AB)^\dagger)^* B^\dagger BC)^{(1,2,3)}\} \ni (B^\dagger BC)^\dagger ((AB)^\dagger ABC(B^\dagger BC)^\dagger)^\dagger (AB)^*$.

⟨42⟩ $\{((A^\dagger)^* BC)^{(1,2,3)}\} \ni (BC)^*((A^* A)^\dagger BC(BC)^*)^\dagger A^\dagger$.

⟨43⟩ $\{(A^\dagger ABC(BC)^\dagger)^{(1,2,3)}\} \ni BC((A^\dagger)^* BC)^\dagger (A^\dagger)^*$.

⟨44⟩ $\{((A^* A)^\dagger BC(BC)^*)^{(1,2,3)}\} \ni ((BC)^\dagger)^*((A^\dagger)^* BC)^\dagger A$.

⟨45⟩ $((A^\dagger)^* BC)^\dagger = (A^\dagger ABC)^\dagger A^*$.

⟨46⟩ $((A^\dagger)^*BC)^\dagger = ((A^*A)^\dagger BC)^\dagger A^\dagger$.

⟨47⟩ $(A^\dagger ABC)^\dagger = ((A^\dagger)^*BC)^\dagger (A^\dagger)^*$.

⟨48⟩ $((A^*A)^\dagger BC)^\dagger = ((A^\dagger)^*BC)^\dagger A$.

⟨49⟩ $A((BC)^\dagger A^\dagger A)^\dagger (BC)^\dagger A^\dagger$ *is an orthogonal projector*.

⟨50⟩ $(A^\dagger)^*((BC)^\dagger (A^*A)^\dagger)^\dagger (BC)^\dagger A^\dagger$ *is an orthogonal projector*.

⟨51⟩ $A^\dagger((BC)^\dagger A^\dagger)^\dagger((BC)^\dagger A^\dagger)A$ *is an orthogonal projector*.

⟨52⟩ $A(BC(BC)^\dagger A^\dagger A)^\dagger A^\dagger$ *is an orthogonal projector*.

⟨53⟩ $(A^\dagger)^*(BC(BC)^*)^\dagger((A^*A)^\dagger(BC(BC)^*)^\dagger)^\dagger A^\dagger$ *is an orthogonal projector*.

⟨54⟩ $((BC)^\dagger A^\dagger)^\dagger((BC)^\dagger A^\dagger)$ *and* $(AA^*)^\dagger$ *commute*.

⟨55⟩ $A((BC)^\dagger A^\dagger A)^\dagger (BC)^\dagger A^\dagger$ *and* $(AA^*)^\dagger$ *commute*.

⟨56⟩ $(A^\dagger)^*((BC)^\dagger (A^*A)^\dagger)^\dagger (BC)^\dagger A^\dagger$ *and* $(AA^*)^\dagger$ *commute*.

⟨57⟩ $A^\dagger((BC)^\dagger A^\dagger)^\dagger((BC)^\dagger A^\dagger)A$ *and* $(A^*A)^\dagger$ *commute*.

⟨58⟩ $\mathscr{R}(AA^*ABC) = \mathscr{R}(ABC)$.

⟨59⟩ $\mathscr{R}((A^\dagger)^*BC) = \mathscr{R}(ABC)$.

⟨60⟩ $r[AA^*ABC,\ ABC] = r(ABC)$.

⟨61⟩ $r[(A^\dagger)^*BC,\ ABC] = r(ABC)$.

⟨62⟩ $r[(A^*AA^*)^\dagger BC,\ (A^\dagger)^*BC] = r((A^\dagger)^*BC)$.

Proof. It follows immediately from replacing B by BC in Lemma 2.10. □

Theorem 3.3. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, *and* $C \in \mathbb{C}^{p\times q}$. *Then the following 62 statements are equivalent*:

⟨1⟩ $\{(ABC)^{(1,2,4)}\} \ni C^\dagger(ABCC^\dagger)^\dagger$.

⟨2⟩ $\{(ABC)^{(1,2,4)}\} \ni C^*(ABCC^*)^\dagger$.

⟨3⟩ $\{(ABCC^\dagger)^{(1,2,4)}\} \ni C(ABC)^\dagger$.

⟨4⟩ $\{(ABCC^*)^{(1,2,4)}\} \ni (C^\dagger)^*(ABC)^\dagger$.

⟨5⟩ $\{(ABC)^{(1,2,4)}\} \ni C^\dagger((AB)^\dagger ABCC^\dagger)^\dagger (AB)^\dagger$.

⟨6⟩ $\{(ABC)^{(1,2,4)}\} \ni C^*((AB)^*ABCC^*)^\dagger (AB)^*$.

⟨7⟩ $\{((AB)^\dagger ABCC^\dagger)^{(1,2,4)}\} \ni C(ABC)^\dagger AB$.

⟨8⟩ $\{(AB)^*ABCC^*)^{(1,2,4)}\} \ni (C^\dagger)^*(ABC)^\dagger((AB)^\dagger)^*$.

⟨9⟩ $(ABC)^\dagger = C^\dagger(ABCC^\dagger)^\dagger$.

⟨10⟩ $ABC = (C^\dagger(ABCC^\dagger)^\dagger)^\dagger$.

⟨11⟩ $ABC = (C(ABC)^\dagger)^\dagger C$.

⟨12⟩ $C(ABC)^\dagger = (ABCC^\dagger)^\dagger$.

⟨13⟩ $C(ABC)^\dagger AB = (ABCC^\dagger)^\dagger AB$.

⟨14⟩ $(ABC)^\dagger ABC = C^\dagger(ABCC^\dagger)^\dagger ABC$.

⟨15⟩ $C(ABC)^\dagger ABC = (ABCC^\dagger)^\dagger ABC$.

⟨16⟩ $(ABC)^\dagger ABCC^\dagger(AB)^\dagger = C^\dagger(AB)^\dagger$.

⟨17⟩ $ABCC^\dagger(AB)^\dagger(C^\dagger(AB)^\dagger)^\dagger = ABC$.

⟨18⟩ $(ABC)^{\dagger}ABC = C^{\dagger}(AB)^{\dagger}(C^{\dagger}(AB)^{\dagger})^{\dagger}$.

⟨19⟩ $(ABC)^{\dagger} = C^{*}(ABCC^{*})^{\dagger}$.

⟨20⟩ $ABC = (C^{*}(ABCC^{*})^{\dagger})^{\dagger}$.

⟨21⟩ $ABC = ((C^{\dagger})^{*}(ABC)^{\dagger})^{\dagger}(C^{\dagger})^{*}$.

⟨22⟩ $(C^{\dagger})^{*}(ABC)^{\dagger} = (ABCC^{*})^{\dagger}$.

⟨23⟩ $(C^{\dagger})^{*}(ABC)^{\dagger}AB = (ABCC^{*})^{\dagger}AB$.

⟨24⟩ $(ABC)^{\dagger}ABC = C^{*}(ABCC^{*})^{\dagger}ABC$.

⟨25⟩ $(C^{\dagger})^{*}(ABC)^{\dagger}ABC = (ABCC^{*})^{\dagger}ABC$.

⟨26⟩ $(C^{\dagger}(AB)^{\dagger}ABC)^{\dagger} = (ABC)^{\dagger}(C^{\dagger}(AB)^{\dagger})^{\dagger}$.

⟨27⟩ $C^{\dagger}(ABCC^{\dagger})^{\dagger}ABC$ *is an orthogonal projector*.

⟨28⟩ $C^{*}(ABCC^{*})^{\dagger}ABC$ *is an orthogonal projector*.

⟨29⟩ $C(ABC)^{\dagger}(ABC)C^{\dagger}$ *is an orthogonal projector*.

⟨30⟩ $C^{\dagger}((AB)^{\dagger}ABCC^{\dagger})^{\dagger}C$ *is an orthogonal projector*.

⟨31⟩ $C^{*}((AB)^{*}ABCC^{*})^{\dagger}(AB)^{*}ABC$ *is an orthogonal projector*.

⟨32⟩ $(ABC)^{\dagger}ABC$ *and* $C^{*}C$ *commute*.

⟨33⟩ $C^{\dagger}(ABCC^{\dagger})^{\dagger}ABC$ *and* $C^{*}C$ *commute*.

⟨34⟩ $C^{*}(ABCC^{*})^{\dagger}ABC$ *and* $C^{*}C$ *commute*.

⟨35⟩ $C(ABC)^{\dagger}(ABC)C^{\dagger}$ *and* CC^{*} *commute*.

⟨36⟩ $C^{\dagger}(AB)^{\dagger}ABC$ *is EP*.

⟨37⟩ $\{(AB(C^{\dagger})^{*})^{(1,2,4)}\} \ni C^{*}(ABCC^{\dagger})^{\dagger}$.

⟨38⟩ $\{(AB(C^{\dagger})^{*})^{(1,2,4)}\} \ni C^{\dagger}(AB(CC^{*})^{\dagger})^{\dagger}$.

⟨39⟩ $\{(ABCC^{\dagger})^{(1,2,4)}\} \ni (C^{\dagger})^{*}(AB(C^{\dagger})^{*})^{\dagger}$.

⟨40⟩ $\{(AB(CC^{*})^{\dagger})^{(1,2,4)}\} \ni C(AB(C^{\dagger})^{*})^{\dagger}$.

⟨41⟩ $\{(AB(C^{\dagger})^{*})^{(1,2,4)}\} \ni C^{*}((AB)^{\dagger}ABCC^{\dagger})^{\dagger}(AB)^{\dagger}$.

⟨42⟩ $\{((AB)^{*}AB(C^{\dagger})^{*})^{(1,2,4)}\} \ni C^{\dagger}(AB(CC^{*})^{\dagger})^{\dagger}(AB)^{*}$.

⟨43⟩ $\{((AB)^{\dagger}ABCC^{\dagger})^{(1,2,4)}\} \ni (C^{\dagger})^{*}(AB(C^{\dagger})^{*})^{\dagger}AB$.

⟨44⟩ $\{(AB)^{*}AB(CC^{*})^{\dagger})^{(1,2,4)}\} \ni C(AB(C^{\dagger})^{*})^{\dagger}((AB)^{\dagger})^{*}$.

⟨45⟩ $(AB(C^{\dagger})^{*})^{\dagger} = C^{*}(ABCC^{\dagger})^{\dagger}$.

⟨46⟩ $(AB(C^{\dagger})^{*}) = C^{\dagger}(AB(CC^{*})^{\dagger})^{\dagger}$.

⟨47⟩ $(ABCC^{\dagger})^{\dagger} = (C^{\dagger})^{*}(AB(C^{\dagger})^{*})^{\dagger}$.

⟨48⟩ $(AB(C^{\dagger})^{*})^{\dagger} = C(AB(CC^{*})^{\dagger})^{\dagger}$.

⟨49⟩ $C^{\dagger}(AB)^{\dagger}(CC^{\dagger}(AB)^{\dagger})^{\dagger}C$ *is an orthogonal projector*.

⟨50⟩ $C^{\dagger}(AB)^{\dagger}((CC^{*})^{\dagger}(AB)^{\dagger})^{\dagger}(C^{\dagger})^{*}$ *is an orthogonal projector*.

⟨51⟩ $C(C^{\dagger}(AB)^{\dagger})(C^{\dagger}(AB)^{\dagger})^{\dagger}C^{\dagger}$ *is an orthogonal projector*.

⟨52⟩ $C^{\dagger}(CC^{\dagger}(AB)^{\dagger}AB)^{\dagger}C$ *is an orthogonal projector*.

⟨53⟩ $C^{\dagger}(((AB)^{*}AB)^{\dagger}(CC^{*})^{\dagger})^{\dagger}((AB)^{*}AB)^{\dagger}(C^{\dagger})^{*}$ *is an orthogonal projector*.

⟨54⟩ $(C^{\dagger}(AB)^{\dagger})(C^{\dagger}(AB)^{\dagger})^{\dagger}$ *and* $(C^{*}C)^{\dagger}$ *commute*.

⟨55⟩ $C^\dagger(AB)^\dagger(CC^\dagger(AB)^\dagger)^\dagger C$ *and* $(C^*C)^\dagger$ *commute.*

⟨56⟩ $C^\dagger(AB)^\dagger((CC^*)^\dagger(AB)^\dagger)^\dagger(C^\dagger)^*$ *and* $(C^*C)^\dagger$ *commute.*

⟨57⟩ $C(C^\dagger(AB)^\dagger)(C^\dagger(AB)^\dagger)^\dagger C^\dagger$ *and* $(CC^*)^\dagger$ *commute.*

⟨58⟩ $\mathscr{R}(C^*CC^*(AB)^*) = \mathscr{R}((ABC)^*)$.

⟨59⟩ $\mathscr{R}(C^\dagger(AB)^\dagger) = \mathscr{R}(C^*(AB)^*)$.

⟨60⟩ $r[(ABCC^*C)^*,\ (ABC)^*] = r(ABC)$.

⟨61⟩ $r[C^\dagger(AB)^\dagger,\ C^*(AB)^*] = r(ABC)$.

⟨62⟩ $r[(CC^*C)^\dagger(AB)^\dagger,\ C^\dagger(AB)^\dagger] = r(C^\dagger(AB)^\dagger)$.

Proof. It follows immediately from replacing A by AB and B by C in Lemma 2.11. □

We next derive a family of statements that are equivalent to the nested ROL in (1.12).

Theorem 3.4. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, *and* $C \in \mathbb{C}^{p\times q}$. *Then the following 137 statements are equivalent*:

⟨1⟩ $(ABC)^\dagger = C^\dagger(A^\dagger ABCC^\dagger)^\dagger A^\dagger$.

⟨2⟩ $C(ABC)^\dagger A = (A^\dagger ABCC^\dagger)^\dagger$.

⟨3⟩ $A(C(ABC)^\dagger A)^\dagger C = ABC$.

⟨4⟩ $C(A(C(ABC)^\dagger A)^\dagger C)^\dagger A = C(ABC)^\dagger A$.

⟨5⟩ $A^\dagger ABCC^\dagger = A^\dagger(C^\dagger(A^\dagger ABCC^\dagger)^\dagger A^\dagger)^\dagger C^\dagger$.

⟨6⟩ $(ABC)^\dagger = C^\dagger(ABCC^\dagger)^\dagger ABC(A^\dagger ABC)^\dagger A^\dagger$.

⟨7⟩ $C(ABC)^\dagger A = (ABCC^\dagger)^\dagger ABC(A^\dagger ABC)^\dagger$.

⟨8⟩ $\{(ABC)^{(1,2,3)}\} \ni (A^\dagger ABC)^\dagger A^\dagger$ *and* $\{(ABC)^{(1,2,4)}\} \ni C^\dagger(ABCC^\dagger)^\dagger$.

⟨9⟩ $\{(ABC)^{(1,2,3)}\} \ni (A^*ABC)^\dagger A^*$ *and* $\{(ABC)^{(1,2,4)}\} \ni C^*(ABCC^*)^\dagger$.

⟨10⟩ $\{(A^\dagger ABC)^{(1,2,3)}\} \ni (ABC)^\dagger A$ *and* $\{(ABCC^\dagger)^{(1,2,4)}\} \ni C(ABC)^\dagger$.

⟨11⟩ $\{(A^*ABC)^{(1,2,3)}\} \ni (ABC)^\dagger(A^\dagger)^*$ *and* $\{(ABCC^*)^{(1,2,4)}\} \ni (C^\dagger)^*(ABC)^\dagger$.

⟨12⟩ $\{(ABC)^{(1,2,3)}\} \ni (BC)^\dagger(A^\dagger ABC(BC)^\dagger)^\dagger A^\dagger$ *and* $\{(ABC)^{(1,2,4)}\} \ni C^\dagger((AB)^\dagger ABCC^\dagger)^\dagger(AB)^\dagger$.

⟨13⟩ $\{(ABC)^{(1,2,3)}\} \ni (BC)^*(A^*ABC(BC)^*)^\dagger A^*$ *and* $\{(ABC)^{(1,2,4)}\} \ni C^*((AB)^*ABCC^*)^\dagger(AB)^*$.

⟨14⟩ $\{(A^\dagger ABC(BC)^\dagger)^{(1,2,3)}\} \ni BC(ABC)^\dagger A$ *and* $\{((AB)^\dagger ABCC^\dagger)^{(1,2,4)}\} \ni C(ABC)^\dagger AB$.

⟨15⟩ $\{(A^*ABC(BC)^*)^{(1,2,3)}\} \ni ((BC)^\dagger)^*(ABC)^\dagger(A^\dagger)^*$ *and* $\{(A^*ABCC^*)^{(1,2,4)}\} \ni (C^\dagger)^*(ABC)^\dagger((AB)^\dagger)^*$.

⟨16⟩ $(ABC)^\dagger = (A^\dagger ABC)^\dagger A^\dagger$ *and* $(ABC)^\dagger = C^\dagger(ABCC^\dagger)^\dagger$.

⟨17⟩ $(A^\dagger ABC)^\dagger = (ABC)^\dagger A$ *and* $(ABCC^\dagger)^\dagger = C(ABC)^\dagger$.

⟨18⟩ $ABC(ABC)^\dagger = ABC(A^\dagger ABC)^\dagger A^\dagger$ *and* $(ABC)^\dagger ABC = C^\dagger(ABCC^\dagger)^\dagger ABC$.

⟨19⟩ $ABC(ABC)^\dagger A = ABC(A^\dagger ABC)^\dagger$ *and* $C(ABC)^\dagger ABC = (ABCC^\dagger)^\dagger ABC$.

⟨20⟩ $(BC)^\dagger A^\dagger ABC(ABC)^\dagger = (BC)^\dagger A^\dagger$ *and* $(ABC)^\dagger ABCC^\dagger(AB)^\dagger = C^\dagger(AB)^\dagger$.

⟨21⟩ $((BC)^\dagger A^\dagger)^\dagger(BC)^\dagger A^\dagger ABC = ABC$ *and* $ABCC^\dagger(AB)^\dagger(C^\dagger(AB)^\dagger)^\dagger = ABC$.

⟨22⟩ $ABC(ABC)^\dagger = ((BC)^\dagger A^\dagger)^\dagger(BC)^\dagger A^\dagger$ *and* $(ABC)^\dagger ABC = C^\dagger(AB)^\dagger(C^\dagger(AB)^\dagger)^\dagger$.

⟨23⟩ $(A^\dagger ABCC^\dagger)^\dagger = (ABCC^\dagger)^\dagger A$ *and* $(A^\dagger ABCC^\dagger)^\dagger = C(A^\dagger ABC)^\dagger$.

⟨24⟩ $(ABCC^\dagger)^\dagger = (A^\dagger ABCC^\dagger)^\dagger A^\dagger$ *and* $(A^\dagger ABC)^\dagger = C^\dagger(A^\dagger ABCC^\dagger)^\dagger$.

⟨25⟩ $(A^\dagger ABC)^\dagger A^\dagger = C^\dagger (ABCC^\dagger)^\dagger$.

⟨26⟩ $(ABCC^\dagger)^\dagger A = C(A^\dagger ABC)^\dagger$.

⟨27⟩ $((A^\dagger)^* B(C^\dagger)^*)^\dagger = C^*(A^\dagger ABCC^\dagger)^\dagger A^*$.

⟨28⟩ $(C^\dagger)^*((A^\dagger)^* B(C^\dagger)^*)^\dagger (A^\dagger)^* = (A^\dagger ABCC^\dagger)^\dagger$.

⟨29⟩ $(A^\dagger)^*((C^\dagger)^*((A^\dagger)^* B(C^\dagger)^*)^\dagger (A^\dagger)^*)^\dagger (C^\dagger)^* = (A^\dagger)^* B(C^\dagger)^*$.

⟨30⟩ $(C^\dagger)^*((A^\dagger)^*((C^\dagger)^*(ABC)^\dagger (A^\dagger)^*)^\dagger (C^\dagger)^*)^\dagger (A^\dagger)^* = (C^\dagger)^*((A^\dagger)^* B(C^\dagger)^*)^\dagger (A^\dagger)^*$.

⟨31⟩ $A^\dagger ABCC^\dagger = A^*(C^*(A^\dagger ABCC^\dagger)^\dagger A^*)^\dagger C^*$.

⟨32⟩ $((A^\dagger)^* B(C^\dagger)^*)^\dagger = C^*((A^\dagger)^* BCC^\dagger)^\dagger (A^\dagger)^* B(C^\dagger)^*(A^\dagger AB(C^\dagger)^*)^\dagger A^*$.

⟨33⟩ $(C^\dagger)^*((A^\dagger)^* B(C^\dagger)^*)^\dagger (A^\dagger)^* = ((A^\dagger)^* BCC^\dagger)^\dagger (A^\dagger)^* B(C^\dagger)^*(A^\dagger AB(C^\dagger)^*)^\dagger$.

⟨34⟩ $((A^\dagger)^* BC)^\dagger = (A^\dagger ABC)^\dagger A^*$ *and* $(AB(C^\dagger)^*)^\dagger = C^*(ABCC^\dagger)^\dagger$.

⟨35⟩ $(A^\dagger ABC)^\dagger = ((A^\dagger)^* BC)^\dagger (A^\dagger)^*$ *and* $(ABCC^\dagger)^\dagger = (C^\dagger)^*(AB(C^\dagger)^*)^\dagger$.

⟨36⟩ $ABC(ABC)^\dagger = AB(C^\dagger)^*(A^\dagger AB(C^\dagger)^*)^\dagger A^\dagger$ *and* $(ABC)^\dagger ABC = C^\dagger((A^\dagger)^* BCC^\dagger)^\dagger (A^\dagger)^* BC$.

⟨37⟩ $ABC(ABC)^\dagger A = AB(C^\dagger)^*(A^\dagger AB(C^\dagger)^*)^\dagger$ *and* $C(ABC)^\dagger ABC = ((A^\dagger)^* BCC^\dagger)^\dagger (A^\dagger)^* BC$.

⟨38⟩ $ABC(ABC)^\dagger = ((B(C^\dagger)^*)^\dagger A^\dagger)^\dagger (B(C^\dagger)^*)^\dagger A^\dagger$ *and* $(ABC)^\dagger ABC = C^\dagger((A^\dagger)^* B)^\dagger (C^\dagger((A^\dagger)^* B)^\dagger)^\dagger$.

⟨39⟩ $(BC)^\dagger A^\dagger ABC((A^\dagger)^* BC)^\dagger = (BC)^\dagger A^*$ *and* $(AB(C^\dagger)^*)^\dagger ABCC^\dagger (AB)^\dagger = C^*(AB)^\dagger$.

⟨40⟩ $((BC)^\dagger A^*)^\dagger (BC)^\dagger A^\dagger ABC = (A^\dagger)^* BC$ *and* $ABCC^\dagger (AB)^\dagger (C^*(AB)^\dagger)^\dagger = AB(C^\dagger)^*$.

⟨41⟩ $(A^\dagger ABCC^\dagger)^\dagger = ((A^\dagger)^* BCC^\dagger)^\dagger (A^\dagger)^*$ *and* $(A^\dagger ABCC^\dagger)^\dagger = (C^\dagger)^*(A^\dagger AB(C^\dagger)^*)^\dagger$.

⟨42⟩ $((A^\dagger)^* BCC^\dagger)^\dagger = (A^\dagger ABCC^\dagger)^\dagger A^*$ *and* $(A^\dagger AB(C^\dagger)^*)^\dagger = C^*(A^\dagger ABCC^\dagger)^\dagger$.

⟨43⟩ $(A^\dagger AB(C^\dagger)^*)^\dagger A^* = C^*((A^\dagger)^* BCC^\dagger)^\dagger$.

⟨44⟩ $((A^\dagger)^* BCC^\dagger)^\dagger (A^\dagger)^* = (C^\dagger)^*(A^\dagger AB(C^\dagger)^*)^\dagger$.

⟨45⟩ $C^*((A^\dagger)^* BCC^\dagger)^\dagger = (A^\dagger AB(C^\dagger)^*)^\dagger A^*$.

⟨46⟩ $(ABC)^\dagger = C^*(A^* ABCC^*)^\dagger A^*$.

⟨47⟩ $(A^* ABCC^*)^\dagger = (C^\dagger)^*(ABC)^\dagger (A^\dagger)^*$.

⟨48⟩ $(ABC)^\dagger = C^*(ABCC^*)^\dagger ABC(A^* ABC)^\dagger A^*$.

⟨49⟩ $(C^\dagger)^*(ABC)^\dagger (A^\dagger)^* = (ABCC^*)^\dagger ABC(A^* ABC)^\dagger$.

⟨50⟩ $(ABC)^\dagger = (A^* ABC)^\dagger A^*$ *and* $(ABC)^\dagger = C^*(ABCC^*)^\dagger$.

⟨51⟩ $(A^* ABC)^\dagger = (ABC)^\dagger (A^\dagger)^*$ *and* $(ABCC^*)^\dagger = (C^\dagger)^*(ABC)^\dagger$.

⟨52⟩ $((A^\dagger)^* BC)^\dagger = ((A^* A)^\dagger BC)^\dagger A^\dagger$ *and* $(AB(C^\dagger)^*)^\dagger = C^\dagger (AB(CC^*)^\dagger)^\dagger$.

⟨53⟩ $((A^* A)^\dagger BC)^\dagger = ((A^\dagger)^* BC)^\dagger A$ *and* $(AB(CC^*)^\dagger)^\dagger = C(AB(C^\dagger)^*)^\dagger$.

⟨54⟩ $((A^\dagger)^* B(C^\dagger)^*)^\dagger = ((A^* A)^\dagger B(C^\dagger)^*)^\dagger A^\dagger$ *and* $((A^\dagger)^* B(C^\dagger)^*)^\dagger = C^\dagger((A^\dagger)^* B(CC^*)^\dagger)^\dagger$.

⟨55⟩ $((A^* A)^\dagger B(C^\dagger)^*)^\dagger = ((A^\dagger)^* B(C^\dagger)^*)^\dagger A$ *and* $((A^\dagger)^* B(CC^*)^\dagger)^\dagger = C((A^\dagger)^* B(C^\dagger)^*)^\dagger$.

⟨56⟩ $((A^\dagger)^* B(C^\dagger)^*)^\dagger = C^\dagger((A^* A)^\dagger B(CC^*)^\dagger)^\dagger A^\dagger$.

⟨57⟩ $((A^* A)^\dagger B(CC^*)^\dagger)^\dagger = C((A^\dagger)^* B(C^\dagger)^*)^\dagger A$.

⟨58⟩ $(A^* ABCC^*)^\dagger = (ABCC^*)^\dagger (A^\dagger)^*$ *and* $(A^* ABCC^*)^\dagger = (C^\dagger)^*(A^* ABC)^\dagger$.

⟨59⟩ $(ABCC^*)^\dagger = (A^* ABCC^*)^\dagger A^*$ *and* $(A^* ABC)^\dagger = C^*(A^* ABCC^*)^\dagger$.

⟨60⟩ $(A^* ABC)^\dagger A^* = C^*(ABCC^*)^\dagger$.

⟨61⟩ $(ABCC^*)^\dagger (A^\dagger)^* = (C^\dagger)^*(A^* ABC)^\dagger$.

⟨62⟩ $(AB(CC^*)^\dagger)^\dagger = (A^*AB(CC^*)^\dagger)^\dagger A^*$ *and* $((A^*A)^\dagger BC)^\dagger = C^*((A^*A)^\dagger BCC^*)^\dagger$.

⟨63⟩ $(A^*AB(CC^*)^\dagger)^\dagger = (AB(CC^*)^\dagger)^\dagger (A^\dagger)^*$ *and* $((A^*A)^\dagger BCC^*)^\dagger = (C^\dagger)^*((A^*A)^\dagger BC)^\dagger$.

⟨64⟩ $((A^\dagger)^* B(CC^*)^\dagger)^\dagger A = C((A^*A)^\dagger B(C^\dagger)^*)^\dagger$.

⟨65⟩ $((A^*A)^\dagger B(C^\dagger)^*)^\dagger A^\dagger = C^\dagger((A^\dagger)^* B(CC^*)^\dagger)^\dagger$.

⟨66⟩ $(A^*ABC)^\dagger = (AA^*ABC)^\dagger A$ *and* $(ABCC^*)^\dagger = C(ABCC^*C)^\dagger$.

⟨67⟩ $(AA^*ABC)^\dagger = (A^*ABC)^\dagger A^\dagger$ *and* $(ABCC^*C)^\dagger = C^\dagger(ABCC^*)^\dagger$.

⟨68⟩ $((A^*A)^\dagger BC)^\dagger = ((A^*AA^*)^\dagger BC)^\dagger (A^\dagger)^*$ *and* $(AB(CC^*)^\dagger)^\dagger = (C^\dagger)^*(AB(C^*CC^*)^\dagger)^\dagger$.

⟨69⟩ $((A^*A)^\dagger BC)^\dagger A^* = ((A^*AA^*)^\dagger BC)^\dagger$ *and* $C^*(AB(CC^*)^\dagger)^\dagger = (AB(C^*CC^*)^\dagger)^\dagger$.

⟨70⟩ $((A^*A)^\dagger B(C^\dagger)^*)^\dagger = ((A^*AA^*)^\dagger B(C^\dagger)^*)^\dagger (A^\dagger)^*$ *and* $((A^\dagger)^* B(CC^*)^\dagger)^\dagger = (C^\dagger)^*((A^\dagger)^* B(C^*CC^*)^\dagger)^\dagger$.

⟨71⟩ $((A^*AA^*)^\dagger B(C^\dagger)^*)^\dagger = ((A^*A)^\dagger B(C^\dagger)^*)^\dagger A^*$ *and* $((A^\dagger)^* B(C^*CC^*)^\dagger)^\dagger = C^*((A^\dagger)^* B(CC^*)^\dagger)^\dagger$.

⟨72⟩ $(A^*ABCC^*)^\dagger = C(A^*ABCC^*C)^\dagger$ *and* $(A^*ABCC^*)^\dagger = (AA^*ABCC^*)^\dagger A$.

⟨73⟩ $(A^*ABCC^*C)^\dagger = C^\dagger(A^*ABCC^*)^\dagger$ *and* $(AA^*ABCC^*)^\dagger = (A^*ABCC^*)^\dagger A^\dagger$.

⟨74⟩ $((A^*A)^\dagger BCC^*)^\dagger = C((A^*A)^\dagger BCC^*C)^\dagger$ *and* $(A^*AB(CC^*)^\dagger)^\dagger = (AA^*AB(CC^*)^\dagger)^\dagger A$.

⟨75⟩ $((A^*A)^\dagger BCC^*C)^\dagger = C^\dagger((A^*A)^\dagger BCC^*)^\dagger$ *and* $(A^*AB(CC^*)^\dagger)^\dagger A^\dagger = (AA^*AB(CC^*)^\dagger)^\dagger$.

⟨76⟩ $(A^*AB(CC^*)^\dagger)^\dagger = (C^\dagger)^*(A^*AB(C^*CC^*)^\dagger)^\dagger$ *and* $((A^*A)^\dagger BCC^*)^\dagger = ((A^*AA^*)^\dagger BCC^*)^\dagger (A^\dagger)^*$.

⟨77⟩ $(A^*AB(C^*CC^*)^\dagger)^\dagger = C^*(A^*AB(CC^*)^\dagger)^\dagger$ *and* $(AA^*AB(CC^*)^\dagger)^\dagger = (A^*AB(CC^*)^\dagger)^\dagger A^*$.

⟨78⟩ $((A^*A)^\dagger B(CC^*)^\dagger)^\dagger = (C^\dagger)^*((A^*A)^\dagger B(C^*CC^*)^\dagger)^\dagger$ *and* $((A^*A)^\dagger B(CC^*)^\dagger)^\dagger = ((A^*AA^*)^\dagger B(CC^*)^\dagger)^\dagger (A^\dagger)^*$.

⟨79⟩ $((A^*A)^\dagger B(C^*CC^*)^\dagger)^\dagger = C^*((A^*A)^\dagger B(CC^*)^\dagger)^\dagger$ *and* $((A^*AA^*)^\dagger B(CC^*)^\dagger)^\dagger = ((A^*A)^\dagger B(CC^*)^\dagger)^\dagger A^*$.

⟨80⟩ $C(A^*ABCC^*C)^\dagger = (AA^*ABCC^*)^\dagger A$.

⟨81⟩ $(A^*ABCC^*C)^\dagger A^\dagger = C^\dagger(AA^*ABCC^*)^\dagger$.

⟨82⟩ $C((A^*A)^\dagger BCC^*C)^\dagger = ((A^*AA^*)^\dagger BCC^*)^\dagger (A^\dagger)^*$.

⟨83⟩ $((A^*A)^\dagger BCC^*C)^\dagger A^* = C^\dagger((A^*AA^*)^\dagger BCC^*)^\dagger$.

⟨84⟩ $(C^\dagger)^*(A^*AB(C^*CC^*)^\dagger)^\dagger = (AA^*AB(CC^*)^\dagger)^\dagger A$.

⟨85⟩ $(A^*AB(C^*CC^*)^\dagger)^\dagger A^\dagger = C^*(AA^*AB(CC^*)^\dagger)^\dagger$.

⟨86⟩ $((CC^*C)^\dagger B^*(A^*A)^\dagger)^\dagger C^\dagger = A^\dagger((CC^*)^\dagger B^*(AA^*A)^\dagger)^\dagger$.

⟨87⟩ $A((CC^*C)^\dagger B^*(A^*A)^\dagger)^\dagger = ((CC^*)^\dagger B^*(AA^*A)^\dagger)^\dagger C$.

⟨88⟩ $(A^*ABCC^*)^\dagger = C(AA^*ABCC^*C)^\dagger A$.

⟨89⟩ $(AA^*ABCC^*C)^\dagger = C^\dagger(A^*ABCC^*)^\dagger A^\dagger$.

⟨90⟩ $((A^*A)^\dagger BCC^*)^\dagger = C((A^*AA^*)^\dagger BCC^*C)^\dagger (A^\dagger)^*$.

⟨91⟩ $((A^*AA^*)^\dagger BCC^*C)^\dagger = C^\dagger((A^*A)^\dagger BCC^*)^\dagger A^*$.

⟨92⟩ $(A^*AB(CC^*)^\dagger)^\dagger = (C^\dagger)^*(AA^*AB(C^*CC^*)^\dagger)^\dagger A$.

⟨93⟩ $(AA^*AB(C^*CC^*)^\dagger)^\dagger = C^*(A^*AB(CC^*)^\dagger)^\dagger (A^\dagger)^*$.

⟨94⟩ $((A^*A)^\dagger B(CC^*)^\dagger)^\dagger = (C^\dagger)^*((A^*AA^*)^\dagger B(C^*CC^*)^\dagger)^\dagger (A^\dagger)^*$.

⟨95⟩ $((A^*AA^*)^\dagger B(C^*CC^*)^\dagger)^\dagger = C^*((A^*A)^\dagger B(CC^*)^\dagger)^\dagger A^*$.

⟨96⟩ $(AA^*ABCC^*C)^\dagger = ((A^*A)^2 BCC^*C)^\dagger A^*$ *and* $(AA^*ABCC^*C)^\dagger = C^*(AA^*AB(CC^*)^2)^\dagger$.

⟨97⟩ $((A^*A)^2 BCC^*C)^\dagger = (AA^*ABCC^*C)^\dagger (A^\dagger)^*$ *and* $(AA^*AB(CC^*)^2)^\dagger = (C^\dagger)^*(AA^*ABCC^*C)^\dagger$.

⟨98⟩ $((A^*A)^2BCC^*C)^\dagger A^* = C^*(AA^*AB(CC^*)^2)^\dagger$.

⟨99⟩ $(AA^*ABCC^*C)^\dagger (A^\dagger)^* = (C^\dagger)^*(AA^*ABCC^*C)^\dagger$.

⟨100⟩ $(AA^*ABCC^*C)^\dagger = C^*((A^*A)^2B(CC^*)^2)^\dagger A^*$.

⟨101⟩ $((A^*A)^2B(CC^*)^2)^\dagger = (C^\dagger)^*(AA^*ABCC^*C)^\dagger (A^\dagger)^*$.

⟨102⟩ $(CC^\dagger B^*A^\dagger A)^\dagger = (C^\dagger B^*A^\dagger A)^\dagger C^\dagger$ *and* $(CC^\dagger B^*A^\dagger A)^\dagger = A^\dagger (CC^\dagger B^*A^\dagger)^\dagger$.

⟨103⟩ $(C^\dagger B^*A^\dagger A)^\dagger = (CC^\dagger B^*A^\dagger A)^\dagger C$ *and* $(CC^\dagger B^*A^\dagger)^\dagger = A(CC^\dagger B^*A^\dagger A)^\dagger$.

⟨104⟩ $(A^*ABCC^\dagger)^\dagger = C(A^*ABC)^\dagger$ *and* $(A^\dagger ABCC^*)^\dagger = (ABCC^*)^\dagger A$.

⟨105⟩ $(A^*ABC)^\dagger = C^\dagger (A^*ABCC^\dagger)^\dagger$ *and* $(ABCC^*)^\dagger = (A^\dagger ABCC^*)^\dagger A^\dagger$.

⟨106⟩ $(C^\dagger B^*(A^*A)^\dagger)^\dagger = ((CC^*)^\dagger B^*(A^*A)^\dagger)^\dagger (C^\dagger)^*$ *and* $((CC^*)^\dagger B^*A^\dagger)^\dagger = (A^\dagger)^*((CC^*)^\dagger B^*(A^*A)^\dagger)^\dagger$.

⟨107⟩ $((A^*A)^\dagger B(C^\dagger)^*)^\dagger = C^\dagger ((A^*A)^\dagger B(CC^*)^\dagger)^\dagger$ *and* $((A^\dagger)^*B(CC^*)^\dagger)^\dagger = ((A^*A)^\dagger B(CC^*)^\dagger)^\dagger A^\dagger$.

⟨108⟩ $((A^*A)^\dagger B(CC^*)^\dagger)^\dagger = C((A^*A)^\dagger B(C^\dagger)^*)^\dagger$ *and* $((A^*A)^\dagger B(CC^*)^\dagger)^\dagger = ((A^\dagger)^*B(CC^*)^\dagger)^\dagger A$.

⟨109⟩ $(ABC(BC)^\dagger A^\dagger)^\dagger = ((BC)^\dagger A^\dagger)^\dagger (ABC)^\dagger$ *and* $(C^\dagger (AB)^\dagger ABC)^\dagger = (ABC)^\dagger (C^\dagger (AB)^\dagger)^\dagger$.

⟨110⟩ $ABC(A^\dagger ABC)^\dagger A^\dagger$ *and* $C^\dagger (ABCC^\dagger)^\dagger ABC$ *are orthogonal projectors.*

⟨111⟩ $ABC(A^*ABC)^\dagger A^*$ *and* $C^*(ABCC^*)^\dagger ABC$ *are orthogonal projectors.*

⟨112⟩ $A^\dagger (ABC)(ABC)^\dagger A$ *and* $C(ABC)^\dagger (ABC)C^\dagger$ *orthogonal projectors.*

⟨113⟩ $ABC(ABC)^\dagger$ *and* AA^* *commute, and* $(ABC)^\dagger ABC$ *and* C^*C *commute.*

⟨114⟩ $ABC(A^\dagger ABC)^\dagger A^\dagger$ *and* AA^* *commute, and* $C^\dagger (ABCC^\dagger)^\dagger ABC$ *and* C^*C *commute.*

⟨115⟩ $ABC(A^*ABC)^\dagger A^*$ *and* AA^* *commute, and* $C^*(ABCC^*)^\dagger ABC$ *and* C^*C *commute.*

⟨116⟩ $A^\dagger (ABC)(ABC)^\dagger A$ *and* A^*A *commute, and* $C(ABC)^\dagger (ABC)C^\dagger$ *and* CC^* *commute.*

⟨117⟩ $A((BC)^\dagger A^\dagger A)^\dagger (BC)^\dagger A^\dagger$ *and* $C^\dagger (AB)^\dagger (CC^\dagger (AB)^\dagger)^\dagger C$ *are orthogonal projectors.*

⟨118⟩ $(A^\dagger)^*((BC)^\dagger (A^*A)^\dagger)^\dagger (BC)^\dagger A^\dagger$ *and* $C^\dagger (AB)^\dagger ((CC^*)^\dagger (AB)^\dagger)^\dagger (C^\dagger)^*$ *are orthogonal projectors.*

⟨119⟩ $A^\dagger ((BC)^\dagger A^\dagger)^\dagger ((BC)^\dagger A^\dagger)A$ *and* $C(C^\dagger (AB)^\dagger)(C^\dagger (AB)^\dagger)^\dagger C^\dagger$ *are orthogonal projectors.*

⟨120⟩ $((BC)^\dagger A^\dagger)^\dagger ((BC)^\dagger A^\dagger)$ *and* $(AA^*)^\dagger$ *commute, and* $(C^\dagger (AB)^\dagger)(C^\dagger (AB)^\dagger)^\dagger$ *and* $(C^*C)^\dagger$ *commute.*

⟨121⟩ $A((BC)^\dagger A^\dagger A)^\dagger (BC)^\dagger A^\dagger$ *and* $(AA^*)^\dagger$ *commute, and* $C^\dagger (AB)^\dagger (CC^\dagger (AB)^\dagger)^\dagger C$ *and* $(C^*C)^\dagger$ *commute.*

⟨122⟩ $(A^\dagger)^*((BC)^\dagger (A^*A)^\dagger)^\dagger (BC)^\dagger A^\dagger$ *and* $(AA^*)^\dagger$ *commute, and* $C^\dagger (AB)^\dagger ((CC^*)^\dagger (AB)^\dagger)^\dagger (C^\dagger)^*$ *and* $(C^*C)^\dagger$ *commute.*

⟨123⟩ $A^\dagger ((BC)^\dagger A^\dagger)^\dagger ((BC)^\dagger A^\dagger)A$ *and* $(A^*A)^\dagger$ *commute, and* $C(C^\dagger (AB)^\dagger)(C^\dagger (AB)^\dagger)^\dagger C^\dagger$ *and* $(CC^*)^\dagger$ *commute.*

⟨124⟩ $ABC(BC)^\dagger A^\dagger$ *and* $C^\dagger (AB)^\dagger ABC$ *are EP.*

⟨125⟩ $\{((A^\dagger)^*BC)^{(1,2,3)}\} \ni (A^\dagger ABC)^\dagger A^*$ *and* $\{(AB(C^\dagger)^*)^{(1,2,4)}\} \ni C^*(ABCC^\dagger)^\dagger$.

⟨126⟩ $\{((A^\dagger)^*BC)^{(1,2,3)}\} \ni ((A^*A)^\dagger BC)^\dagger A^\dagger$ *and* $\{(AB(C^\dagger)^*)^{(1,2,4)}\} \ni C^\dagger (AB(CC^*)^\dagger)^\dagger$.

⟨127⟩ $\{(A^\dagger ABC)^{(1,2,3)}\} \ni ((A^\dagger)^*BC)^\dagger (A^\dagger)^*$ *and* $\{(ABCC^\dagger)^{(1,2,4)}\} \ni (C^\dagger)^*(AB(C^\dagger)^*)^\dagger$.

⟨128⟩ $\{((A^*A)^\dagger BC)^{(1,2,3)}\} \ni ((A^\dagger)^*BC)^\dagger A$ *and* $\{(AB(CC^*)^\dagger)^{(1,2,4)}\} \ni C(AB(C^\dagger)^*)^\dagger$.

⟨129⟩ $\{((A^\dagger)^*BC)^{(1,2,3)}\} \ni (BC)^\dagger (A^\dagger ABC(BC)^\dagger)^\dagger A^*$ *and* $\{(AB(C^\dagger)^*)^{(1,2,4)}\} \ni C^*((AB)^\dagger ABCC^\dagger)^\dagger (AB)^\dagger$.

⟨130⟩ $\{((A^\dagger)^*BC)^{(1,2,3)}\} \ni (BC)^*((A^*A)^\dagger BC(BC)^*)^\dagger A^\dagger$ *and* $\{(AB(C^\dagger)^*)^{(1,2,4)}\} \ni C^\dagger (A^*AB(CC^*)^\dagger)^\dagger (AB)^*$.

⟨131⟩ $\{(A^\dagger ABC(BC)^\dagger)^{(1,2,3)}\} \ni BC((A^\dagger)^*BC)^\dagger (A^\dagger)^*$ *and* $\{((AB)^\dagger ABCC^\dagger)^{(1,2,4)}\} \ni (C^\dagger)^*(AB(C^\dagger)^*)^\dagger AB$.

⟨132⟩ $\{((A^*A)^\dagger BC(BC)^*)^{(1,2,3)}\} \ni ((BC)^\dagger)^*((A^\dagger)^*BC)^\dagger A$ *and* $\{((AB)^*AB(CC^*)^\dagger)^{(1,2,4)}\} \ni C(AB(C^\dagger)^*)^\dagger((AB)^\dagger)^*$.

⟨133⟩ $(AA^*ABC)(AA^*ABC)^\dagger = (ABC)(ABC)^\dagger$ *and* $(ABCC^*C)^\dagger(ABCC^*C) = (ABC)^\dagger(ABC)$.

⟨134⟩ $\mathscr{R}((ABC)^\dagger) = \mathscr{R}(C^\dagger(A^\dagger ABCC^\dagger)^\dagger A^\dagger)$ *and* $\mathscr{R}(((ABC)^\dagger)^*) = \mathscr{R}((C^\dagger(A^\dagger ABCC^\dagger)^\dagger A^\dagger)^*)$.

⟨135⟩ $\mathscr{R}(AA^*ABC) = \mathscr{R}(ABC)$ *and* $\mathscr{R}((ABCC^*C)^*) = \mathscr{R}((ABC)^*)$.

⟨136⟩ $r[AA^*ABC,\ ABC] = r[(ABCC^*C)^*,\ (ABC)^*] = r(ABC)$.

⟨137⟩ $r[(A^*AA^*)^\dagger BC,\ (A^\dagger)^*BC] = r((A^\dagger)^*BC)$ *and* $r[(CC^*C)^\dagger(AB)^*,\ C^\dagger(AB)^*] = r(C^\dagger(AB)^*)$.

Proof. It is easy to verify by the definition of Moore–Penrose generalized inverse and (3.9) that the two facts $(ABC)^\dagger \in \{(ABC)^{(2)}\}$ and $C^\dagger(A^\dagger ABCC^\dagger)^\dagger A^\dagger \in \{(ABC)^{(2)}\}$ hold. In this case, applying (2.12) to $(ABC)^\dagger$ and $C^\dagger(A^\dagger ABCC^\dagger)^\dagger A^\dagger$ leads to the equivalence of ⟨1⟩ and ⟨134⟩. Furthermore, applying (2.11) to the difference $(ABC)^\dagger - C^\dagger(A^\dagger ABCC^\dagger)^\dagger A^\dagger$ and simplifying by (2.13) and Lemma 2.7, we obtain the following rank equalities

$$\begin{aligned}
&r((ABC)^\dagger - C^\dagger(A^\dagger ABCC^\dagger)^\dagger A^\dagger)\\
&= r\begin{bmatrix}(ABC)^\dagger\\ C^\dagger(A^\dagger ABCC^\dagger)^\dagger A^\dagger\end{bmatrix} + r[(ABC)^\dagger,\ C^\dagger(A^\dagger ABCC^\dagger)^\dagger A^\dagger] - r((ABC)^\dagger) - r(C^\dagger(A^\dagger ABCC^\dagger)^\dagger A^\dagger)\\
&= r\begin{bmatrix}(ABC)^*\\ (A^\dagger ABCC^\dagger)^* A^\dagger\end{bmatrix} + r[(ABC)^*,\ C^\dagger(A^\dagger ABCC^\dagger)^*] - r((ABC)^\dagger) - r(A^\dagger ABCC^\dagger)\\
&= r\begin{bmatrix}(ABC)^*\\ C^*B^*A^\dagger\end{bmatrix} + r[(ABC)^*,\ C^\dagger B^*A^*] - 2r(ABC)\\
&= r\begin{bmatrix}(ABC)^*AA^*\\ C^*B^*A^*\end{bmatrix} + r[C^*C(ABC)^*,\ C^*B^*A^*] - 2r(ABC)\\
&= r\begin{bmatrix}ABC\\ ABCC^*C\end{bmatrix} + r[ABC,\ AA^*ABC] - 2r(ABC).
\end{aligned} \tag{3.11}$$

Setting all sides of (3.11) equal to zero and noticing that $r\begin{bmatrix}ABC\\ ABCC^*C\end{bmatrix} \geq r(ABC)$ and $r[ABC,\ AA^*ABC] \geq r(ABC)$ leads to
$r\begin{bmatrix}ABC\\ ABCC^*C\end{bmatrix} = r[ABC,\ AA^*ABC] = r(AA^*ABC) = r(ABCC^*C) = r(ABC)$, thus establishing the equivalences of ⟨1⟩, ⟨135⟩, and ⟨136⟩ by Lemma 2.2.

The equivalence of ⟨136⟩ and ⟨137⟩ follows from Theorem 3.2 ⟨61⟩ and ⟨62⟩ and Theorem 3.3 ⟨61⟩ and ⟨62⟩.

Pre- and post-multiplying the equality in ⟨1⟩ with C and A, respectively, yield the equality in ⟨2⟩. Conversely, pre- and post-multiplying the equality in ⟨2⟩ with $C^\dagger$ and $A^\dagger$, respectively, and simplifying yield the equality in ⟨1⟩.

We first take the Moore–Penrose generalized inverses of both sides of the equality in ⟨2⟩ to yield $(C(ABC)^\dagger A)^\dagger = A^\dagger ABCC^\dagger$. We then pre- and post-multiplying the equality with A and C, respectively, to obtain the equality in ⟨3⟩. Conversely, pre- and post-multiplying the equality in ⟨3⟩ with $A^\dagger$ and $C^\dagger$, respectively, and simplifying yield $(C(ABC)^\dagger A)^\dagger = A^\dagger ABCC^\dagger$, which is equivalent to ⟨2⟩ by taking the Moore–Penrose generalized inverses of both sides of the equality.

Result ⟨3⟩ obviously implies ⟨4⟩ by substitution. Conversely, pre- and post-multiplying the equality in ⟨4⟩ with $C^\dagger$ and $A^\dagger$, respectively, and simplifying yield the equality in ⟨3⟩.

Taking the Moore–Penrose generalized inverses of both sides of the equality in ⟨1⟩, and then pre- and post-multiplying the equality with $A^\dagger$ and $C^\dagger$, respectively, we obtain the equality in ⟨5⟩. Conversely, pre- and post-multiplying the equality in ⟨5⟩ with A and C, respectively, and then taking the Moore–Penrose generalized inverses of both sides of the equality, we obtain ⟨1⟩.

Notice by (3.10) that $C^\dagger(ABCC^\dagger)^\dagger ABC(A^\dagger ABC)^\dagger A^\dagger \in \{(ABC)^{(2)}\}$ holds. Then applying (2.11) to the difference $(ABC)^\dagger - C^\dagger(ABCC^\dagger)^\dagger ABC(A^\dagger ABC)^\dagger A^\dagger$ and simplifying by (2.13) and Lemma 2.7, we obtain

$$\begin{aligned}
& r((ABC)^\dagger - C^\dagger(ABCC^\dagger)^\dagger ABC(A^\dagger ABC)^\dagger A^\dagger) \\
&= r\begin{bmatrix} (ABC)^\dagger \\ C^\dagger(ABCC^\dagger)^\dagger ABC(A^\dagger ABC)^\dagger A^\dagger \end{bmatrix} + r[(ABC)^\dagger,\ C^\dagger(ABCC^\dagger)^\dagger ABC(A^\dagger ABC)^\dagger A^\dagger] \\
&\quad - r((ABC)^\dagger) - r(C^\dagger(ABCC^\dagger)^\dagger ABC(A^\dagger ABC)^\dagger A^\dagger) \\
&= r\begin{bmatrix} (ABC)^* \\ (A^\dagger ABC)^* A^\dagger \end{bmatrix} + r[(ABC)^*,\ C^\dagger(ABCC^\dagger)^*] - 2r(ABC) \\
&= r\begin{bmatrix} (ABC)^* \\ (BC)^* A^\dagger \end{bmatrix} + r[(ABC)^*,\ C^\dagger(AB)^*] - 2r(ABC) \\
&= r\begin{bmatrix} ABC \\ ABCC^*C \end{bmatrix} + r[ABC,\ AA^*ABC] - 2r(ABC).
\end{aligned} \tag{3.12}$$

Setting all sides of (3.12) equal to zero leads to the equivalence of ⟨6⟩ and ⟨136⟩.

Pre- and post-multiplying the equality in ⟨6⟩ with A and C, respectively, and simplifying we obtain the equality in ⟨7⟩. Conversely, pre- and post-multiplying the equality in ⟨7⟩ with $A^\dagger$ and $C^\dagger$, respectively, we obtain ⟨6⟩.

Combining Theorem 3.2 ⟨1⟩–⟨8⟩ and ⟨60⟩ with Theorem 3.3 ⟨1⟩–⟨8⟩ and ⟨60⟩ leads to the equivalences of ⟨8⟩–⟨16⟩ and ⟨136⟩.

Combining Theorem 3.2 ⟨13⟩–⟨18⟩ and ⟨60⟩ with Theorem 3.3 ⟨13⟩–⟨18⟩ and ⟨60⟩ leads to the equivalences of ⟨17⟩–⟨22⟩ and ⟨136⟩.

It is easy to verify that $(ABCC^\dagger)^\dagger A \in \{(A^\dagger ABCC^\dagger)^{(1,2)}\}$ and $C(A^\dagger ABC)^\dagger \in \{(A^\dagger ABCC^\dagger)^{(1,2)}\}$ hold. In this case, applying (2.11) to the two differences $(A^\dagger ABCC^\dagger)^\dagger - (ABCC^\dagger)^\dagger A$ and $(A^\dagger ABCC^\dagger)^\dagger - C(A^\dagger ABC)^\dagger$ and simplifying by (2.13) and Lemma 2.7 yield

$$\begin{aligned}
& r((A^\dagger ABCC^\dagger)^\dagger - (ABCC^\dagger)^\dagger A) \\
&= r\begin{bmatrix} (A^\dagger ABCC^\dagger)^\dagger \\ (ABCC^\dagger)^\dagger A \end{bmatrix} + r[(A^\dagger ABCC^\dagger)^\dagger,\ (ABCC^\dagger)^\dagger A] - r((A^\dagger ABCC^\dagger)^\dagger) - r((ABCC^\dagger)^\dagger A) \\
&= r\begin{bmatrix} (A^\dagger ABCC^\dagger)^* \\ (ABCC^\dagger)^* A \end{bmatrix} + r[(A^\dagger ABCC^\dagger)^*,\ (ABCC^\dagger)^*] - 2r(ABC) \\
&= r\begin{bmatrix} A^\dagger ABCC^\dagger \\ ABCC^\dagger \end{bmatrix} + r[A^\dagger ABCC^\dagger,\ A^*ABCC^\dagger] - 2r(ABC) \\
&= r[ABC,\ AA^*ABC] - r(ABC),
\end{aligned} \tag{3.13}$$

and

$$r((A^\dagger ABCC^\dagger)^\dagger - C(A^\dagger ABC)^\dagger) = r[C^*C(ABC)^*,\ (ABC)^*] - r(ABC). \tag{3.14}$$

Setting all sides of (3.13) and (3.14) equal to zero leads to the equivalence of ⟨23⟩ and ⟨136⟩.

Post- and pre-multiplying the two equalities in ⟨20⟩ with $A^{\dagger}$ and $C^{\dagger}$, respectively, and simplifying yield the two equalities in ⟨24⟩. Conversely, post- and pre-multiplying the two equalities in ⟨24⟩ with A and C, respectively, and simplifying yield the two equalities in ⟨23⟩.

Notice by (3.7) and (3.8) that $(A^{\dagger}ABC)^{\dagger}A^{\dagger} \in \{(ABC)^{(2)}\}$ and $C^{\dagger}(ABCC^{\dagger})^{\dagger} \in \{(ABC)^{(2)}\}$ hold. Hence applying (2.11) to $(A^{\dagger}ABC)^{\dagger}A^{\dagger} - C^{\dagger}(ABCC^{\dagger})^{\dagger}$ and simplifying by (2.13) and Lemma 2.7 yield

$$\begin{aligned}
& r((A^{\dagger}ABC)^{\dagger}A^{\dagger} - C^{\dagger}(ABCC^{\dagger})^{\dagger}) \\
&= r\begin{bmatrix} (A^{\dagger}ABC)^{\dagger}A^{\dagger} \\ C^{\dagger}(ABCC^{\dagger})^{\dagger} \end{bmatrix} + r[(A^{\dagger}ABC)^{\dagger}A^{\dagger},\ C^{\dagger}(ABCC^{\dagger})^{\dagger}] - r((A^{\dagger}ABC)^{\dagger}A^{\dagger}) - r(C^{\dagger}(ABCC^{\dagger})^{\dagger}) \\
&= r\begin{bmatrix} (A^{\dagger}ABC)^{*}A^{\dagger} \\ (ABCC^{\dagger})^{\dagger} \end{bmatrix} + r[(A^{\dagger}ABC)^{\dagger},\ C^{\dagger}(ABCC^{\dagger})^{*}] - 2r(ABC) \\
&= r\begin{bmatrix} (BC)^{*}A^{\dagger} \\ (ABCC^{\dagger})^{*} \end{bmatrix} + r[(A^{\dagger}ABC)^{*},\ C^{\dagger}(AB)^{*}] - 2r(ABC) \\
&= r\begin{bmatrix} (BC)^{*}A^{\dagger} \\ CC^{\dagger}(AB)^{*} \end{bmatrix} + r[(BC)^{*}A^{\dagger}A,\ C^{\dagger}(AB)^{*}] - 2r(ABC) \\
&= r\begin{bmatrix} (BC)^{*}A^{\dagger} \\ C^{*}(AB)^{*} \end{bmatrix} + r[(BC)^{*}A^{*},\ C^{\dagger}(AB)^{*}] - 2r(ABC) \\
&= r\begin{bmatrix} ABC \\ ABCC^{*}C \end{bmatrix} + r[ABC,\ AA^{*}ABC] - 2r(ABC). \qquad (3.15)
\end{aligned}$$

Setting all sides of (3.15) equal to zero leads to the equivalence of ⟨25⟩ and ⟨136⟩.

Post- and pre-multiplying the equality in ⟨25⟩ with A and C, respectively, yield the equality in ⟨26⟩. Conversely, post- and pre-multiplying the equality in ⟨26⟩ with $A^{\dagger}$ and $C^{\dagger}$, respectively, and simplifying yield the equality in ⟨25⟩.

Replacing A and C by $(A^{\dagger})^{*}$ and $(C^{\dagger})^{*}$, respectively, in ⟨1⟩–⟨7⟩ and ⟨16⟩–⟨26⟩ leads to the equivalences of these results and ⟨27⟩–⟨45⟩ through ⟨136⟩ and ⟨137⟩.

It is easy to verify by the definition of Moore–Penrose generalized inverse that $C^{*}(A^{*}ABCC^{*})^{\dagger}A^{*} \in \{(ABC)^{(2)}\}$ holds. In this case, applying (2.11) to the difference $(ABC)^{\dagger} - C^{*}(A^{*}ABCC^{*})^{\dagger}A^{*}$ and simplifying by (2.13) and Lemma 2.7, we obtain

$$\begin{aligned}
& r((ABC)^{\dagger} - C^{*}(A^{*}ABCC^{*})^{\dagger}A^{*}) \\
&= r\begin{bmatrix} (ABC)^{\dagger} \\ C^{*}(A^{*}ABCC^{*})^{\dagger}A^{*} \end{bmatrix} + r[(ABC)^{\dagger},\ C^{*}(A^{*}ABCC^{*})^{\dagger}A^{*}] - r((ABC)^{\dagger}) - r(C^{*}(A^{*}ABCC^{*})^{\dagger}A^{*}) \\
&= r\begin{bmatrix} (ABC)^{*} \\ C^{*}(A^{*}ABCC^{*})^{*} \end{bmatrix} + r[(ABC)^{*},\ C^{*}(A^{*}ABCC^{*})^{*}] - 2r(ABC) \\
&= r\begin{bmatrix} ABC \\ ABCC^{*}C \end{bmatrix} + r[ABC,\ AA^{*}ABC] - 2r(ABC). \qquad (3.16)
\end{aligned}$$

Setting all sides of (3.16) equal to zero leads to the equivalence of ⟨46⟩ and ⟨136⟩.

Pre- and post-multiplying the equality in ⟨46⟩ with $(C^{\dagger})^{*}$ and $(A^{\dagger})^{*}$, respectively, and simplifying yield the equality in ⟨47⟩. Conversely, pre- and post-multiplying the equality in ⟨47⟩ with $C^{\dagger}$ and $A^{\dagger}$, respectively, and simplifying yield the equality in ⟨46⟩.

It is easy to verify by the definition of Moore–Penrose generalized inverse that $C^*(ABCC^*)^\dagger ABC(A^*ABC)^\dagger A^* \in \{(ABC)^{(2)}\}$ holds. Then applying (2.11) to the difference $(ABC)^\dagger - C^*(ABCC^*)^\dagger ABC(A^*ABC)^\dagger A^*$ and simplifying by (2.13) and Lemma 2.7, we obtain

$$\begin{aligned}
& r((ABC)^\dagger - C^*(ABCC^*)^\dagger ABC(A^*ABC)^\dagger A^*) \\
&= r\begin{bmatrix} (ABC)^\dagger \\ C^*(ABCC^*)^\dagger ABC(A^*ABC)^\dagger A^* \end{bmatrix} + r[(ABC)^\dagger,\ C^*(ABCC^*)^\dagger ABC(A^*ABC)^\dagger A^*] \\
&\quad - r((ABC)^\dagger) - r(C^*(ABCC^*)^\dagger ABC(A^*ABC)^\dagger A^*) \\
&= r\begin{bmatrix} (ABC)^* \\ C^*(ABCC^*)^\dagger ABC(A^*ABC)^\dagger A^* \end{bmatrix} + r[(ABC)^*,\ C^*(ABCC^*)^\dagger ABC(A^*ABC)^\dagger A^*] - 2r(ABC) \\
&= r\begin{bmatrix} (ABC)^* \\ C^*(ABCC^*)^* \end{bmatrix} + r[(ABC)^*,\ C^*(ABCC^*)^*] - 2r(ABC) \\
&= r\begin{bmatrix} ABC \\ ABCC^*C \end{bmatrix} + r[ABC,\ AA^*ABC] - 2r(ABC).
\end{aligned} \tag{3.17}$$

Setting all sides of (3.17) equal to zero leads to the equivalence of ⟨48⟩ and ⟨136⟩.

Pre- and post-multiplying the equality in ⟨48⟩ with $(C^\dagger)^*$ and $(A^\dagger)^*$, respectively, and simplifying yield the equality in ⟨49⟩. Conversely, pre- and post-multiplying the equality in ⟨49⟩ with $C^\dagger$ and $A^\dagger$, respectively, and simplifying yield the equality in ⟨48⟩.

The equivalence of ⟨1⟩ and ⟨50⟩ follows from combining Theorem 3.2 ⟨1⟩ and ⟨19⟩ and Theorem 3.3 ⟨1⟩ and ⟨19⟩.

Post- and pre-multiplying the equality in ⟨50⟩ with $(A^\dagger)^*$ and $(C^\dagger)^*$, respectively, and simplifying yield the equality in ⟨51⟩. Conversely, post- and pre-multiplying the equality in ⟨51⟩ with * and C^*, respectively, and simplifying yield the equality in ⟨50⟩.

Replacing A and C with $(A^\dagger)^*$ and $(C^\dagger)^*$ in ⟨50⟩ and ⟨51⟩, respectively, and simplifying yield the equivalences ⟨1⟩ with ⟨52⟩ and ⟨53⟩ through equivalence of ⟨136⟩ and ⟨137⟩.

Replacing A and C with $(A^\dagger)^*$ and $(C^\dagger)^*$ in ⟨50⟩ and ⟨51⟩, simultaneously, and simplifying yield the equivalences ⟨1⟩ with ⟨54⟩ and ⟨55⟩ through equivalence of ⟨136⟩ and ⟨137⟩.

Replacing A and C with $(A^\dagger)^*$ and $(C^\dagger)^*$ in ⟨46⟩ and ⟨47⟩, simultaneously, and simplifying yield the equivalences ⟨1⟩ with ⟨56⟩ and ⟨57⟩ through equivalence of ⟨136⟩ and ⟨137⟩.

It is easy to verify that $(ABCC^*)^\dagger(A^\dagger)^* \in \{(A^*ABCC^*)^{(2)}\}$ and $(C^\dagger)^*(A^*ABC)^\dagger \in \{(A^*ABCC^*)^{(2)}\}$ hold. In this case, applying (2.11) to the difference $(A^*ABCC^*)^\dagger - (ABCC^*)^\dagger(A^\dagger)^*$ and $(A^*ABCC^*)^\dagger - (C^\dagger)^*(A^*ABC)^\dagger$ and simplifying by (2.13) and Lemma 2.7 yields

$$\begin{aligned}
& r((A^*ABCC^*)^\dagger - (ABCC^*)^\dagger(A^\dagger)^*) \\
&= r\begin{bmatrix} (A^*ABCC^*)^\dagger \\ (ABCC^*)^\dagger(A^\dagger)^* \end{bmatrix} + r[(A^*ABCC^*)^\dagger,\ (ABCC^*)^\dagger(A^\dagger)^*] - r((A^*ABCC^*)^\dagger) - r((ABCC^*)^\dagger(A^\dagger)^*) \\
&= r\begin{bmatrix} (A^*ABCC^*)^* \\ (ABCC^*)^*(A^\dagger)^* \end{bmatrix} + r[(A^*ABCC^*)^*,\ (ABCC^*)^*] - 2r(ABC) \\
&= r\begin{bmatrix} (A^*ABCC^*)^*A^* \\ (BC)^*A^* \end{bmatrix} - r(ABC) \\
&= r[ABC,\ AA^*ABC] - r(ABC),
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
&r((A^*ABCC^*)^\dagger - (C^\dagger)^*(A^*ABC)^\dagger)\\
&= r\begin{bmatrix} (A^*ABCC^*)^\dagger \\ (C^\dagger)^*(A^*ABC)^\dagger \end{bmatrix} + r[(A^*ABCC^*)^\dagger,\ (C^\dagger)^*(A^*ABC)^\dagger] - r((A^*ABCC^*)^\dagger) - r((C^\dagger)^*(A^*ABC)^\dagger)\\
&= r\begin{bmatrix} (A^*ABCC^*)^* \\ C(A^*ABC)^* \end{bmatrix} + r[(A^*ABCC^*)^*,\ (C^\dagger)^*(A^*ABC)^*] - 2r(ABC)\\
&= r[C^*(A^*ABCC^*)^*,\ (A^*ABC)^*] - r(ABC)\\
&= r\begin{bmatrix} ABC \\ ABCC^*C \end{bmatrix} - r(ABC).
\end{aligned} \tag{3.19}$$

Setting all sides of (3.18) and (3.19) equal to zero leads to the equivalence of ⟨58⟩ and ⟨136⟩.

Post- and pre-multiplying the equality in ⟨58⟩ with A^* and C^*, respectively, and simplifying yield the equality in ⟨59⟩. Conversely, post- and pre-multiplying the equality in ⟨58⟩ with $(A^\dagger)^*$ and $(C^\dagger)^*$, respectively, and simplifying yield the equality in ⟨59⟩.

It is easy to verify that $(A^*ABC)^\dagger A^* \in \{(ABC)^{(2)}\}$ and $C^*(ABCC^*)^\dagger \in \{(ABC)^{(2)}\}$ hold. Then applying (2.11), (2.13), and Lemma 2.7,

$$\begin{aligned}
&r((A^*ABC)^\dagger A^* - C^*(ABCC^*)^\dagger)\\
&= r\begin{bmatrix} (A^*ABC)^\dagger A^* \\ C^*(ABCC^*)^\dagger \end{bmatrix} + r[(A^*ABC)^\dagger A^*,\ C^*(ABCC^*)^\dagger] - r((A^*ABC)^\dagger A^*) - r(C^*(ABCC^*)^\dagger)\\
&= r\begin{bmatrix} (A^*ABC)^* A^* \\ (ABCC^*)^* \end{bmatrix} + r[(A^*ABC)^*,\ C^*(ABCC^*)^*] - 2r(ABC)\\
&= r\begin{bmatrix} ABC \\ ABCC^*C \end{bmatrix} + r[ABC,\ AA^*ABC] - 2r(ABC).
\end{aligned} \tag{3.20}$$

Setting all sides of (3.20) equal to zero leads to the equivalence of ⟨60⟩ and ⟨136⟩.

Pre- and post-multiplying the equality in ⟨60⟩ with $(C^\dagger)^*$ and $(A^\dagger)^*$, respectively, and simplifying yield the equality in ⟨61⟩. Conversely, post- and pre-multiplying the equality in ⟨61⟩ with C^* and A^*, respectively, and simplifying yield the equality in ⟨61⟩.

Replacing C and A with $(C^\dagger)^*$ and $(A^\dagger)^*$ in the two equalities in ⟨58⟩, respectively, and simplifying yield the equivalence of ⟨1⟩ with ⟨62⟩ through the equivalence of ⟨136⟩ and ⟨137⟩.

Post- and pre-multiplying the two equalities in ⟨62⟩ with $(A^\dagger)^*$ and $(C^\dagger)^*$, respectively, and simplifying yield the two equalities in ⟨63⟩. Conversely, post- and pre-multiplying the equality in ⟨61⟩ with A^* and C^*, respectively, and simplifying yield the equality in ⟨62⟩.

Replacing A and C with $(A^\dagger)^*$ and $(C^\dagger)^*$ in the equality in ⟨61⟩, respectively, and simplifying yield the equivalence ⟨1⟩ with ⟨64⟩ through the equivalence of ⟨136⟩ and ⟨137⟩.

Replacing A and C with $(A^\dagger)^*$ and $(C^\dagger)^*$ in the equality in ⟨60⟩, respectively, and simplifying yield the equivalence ⟨1⟩ with ⟨65⟩ through the equivalence of ⟨136⟩ and ⟨137⟩.

The derivations of (3.11)–(3.20) present typical steps of establishing and simplifying matrix rank equalities associated with nested ROLs in the corresponding statements. The equivalences of ⟨66⟩–⟨108⟩ with ⟨136⟩ and ⟨137⟩ can be established by similar approaches, and the routine verifications are left to the reader.

Combining Theorem 3.2 ⟨26⟩ and ⟨60⟩ and Theorem 3.3 ⟨26⟩ and ⟨60⟩, and comparing them with ⟨136⟩, we obtain the equivalence of ⟨109⟩ and ⟨136⟩.

Combining Theorem 3.2 ⟨27⟩–⟨29⟩ and ⟨60⟩ with Theorem 3.3 ⟨27⟩–⟨29⟩ and ⟨60⟩, and comparing them with ⟨136⟩, we obtain the equivalences of ⟨110⟩–⟨112⟩ and ⟨136⟩.

Combining Theorem 3.2 ⟨32⟩–⟨35⟩ and ⟨60⟩ with Theorem 3.3 ⟨32⟩–⟨35⟩ and ⟨60⟩, and comparing them with ⟨136⟩, we obtain the equivalences of ⟨113⟩–⟨116⟩ and ⟨136⟩.

Combining Theorem 3.2 ⟨49⟩–⟨51⟩ and ⟨60⟩ and Theorem 3.3 ⟨49⟩–⟨51⟩ and ⟨60⟩, and comparing them with ⟨136⟩, we obtain the equivalences of ⟨117⟩–⟨119⟩ and ⟨136⟩.

Combining Theorem 3.2 ⟨54⟩–⟨57⟩ and ⟨60⟩ with Theorem 3.3 ⟨54⟩–⟨57⟩ and ⟨60⟩, and comparing them with ⟨136⟩, we obtain the equivalences of ⟨120⟩–⟨123⟩ and ⟨136⟩.

Combining Theorem 3.2 ⟨36⟩ and ⟨60⟩ and Theorem 3.3 ⟨36⟩ and ⟨60⟩, and comparing them with ⟨136⟩, we obtain the equivalence of ⟨124⟩ and ⟨136⟩.

Combining Theorem 3.2 ⟨37⟩–⟨44⟩ and ⟨60⟩ with Theorem 3.3 ⟨37⟩–⟨44⟩ and ⟨60⟩, and comparing them with ⟨136⟩, we obtain the equivalences of ⟨125⟩–⟨132⟩ and ⟨136⟩.

The equivalence of ⟨133⟩ and ⟨135⟩ follows from Lemma 2.2 (b). □

With a bit more work we can also obtain the following result.

Theorem 3.5. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $C \in \mathbb{C}^{p\times q}$. Then the following 3 statements are equivalent*:

⟨1⟩ $(ABC)^{\dagger} = C^{\dagger}(A^{\dagger}ABCC^{\dagger})^{\dagger}A^{\dagger}$.

⟨2⟩ $(ABC)^{\dagger} = C^{\dagger}(A^{\dagger}AB + BCC^{\dagger} - B)^{\dagger}A^{\dagger}$.

⟨3⟩ $((A^{\dagger})^{*}B(C^{\dagger})^{*})^{\dagger} = C^{*}(A^{\dagger}AB + BCC^{\dagger} - B)^{\dagger}A^{*}$.

Proof. Since $A^{\dagger}A$ and $CC^{\dagger}$ are orthogonal projectors, we obtain by Lemma 2.9 the following matrix identity

$$(A^{\dagger}ABCC^{\dagger})^{\dagger} = (F_ABE_C)^{\dagger} - (B - A^{\dagger}AB - BCC^{\dagger})^{\dagger}.$$

Substituting it into the equality in ⟨1⟩ and simplifying lead to the equivalence of ⟨1⟩ and ⟨2⟩. Replacing A and C in ⟨1⟩ by $(A^{\dagger})^{*}$ and $(C^{\dagger})^{*}$, respectively, yields ⟨3⟩. □

Finally, we present a pair of equivalent facts associated with (1.12).

Corollary 3.6. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $C \in \mathbb{C}^{p\times q}$. Then the following 2 statements are equivalent*:

⟨1⟩ $(ABC)^{\dagger} = C^{\dagger}(A^{\dagger}ABCC^{\dagger})^{\dagger}A^{\dagger}$.

⟨2⟩ $((A^{*}A)^{1/2}B(CC^{*})^{1/2})^{\dagger} = ((CC^{*})^{1/2})^{\dagger}(A^{\dagger}ABCC^{\dagger})^{\dagger}((A^{*}A)^{1/2})^{\dagger}$.

Proof. It is well recognized in matrix theory that the two products $A^{*}A$ and CC^{*} are positive semi-definite matrices, and the two square roots $(A^{*}A)^{1/2}$ and $(CC^{*})^{1/2}$ make sense and are unique. In this situation,

$$\mathscr{R}((A^{*}A)^{1/2}) = \mathscr{R}(A^{*}A) = \mathscr{R}(A^{*}) \text{ and } \mathscr{R}(CC^{*})^{1/2}) = \mathscr{R}(CC^{*}) = \mathscr{R}(C)$$

hold, so that

$$((A^{*}A)^{1/2})^{\dagger}((A^{*}A)^{1/2}) = A^{\dagger}A \text{ and } (CC^{*})^{1/2}((CC^{*})^{1/2})^{\dagger} = CC^{\dagger}$$

Hold. Now applying Theorem 3.4 ⟨1⟩ and ⟨136⟩ to the matrix product $(A^{*}A)^{1/2}B(CC^{*})^{1/2}$, we obtain

$$((A^{*}A)^{1/2}B(CC^{*})^{1/2})^{\dagger} = ((CC^{*})^{1/2})^{\dagger}(((A^{*}A)^{1/2})^{\dagger}((A^{*}A)^{1/2})B(CC^{*})^{1/2}((CC^{*})^{1/2})^{\dagger})^{\dagger}((A^{*}A)^{1/2})^{\dagger}$$

$$\Leftrightarrow ((A^*A)^{1/2}B(CC^*)^{1/2})^\dagger = ((CC^*)^{1/2})^\dagger(A^\dagger ABCC^\dagger)^\dagger((A^*A)^{1/2})^\dagger$$
$$\Leftrightarrow r[(A^*A)^{3/2}B(CC^*)^{1/2},\ (A^*A)^{1/2}B(CC^*)^{1/2}]$$
$$= r[(CC^*)^{3/2}B^*(A^*A)^{1/2},\ (CC^*)^{1/2}B^*(A^*A)^{1/2}] = r((A^*A)^{1/2}B(CC^*)^{1/2}),$$

where by Lemmas 2.4 and 2.7,

$$\begin{aligned} r[(A^*A)^{3/2}B(CC^*)^{1/2},\ (A^*A)^{1/2}B(CC^*)^{1/2}] &= r[(A^*A)^2BCC^*,\ A^*ABCC^*] \\ &= r[AA^*ABC,\ ABC], \\ r[(CC^*)^{3/2}B^*(A^*A)^{1/2},\ (CC^*)^{1/2}B^*(A^*A)^{1/2}] &= r[(CC^*)^2B^*A^*A,\ CC^*B^*A^*A] \\ &= r[C^*C(ABC)^*,\ (ABC)^*], \\ r((A^*A)^{1/2}B(CC^*)^{1/2}) &= r(A^*ABCC^*) = r(ABC). \end{aligned}$$

Combining the above two groups of equalities with Theorem 3.4 ⟨1⟩ and ⟨136⟩ leads to the equivalence of ⟨1⟩ and ⟨2⟩ in the theorem. □

Various consequences can be derived form Theorems 3.2–3.4. For example, (1.12) is reduced to the second equality in (1.3) if $A^\dagger ABCC^\dagger = B$, which includes $A^\dagger A = I_n$ and $CC^\dagger = I_p$ as its special case. Thus we obtain the following results from Theorem 3.4.

Corollary 3.7. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, *and* $C \in \mathbb{C}^{p\times q}$, *and assume that* $A^\dagger ABCC^\dagger = B$ *holds, i.e.,* $\mathscr{R}(B) \subseteq \mathscr{R}(A^*)$ *and* $\mathscr{R}(B^*) \subseteq \mathscr{R}(C)$. *Then the following 137 statements are equivalent:*

⟨1⟩ $(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$.

⟨2⟩ $C(ABC)^\dagger A = B^\dagger$.

⟨3⟩ $A(C(ABC)^\dagger A)^\dagger C = ABC$.

⟨4⟩ $C(A(C(ABC)^\dagger A)^\dagger C)^\dagger A = C(ABC)^\dagger A$.

⟨5⟩ $B = A^\dagger(C^\dagger B^\dagger A^\dagger)^\dagger C^\dagger$.

⟨6⟩ $(ABC)^\dagger = C^\dagger(AB)^\dagger ABC(BC)^\dagger A^\dagger$.

⟨7⟩ $C(ABC)^\dagger A = (AB)^\dagger ABC(BC)^\dagger$.

⟨8⟩ $\{(ABC)^{(1,2,3)}\} \ni (BC)^\dagger A^\dagger$ *and* $\{(ABC)^{(1,2,4)}\} \ni C^\dagger(AB)^\dagger$.

⟨9⟩ $\{(ABC)^{(1,2,3)}\} \ni (A^*ABC)^\dagger A^*$ *and* $\{(ABC)^{(1,2,4)}\} \ni C^*(ABCC^*)^\dagger$.

⟨10⟩ $\{(BC)^{(1,2,3)}\} \ni (ABC)^\dagger A$ *and* $\{(AB)^{(1,2,4)}\} \ni C(ABC)^\dagger$.

⟨11⟩ $\{(A^*ABC)^{(1,2,3)}\} \ni (ABC)^\dagger(A^\dagger)^*$ *and* $\{(ABCC^*)^{(1,2,4)}\} \ni (C^\dagger)^*(ABC)^\dagger$.

⟨12⟩ $\{(ABC)^{(1,2,3)}\} \ni (BC)^\dagger(BC(BC)^\dagger)^\dagger A^\dagger$ *and* $\{(ABC)^{(1,2,4)}\} \ni C^\dagger((AB)^\dagger AB)^\dagger(AB)^\dagger$.

⟨13⟩ $\{(ABC)^{(1,2,3)}\} \ni (BC)^*(A^*ABC(BC)^*)^\dagger A^*$ *and* $\{(ABC)^{(1,2,4)}\} \ni C^*((AB)^*ABCC^*)^\dagger(AB)^*$.

⟨14⟩ $\{(BC(BC)^\dagger)^{(1,2,3)}\} \ni BC(ABC)^\dagger A$ *and* $\{((AB)^\dagger AB)^{(1,2,4)}\} \ni C(ABC)^\dagger AB$.

⟨15⟩ $\{(A^*ABC(BC)^*)^{(1,2,3)}\} \ni ((BC)^\dagger)^*(ABC)^\dagger(A^\dagger)^*$ *and* $\{(A^*ABCC^*)^{(1,2,4)}\} \ni (C^\dagger)^*(ABC)^\dagger((AB)^\dagger)^*$.

⟨16⟩ $(ABC)^\dagger = (BC)^\dagger A^\dagger$ *and* $(ABC)^\dagger = C^\dagger(AB)^\dagger$.

⟨17⟩ $(BC)^\dagger = (ABC)^\dagger A$ *and* $(AB)^\dagger = C(ABC)^\dagger$.

⟨18⟩ $ABC(ABC)^\dagger = ABC(BC)^\dagger A^\dagger$ *and* $(ABC)^\dagger ABC = C^\dagger(AB)^\dagger ABC$.

⟨19⟩ $ABC(ABC)^\dagger A = ABC(BC)^\dagger$ *and* $C(ABC)^\dagger ABC = (AB)^\dagger ABC$.

⟨20⟩ $(BC)^{\dagger}BC(ABC)^{\dagger} = (BC)^{\dagger}A^{\dagger}$ *and* $(ABC)^{\dagger}AB(AB)^{\dagger} = C^{\dagger}(AB)^{\dagger}$.

⟨21⟩ $((BC)^{\dagger}A^{\dagger})^{\dagger}(BC)^{\dagger}BC = ABC$ *and* $AB(AB)^{\dagger}(C^{\dagger}(AB)^{\dagger})^{\dagger} = ABC$.

⟨22⟩ $ABC(ABC)^{\dagger} = ((BC)^{\dagger}A^{\dagger})^{\dagger}(BC)^{\dagger}A^{\dagger}$ *and* $(ABC)^{\dagger}ABC = C^{\dagger}(AB)^{\dagger}(C^{\dagger}(AB)^{\dagger})^{\dagger}$.

⟨23⟩ $B^{\dagger} = (AB)^{\dagger}A$ *and* $B^{\dagger} = C(BC)^{\dagger}$.

⟨24⟩ $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ *and* $(BC)^{\dagger} = C^{\dagger}B^{\dagger}$.

⟨25⟩ $(BC)^{\dagger}A^{\dagger} = C^{\dagger}(AB)^{\dagger}$.

⟨26⟩ $(AB)^{\dagger}A = C(BC)^{\dagger}$.

⟨27⟩ $((A^{\dagger})^{*}B(C^{\dagger})^{*})^{\dagger} = C^{*}B^{\dagger}A^{*}$.

⟨28⟩ $(C^{\dagger})^{*}((A^{\dagger})^{*}B(C^{\dagger})^{*})^{\dagger}(A^{\dagger})^{*} = B^{\dagger}$.

⟨29⟩ $(A^{\dagger})^{*}((C^{\dagger})^{*}((A^{\dagger})^{*}B(C^{\dagger})^{*})^{\dagger}(A^{\dagger})^{*})^{\dagger}(C^{\dagger})^{*} = (A^{\dagger})^{*}B(C^{\dagger})^{*}$.

⟨30⟩ $(C^{\dagger})^{*}((A^{\dagger})^{*}((C^{\dagger})^{*}(ABC)^{\dagger}(A^{\dagger})^{*})^{\dagger}(C^{\dagger})^{*})^{\dagger}(A^{\dagger})^{*} = (C^{\dagger})^{*}((A^{\dagger})^{*}B(C^{\dagger})^{*})^{\dagger}(A^{\dagger})^{*}$.

⟨31⟩ $B = A^{*}(C^{*}B^{\dagger}A^{*})^{\dagger}C^{*}$.

⟨32⟩ $((A^{\dagger})^{*}B(C^{\dagger})^{*})^{\dagger} = C^{*}((A^{\dagger})^{*}B)^{\dagger}(A^{\dagger})^{*}B(C^{\dagger})^{*}(B(C^{\dagger})^{*})^{\dagger}A^{*}$.

⟨33⟩ $(C^{\dagger})^{*}((A^{\dagger})^{*}B(C^{\dagger})^{*})^{\dagger}(A^{\dagger})^{*} = ((A^{\dagger})^{*}B)^{\dagger}(A^{\dagger})^{*}B(C^{\dagger})^{*}(B(C^{\dagger})^{*})^{\dagger}$.

⟨34⟩ $((A^{\dagger})^{*}BC)^{\dagger} = (BC)^{\dagger}A^{*}$ *and* $(AB(C^{\dagger})^{*})^{\dagger} = C^{*}(AB)^{\dagger}$.

⟨35⟩ $(BC)^{\dagger} = ((A^{\dagger})^{*}BC)^{\dagger}(A^{\dagger})^{*}$ *and* $(AB)^{\dagger} = (C^{\dagger})^{*}(AB(C^{\dagger})^{*})^{\dagger}$.

⟨36⟩ $ABC(ABC)^{\dagger} = AB(C^{\dagger})^{*}(B(C^{\dagger})^{*})^{\dagger}A^{\dagger}$ *and* $(ABC)^{\dagger}ABC = C^{\dagger}((A^{\dagger})^{*}B)^{\dagger}(A^{\dagger})^{*}BC$.

⟨37⟩ $ABC(ABC)^{\dagger}A = AB(C^{\dagger})^{*}(B(C^{\dagger})^{*})^{\dagger}$ *and* $C(ABC)^{\dagger}ABC = ((A^{\dagger})^{*}B)^{\dagger}(A^{\dagger})^{*}BC$.

⟨38⟩ $ABC(ABC)^{\dagger} = ((B(C^{\dagger})^{*})^{\dagger}A^{\dagger})^{\dagger}(B(C^{\dagger})^{*})^{\dagger}A^{\dagger}$ *and* $(ABC)^{\dagger}ABC = C^{\dagger}((A^{\dagger})^{*}B)^{\dagger}(C^{\dagger}((A^{\dagger})^{*}B)^{\dagger})^{\dagger}$.

⟨39⟩ $(BC)^{\dagger}BC((A^{\dagger})^{*}BC)^{\dagger} = (BC)^{\dagger}A^{*}$ *and* $(AB(C^{\dagger})^{*})^{\dagger}AB(AB)^{\dagger} = C^{*}(AB)^{\dagger}$.

⟨40⟩ $((BC)^{\dagger}A^{*})^{\dagger}(BC)^{\dagger}BC = (A^{\dagger})^{*}BC$ *and* $AB(AB)^{\dagger}(C^{*}(AB)^{\dagger})^{\dagger} = AB(C^{\dagger})^{*}$.

⟨41⟩ $B^{\dagger} = ((A^{\dagger})^{*}B)^{\dagger}(A^{\dagger})^{*}$ *and* $B^{\dagger} = (C^{\dagger})^{*}(B(C^{\dagger})^{*})^{\dagger}$.

⟨42⟩ $((A^{\dagger})^{*}B)^{\dagger} = B^{\dagger}A^{*}$ *and* $(B(C^{\dagger})^{*})^{\dagger} = C^{*}B^{\dagger}$.

⟨43⟩ $(B(C^{\dagger})^{*})^{\dagger}A^{*} = C^{*}((A^{\dagger})^{*}B)^{\dagger}$.

⟨44⟩ $((A^{\dagger})^{*}B)^{\dagger}(A^{\dagger})^{*} = (C^{\dagger})^{*}(B(C^{\dagger})^{*})^{\dagger}$.

⟨45⟩ $C^{*}((A^{\dagger})^{*}B)^{\dagger} = (B(C^{\dagger})^{*})^{\dagger}A^{*}$.

⟨46⟩ $(ABC)^{\dagger} = C^{*}(A^{*}ABCC^{*})^{\dagger}A^{*}$.

⟨47⟩ $(A^{*}ABCC^{*})^{\dagger} = (C^{\dagger})^{*}(ABC)^{\dagger}(A^{\dagger})^{*}$.

⟨48⟩ $(ABC)^{\dagger} = C^{*}(ABCC^{*})^{\dagger}ABC(A^{*}ABC)^{\dagger}A^{*}$.

⟨49⟩ $(C^{\dagger})^{*}(ABC)^{\dagger}(A^{\dagger})^{*} = (ABCC^{*})^{\dagger}ABC(A^{*}ABC)^{\dagger}$.

⟨50⟩ $(ABC)^{\dagger} = (A^{*}ABC)^{\dagger}A^{*}$ *and* $(ABC)^{\dagger} = C^{*}(ABCC^{*})^{\dagger}$.

⟨51⟩ $(A^{*}ABC)^{\dagger} = (ABC)^{\dagger}(A^{\dagger})^{*}$ *and* $(ABCC^{*})^{\dagger} = (C^{\dagger})^{*}(ABC)^{\dagger}$.

⟨52⟩ $((A^{\dagger})^{*}BC)^{\dagger} = ((A^{*}A)^{\dagger}BC)^{\dagger}A^{\dagger}$ *and* $(AB(C^{\dagger})^{*})^{\dagger} = C^{\dagger}(AB(CC^{*})^{\dagger})^{\dagger}$.

⟨53⟩ $((A^{*}A)^{\dagger}BC)^{\dagger} = ((A^{\dagger})^{*}BC)^{\dagger}A$ *and* $(AB(CC^{*})^{\dagger})^{\dagger} = C(AB(C^{\dagger})^{*})^{\dagger}$.

⟨54⟩ $((A^{\dagger})^{*}B(C^{\dagger})^{*})^{\dagger} = ((A^{*}A)^{\dagger}B(C^{\dagger})^{*})^{\dagger}A^{\dagger}$ *and* $((A^{\dagger})^{*}B(C^{\dagger})^{*})^{\dagger} = C^{\dagger}((A^{\dagger})^{*}B(CC^{*})^{\dagger})^{\dagger}$.

⟨55⟩ $((A^{*}A)^{\dagger}B(C^{\dagger})^{*})^{\dagger} = ((A^{\dagger})^{*}B(C^{\dagger})^{*})^{\dagger}A$ *and* $((A^{\dagger})^{*}B(CC^{*})^{\dagger})^{\dagger} = C((A^{\dagger})^{*}B(C^{\dagger})^{*})^{\dagger}$.

⟨56⟩ $((A^{\dagger})^{*}B(C^{\dagger})^{*})^{\dagger} = C^{\dagger}((A^{*}A)^{\dagger}B(CC^{*})^{\dagger})^{\dagger}A^{\dagger}$.

⟨57⟩ $((A^*A)^\dagger B(CC^*)^\dagger)^\dagger = C((A^\dagger)^* B(C^\dagger)^*)^\dagger A$.

⟨58⟩ $(A^*ABCC^*)^\dagger = (ABCC^*)^\dagger (A^\dagger)^*$ *and* $(A^*ABCC^*)^\dagger = (C^\dagger)^*(A^*ABC)^\dagger$.

⟨59⟩ $(ABCC^*)^\dagger = (A^*ABCC^*)^\dagger A^*$ *and* $(A^*ABC)^\dagger = C^*(A^*ABCC^*)^\dagger$.

⟨60⟩ $(A^*ABC)^\dagger A^* = C^*(ABCC^*)^\dagger$.

⟨61⟩ $(ABCC^*)^\dagger (A^\dagger)^* = (C^\dagger)^*(A^*ABC)^\dagger$.

⟨62⟩ $(AB(CC^*)^\dagger)^\dagger = (A^*AB(CC^*)^\dagger)^\dagger A^*$ *and* $((A^*A)^\dagger BC)^\dagger = C^*((A^*A)^\dagger BCC^*)^\dagger$.

⟨63⟩ $(A^*AB(CC^*)^\dagger)^\dagger = (AB(CC^*)^\dagger)^\dagger (A^\dagger)^*$ *and* $((A^*A)^\dagger BCC^*)^\dagger = (C^\dagger)^*((A^*A)^\dagger BC)^\dagger$.

⟨64⟩ $((A^\dagger)^* B(CC^*)^\dagger)^\dagger A = C((A^*A)^\dagger B(C^\dagger)^*)^\dagger$.

⟨65⟩ $((A^*A)^\dagger B(C^\dagger)^*)^\dagger A^\dagger = C^\dagger((A^\dagger)^* B(CC^*)^\dagger)^\dagger$.

⟨66⟩ $(A^*ABC)^\dagger = (AA^*ABC)^\dagger A$ *and* $(ABCC^*)^\dagger = C(ABCC^*C)^\dagger$.

⟨67⟩ $(AA^*ABC)^\dagger = (A^*ABC)^\dagger A^\dagger$ *and* $(ABCC^*C)^\dagger = C^\dagger(ABCC^*)^\dagger$.

⟨68⟩ $((A^*A)^\dagger BC)^\dagger = ((A^*AA^*)^\dagger BC)^\dagger (A^\dagger)^*$ *and* $(AB(CC^*)^\dagger)^\dagger = (C^\dagger)^*(AB(C^*CC^*)^\dagger)^\dagger$.

⟨69⟩ $((A^*A)^\dagger BC)^\dagger A^* = ((A^*AA^*)^\dagger BC)^\dagger$ *and* $C^*(AB(CC^*)^\dagger)^\dagger = (AB(C^*CC^*)^\dagger)^\dagger$.

⟨70⟩ $((A^*A)^\dagger B(C^\dagger)^*)^\dagger = ((A^*AA^*)^\dagger B(C^\dagger)^*)^\dagger (A^\dagger)^*$ *and* $((A^\dagger)^* B(CC^*)^\dagger)^\dagger = (C^\dagger)^*((A^\dagger)^* B(C^*CC^*)^\dagger)^\dagger$.

⟨71⟩ $((A^*AA^*)^\dagger B(C^\dagger)^*)^\dagger = ((A^*A)^\dagger B(C^\dagger)^*)^\dagger A^*$ *and* $((A^\dagger)^* B(C^*CC^*)^\dagger)^\dagger = C^*((A^\dagger)^* B(CC^*)^\dagger)^\dagger$.

⟨72⟩ $(A^*ABCC^*)^\dagger = C(A^*ABCC^*C)^\dagger$ *and* $(A^*ABCC^*)^\dagger = (AA^*ABCC^*)^\dagger A$.

⟨73⟩ $(A^*ABCC^*C)^\dagger = C^\dagger(A^*ABCC^*)^\dagger$ *and* $(AA^*ABCC^*)^\dagger = (A^*ABCC^*)^\dagger A^\dagger$.

⟨74⟩ $((A^*A)^\dagger BCC^*)^\dagger = C((A^*A)^\dagger BCC^*C)^\dagger$ *and* $(A^*AB(CC^*)^\dagger)^\dagger = (AA^*AB(CC^*)^\dagger)^\dagger A$.

⟨75⟩ $((A^*A)^\dagger BCC^*C)^\dagger = C^\dagger((A^*A)^\dagger BCC^*)^\dagger$ *and* $(A^*AB(CC^*)^\dagger)^\dagger A^\dagger = (AA^*AB(CC^*)^\dagger)^\dagger$.

⟨76⟩ $(A^*AB(CC^*)^\dagger)^\dagger = (C^\dagger)^*(A^*AB(C^*CC^*)^\dagger)^\dagger$ *and* $((A^*A)^\dagger BCC^*)^\dagger = ((A^*AA^*)^\dagger BCC^*)^\dagger (A^\dagger)^*$.

⟨77⟩ $(A^*AB(C^*CC^*)^\dagger)^\dagger = C^*(A^*AB(CC^*)^\dagger)^\dagger$ *and* $(AA^*AB(CC^*)^\dagger)^\dagger = (A^*AB(CC^*)^\dagger)^\dagger A^*$.

⟨78⟩ $((A^*A)^\dagger B(CC^*)^\dagger)^\dagger = (C^\dagger)^*((A^*A)^\dagger B(C^*CC^*)^\dagger)^\dagger$
and $((A^*A)^\dagger B(CC^*)^\dagger)^\dagger = ((A^*AA^*)^\dagger B(CC^*)^\dagger)^\dagger (A^\dagger)^*$.

⟨79⟩ $((A^*A)^\dagger B(C^*CC^*)^\dagger)^\dagger = C^*((A^*A)^\dagger B(CC^*)^\dagger)^\dagger$ *and* $((A^*AA^*)^\dagger B(CC^*)^\dagger)^\dagger = ((A^*A)^\dagger B(CC^*)^\dagger)^\dagger A^*$.

⟨80⟩ $C(A^*ABCC^*C)^\dagger = (AA^*ABCC^*)^\dagger A$.

⟨81⟩ $(A^*ABCC^*C)^\dagger A^\dagger = C^\dagger(AA^*ABCC^*)^\dagger$.

⟨82⟩ $C((A^*A)^\dagger BCC^*C)^\dagger = ((A^*AA^*)^\dagger BCC^*)^\dagger (A^\dagger)^*$.

⟨83⟩ $((A^*A)^\dagger BCC^*C)^\dagger A^* = C^\dagger((A^*AA^*)^\dagger BCC^*)^\dagger$.

⟨84⟩ $(C^\dagger)^*(A^*AB(C^*CC^*)^\dagger)^\dagger = (AA^*AB(CC^*)^\dagger)^\dagger A$.

⟨85⟩ $(A^*AB(C^*CC^*)^\dagger)^\dagger A^\dagger = C^*(AA^*AB(CC^*)^\dagger)^\dagger$.

⟨86⟩ $((CC^*C)^\dagger B^*(A^*A)^\dagger)^\dagger C^\dagger = A^\dagger((CC^*)^\dagger B^*(AA^*A)^\dagger)^\dagger$.

⟨87⟩ $A((CC^*C)^\dagger B^*(A^*A)^\dagger)^\dagger = ((CC^*)^\dagger B^*(AA^*A)^\dagger)^\dagger C$.

⟨88⟩ $(A^*ABCC^*)^\dagger = C(AA^*ABCC^*C)^\dagger A$.

⟨89⟩ $(AA^*ABCC^*C)^\dagger = C^\dagger(A^*ABCC^*)^\dagger A^\dagger$.

⟨90⟩ $((A^*A)^\dagger BCC^*)^\dagger = C((A^*AA^*)^\dagger BCC^*C)^\dagger (A^\dagger)^*$.

⟨91⟩ $((A^*AA^*)^\dagger BCC^*C)^\dagger = C^\dagger((A^*A)^\dagger BCC^*)^\dagger A^*$.

⟨92⟩ $(A^*AB(CC^*)^\dagger)^\dagger = (C^\dagger)^*(AA^*AB(C^*CC^*)^\dagger)^\dagger A$.

⟨93⟩ $(AA^*AB(C^*CC^*)^\dagger)^\dagger = C^*(A^*AB(CC^*)^\dagger)^\dagger(A^\dagger)^*$.

⟨94⟩ $((A^*A)^\dagger B(CC^*)^\dagger)^\dagger = (C^\dagger)^*((A^*AA^*)^\dagger B(C^*CC^*)^\dagger)^\dagger(A^\dagger)^*$.

⟨95⟩ $((A^*AA^*)^\dagger B(C^*CC^*)^\dagger)^\dagger = C^*((A^*A)^\dagger B(CC^*)^\dagger)^\dagger A^*$.

⟨96⟩ $(AA^*ABCC^*C)^\dagger = ((A^*A)^2BCC^*C)^\dagger A^*$ *and* $(AA^*ABCC^*C)^\dagger = C^*(AA^*AB(CC^*)^2)^\dagger$.

⟨97⟩ $((A^*A)^2BCC^*C)^\dagger = (AA^*ABCC^*C)^\dagger(A^\dagger)^*$ *and* $(AA^*AB(CC^*)^2)^\dagger = (C^\dagger)^*(AA^*ABCC^*C)^\dagger$.

⟨98⟩ $((A^*A)^2BCC^*C)^\dagger A^* = C^*((AA^*AB(CC^*)^2)^\dagger$.

⟨99⟩ $(AA^*ABCC^*C)^\dagger(A^\dagger)^* = (C^\dagger)^*(AA^*ABCC^*C)^\dagger$.

⟨100⟩ $(AA^*ABCC^*C)^\dagger = C^*((A^*A)^2B(CC^*)^2)^\dagger A^*$.

⟨101⟩ $((A^*A)^2B(CC^*)^2)^\dagger = (C^\dagger)^*(AA^*ABCC^*C)^\dagger(A^\dagger)^*$.

⟨102⟩ $(B^*)^\dagger = A^\dagger(B^*A^\dagger)^\dagger$ *and* $(B^*)^\dagger = (C^\dagger B^*)^\dagger C^\dagger$.

⟨103⟩ $(B^*A^\dagger)^\dagger = A(B^*)^\dagger$ *and* $(C^\dagger B^*)^\dagger = (B^*)^\dagger C$.

⟨104⟩ $(A^*AB)^\dagger = C(A^*ABC)^\dagger$ *and* $(BCC^*)^\dagger = (ABCC^*)^\dagger A$.

⟨105⟩ $(A^*ABC)^\dagger = C^\dagger(A^*AB)^\dagger$ *and* $(ABCC^*)^\dagger = (BCC^*)^\dagger A^\dagger$.

⟨106⟩ $(C^\dagger B^*(A^*A)^\dagger)^\dagger = ((CC^*)^\dagger B^*(A^*A)^\dagger)^\dagger(C^\dagger)^*$ *and* $((CC^*)^\dagger B^*A^\dagger)^\dagger = (A^\dagger)^*((CC^*)^\dagger B^*(A^*A)^\dagger)^\dagger$.

⟨107⟩ $((A^*A)^\dagger B(C^\dagger)^*)^\dagger = C^\dagger((A^*A)^\dagger B(CC^*)^\dagger)^\dagger$ *and* $((A^\dagger)^*B(CC^*)^\dagger)^\dagger = ((A^*A)^\dagger B(CC^*)^\dagger)^\dagger A^\dagger$.

⟨108⟩ $((A^*A)^\dagger B(CC^*)^\dagger)^\dagger = C((A^*A)^\dagger B(C^\dagger)^*)^\dagger$ *and* $((A^*A)^\dagger B(CC^*)^\dagger)^\dagger = ((A^\dagger)^*B(CC^*)^\dagger)^\dagger A$.

⟨109⟩ $(ABC(BC)^\dagger A^\dagger)^\dagger = ((BC)^\dagger A^\dagger)^\dagger(ABC)^\dagger$ *and* $(C^\dagger(AB)^\dagger ABC)^\dagger = (ABC)^\dagger(C^\dagger(AB)^\dagger)^\dagger$.

⟨110⟩ $ABC(BC)^\dagger A^\dagger$ *and* $C^\dagger(AB)^\dagger ABC$ *are orthogonal projectors.*

⟨111⟩ $ABC(A^*ABC)^\dagger A^*$ *and* $C^*(ABCC^*)^\dagger ABC$ *are orthogonal projectors.*

⟨112⟩ $A^\dagger(ABC)(ABC)^\dagger A$ *and* $C(ABC)^\dagger(ABC)C^\dagger$ *orthogonal projectors.*

⟨113⟩ $ABC(ABC)^\dagger$ *and* AA^* *commute, and* $(ABC)^\dagger ABC$ *and* C^*C *commute.*

⟨114⟩ $ABC(BC)^\dagger A^\dagger$ *and* AA^* *commute, and* $C^\dagger(AB)^\dagger ABC$ *and* C^*C *commute.*

⟨115⟩ $ABC(A^*ABC)^\dagger A^*$ *and* AA^* *commute, and* $C^*(ABCC^*)^\dagger ABC$ *and* C^*C *commute.*

⟨116⟩ $A^\dagger(ABC)(ABC)^\dagger A$ *and* A^*A *commute, and* $C(ABC)^\dagger(ABC)C^\dagger$ *and* CC^* *commute.*

⟨117⟩ $ABC(BC)^\dagger A^\dagger$ *and* $C^\dagger(AB)^\dagger ABC$ *are orthogonal projectors.*

⟨118⟩ $(A^\dagger)^*((BC)^\dagger(A^*A)^\dagger)^\dagger(BC)^\dagger A^\dagger$ *and* $C^\dagger(AB)^\dagger((CC^*)^\dagger(AB)^\dagger)^\dagger(C^\dagger)^*$ *are orthogonal projectors.*

⟨119⟩ $A^\dagger((BC)^\dagger A^\dagger)^\dagger((BC)^\dagger A^\dagger)A$ *and* $C(C^\dagger(AB)^\dagger)(C^\dagger(AB)^\dagger)^\dagger C^\dagger$ *are orthogonal projectors.*

⟨120⟩ $((BC)^\dagger A^\dagger)^\dagger((BC)^\dagger A^\dagger)$ *and* $(AA^*)^\dagger$ *commute, and* $(C^\dagger(AB)^\dagger)(C^\dagger(AB)^\dagger)^\dagger$ *and* $(C^*C)^\dagger$ *commute.*

⟨121⟩ $A((BC)^\dagger)^\dagger(BC)^\dagger A^\dagger$ *and* $(AA^*)^\dagger$ *commute, and* $C^\dagger(AB)^\dagger((AB)^\dagger)^\dagger C$ *and* $(C^*C)^\dagger$ *commute.*

⟨122⟩ $(A^\dagger)^*((BC)^\dagger(A^*A)^\dagger)^\dagger(BC)^\dagger A^\dagger$ *and* $(AA^*)^\dagger$ *commute, and* $C^\dagger(AB)^\dagger((CC^*)^\dagger(AB)^\dagger)^\dagger(C^\dagger)^*$ *and* $(C^*C)^\dagger$ *commute.*

⟨123⟩ $A^\dagger((BC)^\dagger A^\dagger)^\dagger((BC)^\dagger A^\dagger)A$ *and* $(A^*A)^\dagger$ *commute, and* $C(C^\dagger(AB)^\dagger)(C^\dagger(AB)^\dagger)^\dagger C^\dagger$ *and* $(CC^*)^\dagger$ *commute.*

⟨124⟩ $ABC(BC)^\dagger A^\dagger$ *and* $C^\dagger(AB)^\dagger ABC$ *are EP.*

⟨125⟩ $\{((A^\dagger)^*BC)^{(1,2,3)}\} \ni (BC)^\dagger A^*$ *and* $\{(AB(C^\dagger)^*)^{(1,2,4)}\} \ni C^*(AB)^\dagger$.

⟨126⟩ $\{((A^\dagger)^*BC)^{(1,2,3)}\} \ni ((A^*A)^\dagger BC)^\dagger A^\dagger$ *and* $\{(AB(C^\dagger)^*)^{(1,2,4)}\} \ni C^\dagger(AB(CC^*)^\dagger)^\dagger$.

⟨127⟩ $\{(BC)^{(1,2,3)}\} \ni ((A^\dagger)^*BC)^\dagger(A^\dagger)^*$ *and* $\{(AB)^{(1,2,4)}\} \ni (C^\dagger)^*(AB(C^\dagger)^*)^\dagger$.

⟨128⟩ $\{((A^*A)^\dagger BC)^{(1,2,3)}\} \ni ((A^\dagger)^* BC)^\dagger A$ *and* $\{(AB(CC^*)^\dagger)^{(1,2,4)}\} \ni C(AB(C^\dagger)^*)^\dagger$.

⟨129⟩ $\{((A^\dagger)^* BC)^{(1,2,3)}\} \ni (BC)^\dagger (BC(BC)^\dagger)^\dagger A^*$ *and* $\{(AB(C^\dagger)^*)^{(1,2,4)}\} \ni C^*((AB)^\dagger AB)^\dagger (AB)^\dagger$.

⟨130⟩ $\{((A^\dagger)^* BC)^{(1,2,3)}\} \ni (BC)^*((A^*A)^\dagger BC(BC)^*)^\dagger A^\dagger$ *and* $\{(AB(C^\dagger)^*)^{(1,2,4)}\} \ni C^\dagger (A^*AB(CC^*)^\dagger)^\dagger (AB)^*$.

⟨131⟩ $\{(BB^\dagger)^{(1,2,3)}\} \ni B((A^\dagger)^* B)^\dagger (A^\dagger)^*$ *and* $\{(B^\dagger B)^{(1,2,4)}\} \ni (C^\dagger)^*(B(C^\dagger)^*)^\dagger B$.

⟨132⟩ $\{((A^*A)^\dagger BC(BC)^*)^{(1,2,3)}\} \ni ((BC)^\dagger)^*((A^\dagger)^* BC)^\dagger A$
and $\{((AB)^*AB(CC^*)^\dagger)^{(1,2,4)}\} \ni C(AB(C^\dagger)^*)^\dagger ((AB)^\dagger)^*$.

⟨133⟩ $(A^*AB)(A^*AB)^\dagger = BB^\dagger$ *and* $(BCC^*)^\dagger (BCC^*) = B^\dagger B$.

⟨134⟩ $\mathscr{R}(((AB)^\dagger)^*) = \mathscr{R}((B^\dagger A^\dagger)^*)$ *and* $\mathscr{R}((BC)^\dagger) = \mathscr{R}(C^\dagger B^\dagger)$.

⟨135⟩ $\mathscr{R}(A^*AB) = \mathscr{R}(B)$ *and* $\mathscr{R}((BCC^*)^*) = \mathscr{R}(B^*)$.

⟨136⟩ $r[A^*AB,\ B] = r[(BCC^*)^*,\ B^*] = r(B)$.

⟨137⟩ $r[(A^*AA^*)^\dagger B,\ (A^\dagger)^* B] = r(B)$ *and* $r[(CC^*C)^\dagger B^*,\ C^\dagger B^*] = r(B)$.

4. Main results II

The analysis carried out above for the nested ROLs in (1.10)–(1.15) can similarly be done for the nested ROLs in (1.16)–(1.21) by comparing (1.23) and (1.24) and making symbolic replacements of the results in Lemma 3.1, Theorems 3.2–3.5, and Corollary 3.6. The details are readily presented below without proofs.

Lemma 4.1. *[26] Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, *and* $C \in \mathbb{C}^{p\times q}$.

⟨1⟩ *The following 3 matrix set inclusions always hold*

$$\{(ABC)^{(1)}\} \supseteq \{((AB)^{(1)}ABC)^{(1)}(AB)^{(1)}\}, \tag{4.1}$$
$$\{(ABC)^{(1)}\} \supseteq \{(BC)^{(1)}(ABC(BC)^{(1)})^{(1)}\}, \tag{4.2}$$
$$\{(ABC)^{(1)}\} \supseteq \{(BC)^{(1)}((AB)^{(1)}ABC(BC)^{(1)})^{(1)}(AB)^{(1)}\}. \tag{4.3}$$

⟨2⟩ *The following 3 matrix set inclusions always hold*

$$\{(ABC)^{(1,2)}\} \supseteq \{((AB)^{(1,2)}ABC)^{(1,2)}(AB)^{(1,2)}\}, \tag{4.4}$$
$$\{(ABC)^{(1,2)}\} \supseteq \{(BC)^{(1,2)}(ABC(BC)^{(1,2)})^{(1,2)}\}, \tag{4.5}$$
$$\{(ABC)^{(1,2)}\} \supseteq \{(BC)^{(1,2)}((AB)^{(1,2)}ABC(BC)^{(1,2)})^{(1,2)}(AB)^{(1,2)}\}. \tag{4.6}$$

⟨3⟩ *The following 4 matrix set inclusions always hold*

$$\{(ABC)^{(1,2)}\} \ni ((AB)^\dagger ABC)^\dagger (AB)^\dagger, \tag{4.7}$$
$$\{(ABC)^{(1,2)}\} \ni (BC)^\dagger (ABC(BC)^\dagger)^\dagger, \tag{4.8}$$
$$\{(ABC)^{(1,2)}\} \ni (BC)^\dagger ((AB)^\dagger ABC(BC)^\dagger)^\dagger (AB)^\dagger, \tag{4.9}$$
$$\{(ABC)^{(1,2)}\} \ni (BC)^\dagger (ABC(BC)^\dagger)^\dagger ABC((AB)^\dagger ABC)^\dagger (AB)^\dagger. \tag{4.10}$$

⟨4⟩ *The two matrix equalities in* (1.19) *and* (1.20) *always hold. In particular, if* $(AB)^\dagger ABC = B^\dagger BC$ *and* $ABC(BC)^\dagger = ABB^\dagger$, *then the nested ROL* $(ABC)^\dagger = (B^\dagger BC)^\dagger B^\dagger (ABB^\dagger)^\dagger$ *holds.*

Theorem 4.2. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $C \in \mathbb{C}^{p\times q}$. Then the following 62 statements are equivalent*:

⟨1⟩ $\{(ABC)^{(1,2,3)}\} \ni ((AB)^{\dagger}ABC)^{\dagger}(AB)^{\dagger}$.

⟨2⟩ $\{(ABC)^{(1,2,3)}\} \ni ((AB)^{*}ABC)^{\dagger}(AB)^{*}$.

⟨3⟩ $\{((AB)^{\dagger}ABC)^{(1,2,3)}\} \ni (ABC)^{\dagger}AB$.

⟨4⟩ $\{((AB)^{*}ABC)^{(1,2,3)}\} \ni (ABC)^{\dagger}((AB)^{\dagger})^{*}$.

⟨5⟩ $\{(ABC)^{(1,2,3)}\} \ni (B^{\dagger}BC)^{\dagger}((AB)^{\dagger}ABC(B^{\dagger}BC)^{\dagger})^{\dagger}(AB)^{\dagger}$.

⟨6⟩ $\{(ABC)^{(1,2,3)}\} \ni (B^{\dagger}BC)^{*}((AB)^{*}ABC(B^{\dagger}BC)^{*})^{\dagger}(AB)^{*}$.

⟨7⟩ $\{((AB)^{\dagger}ABC(B^{\dagger}BC)^{\dagger})^{(1,2,3)}\} \ni B^{\dagger}BC(ABC)^{\dagger}AB$.

⟨8⟩ $\{((AB)^{*}ABC(B^{\dagger}BC)^{*})^{(1,2,3)}\} \ni ((B^{\dagger}BC)^{\dagger})^{*}(ABC)^{\dagger}((AB)^{\dagger})^{*}$.

⟨9⟩ $(ABC)^{\dagger} = ((AB)^{\dagger}ABC)^{\dagger}(AB)^{\dagger}$.

⟨10⟩ $ABC = (((AB)^{\dagger}ABC)^{\dagger}(AB)^{\dagger})^{\dagger}$.

⟨11⟩ $ABC = AB((ABC)^{\dagger}AB)^{\dagger}$.

⟨12⟩ $(ABC)^{\dagger}AB = ((AB)^{\dagger}ABC)^{\dagger}$.

⟨13⟩ $BC(ABC)^{\dagger}AB = BC((AB)^{\dagger}ABC)^{\dagger}$.

⟨14⟩ $ABC(ABC)^{\dagger} = ABC((AB)^{\dagger}ABC)^{\dagger}(AB)^{\dagger}$.

⟨15⟩ $ABC(ABC)^{\dagger}AB = ABC((AB)^{\dagger}ABC)^{\dagger}$.

⟨16⟩ $(B^{\dagger}BC)^{\dagger}(AB)^{\dagger}ABC(ABC)^{\dagger} = (B^{\dagger}BC)^{\dagger}(AB)^{\dagger}$.

⟨17⟩ $((B^{\dagger}BC)^{\dagger}(AB)^{\dagger})^{\dagger}(B^{\dagger}BC)^{\dagger}(AB)^{\dagger}ABC = ABC$.

⟨18⟩ $ABC(ABC)^{\dagger} = ((B^{\dagger}BC)^{\dagger}(AB)^{\dagger})^{\dagger}(B^{\dagger}BC)^{\dagger}(AB)^{\dagger}$.

⟨19⟩ $(ABC)^{\dagger} = ((AB)^{*}ABC)^{\dagger}(AB)^{*}$.

⟨20⟩ $ABC = (((AB)^{*}ABC)^{\dagger}(AB)^{*})^{\dagger}$.

⟨21⟩ $ABC = ((AB)^{\dagger})^{*}((ABC)^{\dagger}((AB)^{\dagger})^{*})^{\dagger}$.

⟨22⟩ $(ABC)^{\dagger}((AB)^{\dagger})^{*} = ((AB)^{*}ABC)^{\dagger}$.

⟨23⟩ $BC(ABC)^{\dagger}((AB)^{\dagger})^{*} = BC((AB)^{*}ABC)^{\dagger}$.

⟨24⟩ $ABC(ABC)^{\dagger} = ABC((AB)^{*}ABC)^{\dagger}(AB)^{*}$.

⟨25⟩ $ABC(ABC)^{\dagger}((AB)^{\dagger})^{*} = ABC((AB)^{*}ABC)^{\dagger}$.

⟨26⟩ $(ABC(B^{\dagger}BC)^{\dagger}(AB)^{\dagger})^{\dagger} = ((B^{\dagger}BC)^{\dagger}(AB)^{\dagger})^{\dagger}(ABC)^{\dagger}$.

⟨27⟩ $ABC((AB)^{\dagger}ABC)^{\dagger}(AB)^{\dagger}$ *is an orthogonal projector*.

⟨28⟩ $ABC((AB)^{*}ABC)^{\dagger}(AB)^{*}$ *is an orthogonal projector*.

⟨29⟩ $(AB)^{\dagger}(ABC)(ABC)^{\dagger}AB$ *is an orthogonal projector*.

⟨30⟩ $AB((AB)^{\dagger}ABC(B^{\dagger}BC)^{\dagger})^{\dagger}(AB)^{\dagger}$ *is an orthogonal projector*.

⟨31⟩ $ABC(B^{\dagger}BC)^{*}((AB)^{*}ABC(B^{\dagger}BC)^{*})^{\dagger}(AB)^{*}$ *is an orthogonal projector*.

⟨32⟩ $ABC(ABC)^{\dagger}$ *and* $AB(AB)^{*}$ *commute*.

⟨33⟩ $ABC((AB)^{\dagger}ABC)^{\dagger}(AB)^{\dagger}$ *and* $AB(AB)^{*}$ *commute*.

⟨34⟩ $ABC((AB)^{*}ABC)^{\dagger}(AB)^{*}$ *and* $AB(AB)^{*}$ *commute*.

⟨35⟩ $(AB)^{\dagger}(ABC)(ABC)^{\dagger}AB$ *and* $(AB)^{*}AB$ *commute*.

⟨36⟩ $ABC(B^{\dagger}BC)^{\dagger}(AB)^{\dagger}$ *is EP*.

⟨37⟩ $\{(((AB)^\dagger)^* B^\dagger BC)^{(1,2,3)}\} \ni ((AB)^\dagger ABC)^\dagger (AB)^*$.

⟨38⟩ $\{(((AB)^\dagger)^* B^\dagger BC)^{(1,2,3)}\} \ni (((AB)^* AB)^\dagger B^\dagger BC)^\dagger (AB)^\dagger$.

⟨39⟩ $\{((AB)^\dagger ABC)^{(1,2,3)}\} \ni (((AB)^\dagger)^* B^\dagger BC)^\dagger ((AB)^\dagger)^*$.

⟨40⟩ $\{(((AB)^* AB)^\dagger B^\dagger BC)^{(1,2,3)}\} \ni (((AB)^\dagger)^* B^\dagger BC)^\dagger AB$.

⟨41⟩ $\{(((ABB^\dagger)^\dagger)^* BC)^{(1,2,3)}\} \ni (BC)^\dagger ((ABB^\dagger)^\dagger ABC(BC)^\dagger)^\dagger (ABB^\dagger)^*$.

⟨42⟩ $\{(((AB)^\dagger)^* B^\dagger BC)^{(1,2,3)}\} \ni (B^\dagger BC)^* (((AB)^* AB)^\dagger B^\dagger BC(B^\dagger BC)^*)^\dagger (AB)^\dagger$.

⟨43⟩ $\{((AB)^\dagger ABC(B^\dagger BC)^\dagger)^{(1,2,3)}\} \ni B^\dagger BC(((AB)^\dagger)^* BC)^\dagger ((AB)^\dagger)^*$.

⟨44⟩ $\{(((AB)^* AB)^\dagger B^\dagger BC(B^\dagger BC)^*)^{(1,2,3)}\} \ni ((B^\dagger BC)^\dagger)^* (((AB)^\dagger)^* B^\dagger BC)^\dagger AB$.

⟨45⟩ $(((AB)^\dagger)^* B^\dagger BC)^\dagger = ((AB)^\dagger ABC)^\dagger (AB)^*$.

⟨46⟩ $(((AB)^\dagger)^* B^\dagger BC)^\dagger = (((AB)^* AB)^\dagger B^\dagger BC)^\dagger (AB)^\dagger$.

⟨47⟩ $((AB)^\dagger ABC)^\dagger = (((AB)^\dagger)^* B^\dagger BC)^\dagger ((AB)^\dagger)^*$.

⟨48⟩ $(((AB)^* AB)^\dagger B^\dagger BC)^\dagger = (((AB)^\dagger)^* B^\dagger BC)^\dagger AB$.

⟨49⟩ $AB((B^\dagger BC)^\dagger (AB)^\dagger AB)^\dagger (B^\dagger BC)^\dagger (AB)^\dagger$ *is an orthogonal projector*.

⟨50⟩ $((AB)^\dagger)^* ((B^\dagger BC)^\dagger ((AB)^* AB)^\dagger)^\dagger (B^\dagger BC)^\dagger (AB)^\dagger$ *is an orthogonal projector*.

⟨51⟩ $(AB)^\dagger ((B^\dagger BC)^\dagger (AB)^\dagger)^\dagger ((B^\dagger BC)^\dagger (AB)^\dagger) AB$ *is an orthogonal projector*.

⟨52⟩ $AB(B^\dagger BC(B^\dagger BC)^\dagger (AB)^\dagger AB)^\dagger (AB)^\dagger$ *is an orthogonal projector*.

⟨53⟩ $((AB)^\dagger)^* (B^\dagger BC(B^\dagger BC)^*)^\dagger (((AB)^* AB)^\dagger (B^\dagger BC(B^\dagger BC)^*)^\dagger)^\dagger (AB)^\dagger$ *is an orthogonal projector*.

⟨54⟩ $((B^\dagger BC)^\dagger (AB)^\dagger)^\dagger ((B^\dagger BC)^\dagger (AB)^\dagger)$ *and* $(AB(AB)^*)^\dagger$ *commute*.

⟨55⟩ $AB((B^\dagger BC)^\dagger (AB)^\dagger (AB))^\dagger (B^\dagger BC)^\dagger (AB)^\dagger$ *and* $(AB(AB)^*)^\dagger$ *commute*.

⟨56⟩ $((AB)^\dagger)^* ((B^\dagger BC)^\dagger ((AB)^* AB)^\dagger)^\dagger (B^\dagger BC)^\dagger (AB)^\dagger$ *and* $((AB)(AB)^*)^\dagger$ *commute*.

⟨57⟩ $(AB)^\dagger ((B^\dagger BC)^\dagger (AB)^\dagger)^\dagger ((B^\dagger BC)^\dagger (AB)^\dagger) AB$ *and* $((AB)^* AB)^\dagger$ *commute*.

⟨58⟩ $\mathscr{R}((AB)(AB)^* ABC) = \mathscr{R}(ABC)$.

⟨59⟩ $\mathscr{R}(((AB)^\dagger)^* B^\dagger BC) = \mathscr{R}(ABC)$.

⟨60⟩ $r[(AB)(AB)^* ABC,\ ABC] = r(ABC)$.

⟨61⟩ $r[((AB)^\dagger)^* B^\dagger BC,\ ABC] = r(ABC)$.

⟨62⟩ $r[((AB)^* AB(AB)^*)^\dagger B^\dagger BC,\ ((AB)^\dagger)^* B^\dagger BC] = r(((AB)^\dagger)^* B^\dagger BC)$.

Theorem 4.3. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, and $C \in \mathbb{C}^{p\times q}$. Then the following 62 statements are equivalent*:

⟨1⟩ $\{(ABC)^{(1,2,4)}\} \ni (BC)^\dagger (ABC(BC)^\dagger)^\dagger$.

⟨2⟩ $\{(ABC)^{(1,2,4)}\} \ni (BC)^* (ABC(BC)^*)^\dagger$.

⟨3⟩ $\{(ABC(BC)^\dagger)^{(1,2,4)}\} \ni BC(ABC)^\dagger$.

⟨4⟩ $\{(ABC(BC)^*)^{(1,2,4)}\} \ni ((BC)^\dagger)^* (ABC)^\dagger$.

⟨5⟩ $\{(ABC)^{(1,2,4)}\} \ni (BC)^\dagger ((ABB^\dagger)^\dagger ABC(BC)^\dagger)^\dagger (ABB^\dagger)^\dagger$.

⟨6⟩ $\{(ABC)^{(1,2,4)}\} \ni (BC)^* ((ABB^\dagger)^* ABC(BC)^*)^\dagger (ABB^\dagger)^*$.

⟨7⟩ $\{((ABB^\dagger)^\dagger ABC(BC)^\dagger)^{(1,2,4)}\} \ni BC(ABC)^\dagger ABB^\dagger$.

⟨8⟩ $\{(ABB^\dagger)^* ABC(BC)^*)^{(1,2,4)}\} \ni ((BC)^\dagger)^* (ABC)^\dagger ((ABB^\dagger)^\dagger)^*$.

⟨9⟩ $(ABC)^\dagger = (BC)^\dagger (ABC(BC)^\dagger)^\dagger$.

⟨10⟩ $ABC = ((BC)^{\dagger}(ABC(BC)^{\dagger})^{\dagger})^{\dagger}$.

⟨11⟩ $ABC = (BC(ABC)^{\dagger})^{\dagger}BC$.

⟨12⟩ $BC(ABC)^{\dagger} = (ABC(BC)^{\dagger})^{\dagger}$.

⟨13⟩ $BC(ABC)^{\dagger}AB = (ABC(BC)^{\dagger})^{\dagger}AB$.

⟨14⟩ $(ABC)^{\dagger}ABC = (BC)^{\dagger}(ABC(BC)^{\dagger})^{\dagger}ABC$.

⟨15⟩ $BC(ABC)^{\dagger}ABC = (ABC(BC)^{\dagger})^{\dagger}ABC$.

⟨16⟩ $(ABC)^{\dagger}ABC(BC)^{\dagger}(ABB^{\dagger})^{\dagger} = (BC)^{\dagger}(ABB^{\dagger})^{\dagger}$.

⟨17⟩ $ABC(BC)^{\dagger}(ABB^{\dagger})^{\dagger}((BC)^{\dagger}(ABB^{\dagger})^{\dagger})^{\dagger} = ABC$.

⟨18⟩ $(ABC)^{\dagger}ABC = (BC)^{\dagger}(ABB^{\dagger})^{\dagger}((BC)^{\dagger}(ABB^{\dagger})^{\dagger})^{\dagger}$.

⟨19⟩ $(ABC)^{\dagger} = (BC)^{*}(ABC(BC)^{*})^{\dagger}$.

⟨20⟩ $ABC = ((BC)^{*}(ABC(BC)^{*})^{\dagger})^{\dagger}$.

⟨21⟩ $ABC = (((BC)^{\dagger})^{*}(ABC)^{\dagger})^{\dagger}((BC)^{\dagger})^{*}$.

⟨22⟩ $((BC)^{\dagger})^{*}(ABC)^{\dagger} = (ABC(BC)^{*})^{\dagger}$.

⟨23⟩ $((BC)^{\dagger})^{*}(ABC)^{\dagger}AB = (ABC(BC)^{*})^{\dagger}AB$.

⟨24⟩ $(ABC)^{\dagger}ABC = (BC)^{*}(ABC(BC)^{*})^{\dagger}ABC$.

⟨25⟩ $((BC)^{\dagger})^{*}(ABC)^{\dagger}ABC = (ABC(BC)^{*})^{\dagger}ABC$.

⟨26⟩ $((BC)^{\dagger}(ABB^{\dagger})^{\dagger}ABC)^{\dagger} = (ABC)^{\dagger}((BC)^{\dagger}(ABB^{\dagger})^{\dagger})^{\dagger}$.

⟨27⟩ $(BC)^{\dagger}(ABC(BC)^{\dagger})^{\dagger}ABC$ *is an orthogonal projector*.

⟨28⟩ $(BC)^{*}(ABC(BC)^{*})^{\dagger}ABC$ *is an orthogonal projector*.

⟨29⟩ $BC(ABC)^{\dagger}(ABC)(BC)^{\dagger}$ *is an orthogonal projector*.

⟨30⟩ $(BC)^{\dagger}((ABB^{\dagger})^{\dagger}ABC(BC)^{\dagger})^{\dagger}BC$ *is an orthogonal projector*.

⟨31⟩ $(BC)^{*}((ABB^{\dagger})^{*}ABC(BC)^{*})^{\dagger}(ABB^{\dagger})^{*}ABC$ *is an orthogonal projector*.

⟨32⟩ $(ABC)^{\dagger}ABC$ *and* $(BC)^{*}BC$ *commute*.

⟨33⟩ $(BC)^{\dagger}(ABC(BC)^{\dagger})^{\dagger}ABC$ *and* $(BC)^{*}BC$ *commute*.

⟨34⟩ $(BC)^{*}(ABC(BC)^{*})^{\dagger}ABC$ *and* $(BC)^{*}BC$ *commute*.

⟨35⟩ $BC(ABC)^{\dagger}(ABC)(BC)^{\dagger}$ *and* $BC(BC)^{*}$ *commute*.

⟨36⟩ $(BC)^{\dagger}(ABB^{\dagger})^{\dagger}ABC$ *is EP*.

⟨37⟩ $\{(ABB^{\dagger}((BC)^{\dagger})^{*})^{(1,2,4)}\} \ni (BC)^{*}(ABC(BC)^{\dagger})^{\dagger}$.

⟨38⟩ $\{(ABB^{\dagger}((BC)^{\dagger})^{*})^{(1,2,4)}\} \ni (BC)^{\dagger}(ABB^{\dagger}(BC(BC)^{*})^{\dagger})^{\dagger}$.

⟨39⟩ $\{(ABC(BC)^{\dagger})^{(1,2,4)}\} \ni ((BC)^{\dagger})^{*}(ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger}$.

⟨40⟩ $\{(ABB^{\dagger}((BC)(BC)^{*})^{\dagger})^{(1,2,4)}\} \ni BC(ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger}$.

⟨41⟩ $\{(ABB^{\dagger}((BC)^{\dagger})^{*})^{(1,2,4)}\} \ni (BC)^{*}((ABB^{\dagger})^{\dagger}ABC(BC)^{\dagger})^{\dagger}(ABB^{\dagger})^{\dagger}$.

⟨42⟩ $\{((ABB^{\dagger})^{*}ABB^{\dagger}((BC)^{\dagger})^{*})^{(1,2,4)}\} \ni (BC)^{\dagger}(ABB^{\dagger}(BC(BC)^{*})^{\dagger})^{\dagger}(ABB^{\dagger})^{*}$.

⟨43⟩ $\{((ABB^{\dagger})^{\dagger}ABC(BC)^{\dagger})^{(1,2,4)}\} \ni ((BC)^{\dagger})^{*}(ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger}ABB^{\dagger}$.

⟨44⟩ $\{((ABB^{\dagger})^{*}ABB^{\dagger}((BC)(BC)^{*})^{\dagger})^{(1,2,4)}\} \ni BC(ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger}((ABB^{\dagger})^{\dagger})^{*}$.

⟨45⟩ $(ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger} = (BC)^{*}(ABC(BC)^{\dagger})^{\dagger}$.

⟨46⟩ $(ABB^{\dagger}((BC)^{\dagger})^{*}) = (BC)^{\dagger}(ABB^{\dagger}((BC)(BC)^{*})^{\dagger})^{\dagger}$.

⟨47⟩ $(ABC(BC)^{\dagger})^{\dagger} = ((BC)^{\dagger})^{*}(ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger}$.

⟨48⟩ $(ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger} = BC(ABB^{\dagger}((BC)(BC)^{*})^{\dagger})^{\dagger}$.

⟨49⟩ $(BC)^{\dagger}(ABB^{\dagger})^{\dagger}((BC)(BC)^{\dagger}(ABB^{\dagger})^{\dagger})^{\dagger}BC$ *is an orthogonal projector*.

⟨50⟩ $(BC)^{\dagger}(ABB^{\dagger})^{\dagger}(((BC)(BC)^{*})^{\dagger}(ABB^{\dagger})^{\dagger})^{\dagger}((BC)^{\dagger})^{*}$ *is an orthogonal projector*.

⟨51⟩ $BC((BC)^{\dagger}(ABB^{\dagger})^{\dagger})((BC)^{\dagger}(ABB^{\dagger})^{\dagger})^{\dagger}(BC)^{\dagger}$ *is an orthogonal projector*.

⟨52⟩ $(BC)^{\dagger}((BC)(BC)^{\dagger}(ABB^{\dagger})^{\dagger}ABB^{\dagger})^{\dagger}BC$ *is an orthogonal projector*.

⟨53⟩ $(BC)^{\dagger}(((ABB^{\dagger})^{*}ABB^{\dagger})^{\dagger}((BC)(BC)^{*})^{\dagger})^{\dagger}((ABB^{\dagger})^{*}ABB^{\dagger})^{\dagger}((BC)^{\dagger})^{*}$ *is an orthogonal projector*.

⟨54⟩ $((BC)^{\dagger}(ABB^{\dagger})^{\dagger})((BC)^{\dagger}(ABB^{\dagger})^{\dagger})^{\dagger}$ *and* $((BC)^{*}BC)^{\dagger}$ *commute*.

⟨55⟩ $(BC)^{\dagger}(ABB^{\dagger})^{\dagger}((BC)(BC)^{\dagger}(ABB^{\dagger})^{\dagger})^{\dagger}BC$ *and* $((BC)^{*}BC)^{\dagger}$ *commute*.

⟨56⟩ $(BC)^{\dagger}(ABB^{\dagger})^{\dagger}(((BC)(BC)^{*})^{\dagger}(ABB^{\dagger})^{\dagger})^{\dagger}((BC)^{\dagger})^{*}$ *and* $((BC)^{*}BC)^{\dagger}$ *commute*.

⟨57⟩ $BC((BC)^{\dagger}(ABB^{\dagger})^{\dagger})((BC)^{\dagger}(ABB^{\dagger})^{\dagger})^{\dagger}(BC)^{\dagger}$ *and* $((BC)(BC)^{*})^{\dagger}$ *commute*.

⟨58⟩ $\mathscr{R}((BC)^{*}BC(BC)^{*}A^{*}) = \mathscr{R}((BC)^{*}A^{*})$.

⟨59⟩ $\mathscr{R}((BC)^{\dagger}(ABB^{\dagger})^{\dagger}) = \mathscr{R}((BC)^{*}(ABB^{\dagger})^{*})$.

⟨60⟩ $r[(BC)^{*}BC(ABC)^{*},\ (ABC)^{*}] = r(ABC)$.

⟨61⟩ $r[(BC)^{\dagger}(ABB^{\dagger})^{\dagger},\ (ABC)^{*}] = r(ABC)$.

⟨62⟩ $r[(BC(BC)^{*}BC)^{\dagger}(ABB^{\dagger})^{\dagger},\ (BC)^{\dagger}(ABB^{\dagger})^{\dagger}] = r((BC)^{\dagger}(ABB^{\dagger})^{\dagger})$.

Theorem 4.4. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, *and* $C \in \mathbb{C}^{p\times q}$. *Then the following 137 statements are equivalent*:

⟨1⟩ $(ABC)^{\dagger} = (BC)^{\dagger}((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger}(AB)^{\dagger}$.

⟨2⟩ $BC(ABC)^{\dagger}AB = ((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger}$.

⟨3⟩ $AB(BC(ABC)^{\dagger}AB)^{\dagger}BC = ABC$.

⟨4⟩ $BC(AB(BC(ABC)^{\dagger}AB)^{\dagger}BC)^{\dagger}AB = BC(ABC)^{\dagger}AB$.

⟨5⟩ $(AB)^{\dagger}ABC(BC)^{\dagger} = (AB)^{\dagger}((BC)^{\dagger}((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger}(AB)^{\dagger})^{\dagger}(BC)^{\dagger}$.

⟨6⟩ $(ABC)^{\dagger} = (BC)^{\dagger}(ABC(BC)^{\dagger})^{\dagger}ABC((AB)^{\dagger}ABC)^{\dagger}(AB)^{\dagger}$.

⟨7⟩ $BC(ABC)^{\dagger}AB = (ABC(BC)^{\dagger})^{\dagger}ABC((AB)^{\dagger}ABC)^{\dagger}$.

⟨8⟩ $\{(ABC)^{(1,2,3)}\} \ni ((AB)^{\dagger}ABC)^{\dagger}(AB)^{\dagger}$ *and* $\{(ABC)^{(1,2,4)}\} \ni (BC)^{\dagger}(ABC(BC)^{\dagger})^{\dagger}$.

⟨9⟩ $\{(ABC)^{(1,2,3)}\} \ni ((AB)^{*}ABC)^{\dagger}(AB)^{*}$ *and* $\{(ABC)^{(1,2,4)}\} \ni (BC)^{*}(ABC(BC)^{*})^{\dagger}$.

⟨10⟩ $\{(AB)^{\dagger}ABC)^{(1,2,3)}\} \ni (ABC)^{\dagger}AB$ *and* $\{(ABC(BC)^{\dagger})^{(1,2,4)}\} \ni BC(ABC)^{\dagger}$.

⟨11⟩ $\{((AB)^{*}ABC)^{(1,2,3)}\} \ni (ABC)^{\dagger}((AB)^{\dagger})^{*}$ *and* $\{(ABC(BC)^{*})^{(1,2,4)}\} \ni ((BC)^{\dagger})^{*}(ABC)^{\dagger}$.

⟨12⟩ $\{(ABC)^{(1,2,3)}\} \ni (B^{\dagger}BC)^{\dagger}((AB)^{\dagger}ABC(B^{\dagger}BC)^{\dagger})^{\dagger}(AB)^{\dagger}$
and $\{(ABC)^{(1,2,4)}\} \ni (BC)^{\dagger}((ABB^{\dagger})^{\dagger}ABC(BC)^{\dagger})^{\dagger}(ABB^{\dagger})^{\dagger}$.

⟨13⟩ $\{(ABC)^{(1,2,3)}\} \ni (B^{\dagger}BC)^{*}((AB)^{*}ABC(B^{\dagger}BC)^{*})^{\dagger}(AB)^{*}$
and $\{(ABC)^{(1,2,4)}\} \ni (BC)^{*}((ABB^{\dagger})^{*}ABC(BC)^{*})^{\dagger}(ABB^{\dagger})^{*}$.

⟨14⟩ $\{((AB)^{\dagger}ABC(B^{\dagger}BC)^{\dagger})^{(1,2,3)}\} \ni B^{\dagger}BC(ABC)^{\dagger}AB$
and $\{((ABB^{\dagger})^{\dagger}ABC(BC)^{\dagger})^{(1,2,4)}\} \ni BC(ABC)^{\dagger}ABB^{\dagger}$.

⟨15⟩ $\{((AB)^{*}ABC(B^{\dagger}BC)^{*})^{(1,2,3)}\} \ni ((B^{\dagger}BC)^{\dagger})^{*}(ABC)^{\dagger}((AB)^{\dagger})^{*}$
and $\{((AB)^{*}ABC(BC)^{*})^{(1,2,4)}\} \ni ((BC)^{\dagger})^{*}(ABC)^{\dagger}((ABB^{\dagger})^{\dagger})^{*}$.

⟨16⟩ $(ABC)^{\dagger} = ((AB)^{\dagger}ABC)^{\dagger}(AB)^{\dagger}$ *and* $(ABC)^{\dagger} = (BC)^{\dagger}(ABC(BC)^{\dagger})^{\dagger}$.

⟨17⟩ $((AB)^{\dagger}ABC)^{\dagger} = (ABC)^{\dagger}AB$ *and* $(ABC(BC)^{\dagger})^{\dagger} = BC(ABC)^{\dagger}$.

⟨18⟩ $ABC(ABC)^{\dagger} = ABC((AB)^{\dagger}ABC)^{\dagger}(AB)^{\dagger}$ *and* $(ABC)^{\dagger}ABC = (BC)^{\dagger}(ABC(BC)^{\dagger})^{\dagger}ABC$.

⟨19⟩ $ABC(ABC)^{\dagger}AB = ABC((AB)^{\dagger}ABC)^{\dagger}$ *and* $BC(ABC)^{\dagger}ABC = (ABC(BC)^{\dagger})^{\dagger}ABC$.

⟨20⟩ $(B^{\dagger}BC)^{\dagger}(AB)^{\dagger}ABC(ABC)^{\dagger} = (B^{\dagger}BC)^{\dagger}(AB)^{\dagger}$ *and* $(ABC)^{\dagger}ABC(BC)^{\dagger}(ABB^{\dagger})^{\dagger} = (BC)^{\dagger}(ABB^{\dagger})^{\dagger}$.

⟨21⟩ $((B^{\dagger}BC)^{\dagger}(AB)^{\dagger})^{\dagger}(B^{\dagger}BC)^{\dagger}(AB)^{\dagger}ABC = ABC$ *and* $ABC(BC)^{\dagger}(ABB^{\dagger})^{\dagger}((BC)^{\dagger}(ABB^{\dagger})^{\dagger})^{\dagger} = ABC$.

⟨22⟩ $ABC(ABC)^{\dagger} = ((B^{\dagger}BC)^{\dagger}(AB)^{\dagger})^{\dagger}(B^{\dagger}BC)^{\dagger}(AB)^{\dagger}$
and $(ABC)^{\dagger}ABC = (BC)^{\dagger}(ABB^{\dagger})^{\dagger}((BC)^{\dagger}(ABB^{\dagger})^{\dagger})^{\dagger}$.

⟨23⟩ $((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger} = (ABC(BC)^{\dagger})^{\dagger}AB$ *and* $((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger} = BC((AB)^{\dagger}ABC)^{\dagger}$.

⟨24⟩ $(ABC(BC)^{\dagger})^{\dagger} = ((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger}(AB)^{\dagger}$ *and* $((AB)^{\dagger}ABC)^{\dagger} = (BC)^{\dagger}((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger}$.

⟨25⟩ $((AB)^{\dagger}ABC)^{\dagger}(AB)^{\dagger} = (BC)^{\dagger}(ABC(BC)^{\dagger})^{\dagger}$.

⟨26⟩ $(ABC(BC)^{\dagger})^{\dagger}AB = BC((AB)^{\dagger}ABC)^{\dagger}$.

⟨27⟩ $(((AB)^{\dagger})^{*}B^{\dagger}((BC)^{\dagger})^{*})^{\dagger} = (BC)^{*}((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger}(AB)^{*}$.

⟨28⟩ $((BC)^{\dagger})^{*}(((AB)^{\dagger})^{*}B^{\dagger}((BC)^{\dagger})^{*})^{\dagger}((AB)^{\dagger})^{*} = ((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger}$.

⟨29⟩ $((AB)^{\dagger})^{*}(((BC)^{\dagger})^{*}(((AB)^{\dagger})^{*}B^{\dagger}((BC)^{\dagger})^{*})^{\dagger}((AB)^{\dagger})^{*})^{\dagger}((BC)^{\dagger})^{*} = ((AB)^{\dagger})^{*}B^{\dagger}((BC)^{\dagger})^{*}$.

⟨30⟩ $((BC)^{\dagger})^{*}(((AB)^{\dagger})^{*}(((BC)^{\dagger})^{*}(ABC)^{\dagger}((AB)^{\dagger})^{*})^{\dagger}((BC)^{\dagger})^{*})^{\dagger}((AB)^{\dagger})^{*}$
$= ((BC)^{\dagger})^{*}(((AB)^{\dagger})^{*}B^{\dagger}((BC)^{\dagger})^{*})^{\dagger}((AB)^{\dagger})^{*}$.

⟨31⟩ $(AB)^{\dagger}ABC(BC)^{\dagger} = (AB)^{*}((BC)^{*}((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger}(AB)^{*})^{\dagger}(BC)^{*}$.

⟨32⟩ $(((AB)^{\dagger})^{*}B^{\dagger}((BC)^{\dagger})^{*})^{\dagger} = (BC)^{*}(((AB)^{\dagger})^{*}B^{\dagger}BC(BC)^{\dagger})^{\dagger}((AB)^{\dagger})^{*}B^{\dagger}((BC)^{\dagger})^{*}$
$\times((AB)^{\dagger}ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger}(AB)^{*}$.

⟨33⟩ $((BC)^{\dagger})^{*}(((AB)^{\dagger})^{*}B^{\dagger}((BC)^{\dagger})^{*})^{\dagger}((AB)^{\dagger})^{*}$
$= (((AB)^{\dagger})^{*}B^{\dagger}BC(BC)^{\dagger})^{\dagger}((ABB^{\dagger})^{\dagger})^{*}B^{\dagger}((BC)^{\dagger})^{*}((AB)^{\dagger}ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger}$.

⟨34⟩ $(((AB)^{\dagger})^{*}B^{\dagger}BC)^{\dagger} = ((AB)^{\dagger}ABC)^{\dagger}(AB)^{*}$ *and* $(ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger} = (BC)^{*}(ABC(BC)^{\dagger})^{\dagger}$.

⟨35⟩ $((AB)^{\dagger}ABC)^{\dagger} = (((AB)^{\dagger})^{*}B^{\dagger}BC)^{\dagger}((AB)^{\dagger})^{*}$ *and* $(ABC(BC)^{\dagger})^{\dagger} = ((BC)^{\dagger})^{*}(ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger}$.

⟨36⟩ $ABC(ABC)^{\dagger} = ABB^{\dagger}((BC)^{\dagger})^{*}((AB)^{\dagger}ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger}(AB)^{\dagger}$
and $(ABC)^{\dagger}ABC = (BC)^{\dagger}(((AB)^{\dagger})^{*}B^{\dagger}BC(BC)^{\dagger})^{\dagger}((AB)^{\dagger})^{*}B^{\dagger}BC$.

⟨37⟩ $ABC(ABC)^{\dagger}AB = ABB^{\dagger}((BC)^{\dagger})^{*}((AB)^{\dagger}ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger}$
and $BC(ABC)^{\dagger}ABC = (((AB)^{\dagger})^{*}B^{\dagger}BC(BC)^{\dagger})^{\dagger}((AB)^{\dagger})^{*}B^{\dagger}BC$.

⟨38⟩ $ABC(ABC)^{\dagger} = ((B^{\dagger}((BC)^{\dagger})^{*})^{\dagger}(AB)^{\dagger})^{\dagger}(B^{\dagger}((BC)^{\dagger})^{*})^{\dagger}(AB)^{\dagger}$
and $(ABC)^{\dagger}ABC = (BC)^{\dagger}(((AB)^{\dagger})^{*}B^{\dagger})^{\dagger}((BC)^{\dagger}(((AB)^{\dagger})^{*}B^{\dagger})^{\dagger})^{\dagger}$.

⟨39⟩ $(B^{\dagger}BC)^{\dagger}(AB)^{\dagger}ABC(((AB)^{\dagger})^{*}B^{\dagger}BC)^{\dagger} = (B^{\dagger}BC)^{\dagger}(AB)^{*}$
and $(ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger}ABC(BC)^{\dagger}(ABB^{\dagger})^{\dagger} = (BC)^{*}(ABB^{\dagger})^{\dagger}$.

⟨40⟩ $((B^{\dagger}BC)^{\dagger}(AB)^{*})^{\dagger}(B^{\dagger}BC)^{\dagger}(AB)^{\dagger}ABC = ((AB)^{\dagger})^{*}B^{\dagger}BC$
and $ABC(BC)^{\dagger}(ABB^{\dagger})^{\dagger}((BC)^{*}(ABB^{\dagger})^{\dagger})^{\dagger} = ABB^{\dagger}((BC)^{\dagger})^{*}$.

⟨41⟩ $((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger} = (((AB)^{\dagger})^{*}B^{\dagger}BC(BC)^{\dagger})^{\dagger}((AB)^{\dagger})^{*}$
and $((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger} = ((BC)^{\dagger})^{*}((AB)^{\dagger}ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger}$.

⟨42⟩ $(((AB)^{\dagger})^{*}B^{\dagger}BC(BC)^{\dagger})^{\dagger} = ((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger}(AB)^{*}$
and $((AB)^{\dagger}ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger} = (BC)^{*}((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger}$.

⟨43⟩ $((AB)^\dagger ABB^\dagger((BC)^\dagger)^*)^\dagger(AB)^* = (BC)^*(((AB)^\dagger)^* B^\dagger BC(BC)^\dagger)^\dagger$.

⟨44⟩ $(((AB)^\dagger)^* B^\dagger BC(BC)^\dagger)^\dagger((AB)^\dagger)^* = ((BC)^\dagger)^*((AB)^\dagger ABB^\dagger((BC)^\dagger)^*)^\dagger$.

⟨45⟩ $(BC)^*(((AB)^\dagger)^* B^\dagger BC(BC)^\dagger)^\dagger = ((AB)^\dagger ABB^\dagger((BC)^\dagger)^*)^\dagger(AB)^*$.

⟨46⟩ $(ABC)^\dagger = (BC)^*((AB)^* ABC(BC)^*)^\dagger(AB)^*$.

⟨47⟩ $((AB)^* ABC(BC)^*)^\dagger = ((BC)^\dagger)^*(ABC)^\dagger((AB)^\dagger)^*$.

⟨48⟩ $(ABC)^\dagger = (BC)^*(ABC(BC)^*)^\dagger ABC((AB)^* ABC)^\dagger(AB)^*$.

⟨49⟩ $((BC)^\dagger)^*(ABC)^\dagger((AB)^\dagger)^* = (ABC(BC)^*)^\dagger ABC((AB)^* ABC)^\dagger$.

⟨50⟩ $(ABC)^\dagger = ((AB)^* ABC)^\dagger(AB)^*$ *and* $(ABC)^\dagger = (BC)^*(ABC(BC)^*)^\dagger$.

⟨51⟩ $((AB)^* ABC)^\dagger = (ABC)^\dagger((AB)^\dagger)^*$ *and* $(ABC(BC)^*)^\dagger = ((BC)^\dagger)^*(ABC)^\dagger$.

⟨52⟩ $(((AB)^\dagger)^* B^\dagger BC)^\dagger = (((AB)^*(AB))^\dagger B^\dagger BC)^\dagger(AB)^\dagger$
and $(ABB^\dagger((BC)^\dagger)^*)^\dagger = (BC)^\dagger(ABB^\dagger((BC)(BC)^*)^\dagger)^\dagger$.

⟨53⟩ $(((AB)^*(AB))^\dagger B^\dagger BC)^\dagger = (((AB)^\dagger)^* B^\dagger BC)^\dagger AB$
and $(ABB^\dagger((BC)(BC)^*)^\dagger)^\dagger = BC(ABB^\dagger((BC)^\dagger)^*)^\dagger$.

⟨54⟩ $(((AB)^\dagger)^* B^\dagger((BC)^\dagger)^*)^\dagger = (((AB)^* AB)^\dagger B^\dagger((BC)^\dagger)^*)^\dagger(AB)^\dagger$
and $(((AB)^\dagger)^* B^\dagger((BC)^\dagger)^*)^\dagger = (BC)^\dagger(((AB)^\dagger)^* B^\dagger(BC(BC)^*)^\dagger)^\dagger$.

⟨55⟩ $(((AB)^* AB)^\dagger B^\dagger((BC)^\dagger)^*)^\dagger = (((AB)^\dagger)^* B^\dagger((BC)^\dagger)^*)^\dagger AB$
and $(((AB)^\dagger)^* B^\dagger(BC(BC)^*)^\dagger)^\dagger = BC(((AB)^\dagger)^* B^\dagger((BC)^\dagger)^*)^\dagger$.

⟨56⟩ $(((AB)^\dagger)^* B^\dagger((BC)^\dagger)^*)^\dagger = (BC)^\dagger(((AB)^* AB)^\dagger B^\dagger(BC(BC)^*)^\dagger)^\dagger(AB)^\dagger$.

⟨57⟩ $(((AB)^* AB)^\dagger B^\dagger((BC)(BC)^*)^\dagger)^\dagger = BC(((AB)^\dagger)^* B^\dagger((BC)^\dagger)^*)^\dagger AB$.

⟨58⟩ $((AB)^* ABC(BC)^*)^\dagger = (ABC(BC)^*)^\dagger((AB)^\dagger)^*$
and $((AB)^* ABC(BC)^*)^\dagger = ((BC)^\dagger)^*((AB)^* ABC)^\dagger$.

⟨59⟩ $((AB)^* ABC)^\dagger = (BC)^*((AB)^* ABC(BC)^*)^\dagger$ *and* $(ABC(BC)^*)^\dagger = ((AB)^* ABC(BC)^*)^\dagger(AB)^*$.

⟨60⟩ $((AB)^* ABC)^\dagger(AB)^* = (BC)^*(ABC(BC)^*)^\dagger$.

⟨61⟩ $(ABC(BC)^*)^\dagger((AB)^\dagger)^* = ((BC)^\dagger)^*((AB)^* ABC)^\dagger$.

⟨62⟩ $(((AB)^* AB)^\dagger B^\dagger BC)^\dagger = (BC)^*(((AB)^* AB)^\dagger B^\dagger BC(BC)^*)^\dagger$
and $(ABB^\dagger(BC(BC)^*)^\dagger)^\dagger = ((AB)^* ABB^\dagger(BC(BC)^*)^\dagger)^\dagger(AB)^*$.

⟨63⟩ $(((AB)^* AB)^\dagger B^\dagger BC(BC)^*)^\dagger = ((BC)^\dagger)^*(((AB)^* AB)^\dagger B^\dagger BC)^\dagger$
and $((AB)^* ABB^\dagger(BC(BC)^*)^\dagger)^\dagger = (ABB^\dagger(BC(BC)^*)^\dagger)^\dagger((AB)^\dagger)^*$.

⟨64⟩ $(((AB)^\dagger)^* B^\dagger(BC(BC)^*)^\dagger)^\dagger AB = BC(((AB)^* AB)^\dagger B^\dagger((BC)^\dagger)^*)^\dagger$.

⟨65⟩ $(((AB)^* AB)^\dagger B^\dagger((BC)^\dagger)^*)^\dagger(AB)^\dagger = (BC)^\dagger(((AB)^\dagger)^* B^\dagger(BC(BC)^*)^\dagger)^\dagger$.

⟨66⟩ $((AB)^* ABC)^\dagger = (AB(AB)^* ABC)^\dagger AB$ *and* $(ABC(BC)^*)^\dagger = BC(ABC(BC)^* BC)^\dagger$.

⟨67⟩ $(AB(AB)^* ABC)^\dagger = ((AB)^* ABC)^\dagger(AB)^\dagger$ *and* $(ABC(BC)^* BC)^\dagger = (BC)^\dagger(ABC(BC)^*)^\dagger$.

⟨68⟩ $(((AB)^* AB)^\dagger B^\dagger BC)^\dagger = (((AB)^* AB(AB)^*)^\dagger B^\dagger BC)^\dagger((AB)^\dagger)^*$
and $(ABB^\dagger(BC(BC)^*)^\dagger)^\dagger = ((BC)^\dagger)^*(ABB^\dagger((BC)^* BC(BC)^*)^\dagger)^\dagger$.

⟨69⟩ $(((AB)^* AB)^\dagger B^\dagger BC)^\dagger(AB)^* = (((AB)^* AB(AB)^*)^\dagger B^\dagger BC)^\dagger$
and $(BC)^*(ABB^\dagger(BC(BC)^*)^\dagger)^\dagger = (ABB^\dagger((BC)^* BC(BC)^*)^\dagger)^\dagger$.

⟨70⟩ $(((AB)^* AB)^\dagger B^\dagger((BC)^\dagger)^*)^\dagger = (((AB)^* AB(AB)^*)^\dagger B^\dagger((BC)^\dagger)^*)^\dagger((AB)^\dagger)^*$
and $(((AB)^\dagger)^* B^\dagger(BC(BC)^*)^\dagger)^\dagger = ((BC)^\dagger)^*(((AB)^\dagger)^* B^\dagger((BC)^* BC(BC)^*)^\dagger)^\dagger$.

⟨71⟩ $(((AB)^*AB(AB)^*)^\dagger B^\dagger((BC)^\dagger)^*)^\dagger = (((AB)^*AB)^\dagger B^\dagger((BC)^\dagger)^*)^\dagger(AB)^*$ *and* $(((AB)^\dagger)^*B^\dagger((BC)^*BC(BC)^*)^\dagger)^\dagger = (BC)^*(((AB)^\dagger)^*B^\dagger(BC(BC)^*)^\dagger)^\dagger$.

⟨72⟩ $((AB)^*ABC(BC)^*)^\dagger = BC((AB)^*ABC(BC)^*BC)^\dagger$ *and* $((AB)^*ABC(BC)^*)^\dagger = (AB(AB)^*ABC(BC)^*)^\dagger AB$.

⟨73⟩ $((AB)^*ABC(BC)^*BC)^\dagger = (BC)^\dagger((AB)^*ABC(BC)^*)^\dagger$ *and* $(AB(AB)^*ABC(BC)^*)^\dagger = ((AB)^*ABC(BC)^*)^\dagger(AB)^\dagger$.

⟨74⟩ $(((AB)^*AB)^\dagger B^\dagger BC(BC)^*)^\dagger = BC(((AB)^*AB)^\dagger B^\dagger BC(BC)^*BC)^\dagger$ *and* $((AB)^*ABB^\dagger(BC(BC)^*)^\dagger)^\dagger = (AB(AB)^*ABB^\dagger(BC(BC)^*)^\dagger)^\dagger AB$.

⟨75⟩ $(((AB)^*AB)^\dagger B^\dagger BC(BC)^*BC)^\dagger = (BC)^\dagger(((AB)^*AB)^\dagger B^\dagger BC(BC)^*)^\dagger$ *and* $((AB)^*ABB^\dagger(BC(BC)^*)^\dagger)^\dagger(AB)^\dagger = (AB(AB)^*ABB^\dagger((BC)(BC)^*)^\dagger)^\dagger$.

⟨76⟩ $((AB)^*ABB^\dagger(BC(BC)^*)^\dagger)^\dagger = ((BC)^\dagger)^*((AB)^*ABB^\dagger((BC)^*BC(BC)^*)^\dagger)^\dagger$ *and* $(((AB)^*AB)^\dagger B^\dagger BC(BC)^*)^\dagger = (((AB)^*AB(AB)^*)^\dagger B^\dagger BC(BC)^*)^\dagger((AB)^\dagger)^*$.

⟨77⟩ $((AB)^*ABB^\dagger((BC)^*BC(BC)^*)^\dagger)^\dagger = (BC)^*((AB)^*ABB^\dagger(BC(BC)^*)^\dagger)^\dagger$ *and* $(AB(AB)^*ABB^\dagger(BC(BC)^*)^\dagger)^\dagger = ((AB)^*ABB^\dagger(BC(BC)^*)^\dagger)^\dagger(AB)^*$.

⟨78⟩ $(((AB)^*AB)^\dagger B^\dagger(BC(BC)^*)^\dagger)^\dagger = ((BC)^\dagger)^*(((AB)^*AB)^\dagger B^\dagger((BC)^*BC(BC)^*)^\dagger)^\dagger$ *and* $(((AB)^*AB)^\dagger B^\dagger(BC(BC)^*)^\dagger)^\dagger = (((AB)^*AB(AB)^*)^\dagger B^\dagger(BC(BC)^*)^\dagger)^\dagger((AB)^\dagger)^*$.

⟨79⟩ $(((AB)^*AB)^\dagger B^\dagger((BC)^*BC(BC)^*)^\dagger)^\dagger = (BC)^*(((AB)^*AB)^\dagger B^\dagger(BC(BC)^*)^\dagger)^\dagger$ *and* $(((AB)^*AB(AB)^*)^\dagger B^\dagger(BC(BC)^*)^\dagger)^\dagger = (((AB)^*AB)^\dagger B^\dagger(BC(BC)^*)^\dagger)^\dagger(AB)^*$.

⟨80⟩ $BC((AB)^*ABC(BC)^*BC)^\dagger = (AB(AB)^*ABC(BC)^*)^\dagger AB$.

⟨81⟩ $((AB)^*ABC(BC)^*BC)^\dagger(AB)^\dagger = (BC)^\dagger(AB(AB)^*ABC(BC)^*)^\dagger$.

⟨82⟩ $BC(((AB)^*AB)^\dagger B^\dagger BC(BC)^*BC)^\dagger = (((AB)^*AB(AB)^*)^\dagger B^\dagger BC(BC)^*)^\dagger((AB)^\dagger)^*$.

⟨83⟩ $(((AB)^*AB)^\dagger B^\dagger BC(BC)^*BC)^\dagger(AB)^* = (BC)^\dagger(((AB)^*AB(AB)^*)^\dagger B^\dagger BC(BC)^*)^\dagger$.

⟨84⟩ $((BC)^\dagger)^*((AB)^*ABB^\dagger((BC)^*BC(BC)^*)^\dagger)^\dagger = (AB(AB)^*ABB^\dagger(BC(BC)^*)^\dagger)^\dagger AB$.

⟨85⟩ $((AB)^*ABB^\dagger((BC)^*BC(BC)^*)^\dagger)^\dagger(AB)^\dagger = (BC)^*(AB(AB)^*ABB^\dagger(BC(BC)^*)^\dagger)^\dagger$.

⟨86⟩ $(((BC)(BC)^*BC)^\dagger(B^\dagger)^*((AB)^*AB)^\dagger)^\dagger(BC)^\dagger = (AB)^\dagger((BC(BC)^*)^\dagger(B^\dagger)^*(AB(AB)^*AB)^\dagger)^\dagger$.

⟨87⟩ $AB((BC(BC)^*BC)^\dagger(B^\dagger)^*((AB)^*AB)^\dagger)^\dagger = ((BC(BC)^*)^\dagger(B^\dagger)^*(AB(AB)^*AB)^\dagger)^\dagger BC$.

⟨88⟩ $((AB)^*ABC(BC)^*)^\dagger = BC(AB(AB)^*ABC(BC)^*BC)^\dagger AB$.

⟨89⟩ $(AB(AB)^*ABC(BC)^*BC)^\dagger = (BC)^\dagger((AB)^*ABC(BC)^*)^\dagger(AB)^\dagger$.

⟨90⟩ $(((AB)^*AB)^\dagger B^\dagger BC(BC)^*)^\dagger = BC(((AB)^*AB(AB)^*)^\dagger B^\dagger BC(BC)^*BC)^\dagger((AB)^\dagger)^*$.

⟨91⟩ $(((AB)^*AB(AB)^*)^\dagger B^\dagger BC(BC)^*BC)^\dagger = (BC)^\dagger(((AB)^*AB)^\dagger B^\dagger BC(BC)^*)^\dagger(AB)^*$.

⟨92⟩ $((AB)^*ABB^\dagger(BC(BC)^*)^\dagger)^\dagger = ((BC)^\dagger)^*(AB(AB)^*ABB^\dagger((BC)^*BC(BC)^*)^\dagger)^\dagger AB$.

⟨93⟩ $(AB(AB)^*ABB^\dagger((BC)^*BC(BC)^*)^\dagger)^\dagger = (BC)^*((AB)^*ABB^\dagger(BC(BC)^*)^\dagger)^\dagger((AB)^\dagger)^*$.

⟨94⟩ $(((AB)^*AB)^\dagger B^\dagger(BC(BC)^*)^\dagger)^\dagger = ((BC)^\dagger)^*(((AB)^*AB(AB)^*)^\dagger B^\dagger((BC)^*BC(BC)^*)^\dagger)^\dagger((AB)^\dagger)^*$.

⟨95⟩ $(((AB)^*AB(AB)^*)^\dagger B^\dagger((BC)^*BC(BC)^*)^\dagger)^\dagger = (BC)^*(((AB)^*AB)^\dagger B^\dagger((BC)(BC)^*)^\dagger)^\dagger(AB)^*$.

⟨96⟩ $(AB(AB)^*ABC(BC)^*BC)^\dagger = (((AB)^*AB)^2B^\dagger BC(BC)^*BC)^\dagger(AB)^*$ *and* $(AB(AB)^*ABC(BC)^*BC)^\dagger = (BC)^*(AB(AB)^*ABB^\dagger(BC(BC)^*)^2)^\dagger$.

⟨97⟩ $(((AB)^*AB)^2B^\dagger BC(BC)^*BC)^\dagger = (AB(AB)^*ABC(BC)^*BC)^\dagger((AB)^\dagger)^*$ *and* $(AB(AB)^*ABB^\dagger(BC(BC)^*)^2)^\dagger = ((BC)^\dagger)^*(AB(AB)^*ABC(BC)^*BC)^\dagger$.

⟨98⟩ $(((AB)^*AB)^2B^\dagger BC(BC)^*BC)^\dagger(AB)^* = (BC)^*(AB(AB)^*ABB^\dagger(BC(BC)^*)^2)^\dagger$.

⟨99⟩ $(AB(AB)^*ABC(BC)^*BC)^\dagger((AB)^\dagger)^* = ((BC)^\dagger)^*(AB(AB)^*ABC(BC)^*BC)^\dagger$.

⟨100⟩ $(AB(AB)^*ABC(BC)^*BC)^\dagger = (BC)^*(((AB)^*AB)^2B^\dagger(BC(BC)^*)^2)^\dagger(AB)^*$.

⟨101⟩ $(((AB)^*AB)^2B^\dagger(BC(BC)^*)^2)^\dagger = ((BC)^\dagger)^*(AB(AB)^*ABC(BC)^*BC)^\dagger((AB)^\dagger)^*$.

⟨102⟩ $(BC(BC)^\dagger(B^\dagger)^*(AB)^\dagger AB)^\dagger = ((BC)^\dagger(B^\dagger)^*(AB)^\dagger AB)^\dagger(BC)^\dagger$ *and* $(BC(BC)^\dagger(B^\dagger)^*(AB)^\dagger AB)^\dagger = (AB)^\dagger(BC(BC)^\dagger(B^\dagger)^*(AB)^\dagger)^\dagger$.

⟨103⟩ $(BC(BC)^\dagger(B^\dagger)^*(AB)^\dagger AB)^\dagger BC = ((BC)^\dagger(B^\dagger)^*(AB)^\dagger AB)^\dagger$ *and* $AB(BC(BC)^\dagger(B^\dagger)^*(AB)^\dagger AB)^\dagger = (BC(BC)^\dagger(B^\dagger)^*(AB)^\dagger)^\dagger$.

⟨104⟩ $((AB)^*ABC(BC)^\dagger)^\dagger = BC((AB)^*ABC)^\dagger$ *and* $((AB)^\dagger ABC(BC)^*)^\dagger = (ABC(BC)^*)^\dagger AB$.

⟨105⟩ $((AB)^*ABC)^\dagger = (BC)^\dagger((AB)^*ABC(BC)^\dagger)^\dagger$ *and* $(ABC(BC)^*)^\dagger = ((AB)^\dagger ABC(BC)^*)^\dagger(AB)^\dagger$.

⟨106⟩ $((BC)^\dagger(B^\dagger)^*((AB)^*AB)^\dagger)^\dagger = (((BC)(BC)^*)^\dagger(B^\dagger)^*((AB)^*AB)^\dagger)^\dagger((BC)^\dagger)^*$ *and* $((BC(BC)^*)^\dagger(B^\dagger)^*(AB)^\dagger)^\dagger = ((AB)^\dagger)^*((BC(BC)^*)^\dagger(B^\dagger)^*((AB)^*AB)^\dagger)^\dagger$.

⟨107⟩ $(((AB)^*AB)^\dagger B^\dagger((BC)^\dagger)^*)^\dagger = (BC)^\dagger(((AB)^*AB)^\dagger B^\dagger((BC)(BC)^*)^\dagger)^\dagger$ *and* $(((AB)^\dagger)^*B^\dagger(BC(BC)^*)^\dagger)^\dagger = (((AB)^*AB)^\dagger B^\dagger(BC(BC)^*)^\dagger)^\dagger(AB)^\dagger$.

⟨108⟩ $(((AB)^*AB)^\dagger B^\dagger(BC(BC)^*)^\dagger)^\dagger = BC(((AB)^*AB)^\dagger B^\dagger((BC)^\dagger)^*)^\dagger$ *and* $(((AB)^*AB)^\dagger B^\dagger(BC(BC)^*)^\dagger)^\dagger = (((AB)^\dagger)^*B^\dagger(BC(BC)^*)^\dagger)^\dagger AB$.

⟨109⟩ $(ABC(B^\dagger BC)^\dagger(AB)^\dagger)^\dagger = ((B^\dagger BC)^\dagger(AB)^\dagger)^\dagger(ABC)^\dagger$ *and* $(BC)^\dagger(ABB^\dagger)^\dagger ABC)^\dagger = (ABC)^\dagger((BC)^\dagger(ABB^\dagger)^\dagger)^\dagger$.

⟨110⟩ $ABC((AB)^\dagger ABC)^\dagger(AB)^\dagger$ *and* $(BC)^\dagger(ABC(BC)^\dagger)^\dagger ABC$ *are orthogonal projectors*.

⟨111⟩ $ABC((AB)^*ABC)^\dagger(AB)^*$ *and* $(BC)^*(ABC(BC)^*)^\dagger ABC$ *are orthogonal projectors*.

⟨112⟩ $(AB)^\dagger(ABC)(ABC)^\dagger AB$ *and* $BC(ABC)^\dagger(ABC)(BC)^\dagger$ *orthogonal projectors*.

⟨113⟩ $ABC(ABC)^\dagger$ *and* $AB(AB)^*$ *commute, and* $(ABC)^\dagger ABC$ *and* $(BC)^*BC$ *commute*.

⟨114⟩ $ABC((AB)^\dagger ABC)^\dagger(AB)^\dagger$ *and* $AB(AB)^*$ *commute, and* $(BC)^\dagger(ABC(BC)^\dagger)^\dagger ABC$ *and* $(BC)^*BC$ *commute*.

⟨115⟩ $ABC((AB)^*ABC)^\dagger(AB)^*$ *and* $AB(AB)^*$ *commute, and* $(BC)^*(ABC(BC)^*)^\dagger ABC$ *and* $(BC)^*BC$ *commute*.

⟨116⟩ $(AB)^\dagger ABC(ABC)^\dagger AB$ *and* $(AB)^*AB$ *commute, and* $BC(ABC)^\dagger ABC(BC)^\dagger$ *and* $BC(BC)^*$ *commute*.

⟨117⟩ $AB((B^\dagger BC)^\dagger(AB)^\dagger AB)^\dagger(B^\dagger BC)^\dagger(AB)^\dagger$ *and* $(BC)^\dagger(ABB^\dagger)^\dagger(BC(BC)^\dagger(ABB^\dagger)^\dagger)^\dagger BC$ *are orthogonal projectors*.

⟨118⟩ $((AB)^\dagger)^*((B^\dagger BC)^\dagger((AB)^*AB)^\dagger)^\dagger(B^\dagger BC)^\dagger(AB)^\dagger$ *and* $(BC)^\dagger(ABB^\dagger)^\dagger((BC(BC)^*)^\dagger(ABB^\dagger)^\dagger)^\dagger((BC)^\dagger)^*$ *are orthogonal projectors*.

⟨119⟩ $(AB)^\dagger((B^\dagger BC)^\dagger(AB)^\dagger)^\dagger(B^\dagger BC)^\dagger(AB)^\dagger AB$ *and* $BC(BC)^\dagger(ABB^\dagger)^\dagger((BC)^\dagger(ABB^\dagger)^\dagger)^\dagger(BC)^\dagger$ *are orthogonal projectors*.

⟨120⟩ $((B^\dagger BC)^\dagger(AB)^\dagger)^\dagger(B^\dagger BC)^\dagger(AB)^\dagger$ *and* $(AB(AB)^*)^\dagger$ *commute, and* $(BC)^\dagger(ABB^\dagger)^\dagger((BC)^\dagger(ABB^\dagger)^\dagger)^\dagger$ *and* $((BC)^*BC)^\dagger$ *commute*.

⟨121⟩ $AB((B^\dagger BC)^\dagger(AB)^\dagger AB)^\dagger(B^\dagger BC)^\dagger(AB)^\dagger$ *and* $(AB(AB)^*)^\dagger$ *commute, and* $(BC)^\dagger(ABB^\dagger)^\dagger(BC(BC)^\dagger(ABB^\dagger)^\dagger)^\dagger BC$ *and* $((BC)^*BC)^\dagger$ *commute*.

⟨122⟩ $((AB)^{\dagger})^{*}((B^{\dagger}BC)^{\dagger}((AB)^{*}AB)^{\dagger})^{\dagger}(B^{\dagger}BC)^{\dagger}(AB)^{\dagger}$ *and* $(AB(AB)^{*})^{\dagger}$ *commute*, *and* $(BC)^{\dagger}(ABB^{\dagger})^{\dagger}((BC(BC)^{*})^{\dagger}(ABB^{\dagger})^{\dagger})^{\dagger}((BC)^{\dagger})^{*}$ *and* $((BC)^{*}BC)^{\dagger}$ *commute*.

⟨123⟩ $(AB)^{\dagger}((B^{\dagger}BC)^{\dagger}(AB)^{\dagger})^{\dagger}((B^{\dagger}BC)^{\dagger}(AB)^{\dagger})AB$ *and* $((AB)^{*}AB)^{\dagger}$ *commute*, *and* $BC((BC)^{\dagger}(ABB^{\dagger})^{\dagger})((BC)^{\dagger}(ABB^{\dagger})^{\dagger})^{\dagger}(BC)^{\dagger}$ *and* $(BC(BC)^{*})^{\dagger}$ *commute*.

⟨124⟩ $ABC(B^{\dagger}BC)^{\dagger}(AB)^{\dagger}$ *and* $(BC)^{\dagger}(ABB^{\dagger})^{\dagger}ABC$ *are EP*.

⟨125⟩ $\{((AB)^{\dagger})^{*}B^{\dagger}BC)^{(1,2,3)}\} \ni ((AB)^{\dagger}ABC)^{\dagger}(AB)^{*}$ *and* $\{(ABB^{\dagger}((BC)^{\dagger})^{*})^{(1,2,4)}\} \ni (BC)^{*}(ABC(BC)^{\dagger})^{\dagger}$.

⟨126⟩ $\{((AB)^{\dagger})^{*}B^{\dagger}BC)^{(1,2,3)}\} \ni (((AB)^{*}AB)^{\dagger}B^{\dagger}BC)^{\dagger}(AB)^{\dagger}$ *and* $\{(ABB^{\dagger}((BC)^{\dagger})^{*})^{(1,2,4)}\} \ni (BC)^{\dagger}(ABB^{\dagger}(BC(BC)^{*})^{\dagger})^{\dagger}$.

⟨127⟩ $\{((AB)^{\dagger}ABC)^{(1,2,3)}\} \ni (((AB)^{\dagger})^{*}B^{\dagger}BC)^{\dagger}((AB)^{\dagger})^{*}$ *and* $\{(ABC(BC)^{\dagger})^{(1,2,4)}\} \ni ((BC)^{\dagger})^{*}(ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger}$.

⟨128⟩ $\{(((AB)^{*}AB)^{\dagger}B^{\dagger}BC)^{(1,2,3)}\} \ni (((AB)^{\dagger})^{*}B^{\dagger}BC)^{\dagger}AB$ *and* $\{(ABB^{\dagger}(BC(BC)^{*})^{\dagger})^{(1,2,4)}\} \ni BC(ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger}$.

⟨129⟩ $\{((AB)^{\dagger})^{*}B^{\dagger}BC)^{(1,2,3)}\} \ni (B^{\dagger}BC)^{\dagger}((AB)^{\dagger}ABC(B^{\dagger}BC)^{\dagger})^{\dagger}(AB)^{*}$ *and* $\{(ABB^{\dagger}((BC)^{\dagger})^{*})^{(1,2,4)}\} \ni (BC)^{*}((ABB^{\dagger})^{\dagger}ABC(BC)^{\dagger})^{\dagger}(ABB^{\dagger})^{\dagger}$.

⟨130⟩ $\{((AB)^{\dagger})^{*}B^{\dagger}BC)^{(1,2,3)}\} \ni (B^{\dagger}BC)^{*}(((AB)^{*}AB)^{\dagger}B^{\dagger}BC(B^{\dagger}BC)^{*})^{\dagger}(AB)^{\dagger}$ *and* $\{(ABB^{\dagger}((BC)^{\dagger})^{*})^{(1,2,4)}\} \ni (BC)^{\dagger}((ABB^{\dagger})^{*}ABB^{\dagger}(BC(BC)^{*})^{\dagger})^{\dagger}(ABB^{\dagger})^{*}$.

⟨131⟩ $\{((AB)^{\dagger}ABC(B^{\dagger}BC)^{\dagger})^{(1,2,3)}\} \ni B^{\dagger}BC(((AB)^{\dagger})^{*}B^{\dagger}BC)^{\dagger}((AB)^{\dagger})^{*}$ *and* $\{((ABB^{\dagger})^{\dagger}ABC(BC)^{\dagger})^{(1,2,4)}\} \ni ((BC)^{\dagger})^{*}(ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger}ABB^{\dagger}$.

⟨132⟩ $\{(((AB)^{*}AB)^{\dagger}B^{\dagger}BC(B^{\dagger}BC)^{*})^{(1,2,3)}\} \ni ((B^{\dagger}BC)^{\dagger})^{*}(((AB)^{\dagger})^{*}B^{\dagger}BC)^{\dagger}AB$ *and* $\{((ABB^{\dagger})^{*}ABB^{\dagger}(BC(BC)^{*})^{\dagger})^{(1,2,4)}\} \ni BC(ABB^{\dagger}((BC)^{\dagger})^{*})^{\dagger}((ABB^{\dagger})^{\dagger})^{*}$.

⟨133⟩ $(AB(AB)^{*}ABC)(AB(AB)^{*}ABC)^{\dagger} = (ABC)(ABC)^{\dagger}$ *and* $(ABC(BC)^{*}BC)^{\dagger}(ABC(BC)^{*}BC) = (ABC)^{\dagger}(ABC)$.

⟨134⟩ $\mathscr{R}((ABC)^{\dagger}) = \mathscr{R}((BC)^{\dagger}((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger}(AB)^{\dagger})$ *and* $\mathscr{R}(((ABC)^{\dagger})^{*}) = \mathscr{R}(((BC)^{\dagger}((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger}(AB)^{\dagger})^{*})$.

⟨135⟩ $\mathscr{R}(AB(AB)^{*}ABC) = \mathscr{R}(ABC)$ *and* $\mathscr{R}((BC)^{*}BC(ABC)^{*}) = \mathscr{R}((ABC)^{*})$.

⟨136⟩ $r[AB(AB)^{*}ABC,\ ABC] = r[(BC)^{*}BC(ABC)^{*},\ (ABC)^{*}] = r(ABC)$.

⟨137⟩ $r[((AB)^{*}AB(AB)^{*})^{\dagger}B^{\dagger}BC,\ ((AB)^{\dagger})^{*}B^{\dagger}BC] = r(((AB)^{\dagger})^{*}B^{\dagger}BC)$ *and* $r[(BC(BC)^{*}BC)^{\dagger}(ABB^{\dagger})^{*},\ (BC)^{\dagger}(ABB^{\dagger})^{*}] = r((BC)^{\dagger}(ABB^{\dagger})^{*})$.

Theorem 4.5. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, *and* $C \in \mathbb{C}^{p\times q}$. *Then the following three statements are equivalent*:

⟨1⟩ $(ABC)^{\dagger} = (BC)^{\dagger}((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger}(AB)^{\dagger}$.

⟨2⟩ $(ABC)^{\dagger} = (BC)^{\dagger}((AB)^{\dagger}ABB^{\dagger} + B^{\dagger}BC(BC)^{\dagger} - B^{\dagger})^{\dagger}(AB)^{\dagger}$.

⟨3⟩ $(((AB)^{\dagger})^{*}B^{\dagger}((BC)^{\dagger})^{*})^{\dagger} = (BC)^{*}((AB)^{\dagger}ABB^{\dagger} + B^{\dagger}BC(BC)^{\dagger} - B^{\dagger})^{\dagger}(AB)^{*}$.

Corollary 4.6. *Let* $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$, *and* $C \in \mathbb{C}^{p\times q}$. *Then the following two statements are equivalent*:

⟨1⟩ $(ABC)^{\dagger} = (BC)^{\dagger}((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger}(AB)^{\dagger}$.

⟨2⟩ $(((AB)^{*}AB)^{1/2}B^{\dagger}(BC(BC)^{*})^{1/2})^{\dagger} = ((BC(BC)^{*})^{1/2})^{\dagger}((AB)^{\dagger}ABC(BC)^{\dagger})^{\dagger}(((AB)^{*}AB)^{1/2})^{\dagger}$.

If $(AB)^{\dagger}ABC(BC)^{\dagger} = B^{\dagger}$, then (1.18) is reduced to $(ABC)^{\dagger} = (BC)^{\dagger}B(AB)^{\dagger}$. In this case, it is easy to obtain a group of equivalent statements associated with the nested ROL from Theorem 4.4, which we leave for the reader.

As applications of the results in Sections 3 and 4, we are able to establish and simplify many other types of matrix expressions and matrix equalities that involve generalized inverses. Here, we mention a convenient way to rewrite the sum $A + B$, where $A, B \in \mathbb{C}^{m\times n}$, as the following products of triple block matrices:

$$A + B = [I_m,\ I_m]\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} I_n \\ I_n \end{bmatrix}, \qquad A + B = [A,\ B]\begin{bmatrix} A^{\dagger} & 0 \\ 0 & B^{\dagger} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix},$$
$$A + B = [AA^{\dagger},\ BB^{\dagger}]\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\begin{bmatrix} A^{\dagger}A \\ B^{\dagger}B \end{bmatrix}, \qquad A + B = \frac{1}{2}[I_m,\ I_m]\begin{bmatrix} A & B \\ B & A \end{bmatrix}\begin{bmatrix} I_n \\ I_n \end{bmatrix}.$$

In these situations, applying the preceding theorems and corollaries to the triple matrix products on the right-hand sides of the four matrix equalities will correspondingly yield several groups of results on the generalized inverses of the sum of two matrices. But we omit them here due to space limits.

5. Conclusions

We have collected and established a myriad of known and novel nested ROLs for generalized inverses of triple matrix matrices. These ROLs and their variations can be classified as concrete examples in the establishment and characterization of various matrix equalities of generalized inverses, so that they provide highly informative accounts of a variety of current researches concerning equalities for matrices and their generalized inverses, and of course can be used as analytic tools to deal adequately with various theoretical and computational problems in the theory of generalized inverses of matrices.

As demonstrated in the preceding sections, ROLs for generalized inverses of multiple products of singular matrices can reasonably be constructed in numerous regular and nested forms, which seem quite complicated in contrast with the ordinary inverses of nonsingular matrices. To illustrate, we present several examples of nested ROLs for the Moore–Penrose generalized inverses of products of four matrices as follows

$$\begin{aligned}
&(ABCD)^{\dagger} = (CD)^{\dagger}C(BC)^{\dagger}B(AB)^{\dagger},\\
&(ABCD)^{\dagger} = D^{\dagger}(C^{\dagger}CDD^{\dagger})^{\dagger}C^{\dagger}(B^{\dagger}BCC^{\dagger})^{\dagger}B^{\dagger}(A^{\dagger}ABB^{\dagger})^{\dagger}A^{\dagger},\\
&(ABCD)^{\dagger} = D^{\dagger}(CDD^{\dagger})^{\dagger}((A^{\dagger}AB)^{\dagger}BC(CDD^{\dagger})^{\dagger})^{\dagger}(A^{\dagger}AB)^{\dagger}A^{\dagger},\\
&(ABCD)^{\dagger} = (BCD)^{\dagger}((ABC)^{\dagger}ABCD(BCD)^{\dagger})^{\dagger}(ABC)^{\dagger},\\
&(ABCD)^{\dagger} = D^{\dagger}(CDD^{\dagger})^{\dagger}(B(CDD^{\dagger})^{\dagger})^{\dagger}(((A^{\dagger}AB)^{\dagger}BC)^{\dagger}(A^{\dagger}AB)^{\dagger}BC(CDD^{\dagger})^{\dagger}(BC(CDD^{\dagger})^{\dagger})^{\dagger})^{\dagger}\\
&\qquad\qquad \times ((A^{\dagger}AB)^{\dagger}BC)^{\dagger}(A^{\dagger}AB)^{\dagger}A^{\dagger}.
\end{aligned}$$

Recall that classification becomes a common theme across all areas of mathematics as various problems, formulas, results, and facts in each field increase gradually. Thus, the equivalence classifications of the ROLs have naturally been proposed and have become one of the challenging but fruitful working areas in the theory of generalized inverses. Nevertheless, the past several decades have seen magnificent breakthroughs via successful adaptation of the matrix rank method and the block matrix method in the investigation of ROLs. By now, the classification program has proven a resounding success in the establishment of matrix equalities composed of generalized inverses, in particular, it has been realized that all ROLs for generalized inverses of multiple matrix products can be divided into certain groups, for which we can approach jointly and obtain many equivalent facts with great efficiency.

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Conflict of interest

The author declares no conflict of interest.

References

1. A. Ben-Israel, T. N. E. Greville, *Generalized inverses: theory and applications*, 2nd Eds., New York: Springer, 2003.
2. S. L. Campbell, C. D. Meyer, *Generalized inverses of linear transformations*, Philadelphia: SIAM, 2009.
3. N. Č. Dinčić, D. S. Djordjević, Basic reverse order law and its equivalencies, *Aequat. Math.*, **85** (2013), 505–517.
4. N. Č. Dinčić, D. S. Djordjević, D. Mosić, Mixed-type reverse order law and its equivalents, *Studia Math.*, **204** (2011), 123–136.
5. I. Erdelyi, On the "reverse order law" related to the generalized inverse of matrix products, *J. Assoc. Comp. Mach.*, **13** (1966), 439–443.
6. I. Erdelyi, Partial isometries closed under multiplication on Hilbert spaces, *J. Math. Anal. Appl.*, **22** (1968), 546–551.
7. A. M. Galperin, Z. Waksman, On pseudo inverse of operator products, *Linear Algebra Appl.*, **33** (1980), 123–131.
8. T. N. E. Greville, Note on the generalized inverse of a matrix product, *SIAM Rev.*, **8** (1966), 518–521.
9. R. E. Hartwig, The reverse order law revisited, *Linear Algebra Appl.*, **76** (1986), 241–246.
10. R. E. Hartwig, K. Spindelböck, Matrices for which A^* and $A^\dagger$ can commute, *Linear Multilinear Algebra*, **14** (1983), 241–256.
11. S. Izumino, The product of operators with closed range and an extension of the reverse order law, *Tôhoku Math. J.*, **34** (1982), 43–52.
12. Y. Liu, Y. Tian, A mixed-type reverse-order law for generalized inverse of a triple matrix product (in Chinese), *Acta Math. Sinica*, **52** (2009), 197–204.

13. G. Marsaglia, G. P. H. Styan, Equalities and inequalities for ranks of matrices, *Linear Multilinear Algebra*, **2** (1974), 269–292.
14. D. Mosić, N. Č. Dinčić, Reverse order law $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$ in rings with involution, *Filomat*, **28** (2014), 1791–1815.
15. D. Mosić, D. S. Djordjević, Reverse order law for the Moore–Penrose inverse in C^*-algebras, *Electron. J. Linear Algebra*, **22** (2011), 92–111.
16. C. R. Rao, S. K. Mitra, *Generalized inverse of matrices and its applications*, New York: Wiley, 1971.
17. R. Penrose, A generalized inverse for matrices, *Proc. Cambridge Phil. Soc.*, **51** (1955), 406–413.
18. Y. Tian, Reverse order laws for the generalized inverses of multiple matrix products, *Linear Algebra Appl.*, **211** (1994), 85–100.
19. Y. Tian, Rank equalities related to outer inverses of matrices and applications, *Linear Multilinear Algebra*, **49** (2002), 269–288.
20. Y. Tian, Using rank formulas to characterize equalities for Moore–Penrose inverses of matrix products, *Appl. Math. Comput.*, **147** (2004), 581–600.
21. Y. Tian, The reverse-order law $(AB)^\dagger = B^\dagger(A^\dagger ABB^\dagger)^\dagger A^\dagger$ and its equivalent equalities, *J. Math. Kyoto Univ.*, **45** (2005), 841–850.
22. Y. Tian, The equivalence between $(AB)^\dagger = B^\dagger A^\dagger$ and other mixed-type reverse-order laws, *Int. J. Math. Edu. Sci. Tech.*, **37** (2007), 331–339.
23. Y. Tian, Some mixed-type reverse-order laws for the Moore–Penrose inverse of a triple matrix product, *Rocky Mt. J. Math.*, **37** (2007), 1327–1347.
24. Y. Tian, On relationships between two linear subspaces and two orthogonal projectors, *Spec. Matrices*, **7** (2019), 142–212.
25. Y. Tian, Miscellaneous reverse order laws for generalized inverses of matrix products with applications, *Adv. Oper. Theory*, **5** (2020), 1889–1942.
26. Y. Tian, Two groups of mixed reverse order laws for generalized inverses of two and three matrix products, *Comp. Appl. Math.*, **39** (2020), 181.
27. Y. Tian, A family of 512 reverse order laws for generalized inverses of a matrix product: a review, *Heliyon*, **6** (2020), e04924.
28. Y. Tian, S. Cheng, Some identities for Moore–Penrose inverses of matrix products, *Linear Multilinear Algebra*, **52** (2004), 405–420.
29. Y. Tian, Y. Liu, On a group of mixed-type reverse-order laws for generalized inverses of a triple matrix product with applications, *Electron. J. Linear Algebra*, **16** (2007), 73–89.

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