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Research article

Nonlocal fractional $p(\cdot)$-Kirchhoff systems with variable-order: Two and three solutions

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Abstract: In this article, we consider the following nonlocal fractional Kirchhoff-type elliptic systems

$$\begin{cases} -M_1\left(\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\eta(x)-\eta(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega\frac{|\eta|^{\overline{p}(x)}}{\overline{p}(x)}dx\right)\left(\Delta^{s(\cdot)}_{p(\cdot)}\eta-|\eta|^{\overline{p}(x)}\eta\right) \\ \quad =\lambda F_\eta(x,\eta,\xi)+\mu G_\eta(x,\eta,\xi),\quad x\in\Omega, \\ -M_2\left(\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi(x)-\xi(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega\frac{|\xi|^{\overline{p}(x)}}{\overline{p}(x)}dx\right)\left(\Delta^{s(\cdot)}_{p(\cdot)}\xi-|\xi|^{\overline{p}(x)}\xi\right) \\ \quad =\lambda F_\xi(x,\eta,\xi)+\mu G_\xi(x,\eta,\xi),\quad x\in\Omega, \\ \eta=\xi=0,\quad x\in\mathbb{R}^N\backslash\Omega, \end{cases}$$

where $M_1(t), M_2(t)$ are the models of Kirchhoff coefficient, Ω is a bounded smooth domain in $\mathbb{R}^N$, $(-\Delta)^{s(\cdot)}_{p(\cdot)}$ is a fractional Laplace operator, λ,μ are two real parameters, F, G are continuous differentiable functions, whose partial derivatives are $F_\eta, F_\xi, G_\eta, G_\xi$. With the help of direct variational methods, we study the existence of solutions for nonlocal fractional $p(\cdot)$-Kirchhoff systems with variable-order, and obtain at least two and three weak solutions based on Bonanno's and Ricceri's critical points theorem. The outstanding feature is the case that the Palais-Smale condition is not requested. The major difficulties and innovations are nonlocal Kirchhoff functions with the presence of the Laplace operator involving two variable parameters.

Keywords: Kirchhoff systems; $p(\cdot)$-Laplace operator; variational methods; Bonanno's critical points theorem; Ricceri's critical points theorem
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1. Introduction and the main results

The study of the existence and multiplicity solutions for nonlocal elliptic systems boundary value problems and variational problems has attracted intense research interests for several decades. In this paper, we investigate the existence of nontrivial solutions for a class of $p(\cdot)$-Kirchhoff systems

$$\begin{cases} -M_1\left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\eta(x)-\eta(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+p(x,y)s(x,y)}}dxdy + \int_\Omega \frac{|\eta|^{\overline{p}(x)}}{\overline{p}(x)}dx\right)\left(\Delta_{p(\cdot)}^{s(\cdot)}\eta - |\eta|^{\overline{p}(x)}\eta\right) \\ \quad = \lambda F_\eta(x,\eta,\xi) + \mu G_\eta(x,\eta,\xi), \quad x\in\Omega, \\ -M_2\left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\xi(x)-\xi(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+p(x,y)s(x,y)}}dxdy + \int_\Omega \frac{|\xi|^{\overline{p}(x)}}{\overline{p}(x)}dx\right)\left(\Delta_{p(\cdot)}^{s(\cdot)}\xi - |\xi|^{\overline{p}(x)}\xi\right) \\ \quad = \lambda F_\xi(x,\eta,\xi) + \mu G_\xi(x,\eta,\xi), \quad x\in\Omega, \\ \eta = \xi = 0, \quad x\in\mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.1}$$

where $M_1(t), M_2(t) : [0,+\infty) \to [0,+\infty)$ are the models of Kirchhoff coefficient, Ω is a bounded smooth domain in $\mathbb{R}^N$, λ,μ are two real parameters. F, G are differentiable and measurable functions in $\mathbb{R}^2$ for all $x\in\Omega$, F_η, F_ξ are the partial derivatives of F, G_η, G_ξ are the partial derivatives of G, whose assumptions will be introduced later.

The fractional $p(\cdot)$-Laplace operator $(-\Delta)_{p(\cdot)}^{s(\cdot)}$ with variable $s(\cdot)$-order is defined as

$$(-\Delta)_{p(\cdot)}^{s(\cdot)}\upsilon(x) := P.V.\int_{\mathbb{R}^N} \frac{|\upsilon(x)-\upsilon(y)|^{p(x,y)-2}(\upsilon(x)-\upsilon(y))}{|x-y|^{N+p(x,y)s(x,y)}}dy, \text{ for all } x\in\mathbb{R}^N, \tag{1.2}$$

where $v \in C_0^\infty(\mathbb{R}^N)$ and $P.V.$ stands for the Cauchy principal value. $(-\Delta)_{p(\cdot)}^{s(\cdot)}$ is a nonlocal operator of elliptic type, which is connected with the Sobolev space of variable exponent. Concerning this kind of operator problems, here we just list a few pieces of literatures, see [1–4]. Especially, Biswas et al. [5] firstly proved a continuous embedding result and Hardy-Littlewood-Sobolev-type result, and then the existence and multiplicity of solutions were obtained by variational approaches. When $s(\cdot)\equiv$ constant and $p(\cdot)\equiv$ constant, $(-\Delta)_{p(\cdot)}^{s(\cdot)}$ in (1.2) reduce to the usual fractional Laplace operator $(-\Delta)_p^s$, see [6–8] for the essential knowledge.

Throughout this paper, $s(\cdot), p(\cdot) \in C_+(\mathfrak{D})$ are two continuous functions that the following assumptions are satisfied.

(S): $s(\cdot) : \overline{\Omega}\times\overline{\Omega} \to (0,1)$ is symmetric, namely, $s(x,y) = s(y,x)$ for any $(x,y)\in\overline{\Omega}\times\overline{\Omega}$ with $\overline{s}(x) = s(x,x)$;

(P): $p(\cdot) : \overline{\Omega}\times\overline{\Omega} \to (1,+\infty)$ is symmetric, namely, $p(x,y) = p(y,x)$ for any $(x,y)\in\overline{\Omega}\times\overline{\Omega}$ with $\overline{p}(x) = p(x,x)$.

Kirchhoff in [9] introduced the following Kirchhoff equation

$$\rho\frac{\partial^2\xi(x)}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial\xi(x)}{\partial t}\right|^2 dx\right)\frac{\partial^2\xi(x)}{\partial x^2} = 0, \tag{1.3}$$

where ρ, p_0, h, E, L with physical meaning are constants. A characteristic of Eq (1.3) is the fact that it contains a nonlocal item $\frac{p_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial\xi(x)}{\partial t}\right|^2 dx$, and then this type of equation is called nonlocal problem.

From then on, the existence, multiplicity, uniqueness, and regularity of solutions for various Kirchhoff-type equations have been studied extensively, such as, see [10–14] for further details.

The continuous Kirchhoff terms $M_i(t) : R_0^+ \to R^+, (i = 1, 2)$ are strictly increasing functions, which the following conditions are satisfied.

(M): There exist $m_i = m_i(\iota) > 0$ and $M_i = M_i(\iota) > 0, (i = 1, 2)$ for any $\iota > 0$ such that

$$M_i \geq M_i(t) \geq m_i \text{ for all } t > \iota, (i = 1, 2)$$

and put

$$\widetilde{M}_i(t) = \int_0^t M_i(\varsigma) d\varsigma \text{ for all } t \in \mathbb{R}_0^+, (i = 1, 2).$$

In recent years, a multitude of scholars has devoted themselves to the study of Kirchhoff-type systems. When $M_1(t) = 1$ and $M_2(t) = 1$, Chen et al. in [15] consider the nontrivial solutions for the following elliptic systems.

$$\begin{cases} (-\Delta)_p^s \eta = \lambda |\eta|^{q-2}\eta + \frac{2\alpha}{\alpha+\beta}|\eta|^{\alpha-2}\eta|\xi|^{\beta}, & x \in \Omega, \\ (-\Delta)_p^s \xi = \lambda |\xi|^{q-2}\xi + \frac{2\beta}{\alpha+\beta}|\xi|^{\beta-2}\xi|\eta|^{\alpha}, & x \in \Omega, \\ \eta = \xi = 0, \quad x \in \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.4}$$

by utilizing Nehari manifold method and Fibering maps, they studied the existence of weak solutions for this kind of problem (1.4). Moreover, it has been applied in the local case $s = 1$ in [16].

In the famous literature [17], the three critical points theorem was established by Ricceri. Starting from this paper, Marano and Motreanu in [18] extended the result of Ricceri to non-differentiable functionals. Subsequently, Fan and Deng in [19] studied the version of Ricceri's result including variables exponents. Ricceri's result in [20] has been successfully applied to Sobolev spaces $W_0^{1,p}(\Omega)$, and then at least three solutions are obtained. Furthermore, Bonanno in [21] established the existence of two intervals of positive real parameters λ for which the functional $\Phi - \lambda J$ has three critical points, and applied the result to obtain two critical points.

By using three critical points theorem, Azroul et al. [22] discussed the fractional p-Laplace systems with bounded domain

$$\begin{cases} M_1\left([\eta]_{s,p}^p\right)(-\Delta)_p^s \eta = \lambda F_\eta(x, \eta, \xi) + \mu G_\eta(x, \eta, \xi), & x \in \Omega, \\ M_2\left([\xi]_{s,p}^p\right)(-\Delta)_p^s \xi = \lambda F_\xi(x, \eta, \xi) + \mu G_\xi(x, \eta, \xi), & x \in \Omega, \\ \eta = \xi = 0, \quad x \in \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.5}$$

thus, the existence and multiplicity of solutions were obtained by Azroul et al. In addition, there are many scholars who have used different methods to study the existence of elliptic systems on bounded and unbounded regions, for instance, see [23–25] for details.

With respect to the fractional $p(\cdot)$-Laplace operators, Azroul et al. [26] dealt with the class of Kirchhoff type elliptic systems in nonlocal fractional Sobolev spaces with variable exponents and constant order

$$\begin{cases} M\left(I_{s,p(x,y)}(\eta)\right)\left((-\Delta)^{s}_{p(x,\cdot)}\eta + |\eta|^{\overline{p}(x)-2}\eta\right) \\ \quad = \lambda F_{\eta}(x,\eta,\xi) + \mu G_{\eta}(x,\eta,\xi), \text{ in } \Omega, \\ M\left(I_{s,q(x,y)}(\xi)\right)\left((-\Delta)^{s}_{q(x,\cdot)}\xi + |\xi|^{\overline{q}(x)-2}\xi\right) \\ \quad = \lambda F_{\xi}(x,\eta,\xi) + \mu G_{\xi}(x,\eta,\xi), \text{ in } \Omega, \\ \eta = \xi = 0, \text{ in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.6}$$

where

$$I_{s,r(x,y)}(\omega) = \int_{\Omega\times\Omega} \frac{|\omega(x)-\omega(y)|^{r(x,y)}}{r(x,y)|x-y|^{N+r(x,y)s}}dxdy + \int_{\Omega} \frac{|\omega(x)|^{\overline{r}(x)}}{\overline{r}(x)}dx.$$

Based on the three critical points theorem introduced by Ricceri and on the theory of fractional Sobolev spaces with variable exponents, the existence of weak solutions for a nonlocal fractional elliptic system of $(p(x,\cdot), q(x,\cdot))$-Kirchhoff type with homogeneous Dirichlet boundary conditions was obtained. By using Ekeland's variational principle and dual fountain theorem, Bu et al. in [27] obtained some new existence and multiplicity of negative energy solutions for the fractional $p(\cdot)$-Laplace operators with constant order without the Ambrosetti-Rabinowitz condition.

Previous studies have shown that the fractional $p(\cdot)$-Laplace operators with variable-order are much more complex and difficult than p-Laplace operators. The investigation of these problems has captured the attention of a host of scholars. For example, Wu et al. in [28] considered the fractional Kirchhoff systems with a bounded set Ω in $\mathbb{R}^N$, as follows:

$$\begin{cases} M_1\left(\int\int_{\mathbb{R}^{2N}} \frac{|\eta(x)-\eta(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+p(x,y)s(x,y)}}dxdy\right)(-\Delta)^{s(\cdot)}_{p(\cdot)}\eta(x) \\ \quad = f(\eta,\xi) + a(x), \quad x \in \Omega, \\ M_2\left(\int\int_{\mathbb{R}^{2N}} \frac{|\xi(x)-\xi(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+p(x,y)s(x,y)}}dxdy\right)(-\Delta)^{s(\cdot)}_{p(\cdot)}\xi(x) \\ \quad = g(\eta,\xi) + b(x), \quad x \in \Omega, \\ \eta = \xi = 0, \quad x \in \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.7}$$

by applying Ekeland variational principle, they obtained the existence of a solution for this class of problem.

When $\mu = 0$, problem (1.1) reduces to the following fractional Kirchhoff-type elliptic systems

$$\begin{cases} -M_1\left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\eta(x)-\eta(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+p(x,y)s(x,y)}}dxdy + \int_{\Omega}\frac{|\eta|^{\overline{p}(x)}}{\overline{p}(x)}dx\right)\left(\Delta^{s(\cdot)}_{p(\cdot)}\eta - |\eta|^{\overline{p}(x)}\eta\right) \\ \quad = \lambda F_{\eta}(x,\eta,\xi), \ x \in \Omega, \\ -M_2\left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\xi(x)-\xi(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+p(x,y)s(x,y)}}dxdy + \int_{\Omega}\frac{|\xi|^{\overline{p}(x)}}{\overline{p}(x)}dx\right)\left(\Delta^{s(\cdot)}_{p(\cdot)}\xi - |\xi|^{\overline{p}(x)}\xi\right) \\ \quad = \lambda F_{\xi}(x,\eta,\xi), \quad x \in \Omega, \\ \eta = \xi = 0, \quad x \in \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.8}$$

Motivated by the above cited works, we take into account the nonlocal fractional Kirchhoff-type elliptic systems with variable-order. Our aims are to establish the existence of at least three solutions for problem (1.1) by utilizing Ricceri's result in [29] and obtain the existence of at least two solutions for problem (1.8) with the help of the multiple critical points theorem in [37]. The primary consideration of the paper is an extension of the results found in the literatures and our results are new to the Kirchhoff-type systems in some ways.

For simplicity, C_j $(j = 1, 2, ..., N)$ are used in various places to denote distinct constants, $i = 1, 2$, and we denote

$$C_+(\mathfrak{D}) := \{\mathcal{H}(\cdot) \in C(\mathfrak{D}, \mathbb{R}) : \ 1 < \mathcal{H}^-(\cdot) \leq \mathcal{H}(\cdot) \leq \mathcal{H}^+(\cdot) < +\infty\},$$

where $\mathcal{H}(\cdot)$ is a real-valued function and

$$\mathcal{H}^-(\cdot) := \min_{\mathfrak{D}} \mathcal{H}(\cdot), \quad \mathcal{H}^+(\cdot) := \max_{\mathfrak{D}} \mathcal{H}(\cdot).$$

$F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a C^1-function, whose partial derivatives are F_η, F_ξ, which satisfy the following conditions.

(F1): For some positive constant C, there exist $\alpha(x), \beta(x) \in C_+(\mathfrak{D})$ and $2 + \alpha^+ + \beta^+ < p^-$ such that

$$|F_s(x, s, t)| \leq C|s|^{\alpha(x)}|t|^{\beta(x)+1}, \ |F_t(x, s, t)| \leq C|s|^{\alpha(x)+1}|t|^{\beta(x)} \ \text{ for all } (x, s, t) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R};$$

(F2): $F(x, s, t) > 0$ for any $(x, s, t) \in \overline{\Omega} \times [1, +\infty) \times [1, +\infty)$, and $F(x, s, t) < 0$ for any $(x, s, t) \in \overline{\Omega} \times (0, 1) \times (0, 1)$, $F(x, 0, 0) = 0$ for a.e. $x \in \overline{\Omega}$.

$G : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a C^1-function, whose partial derivatives are G_η, G_ξ, which satisfy assumptions, as follows:

(G): $\sup_{|s| \leq \sigma, |t| \leq \sigma}(|G_\eta(x, s, t)| + |G_\xi(x, s, t)|) \in L^1(\Omega)$ for all $\sigma > 0$.

Definition 1. We say that $(\eta, \xi) \in X_0$ is a (weak) solution of nonlocal Kirchhoff systems (1.1), if

$$\begin{aligned}
&M_1\left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^{p(x,y)}}{p(x, y)|x - y|^{N+p(x,y)s(x,y)}} dxdy + \int_\Omega \frac{|\eta|^{\overline{p}(x)}}{\overline{p}(x)} dx\right) \\
&\quad\times\left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\eta(x) - \eta(y)|^{p(x,y)-2}(\eta(x) - \eta(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+p(x,y)s(x,y)}} dxdy + \int_\Omega |\eta|^{\overline{p}(x)}\eta\varphi dx\right) + \\
&M_2\left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\xi(x) - \xi(y)|^{p(x,y)}}{p(x, y)|x - y|^{N+p(x,y)s(x,y)}} dxdy + \int_\Omega \frac{|\xi|^{\overline{p}(x)}}{\overline{p}(x)} dx\right) \\
&\quad\times\left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\xi(x) - \xi(y)|^{p(x,y)-2}(\xi(x) - \xi(y))(\psi(x) - \psi(y))}{|x - y|^{N+p(x,y)s(x,y)}} dxdy + \int_\Omega |\xi|^{\overline{p}(x)}\xi\psi dx\right) \\
&=\lambda \int_\Omega F_\eta(x, \eta, \xi)\varphi dx + \lambda \int_\Omega F_\xi(x, \eta, \xi)\psi dx \\
&\quad+ \mu \int_\Omega G_\eta(x, \eta, \xi)\varphi dx + \mu \int_\Omega G_\xi(x, \eta, \xi)\psi dx,
\end{aligned} \tag{1.9}$$

for any $(\varphi, \psi) \in X_0$, and we will introduce X_0 in Section 2.

Define the corresponding functional $I : X_0 \to \mathbb{R}$ associated with Kirchhoff systems (1.1), by

$$I(\eta, \xi) := \Phi(\eta, \xi) + \lambda\Psi(\eta, \xi) + \mu J(\eta, \xi), \tag{1.10}$$

for all $(\eta, \xi) \in X_0$, where

$$J(\eta, \xi) = -\int_\Omega G(x, \eta, \xi)dx, \quad \Psi(\eta, \xi) = -\int_\Omega F(x, \eta, \xi)dx, \tag{1.11}$$

$$\Phi(\eta, \xi) = \widetilde{M}_1\left(\delta_{p(\cdot)}(\eta)\right) + \widetilde{M}_2\left(\delta_{p(\cdot)}(\xi)\right), \tag{1.12}$$

$$\delta_{p(\cdot)}(\upsilon) = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\upsilon(x) - \upsilon(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+p(x,y)s(x,y)}}dxdy + \int_\Omega \frac{|\upsilon|^{\overline{p}(x)}}{\overline{p}(x)}dx. \tag{1.13}$$

The functions $\Phi, \Psi, J : X_0 \to \mathbb{R}$ are well defined, and we define their Gâteaux derivatives at $(\eta, \xi) \in X_0$, by

$$\langle J'(\eta, \xi), (\varphi, \psi)\rangle = -\int_\Omega G_\eta(x, \eta, \xi)\varphi dx - \int_\Omega G_\xi(x, \eta, \xi)\psi dx, \tag{1.14}$$

$$\langle \Psi'(\eta, \xi), (\varphi, \psi)\rangle = -\int_\Omega F_\eta(x, \eta, \xi)\varphi dx - \int_\Omega F_\xi(x, \eta, \xi)\psi dx, \tag{1.15}$$

$$\langle \Phi'(\eta, \xi), (\varphi, \psi)\rangle = M_1\left(\delta_{p(\cdot)}(\eta)\right)\langle \eta, \varphi\rangle + M_2\left(\delta_{p(\cdot)}(\xi)\right)\langle \xi, \psi\rangle, \tag{1.16}$$

for all $(\varphi, \psi) \in X_0$, where

$$\langle \upsilon, \phi\rangle = \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\upsilon(x) - \upsilon(y)|^{p(x,y)-2}(\upsilon(x) - \upsilon(y))(\phi(x) - \phi(y))}{|x-y|^{N+p(x,y)s(x,y)}}dxdy + \int_\Omega |\upsilon|^{\overline{p}(x)}\upsilon\phi dx, \tag{1.17}$$

for all $(\upsilon, \phi) \in X_0$.

Hence, $(\eta, \xi) \in X_0$ is a (weak) solution of Kirchhoff systems (1.1) if and only if (η, ξ) is a critical point of the functional I, that is

$$I'(\eta, \xi) = \Phi'(\eta, \xi) + \lambda\Psi'(\eta, \xi) + \mu J'(\eta, \xi) = 0. \tag{1.18}$$

Definition 2. For $s^+p^+ < N$, and denote by $\mathcal{A}$: there exists a kind of functions $\mathcal{F} : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ such that two Carathéodory functions $\mathcal{F}_\eta = \frac{\partial\mathcal{F}}{\partial\eta}$ and $\mathcal{F}_\xi = \frac{\partial\mathcal{F}}{\partial\xi}$, satisfying

$$\sup_{(x,s,t)\in\Omega\times\mathbb{R}\times\mathbb{R}} \frac{\mathcal{F}_s(x,s,t)}{1 + |s|^{\vartheta(x)-1} + |t|^{\vartheta(x)-1}} < +\infty$$
$$\text{and} \sup_{(x,s,t)\in\Omega\times\mathbb{R}\times\mathbb{R}} \frac{\mathcal{F}_t(x,s,t)}{1 + |s|^{\vartheta(x)-1} + |t|^{\vartheta(x)-1}} < +\infty, \tag{1.19}$$

for any $\vartheta(x) \in [1, p_s^*(x))$.

Now, let us show our results in this article.

Theorem 1.1. *For $s(\cdot), p(\cdot) \in C_+(\mathfrak{D})$ with $s^+p^+ < N$ and $F \in \mathcal{A}$, assume that (S), (P), (M), (F1) and (F2) are satisfied. There exist three constants $a, c_1, c_2 > 0$ with $0 < \gamma \leq 1 < c_1 < c_2$ such that*

$$M^+A(c_1) < m_-H(c_1, a), \quad M^+A(c_2) < m_-H(c_1, a). \tag{1.20}$$

Then, for any

$$\lambda \in \left(\frac{M^+\gamma}{C_\phi H(c_1, a)}, \frac{m_-\gamma}{C_\phi} \min\left\{ \frac{1}{A(c_1)}, \frac{1}{A(c_2)} \right\} \right),$$

there exists a positive real number ρ such that the system (1.8) has at least two weak solutions $w_j = (\eta_j, \xi_j) \in X_0 (j = 1, 2)$ whose norms $\|w_j\|$ in X_0 are less than some positive constant ρ.

Theorem 1.2. *For $s(\cdot), p(\cdot) \in C_+(\mathfrak{D})$ with $s^+p^+ < N$ and $F \in \mathcal{A}$, assume that (S), (P), (M), (F1) and (F2) are satisfied. Then there exists an open interval $\Lambda \subseteq (0, +\infty)$ and a positive real number ρ with the following property: For each $\lambda \in \Lambda$ and for two Carathéodory functions G_η, G_ξ :$\Omega \times \mathbb{R}^2 \to \mathbb{R}$ satisfying (G), there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, problem (1.1) has at least three weak solutions $w_j = (\eta_j, \xi_j) \in X_0 (j = 1, 2, 3)$ whose norms $\|w_j\|$ in X_0 are less than some positive constant ρ.*

Remark 1.1. Existence results for the Kirchhoff-type elliptic systems with both boundary value problems and variational problems were obtained according to using critical points theorem by Ricceri and Bonanno, respectively, where the condition of Palais-Smale is not satisfied.

Remark 1.2. The nonlocal Kirchhoff coefficient $M_1(t)$, $M_2(t)$ stir up some of the fundamental difficulties. To deal with these difficulties, we suppose that $M_1(t)$, $M_2(t)$ are strictly increasing functions, and then prove that the function Φ' is a homeomorphism.

The remaining of this article is organized as follows: Some fundamental results about the fractional Lebesgue spaces and Sobolve spaces are given in Section 2. In Section 3, in order to use critical point theory, we prove some technical lemmas. Theorem 1.1 and Theorem 1.2 are proved in Section 4. Finally, we make a conclusion in Section 5.

2. Preliminary

2.1. Variable exponents Lebesgue spaces

To study Laplacian problems with variable exponents, we need to recall a slice of preliminary theories on generalized Lebesgue spaces $L^{\vartheta(x)}(\Omega)$ and give some necessary lemmas and propositions.

For any $\vartheta(x) \in C_+(\mathfrak{D})$, the generalized Lebesgue spaces with variable exponents is defined by

$$L^{\vartheta(x)}(\Omega) = \left\{ \xi \mid \xi : \Omega \to \mathbb{R} \text{ is a measurable function and } \int_\Omega |\xi|^{\vartheta(x)} dx < \infty \right\}$$

with respect to the norm

$$\|\xi\|_{\vartheta(x)} = \inf\left\{ \chi > 0 : \int_\Omega \left| \frac{\xi}{\chi} \right|^{\vartheta(x)} dx \leq 1 \right\},$$

then, the spaces $(L^{\vartheta(x)}(\Omega), \|\cdot\|_{\vartheta(x)})$ is a separable and reflexive Banach space, see [30, 31].

Let $\vartheta(x)$ be the conjugate exponent of $\widetilde{\vartheta}(x)$, namely

$$\frac{1}{\vartheta(x)} + \frac{1}{\widetilde{\vartheta}(x)} = 1, \text{ for all } x \in \Omega.$$

Lemma 2.1. *(see [31]) Assume that* $\xi \in L^{\vartheta(x)}(\Omega)$ *and* $\eta \in L^{\widetilde{\vartheta}(x)}(\Omega)$*, then*

$$\Big|\int_{\Omega} \xi\eta dx\Big| \leq \Big(\frac{1}{\vartheta^-} + \frac{1}{\widetilde{\vartheta}^-}\Big)\|\xi\|_{\vartheta(x)}\|\eta\|_{\widetilde{\vartheta}(x)} \leq 2\|\xi\|_{\vartheta(x)}\|\eta\|_{\widetilde{\vartheta}(x)}.$$

Proposition 2.1. *(see [30, 32]) If we define*

$$\rho_{\vartheta(x)}(\xi) = \int_{\Omega} |\xi|^{\vartheta(x)} dx,$$

then for all $\xi_n, \xi \in L^{\vartheta(x)}(\Omega)$*, the following properties are possessed.*

$$\begin{aligned}
&(1)\ \ \|\xi\|_{\vartheta(x)} > 1 \Rightarrow \|\xi\|_{\vartheta(x)}^{\vartheta^-} \leq \rho_{\vartheta(x)}(\xi) \leq \|\xi\|_{\vartheta(x)}^{\vartheta^+},\\
&(2)\ \ \|\xi\|_{\vartheta(x)} < 1 \Rightarrow \|\xi\|_{\vartheta(x)}^{\vartheta^+} \leq \rho_{\vartheta(x)}(\xi) \leq \|\xi\|_{\vartheta(x)}^{\vartheta^-},\\
&(3)\ \ \|\xi\|_{\vartheta(x)} < 1\ (resp. = 1, > 1) \Leftrightarrow \rho_{\vartheta(x)}(\xi) < 1\ (resp. = 1, > 1)),\\
&(4)\ \ \|\xi\|_{\vartheta(x)} \to 0\ (resp. \to +\infty) \Leftrightarrow \rho_{\vartheta(x)}(\xi_n) \to 0\ (resp. \to +\infty),\\
&(5)\ \ \lim_{n\to\infty} |\xi_n - \xi|_{\vartheta(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{\vartheta(x)}(\xi_n - \xi) = 0.
\end{aligned}$$

Remark 2.1. Note that for any function $\vartheta_1(x),\ \vartheta_2(x) \in C_+(\overline{\Omega})$ and $\vartheta_1(x) < \vartheta_2(x)$, there exists an embedding $L^{\vartheta_2(x)}(\Omega) \hookrightarrow L^{\vartheta_1(x)}(\Omega)$ for any $x \in \overline{\Omega}$. Especially, when $\vartheta(x) \equiv$ constant, the results of Proposition 2.1 still hold.

2.2. Fractional Sobolev spaces with variable-order

From now on, we briefly review a slice of essential lemmas and propositions about the Sobolev spaces, which will be used later. The readers are invited to consult [33–35] and the references therein.

The fractional Sobolev spaces $W^{s(\cdot),p(\cdot)}(\Omega)$ is defined as

$$W = W^{s(\cdot),p(\cdot)}(\Omega) := \left\{\xi \in L^{\overline{p}(\cdot)}(\Omega) : \int_{\Omega\times\Omega} \frac{|\xi(x)-\xi(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}} dxdy < \infty\right\}$$

and it can be endowed with the norm

$$\|\xi\|_W = \|\xi\|_{\overline{p}(\cdot)} + [\xi]_{s(\cdot),p(\cdot)} \ \text{ for all } \xi \in W,$$

where

$$[\xi]_{s(\cdot),p(\cdot)} = \inf\left\{\chi > 0 : \int_{\Omega\times\Omega} \frac{|\xi(x)-\xi(y)|^{p(x,y)}}{\chi^{p(x,y)}|x-y|^{N+p(x,y)s(x,y)}} dxdy < 1\right\},$$

then, the spaces $(W, \|\cdot\|_W)$ is a separable and reflexive Banach space, see [2, 5] for a more detailed.

We define the new fractional Sobolev spaces W' concerning variable exponent and variable-order for some $\chi > 0$.

$$W' = \left\{\xi : \mathbb{R}^N \to \mathbb{R} : \xi|_{\Omega} \in L^{\overline{p}(\cdot)}(\Omega) : \text{ and } \int_Q \frac{|\xi(x)-\xi(y)|^{p(x,y)}}{\chi^{p(x,y)}|x-y|^{N+p(x,y)s(x,y)}} dxdy < \infty\right\},$$

where $Q = \mathbb{R}^{2N}\backslash(\Omega^c \times \Omega^c)$ and it can also be endowed with the norm

$$\|\xi\|_{W'} = \|\xi\|_{\overline{p}(\cdot)} + [\xi]_{W'} \ \text{ for all } \xi \in W',$$

where

$$[\xi]_{W'} = \inf\left\{\chi > 0 : \int_{Q} \frac{|\xi(x) - \xi(y)|^{p(x,y)}}{\chi^{p(x,y)}|x - y|^{N+p(x,y)s(x,y)}} dxdy < 1\right\}.$$

Remark 2.2. Note that the norm $\|\cdot\|_W$ is different from $\|\cdot\|_{W'}$ for the reason that $\Omega \times \Omega \subset Q$ and $\Omega \times \Omega \neq Q$.

Let

$$W_0 = \left\{\xi \in W' : \xi = 0,\ a.e. \text{ in } \mathbb{R}^N \setminus \Omega\right\}$$

with the norm

$$\|\xi\|_{W_0} = \inf\left\{\chi > 0 : \int_{Q} \frac{|\xi(x) - \xi(y)|^{p(x,y)}}{\chi^{p(x,y)}|x - y|^{N+p(x,y)s(x,y)}} dxdy = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\xi(x) - \xi(y)|^{p(x,y)}}{\chi^{p(x,y)}|x - y|^{N+p(x,y)s(x,y)}} dxdy < 1\right\}.$$

W_0^* denotes the dual spaces of W_0.

In what follows, X_0 will denote the Cartesian product of two fractional Sobolev spaces W_0 and W_0, i.e. $X_0 = W_0 \times W_0$. Defined the norm

$$\|(\eta, \xi)\|_{X_0} = \|\eta\|_{W_0} + \|\xi\|_{W_0},$$

where $\|\eta\|_{W_0},\ \|\xi\|_{W_0}$ is the norm of W_0.

Theorem 2.1. *(see [2]) Let $s(\cdot),\ p(\cdot) \in C_+(\mathfrak{D})$ satisfy (S) and (P), with $N > p(x,y)s(x,y)$ for all $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. Let $\phi(x) \in C_+(\mathfrak{D})$ satisfy*

$$1 < \phi^- = \min_{x \in \overline{\Omega}} \phi(x) \leq \phi(x) < p_s^*(x) = \frac{N\overline{p}(x)}{N - \overline{p}(x)\overline{s}(x)}, \quad \text{for any } x \in \overline{\Omega},$$

where $\overline{p}(x) = p(x,x)$ and $\overline{s}(x) = s(x,x)$. Then, there exists a constant $C_\phi = C_\phi(N, s, p, \phi, \Omega) > 0$ such that

$$\|\xi\|_{\phi(\cdot)} \leq C_\phi \|\xi\|_{W_0}$$

for any $\xi \in W_0$. Moreover, the embedding $W_0 \hookrightarrow L^{\phi(\cdot)}(\Omega)$ is compact.

Proposition 2.2. *(see [36]) If we define*

$$\rho_{p(\cdot)}^{s(\cdot)}(\xi) = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\xi(x) - \xi(y)|^{p(x,y)}}{|x - y|^{N+p(x,y)s(x,y)}} dxdy + \int_\Omega |\xi|^{\overline{p}(x)} dx,$$

then for all $\xi_n, \xi \in W_0$, the following properties hold.

(1) $\|\xi\|_{W_0} < 1\ (resp. = 1, > 1) \Leftrightarrow \rho_{p(\cdot)}^{s(\cdot)}(\xi) < 1\ (resp. = 1, > 1)$,

(2) $\|\xi\|_{W_0} < 1 \Rightarrow \|\xi\|_{W_0}^{p^+} \leq \rho_{p(\cdot)}^{s(\cdot)}(\xi) \leq \|\xi\|_{W_0}^{p^-}$,

(3) $\|\xi\|_{W_0} > 1 \Rightarrow \|\xi\|_{W_0}^{p^-} \leq \rho_{p(\cdot)}^{s(\cdot)}(\xi) \leq \|\xi\|_{W_0}^{p^+}$,

(4) $\lim_{n\to\infty} \|\xi_n\|_{W_0} = 0\ (resp. \to +\infty) \Leftrightarrow \lim_{n\to\infty} \rho_{p(\cdot)}^{s(\cdot)}(\xi_n) = 0\ (resp. \to +\infty)$,

(5) $\lim_{n\to\infty} \|\xi_n - \xi\|_{W_0} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(\cdot)}^{s(\cdot)}(\xi_n - \xi) = 0$.

In order to prove our main results, we present two multiple critical theorems: The first ensures the existence of two critical points, while the second establishes the existence of three critical points due to Ricceri.

Theorem 2.2. *(see [37]) Let X be a reflexive real Banach space, and let $\Phi, \Psi : X \to R$ be two sequentially weakly lower semicontinuous functions. Assume that Φ is (strongly) continuous and satisfies* $\lim_{\|\eta\|\to\infty} \Phi(\eta) = +\infty$. *Assume also that there exist two constants r_1 and r_2 such that*

$$\begin{aligned} (i) &\quad \inf_X \Phi < r_1 < r_2, \\ (ii) &\quad \varphi_1(r_1) < \varphi_2(r_1, r_2), \\ (iii) &\quad \varphi_1(r_2) < \varphi_2(r_1, r_2), \end{aligned}$$

where

$$\varphi_1(r_i) = \inf_{\eta\in\Phi^{-1}(-\infty,r_i)} \frac{\Psi(\eta) - \inf_{\eta\in\overline{\Phi^{-1}(-\infty,r_i)}^{\omega}} \Psi(\eta)}{r_i - \Phi(\eta)},$$

$$\varphi_2(r_1, r_2) = \inf_{\eta\in\Phi^{-1}(-\infty,r_1)} \sup_{\xi\in\Phi^{-1}[r_1,r_2)} \frac{\Psi(\eta) - \Psi(\xi)}{\Phi(\xi) - \Phi(\eta)},$$

for $i = 1, 2$ and $\overline{\Phi^{-1}(-\infty, r_i)}^{\omega}$ is the closure of $\Phi^{-1}(-\infty, r_i)$ in the weak topology. Then, for each

$$\lambda \in \left(\frac{1}{\varphi_2(r_1, r_2)}, \min\left\{\frac{1}{\varphi_1(r_1)}, \frac{1}{\varphi_2(r_2)}\right\}\right),$$

the functional $\Phi + \lambda\Psi$ has two local minima which lie in $\Phi^{-1}(-\infty, r_1)$ and $\Phi^{-1}[r_1, r_2)$, respectively.

Theorem 2.3. *(see [29]) Let X be a reflexive real Banach space $\Phi : X \to R$ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^* and Φ is bounded on each bounded subset of X; $\Psi : X \to R$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, $I \subseteq R$ an interval. Assume that*

$$\lim_{\|\eta\|\to\infty} (\Phi(\eta) + \lambda\Psi(\eta)) = +\infty,$$

for $\lambda \in I$, and that there exists $r \in \mathbb{R}$ and $\eta_0, \eta_1 \in X$ such that

$$\Phi(\eta_0) < r < \Phi(\eta_1),$$

$$\inf_{\eta\in\Phi^{-1}((-\infty,r])} \Psi(\eta) > \frac{(\Phi(\eta_1) - r)\Psi(\eta_0) + (r - \Phi(\eta_0))\Psi(\eta_1)}{\Phi(\eta_1) - \Phi(\eta_0)}.$$

Then there exists an open interval $\Lambda \subseteq I$ and a positive real number ρ with the following property: For every $\lambda \in \Lambda$ and every C^1-functional $J : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$ the equation

$$\Phi'(\eta) + \lambda\Psi'(\eta) + \mu J'(\eta) = 0$$

has at least three solutions in X whose norms are less than δ.

3. Some technical lemmas

In this section, in order to use critical point theory for Kirchhoff systems (1.1), we need the following crucial lemmas, which will play an important role in the proof of our results.

Lemma 3.1. *Assume that the functions $G, F \in \mathcal{A}$, then $J, \Psi \in C^1(X_0, \mathbb{R})$ and their derivatives are defined as (1.14) and (1.15) for all $(\varphi, \psi) \in X_0$. Moreover $J', \Psi' : X_0 \to \mathbb{R}$ is compact.*

Proof. Here we follow the approach in [22], for completeness, we give the proof process. Suppose that $s^+p^+ < N$, we prove that $J'(\eta, \xi)$ is continuous operator. Set $\{(\eta_n, \xi_n)\} \subset X_0$ with $(\eta_n, \xi_n) \to (\eta, \xi)$ strongly in X_0. According to Theorem 2.1, we obtain

$$(\eta_n, \xi_n) \to (\eta, \xi) \text{ in } L^{\vartheta(x)}(\Omega) \times L^{\vartheta(x)}(\Omega).$$

So, for a subsequence denoted by $\{(\eta_n, \xi_n)\}$, there exist functions $\widetilde{\eta}, \widetilde{\xi} \in L^{\vartheta(x)}(\Omega)$ such that

$$\eta_n \to \eta\ ,\ \xi_n \to \xi\ \ a.e. \text{ in } \Omega,$$

$$|\eta_n| \leq \widetilde{\eta}\ ,\ |\xi_n| \leq \widetilde{\xi}\ \ a.e. \text{ in } \Omega,$$

for all $n \in \mathbb{N}$.

Fix $(\widetilde{\eta}, \widetilde{\xi}) \in X_0$ with $\|(\widetilde{\eta}, \widetilde{\xi})\|_{X_0} \leq 1$. Since $G \in \mathcal{A}$, combining Lemma 2.1 with Theorem 2.1, we obtain

$$\begin{aligned}
&\left|\langle J'(\eta_n, \xi_n) - J'(\eta, \xi), (\widetilde{\eta}, \widetilde{\xi})\rangle\right| \\
\leq &\left|\int_\Omega \left[G_\eta(x, \eta_n, \xi_n)dx - G_\eta(x, \eta, \xi)\right]\widetilde{\eta}dx\right. \\
&+ \left.\int_\Omega \left[G_\xi(x, \eta_n, \xi_n)dx - G_\xi(x, \eta, \xi)\right]\widetilde{\xi}dx\right| \\
\leq &C_3\|G_\eta(x, \eta_n, \xi_n) - G_\eta(x, \eta, \xi)\|_{L^{\vartheta'(x)}(\Omega)}\|\widetilde{\eta}\|_{L^{\vartheta(x)}(\Omega)} \\
&+ C_3\|G_\xi(x, \eta_n, \xi_n) - G_\xi(x, \eta, \xi)\|_{L^{\vartheta'(x)}(\Omega)}\|\widetilde{\xi}\|_{L^{\vartheta(x)}(\Omega)} \\
\leq &C_4\|G_\eta(x, \eta_n, \xi_n) - G_\eta(x, \eta, \xi)\|_{L^{\vartheta'(x)}(\Omega)}\|\widetilde{\eta}\|_{W_0} \\
&+ C_4\|G_\xi(x, \eta_n, \xi_n) - G_\xi(x, \eta, \xi)\|_{L^{\vartheta'(x)}(\Omega)}\|\widetilde{\xi}\|_{W_0}. \\
\leq &C_4\Big(\|G_\eta(x, \eta_n, \xi_n) - G_\eta(x, \eta, \xi)\|_{L^{\vartheta'(x)}(\Omega)} \\
&+\|G_\xi(x, \eta_n, \xi_n) - G_\xi(x, \eta, \xi)\|_{L^{\vartheta'(x)}(\Omega)}\Big)\|(\widetilde{\eta}, \widetilde{\xi})\|_{X_0}.
\end{aligned}$$

Consequently, for $\|(\widetilde{\eta}, \widetilde{\xi})\|_{X_0} \leq 1$, we get

$$\begin{aligned}
&\left\|\langle J'(\eta_n, \xi_n) - J'(\eta, \xi), (\widetilde{\eta}, \widetilde{\xi})\rangle\right\|_{W_0^*} \\
\leq &C_4\|G_\eta(x, \eta_n, \xi_n) - G_\xi(x, \eta, \xi)\|_{L^{\vartheta'(x)}(\Omega)} \\
&+ C_4\|G_\xi(x, \eta_n, \xi_n) - G_\xi(x, \eta, \xi)\|_{L^{\vartheta'(x)}(\Omega)}.
\end{aligned}$$

According to Definition 2, we deduce

$$G_\eta(x, \eta_n, \xi_n) - G_\eta(x, \eta, \xi) \to 0 \text{ as } n \to \infty\ \ a.e.\ x \in \Omega,$$

$$G_\xi(x,\eta_n,\xi_n) - G_\xi(x,\eta,\xi) \to 0 \text{ as } n \to \infty \ \ a.e.\ x \in \Omega$$

and

$$|G_\eta(x,\eta_n,\xi_n)| \leq C_5(1 + |\overline{\eta}|^{\vartheta(x)-1} + |\overline{\xi}|^{\vartheta(x)-1}),$$
$$|G_\xi(x,\eta_n,\xi_n)| \leq C_5(1 + |\overline{\eta}|^{\vartheta(x)-1} + |\overline{\xi}|^{\vartheta(x)-1}),$$

using the dominate convergence theorem, we have

$$||G_\eta(x,\eta_n,\xi_n) - G_\eta(x,\eta,\xi))||_{L^{\vartheta'(x)}(\Omega)} \to 0 \text{ as } n \to \infty \ \ a.e.\ x \in \Omega,$$
$$||G_\xi(x,\eta_n,\xi_n) - G_\xi(x,\eta,\xi))||_{L^{\vartheta'(x)}(\Omega)} \to 0 \text{ as } n \to \infty \ \ a.e.\ x \in \Omega.$$

So, this prove that $J'(\eta,\xi)$ is continuous operator.

We show the operator $J'(\eta,\xi)$ is compact. Let $\{(\eta_n,\xi_n)\}$ be a bounded subsequence in X_0. Arguing in the same way as above, we obtain that the sequence $\{J'(\eta_n,\xi_n)\}$ converges strongly, therefor the operator $J'(\eta,\xi)$ is compact.

Similarly, we also deduce that $\Psi'(\eta,\xi)$ is continuous and compact. □

Lemma 3.2. *Assume that (M) is satisfied. Then*

(i) Φ *is sequentially weakly lower semicontinuous and bounded on each bounded subset,*
(ii) $\Phi' : X_0 \to X_0^*$ *is a strictly monotone and continuous operator,*
(iii) $\Phi' : X_0 \to X_0^*$ *is a homeomorphism.*

Proof. (i) Since $\widetilde{M}_i'(t) > m_i > 0$, $\widetilde{M}_i(t)$ are increasing function on $[0,+\infty)$. Argue in a similar way from [38, Lemma 2.4], the operators $\eta \mapsto \delta'_{p(\cdot)}(\eta)$ and $\xi \mapsto \delta'_{p(\cdot)}(\xi)$ are strictly monotone. From [39, Proposition 25.10], $\delta_{p(\cdot)}(\eta)$ and $\delta_{p(\cdot)}(\xi)$ are strictly convex. Set $\{(\eta_n,\xi_n)\} \subset X_0$ be a subsequence such that

$$\eta_n \to \eta\,, \xi_n \to \xi \text{ in } W_0.$$

Based on the convexity of $\delta_{p(\cdot)}(\eta)$ and $\delta_{p(\cdot)}(\xi)$, we obtain

$$\delta_{p(\cdot)}(\eta_n) - \delta_{p(\cdot)}(\eta) \geq \langle \delta'_{p(\cdot)}(\eta), \eta_n - \eta \rangle \tag{3.1}$$

and

$$\delta_{p(\cdot)}(\xi_n) - \delta_{p(\cdot)}(\xi) \geq \langle \delta'_{p(\cdot)}(\xi), \xi_n - \xi \rangle. \tag{3.2}$$

Thus, we have

$$\delta_{p(\cdot)}(\eta) \leq \liminf_{n\to\infty} \delta_{p(\cdot)}(\eta_n) \tag{3.3}$$

and

$$\delta_{p(\cdot)}(\xi) \leq \liminf_{n\to\infty} \delta_{p(\cdot)}(\xi_n), \tag{3.4}$$

namely, the operators $\eta \mapsto \delta_{p(\cdot)}(\eta)$ and $\xi \mapsto \delta_{p(\cdot)}(\xi)$ are sequentially weakly lower semicontinuous.

On the other side, since $t \mapsto \widetilde{M}_i(t)$ are continuous and monotonous functions, we obtain

$$\begin{aligned}\liminf_{n\to\infty}\Phi(\eta_n,\xi_n) =&\liminf_{n\to\infty}\widetilde{M}_1(\delta_{p(\cdot)}(\eta_n))+\liminf_{n\to\infty}\widetilde{M}_2(\delta_{p(\cdot)}(\xi_n))\\ \geq&\widetilde{M}_1\left(\liminf_{n\to\infty}\delta_{p(\cdot)}(\eta_n)\right)+\widetilde{M}_2\left(\liminf_{n\to\infty}\delta_{p(\cdot)}(\xi_n)\right)\\ \geq&\widetilde{M}_1\left(\delta_{p(\cdot)}(\eta)\right)+\widetilde{M}_2\left(\delta_{p(\cdot)}(\xi)\right)=\Phi(\eta,\xi).\end{aligned}\tag{3.5}$$

Consequently, the operator Φ is sequentially weakly lower semicontinuous.

Right now, we claim that Φ is bounded on each bounded subset of X_0. Set $\{(\eta_n,\xi_n)\}\subset X_0$ be a bounded subsequence. By Proposition 2.2, there exist constants $C_6, C_7>0$ so that

$$\delta_{p(\cdot)}(\eta)\leq C_6 \text{ and } \delta_{p(\cdot)}(\xi)\leq C_7.$$

Since $\widetilde{M}_1$ and $\widetilde{M}_2$ is monotone, we have

$$\begin{aligned}\Phi(\eta_n,\xi_n)=&\widetilde{M}_1(\delta_{p(\cdot)}(\eta_n))+\widetilde{M}_2(\delta_{p(\cdot)}(\xi_n))\\ \leq&\widetilde{M}_1(C_6)+\widetilde{M}_2(C_7).\end{aligned}\tag{3.6}$$

Hence, the operator Φ is bounded.

(ii) Let (M) is satisfied, then $\Phi\in C^1(X_0,\mathbb{R})$, and its derivatives are defined by (1.16), we have

$$\langle\Phi'(\eta,\xi),(\varphi,\psi)\rangle=\langle\Phi_\eta(\eta),\varphi\rangle+\langle\Phi_\xi(\xi),\psi\rangle,\tag{3.7}$$

where

$$\begin{aligned}\langle\Phi_\eta(\eta),\varphi\rangle=&M_1\left(\delta_{p(\cdot)}(\eta)\right)\langle\eta,\varphi\rangle\\ \text{and }\langle\Phi_\xi(\xi),\psi\rangle=&M_2\left(\delta_{p(\cdot)}(\xi)\right)\langle\xi,\psi\rangle,\end{aligned}\tag{3.8}$$

for all $(\eta,\xi),(\varphi,\psi)\in X_0$. Therefore, $\Phi'(\eta,\xi)\in X_0^*$, where X_0^* denotes the dual spaces of X_0.

In the first place, we show that $\Phi': X_0\to X_0^*$ is a strictly monotone operator. Since $\Phi_\eta(\eta)$ and $\Phi_\xi(\xi)$ are strictly monotone operators (see [40],Theorem 2.1), Φ' is a strictly monotone operator.

Next, we claim that $\Phi': X_0\to X_0^*$ is a continuous operator. Set $\{(\eta_n,\xi_n)\}\subset X_0$ be a sequence, which converges strongly to (η,ξ) in X_0. So, for a subsequence denoted by $\{(\eta_n,\xi_n)\}$, we suppose

$$\eta_n\to\eta\,,\xi_n\to\xi\ \ a.e. \text{ in }\Omega.$$

Then, the sequence

$$\left\{\frac{|\eta_n(x)-\eta_n(y)|^{p(x,y)-2}(\eta_n(x)-\eta_n(y))}{|x-y|^{(N+p(x,y)s(x,y))/p'(x,y)}}\right\}\tag{3.9}$$

and

$$\left\{\frac{|\xi_n(x)-\xi_n(y)|^{p(x,y)-2}(\xi_n(x)-\xi_n(y))}{|x-y|^{(N+p(x,y)s(x,y))/p'(x,y)}}\right\}\tag{3.10}$$

are bounded in $L^{p'}(\mathbb{R}^N\times\mathbb{R}^N)$, and we have

$$\frac{|\eta_n(x)-\eta_n(y)|^{p(x,y)-2}(\eta_n(x)-\eta_n(y))}{|x-y|^{(N+p(x,y)s(x,y))/p'(x,y)}} \to \frac{|\eta(x)-\eta(y)|^{p(x,y)-2}(\eta(x)-\eta(y))}{|x-y|^{(N+p(x,y)s(x,y))/p'(x,y)}} \tag{3.11}$$

and

$$\frac{|\xi_n(x)-\xi_n(y)|^{p(x,y)-2}(\xi_n(x)-\xi_n(y))}{|x-y|^{(N+p(x,y)s(x,y))/p'(x,y)}} \to \frac{|\xi(x)-\xi(y)|^{p(x,y)-2}(\xi(x)-\xi(y))}{|x-y|^{(N+p(x,y)s(x,y))/p'(x,y)}}. \tag{3.12}$$

According to the Brezis-Lieb lemma [41], combining (3.11) with (3.12) implies

$$\begin{aligned}&\int_{\mathbb{R}^N\times\mathbb{R}^N}\left|\frac{|\eta_n(x)-\eta_n(y)|^{p(x,y)-2}(\eta_n(x)-\eta_n(y))}{|x-y|^{(N+p(x,y)s(x,y))/p'(x,y)}} - \frac{|\eta(x)-\eta(y)|^{p(x,y)-2}(\eta(x)-\eta(y))}{|x-y|^{(N+p(x,y)s(x,y))/p'(x,y)}}\right|^{p'(x,y)} dxdy\\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N\times\mathbb{R}^N}\left(\frac{|\eta_n(x)-\eta_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}} - \frac{|\eta(x)-\eta(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}\right)dxdy\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N\times\mathbb{R}^N}\left|\frac{|\xi_n(x)-\xi_n(y)|^{p(x,y)-2}(\xi_n(x)-\xi_n(y))}{|x-y|^{(N+p(x,y)s(x,y))/p'(x,y)}} - \frac{|\xi(x)-\xi(y)|^{p(x,y)-2}(\xi(x)-\xi(y))}{|x-y|^{(N+p(x,y)s(x,y))/p'(x,y)}}\right|^{p'(x,y)} dxdy\\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N\times\mathbb{R}^N}\left(\frac{|\xi_n(x)-\xi_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}} - \frac{|\xi(x)-\xi(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}\right)dxdy.\end{aligned} \tag{3.14}$$

Based on the fact that $\eta_n \to \eta$, $\xi_n \to \xi$ strongly in X_0 yields

$$\lim_{n\to\infty}\int_{\mathbb{R}^N\times\mathbb{R}^N}\left(\frac{|\eta_n(x)-\eta_n(y)|^{p(x,y)-2}(\eta_n(x)-\eta_n(y))}{|x-y|^{(N+p(x,y)s(x,y))/p'(x,y)}} - \frac{|\eta(x)-\eta(y)|^{p(x,y)-2}(\eta(x)-\eta(y))}{|x-y|^{(N+p(x,y)s(x,y))/p'(x,y)}}\right)dxdy = 0 \tag{3.15}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N\times\mathbb{R}^N}\left(\frac{|\xi_n(x)-\xi_n(y)|^{p(x,y)-2}(\xi_n(x)-\xi_n(y))}{|x-y|^{(N+p(x,y)s(x,y))/p'(x,y)}} - \frac{|\xi(x)-\xi(y)|^{p(x,y)-2}(\xi(x)-\xi(y))}{|x-y|^{(N+p(x,y)s(x,y))/p'(x,y)}}\right)dxdy = 0. \tag{3.16}$$

Furthermore, $M_1(t)$ and $M_2(t)$ are continuous functions, which imply

$$M_1\left(\delta_{p(\cdot)}(\eta_n)\right) \to M_1\left(\delta_{p(\cdot)}(\eta)\right) \tag{3.17}$$

and

$$M_2\left(\delta_{p(\cdot)}(\xi_n)\right) \to M_2\left(\delta_{p(\cdot)}(\xi)\right). \tag{3.18}$$

With the help of Lemma 2.1, we get

$$\begin{aligned}\left|\int_\Omega |\eta_n|^{\overline{p}(x)-2}\eta_n(\eta_n-\eta)dx\right| &\leq \int_\Omega |\eta_n|^{\overline{p}(x)-1}|\eta_n-\eta|dx\\ &\leq \||\eta_n|^{\overline{p}(x)-1}|_{\frac{\overline{p}(x)}{\overline{p}(x)-1}}|\eta_n-\eta|_{\overline{p}(x)}\\ &\to 0,\end{aligned}$$

as $n \to \infty$, and then

$$\lim_{n\to\infty}\int_\Omega |\eta_n|^{\overline{p}(x)-2}\eta_n(\eta_n-\eta)dx = 0. \tag{3.19}$$

Arguing in the similar way as above, we have

$$\lim_{n\to\infty}\int_\Omega |\xi_n|^{\overline{p}(x)-2}\xi_n(\xi_n-\xi)dx = 0. \tag{3.20}$$

Thus, from (3.15)–(3.20), we obtain

$$\begin{aligned}&\|\Phi'(\eta_n,\xi_n)-\Phi'(\eta,\xi)\|_{X_0^*}\\ =&\sup_{(\varphi,\psi)\in X_0,\|(\varphi,\psi)\|_{X_0}\leq 1}|\langle\Phi'(\eta_n,\xi_n),(\varphi,\psi)\rangle-\langle\Phi'(\eta,\xi),(\varphi,\psi)\rangle|\\ \to&0.\end{aligned} \tag{3.21}$$

(iii) Since the operator Φ' is a strictly monotone , it follows that Φ' is injective. Set $(\eta,\xi)\in X_0$ be such that $\|(\eta,\xi)\|_{X_0} > 1$, from (M) and Proposition 2.2, we have

$$\begin{aligned}&\frac{\langle\Phi'(\eta,\xi),(\eta,\xi)\rangle}{\|(\eta,\xi)\|_{X_0}}\\ =&\frac{M_1\left(\delta_{p(\cdot)}(\eta)\right)\langle\eta,\eta\rangle+M_2\left(\delta_{p(\cdot)}(\xi)\right)\langle\xi,\xi\rangle}{\|(\eta,\xi)\|_{X_0}}\\ \geq&\frac{m_1\rho_{p(\cdot)}^{s(\cdot)}(\eta)+m_2\rho_{p(\cdot)}^{s(\cdot)}(\xi)}{\|(\eta,\xi)\|_{X_0}}\\ \geq&\min(m_1,m_2)\frac{\min\left(\|\eta\|_{W_0}^{p^+},\|\eta\|_{W_0}^{p^-}\right)+\min\left(\|\xi\|_{W_0}^{p^+},\|\xi\|_{W_0}^{p^-}\right)}{\|\eta\|_{W_0}+\|\xi\|_{W_0}}.\end{aligned} \tag{3.22}$$

Thus, (3.22) implies that

$$\lim_{\|(\eta,\xi)\|\to\infty}\frac{\langle\Phi'(\eta,\xi),(\eta,\xi)\rangle}{\|(\eta,\xi)\|_{X_0}} = +\infty. \tag{3.23}$$

Consequently, Φ' is coercive operator, thanks to the Minty-Browder Theorem (see [39],Theorem 26A), Φ' is a surjection. Due to its monotonicity, Φ' is an injection. So, $(\Phi')^{-1}$ exists.

Let us first prove that Φ' satisfies property:

$$(Q):\text{ if } (\eta_n,\xi_n)\rightharpoonup(\eta,\xi) \text{ and } \Phi'(\eta_n,\xi_n)\to\Phi'(\eta,\xi), \text{ then } (\eta_n,\xi_n)\to(\eta,\xi).$$

Indeed, set $(\eta_n,\xi_n)\rightharpoonup(\eta,\xi)$ in X_0 , $\Phi'(\eta_n,\xi_n)\to\Phi'(\eta,\xi)$ in X_0^* and Theorem 2.1, which implies

$$\eta_n\to\eta,\ \xi_n\to\xi\ \ a.e.\ x\in\Omega.$$

Based on Fatou's Lemma yields

$$\begin{aligned}&\liminf_{n\to\infty}\left(\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\eta_n(x)-\eta_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega|\eta_n|^{\overline{p}(x)}dx\right)\\ \geq&\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\eta(x)-\eta(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega|\eta|^{\overline{p}(x)}dx\end{aligned}\tag{3.24}$$

and

$$\begin{aligned}&\liminf_{n\to\infty}\left(\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi_n(x)-\xi_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega|\xi_n|^{\overline{p}(x)}dx\right)\\ \geq&\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi(x)-\xi(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega|\xi|^{\overline{p}(x)}dx.\end{aligned}\tag{3.25}$$

Applying Young's inequality, there exist $C_8, C_9>0$ such that

$$\begin{aligned}o_n=&\langle\Phi_{\eta_n},\eta_n-\eta\rangle+\langle\Phi_{\xi_n},\xi_n-\xi\rangle\\ =&M_1\left(\delta_{p(\cdot)}(\eta_n)\right)\left(\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\eta_n(x)-\eta_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega|\eta_n|^{\overline{p}(x)}dx\right.\\ &\left.-\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\eta_n(x)-\eta_n(y)|^{p(x,y)-2}(\eta_n(x)-\eta_n(y))(\eta(x)-\eta(y))}{|x-y|^{N+p(x,y)s(x,y)}}dxdy-\int_\Omega|\eta_n|^{\overline{p}(x)-2}\eta_n\eta dx\right)\\ &+M_2\left(\delta_{p(\cdot)}(\xi_n)\right)\left(\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi_n(x)-\xi_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega|\xi_n|^{\overline{p}(x)}dx\right.\\ &\left.-\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi_n(x)-\xi_n(y)|^{p(x,y)-2}(\xi_n(x)-\xi_n(y))(\xi(x)-\xi(y))}{|x-y|^{N+p(x,y)s(x,y)}}dxdy-\int_\Omega|\xi_n|^{\overline{p}(x)-2}\xi_n\xi dx\right)\\ \geq&m_1\left(\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\eta_n(x)-\eta_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega|\eta_n|^{\overline{p}(x)}dx\right.\\ &\left.-\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\eta_n(x)-\eta_n(y)|^{p(x,y)-1}(\eta(x)-\eta(y))}{|x-y|^{N+p(x,y)s(x,y)}}dxdy-\int_\Omega|\eta_n|^{\overline{p}(x)-1}\eta dx\right)\\ &+m_2\left(\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi_n(x)-\xi_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega|\xi_n|^{\overline{p}(x)}dx\right.\\ &\left.-\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi_n(x)-\xi_n(y)|^{p(x,y)-1}(\xi(x)-\xi(y))}{|x-y|^{N+p(x,y)s(x,y)}}dxdy-\int_\Omega|\xi_n|^{\overline{p}(x)-1}\xi dx\right)\end{aligned}$$

$$
\begin{aligned}
\geq & m_1\Bigg(C_8\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\eta_n(x)-\eta_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+C_8\int_\Omega|\eta_n|^{\overline{p}(x)}dx\\
&-C_9\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\eta(x)-\eta(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy-C_9\int_\Omega|\eta|^{\overline{p}(x)}dx\Bigg)\\
&+m_2\Bigg(C_8\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi_n(x)-\xi_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+C_8\int_\Omega|\xi_n|^{\overline{p}(x)}dx\\
&-C_9\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi(x)-\xi(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy-C_9\int_\Omega|\xi|^{\overline{p}(x)}dx\Bigg). \qquad (3.26)
\end{aligned}
$$

The passage to the liminf implies in the above inequality, we get

$$
\begin{aligned}
0\geq & m_1\Bigg[\liminf_{n\to\infty}\Bigg(\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\eta_n(x)-\eta_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega|\eta_n|^{\overline{p}(x)}dx\Bigg)\\
&-\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\eta(x)-\eta(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy-\int_\Omega|\eta|^{\overline{p}(x)}dx\Bigg]\\
&+m_2\Bigg[\liminf_{n\to\infty}\Bigg(\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi_n(x)-\xi_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega|\xi_n|^{\overline{p}(x)}dx\Bigg)\\
&-\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi(x)-\xi(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy-\int_\Omega|\xi|^{\overline{p}(x)}dx\Bigg]. \qquad (3.27)
\end{aligned}
$$

This and combining (3.24) with (3.25), we obtain

$$
\begin{aligned}
&\liminf_{n\to\infty}\Bigg(\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\eta_n(x)-\eta_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega|\eta_n|^{\overline{p}(x)}dx\Bigg)\\
=&\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\eta(x)-\eta(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega|\eta|^{\overline{p}(x)}dx \qquad (3.28)
\end{aligned}
$$

and

$$
\begin{aligned}
&\liminf_{n\to\infty}\Bigg(\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi_n(x)-\xi_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega|\xi_n|^{\overline{p}(x)}dx\Bigg)\\
=&\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi(x)-\xi(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega|\xi|^{\overline{p}(x)}dx. \qquad (3.29)
\end{aligned}
$$

Then, for a subsequence, we have

$$
\begin{aligned}
&\lim_{n\to\infty}\Bigg(\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\eta_n(x)-\eta_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega|\eta_n|^{\overline{p}(x)}dx\Bigg)\\
=&\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\eta(x)-\eta(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega|\eta|^{\overline{p}(x)}dx \qquad (3.30)
\end{aligned}
$$

and

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi_n(x)-\xi_n(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega|\xi_n|^{\overline{p}(x)}dx\right)$$
$$=\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|\xi(x)-\xi(y)|^{p(x,y)}}{|x-y|^{N+p(x,y)s(x,y)}}dxdy+\int_\Omega|\xi|^{\overline{p}(x)}dx, \tag{3.31}$$

that is

$$\lim_{n\to\infty}\rho_{p(\cdot)}^{s(\cdot)}(\eta_n)=\rho_{p(\cdot)}^{s(\cdot)}(\eta),\ \lim_{n\to\infty}\rho_{p(\cdot)}^{s(\cdot)}(\xi_n)=\rho_{p(\cdot)}^{s(\cdot)}(\xi). \tag{3.32}$$

Since $\eta_n\to\eta$, $\xi_n\to\xi$ in X_0 , Proposition 2.2 imply that

$$\lim_{n\to\infty}\rho_{p(\cdot)}^{s(\cdot)}(\eta_n-\eta)=0,\ \lim_{n\to\infty}\rho_{p(\cdot)}^{s(\cdot)}(\xi_n-\xi)=0, \tag{3.33}$$

that is

$$\lim_{n\to\infty}\|\eta_n-\eta\|_{W_0}=0,\ \lim_{n\to\infty}\|\xi_n-\xi\|_{W_0}=0. \tag{3.34}$$

Consequently

$$\|(\eta_n,\xi_n)-(\eta,\xi)\|=\|\eta_n-\eta\|_{W_0}+\|\xi_n-\xi\|_{W_0}\to 0. \tag{3.35}$$

Next, we prove that $(\Phi')^{-1}$ is continuous operator. Let $\{(f_n,g_n)\}\subset X_0^*$ such that $f_n\to f_0$ in X_0^* and $g_n\to g_0$ in X_0^*. Set $(\eta_n,\xi_n),(\eta,\xi)\in X_0$ such that

$$\Phi_\eta^{-1}(f_n)=\eta_n,\ \Phi_\eta^{-1}(f_0)=\eta,\ \Phi_\xi^{-1}(g_n)=\xi_n,\ \Phi_\xi^{-1}(g_0)=\xi. \tag{3.36}$$

According to coercivity of Φ', we conclude that (η_n,ξ_n) is bounded. Then, up to subsequence $(\eta_n,\xi_n)\rightharpoonup(\widetilde{\eta_0},\widetilde{\xi_0})$, which implies

$$\begin{aligned}&\lim_{n\to\infty}\langle\Phi'(\eta_n,\xi_n)-\Phi'(\eta,\xi),(\eta_n,\xi_n)-(\widetilde{\eta_0},\widetilde{\xi_0})\rangle\\&=\lim_{n\to\infty}\langle(f_n,g_n)-(f_0,g_0),(\eta_n,\xi_n)-(\widetilde{\eta_0},\widetilde{\xi_0})\rangle\\&=0.\end{aligned} \tag{3.37}$$

Combining the property of (Q) and the continuity of Φ', we obtain

$$(\eta_n,\xi_n)\to(\widetilde{\eta_0},\widetilde{\xi_0})\ \text{ in }\ X_0,\ \Phi'(\eta_n,\xi_n)\to\Phi'(\widetilde{\eta_0},\widetilde{\xi_0})=\Phi'(\eta,\xi)\ \text{ in }\ X_0^*. \tag{3.38}$$

Since Φ' is an injection, we deduce that $(\eta,\xi)=(\widetilde{\eta_0},\widetilde{\xi_0})$. □

4. Proof of the main results

In this subsection, we firstly prove Theorem 1.1 by applying Theorem 2.2.

Proof of Theorem 1.1. Let $X=X_0$, Ψ and Φ are given as (1.11) and (1.12), respectively. Note that Ψ' is a compact derivative from Lemma 3.1, Lemma 3.2 ensures that Φ is a weakly lower semicontinuous and bounded operator in X_0, and Φ' admits a continuous inverse operator $\Phi':X_0^*\to X_0$.

Let $(\eta_*, \xi_*) = (0,0)$, then $\Phi(0,0) = \Psi(0,0) = 0$. There exist a point $x_0 \in \Omega$ and pick two positive constants $\Re_2, \Re_1 (\Re_2 > \Re_1)$ such that $B(x_0, \Re_2) \subset \Omega$. Set a, c be positive constants and define the function $\omega(x)$ by

$$\omega(x) = \begin{cases} 0, & x \in \overline{\Omega} \setminus B(x_0, \Re_2), \\ \frac{a}{\Re_2 - \Re_1} \left\{ \Re_2 - \left[\Sigma_{i=1}^N (x^i - x_0^i) \right]^{1/2} \right\}, & x \in B(x_0, \Re_2) \setminus B(x_0, \Re_1), \\ a, & x \in B(x_0, \Re_1), \end{cases} \tag{4.1}$$

where $B(x_0, \Re)$ stands for the open ball in $\mathbb{R}^N$ of radius $\Re$ centered at x_0, then $(\omega(x), \omega(x)) \in X_0$. Denote

$$\begin{aligned} A(c) &= \int_\Omega \sup_{(s,t) \in R \times R : |s|^{p(\cdot)} + |t|^{p(\cdot)} \le c} F(x, s, t), \\ K(a) &= \int_{B(x_0, \Re_2) \setminus B(x_0, \Re_1)} F(x, \omega(x), \omega(x)) + \int_{B(x_0, \Re_1)} F(x, a, a), \\ H(c, a) &= K(a) - A(c),\ m_- = \min\left\{ \frac{m_1}{p^+}, \frac{m_2}{p^+} \right\},\ M^+ = \max\left\{ \frac{M_1}{p^-}, \frac{M_2}{p^-} \right\}. \end{aligned} \tag{4.2}$$

Under condition (M) and by a simple computation, we obtain

$$\begin{aligned} & m_- \left(\min\{ \|\eta(x)\|_{W_0}^{p^+} + \|\xi(x)\|_{W_0}^{p^+}, \|\eta(x)\|_{W_0}^{p^-} + \|\xi(x)\|_{W_0}^{p^-} \} \right) \\ \le & \Phi(\eta, \xi) \\ \le & M^+ \left(\max\{ \|\eta(x)\|_{W_0}^{p^+} + \|\xi(x)\|_{W_0}^{p^+}, \|\eta(x)\|_{W_0}^{p^-} + \|\xi(x)\|_{W_0}^{p^-} \} \right). \end{aligned} \tag{4.3}$$

When $\|\eta\|_{W_0} \to \infty, \|\xi\|_{W_0}$ bounded ($\|\xi\|_{W_0} \to \infty, \|\eta\|_{W_0}$ bounded) and $\|\eta\|_{W_0} \to \infty, \|\xi\|_{W_0} \to \infty$, we have

$$\|(\eta, \xi)\|_{X_0} \to \infty,$$

this implies that

$$\lim_{\|(\eta,\xi)\|_{X_0} \to \infty} \Phi(\eta, \xi) = +\infty. \tag{4.4}$$

Fix γ such that $0 < \gamma \le 1 < c_1 < c_2$ and set

$$r_1 = \frac{m_- c_1 \gamma}{C_\phi}, \quad r_2 = \frac{M^+ c_2 \gamma}{C_\phi}. \tag{4.5}$$

By virtue of (4.1), set $\eta_0 = \xi_0 = \omega$ for $(\eta_0, \xi_0) \in X_0$ and

$$\|\eta_0\|_{W_0} = \|\omega\|_{W_0}, \quad \|\xi_0\|_{W_0} = \|\omega\|_{W_0}. \tag{4.6}$$

Consequently, we have

$$r_1 < \Phi(\eta_0, \xi_0) < r_2. \tag{4.7}$$

Thus, (4.7) implies that

$$\begin{aligned}\varphi_2(r_1, r_2) &= \inf_{(\eta,\xi)\in\Phi^{-1}(-\infty,r_1)} \sup_{(\eta_1,\xi_1)\in\Phi^{-1}[r_1,r_2)} \frac{\Psi(\eta,\xi) - \Psi(\eta_1,\xi_1)}{\Phi(\eta_1,\xi_1) - \Phi(\eta,\xi)} \\ &\geq \inf_{(\eta,\xi)\in\Phi^{-1}(-\infty,r_1)} \frac{\Psi(\eta,\xi) - \Psi(\eta_0,\xi_0)}{\Phi(\eta_0,\xi_0) - \Phi(\eta,\xi)}.\end{aligned} \tag{4.8}$$

According to (F2), (1.20) and (4.1), we obtain

$$\begin{aligned}\int_\Omega F(x,\eta_0,\xi_0)dx = K(a) &> H(c_1,a) \\ &> \frac{M^+}{m_-}A(c_1) > A(c_1) \\ &= \int_\Omega \sup_{(s,t)\in R\times R:|s|^{p(\cdot)}+|t|^{p(\cdot)}\leq c_1} F(x,s,t)dx.\end{aligned} \tag{4.9}$$

For each $(\eta,\xi) \in X_0$ with $\Phi(\eta,\xi) \leq r_1$, we conclude

$$\begin{aligned}&|\eta(x)|^{p(\cdot)} + |\xi(x)|^{p(\cdot)} \\ \leq &C_\phi \left(\min\{\|\eta(x)\|_{W_0}^{p^+} + \|\xi(x)\|_{W_0}^{p^+}, \|\eta(x)\|_{W_0}^{p^-} + \|\xi(x)\|_{W_0}^{p^-}\}\right) \\ \leq &\frac{C_\phi r_1}{m_-} = c_1\gamma \\ \leq &c_1, \text{ for all } x \in \Omega.\end{aligned} \tag{4.10}$$

Fix γ_0 such that $0 < \max\{\|\eta_0\|_{W_0}^{p^+} + \|\xi_0\|_{W_0}^{p^+}, \|\eta_0\|_{W_0}^{p^-} + \|\xi_0\|_{W_0}^{p^-}\} < \gamma_0 < \gamma$. Thus, combining with (4.9) and (4.10), we have

$$\begin{aligned}&\frac{\Psi(\eta,\xi) - \Psi(\eta_0,\xi_0)}{\Phi(\eta_0,\xi_0) - \Phi(\eta,\xi)} \\ = &\frac{\int_\Omega F(x,\eta_0,\xi_0)dx - \int_\Omega F(x,\eta,\xi)dx}{\Phi(\eta_0,\xi_0) - \Phi(\eta,\xi)} \\ \geq &\frac{\int_\Omega F(x,\eta_0,\xi_0)dx - \int_\Omega \sup_{(\eta,\xi)\in R\times R:|\eta|^{p(\cdot)}+|\xi|^{p(\cdot)}\leq c_1} F(x,\eta,\xi)dx}{\Phi(\eta_0,\xi_0) - \Phi(\eta,\xi)} \\ \geq &\frac{\int_\Omega F(x,\eta_0,\xi_0)dx - \int_\Omega \sup_{(\eta,\xi)\in R\times R:|\eta|^{p(\cdot)}+|\xi|^{p(\cdot)}\leq c_1} F(x,\eta,\xi)dx}{\Phi(\eta_0,\xi_0)} \\ \geq &\frac{\int_\Omega F(x,\eta_0,\xi_0)dx - \int_\Omega \sup_{(\eta,\xi)\in R\times R:|\eta|^{p(\cdot)}+|\xi|^{p(\cdot)}\leq c_1} F(x,\eta,\xi)dx}{M^+(\|\eta_0\|_{W_0}^{p(\cdot)} + \|\xi_0\|_{W_0}^{p(\cdot)})} \\ > &\frac{C_\phi}{M^+\gamma}H(c_1,a).\end{aligned} \tag{4.11}$$

From (4.8) and (4.11), we deduce

$$\varphi_2(r_1,r_2) > \frac{C_\phi}{M^+\gamma}H(c_1,a). \tag{4.12}$$

Similarly, for each $(\eta,\xi) \in X_0$ such that $\Phi(\eta,\xi) \le r$, we get

$$\begin{aligned} & |\eta(x)|^{p(\cdot)} + |\xi(x)|^{p(\cdot)} \\ \le & C_\phi \left(\min\{\|\eta(x)\|_{W_0}^{p^+} + \|\xi(x)\|_{W_0}^{p^+}, \|\eta(x)\|_{W_0}^{p^-} + \|\xi(x)\|_{W_0}^{p^-}\}\right) \\ \le & \frac{C_\phi r}{m_-}, \text{ for all } x \in \Omega. \end{aligned} \tag{4.13}$$

According to Φ being sequentially weakly lower semicontinuous, then $\overline{\Phi^{-1}(-\infty,r)^\omega} = \Phi^{-1}(-\infty,r]$. Consequently, we have

$$\begin{aligned} & \inf_{(\eta,\xi)\in\Phi^{-1}(-\infty,r)} \frac{\Psi(\eta,\xi) - \inf_{(\eta,\xi)\in\overline{\Phi^{-1}(-\infty,r)^\omega}} \Psi(\eta,\xi)}{r - \Phi(\eta,\xi)} \\ \le & \frac{\Psi(0,0) - \inf_{(\eta,\xi)\in\overline{\Phi^{-1}(-\infty,r)^\omega}} \Psi(\eta,\xi)}{r - \Phi(0,0)} \\ \le & \frac{-\inf_{(\eta,\xi)\in\overline{\Phi^{-1}(-\infty,r)^\omega}} \Psi(\eta,\xi)}{r} \\ \le & \frac{\int_\Omega \sup_{(\eta,\xi)\in R\times R: |\eta|^{p(\cdot)}+|\xi|^{p(\cdot)} \le \frac{C_\phi r}{m_-}} F(x,\eta,\xi)dx}{r}, \end{aligned} \tag{4.14}$$

this implies

$$\varphi_i(r_i) \le \frac{A(c_i)}{r_i} = \frac{C_\phi}{m_- c_i \gamma} A(c_i) < \frac{C_\phi}{M^+\gamma} H(c_1,a). \tag{4.15}$$

Thus, by virtue of (4.12) and (4.15), we obtain

$$\varphi_i(r_i) < \frac{C_\phi}{M^+\gamma} H(c_1,a) < \varphi_2(r_1,r_2). \tag{4.16}$$

Therefore, the conditions of Theorem 2.2 are satisfied. Thus, the functional $\Phi + \lambda\Psi$ has two local minima $(\eta_i,\xi_i) \in X_0$, which lie in $\Phi^{-1}(-\infty,r_i)$, respectively. Since $I = \Phi + \lambda\Psi \in C^1(X_0,\mathbb{R})$, $(\eta_i,\xi_i) \in X_0$ are the two solutions of the following equation

$$\Phi'(\eta_i,\xi_i) + \lambda\Psi'(\eta_i,\xi_i) = 0. \tag{4.17}$$

That is, $(\eta_i,\xi_i) \in X_0$ are two weak solutions of the nonlocal fractional Kirchhoff-type elliptic systems (1.8). Since $\Phi(\eta_i,\xi_i) < r_i$, combining Theorem 2.1, we obtain

$$\begin{aligned} & |\eta_i(x)|^{p(\cdot)} + |\xi_i(x)|^{p(\cdot)} \\ \le & C_\phi \left(\min\{\|\eta_i(x)\|_{W_0}^{p^+} + \|\xi_i(x)\|_{W_0}^{p^+}, \|\eta_i(x)\|_{W_0}^{p^-} + \|\xi_i(x)\|_{W_0}^{p^-}\}\right) \\ \le & \frac{C_\phi r_2}{m_-} \le c_2, \text{ for all } x \in \Omega, \end{aligned} \tag{4.18}$$

which implies that there is a real number $\rho > 0$ so that $w_j = (\eta_j,\xi_j) \in X_0$ and $\|w_j\| < \rho$. □

In what follows, we prove Theorem 1.2 Theorem by using Theorem 2.3.

Proof of Theorem 1.2. Let $X = X_0$, Ψ and Φ are given by (1.11) and (1.12), respectively. Note that Ψ' is a compact derivative from Lemma 3.1, Lemma 3.2 ensures that Φ is a weakly lower semicontinuous and bounded operator in X_0, and Φ' admits a continuous inverse operator $\Phi' : X_0^* \to X_0$. Moreover,

$$\lim_{\|(\eta,\xi)\|_{X_0}\to\infty} \Phi(\eta,\xi) + \lambda\Psi(\eta,\xi) = +\infty \tag{4.19}$$

for any $\lambda \in (0, +\infty)$. Indeed,

$$\begin{aligned}
\Phi(\eta,\xi) = &\widetilde{M}_1\left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\eta(x)-\eta(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+p(x,y)s(x,y)}}dxdy + \int_\Omega \frac{|\eta|^{\overline{p}(x)}}{\overline{p}(x)}dx\right)\\
&+ \widetilde{M}_2\left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\xi(x)-\xi(y)|^{p(x,y)}}{p(x,y)|x-y|^{N+p(x,y)s(x,y)}}dxdy + \int_\Omega \frac{|\xi|^{\overline{p}(x)}}{\overline{p}(x)}dx\right)\\
= &\widetilde{M}_1\left(\delta_{p(\cdot)}(\eta)\right) + \widetilde{M}_2\left(\delta_{p(\cdot)}(\xi)\right)\\
\geq &\frac{m_1}{p^+}\rho_{p(\cdot)}^{s(\cdot)}(\eta) + \frac{m_2}{p^+}\rho_{p(\cdot)}^{s(\cdot)}(\xi)\\
\geq &\frac{m_1}{p^+}\min\left(\|\eta\|_{W_0}^{p^-}, \|\eta\|_{W_0}^{p^+}\right) + \frac{m_2}{p^+}\min\left(\|\xi\|_{W_0}^{p^-}, \|\xi\|_{W_0}^{p^+}\right).
\end{aligned} \tag{4.20}$$

By virtue of (F1), we obtain

$$|F(x,\eta,\xi)| \leq C_{10}|\eta|^{\alpha(x)+1}|\xi|^{\beta(x)+1} \text{ for all } (x,\eta,\xi) \in \overline{\Omega}\times\mathbb{R}\times\mathbb{R}, \tag{4.21}$$

consequently,

$$\begin{aligned}
\Psi(\eta,\xi) = &-\int_\Omega F(x,\eta,\xi)dx \geq -C_{10}\int_\Omega |\eta|^{\alpha(x)+1}|\xi|^{\beta(x)+1}dx\\
&\geq -C_{10}|\Omega|\max\left(\|\eta\|_\infty^{1+\alpha^-}, \|\eta\|_\infty^{1+\alpha^+}\right)\max\left(\|\xi\|_\infty^{1+\beta^-}, \|\xi\|_\infty^{1+\beta^+}\right).
\end{aligned} \tag{4.22}$$

Since W_0 are continuously embedded in $C(\overline{\Omega})$, there exists a constant C_{11} such that

$$\Psi(\eta,\xi) \geq -C_{10}C_{11}|\Omega|\max\left(\|\eta\|_{W_0}^{1+\alpha^-}, \|\eta\|_{W_0}^{1+\alpha^+}\right)\max\left(\|\xi\|_{W_0}^{1+\beta^-}, \|\xi\|_{W_0}^{1+\beta^+}\right). \tag{4.23}$$

When $\|\eta\|_{W_0} \to \infty$, $\|\xi\|_{W_0}$ bounded ($\|\xi\|_{W_0} \to \infty$, $\|\eta\|_{W_0}$ bounded) and $\|\eta\|_{W_0} \to \infty$, $\|\xi\|_{W_0} \to \infty$, we have

$$\|(\eta,\xi)\|_{X_0} \to \infty. \tag{4.24}$$

According to $2 + \alpha^+ + \beta^+ < p^-$, there exist $p_1 < p^-$ such that $\frac{1+\alpha^+}{p_1} + \frac{1+\beta^+}{p_1} = 1$. Therefore, from (4.23) and Young's inequality, we deduce

$$\Psi(\eta,\xi) \geq -C_{10}C_{11}|\Omega|\left(\frac{1+\alpha^+}{p_1}\|\eta\|_{W_0}^{p_1} + \frac{1+\beta^+}{p_1}\max\left(\|\eta\|_{W_0}^{p_1}, 1\right)\right). \tag{4.25}$$

Hence, for $\lambda > 0$ and $p_1 < p^-$, the combination of (4.20), (4.24) and (4.25) implies that

$$\lim_{\|(\eta,\xi)\|_{X_0}\to\infty} \Phi(\eta,\xi) + \lambda\Psi(\eta,\xi) = +\infty. \tag{4.26}$$

In view of (F2), we choose $\delta > 1$ such that

$$F(x, s, t) > 0 \tag{4.27}$$

for all $s, t > \delta,\ x \in \Omega$, and then

$$F(x, s, t) > 0 = F(x, 0, 0) > F(x, \tau_1, \tau_2) \tag{4.28}$$

for all $\tau_1, \tau_2 \in [0, 1)$. Set a_0, b_1 be two positive real numbers such that $a_0 < \min(1, C_\phi)$ with C_ϕ defined by Theorem 2.1 and $b_1 > \delta$ with $b_1^{p^-}|\Omega| > 1$. From (4.28), we have

$$\int_\Omega F(x, b_1, b_1) > 0 = F(x, 0, 0) > \int_\Omega \sup_{0\le s,t\le a_0} F(x, s, t). \tag{4.29}$$

Let

$$r := \min\left(\frac{m_1}{p^+}\left(\frac{a_0}{C_\phi}\right)^{p^+}, \frac{m_2}{p^+}\left(\frac{a_0}{C_\phi}\right)^{p^+}\right). \tag{4.30}$$

Choosing $(\eta_0(x), \xi_0(x)) = (0, 0), (\eta_1(x), \xi_1(x)) = (b_1, b_1)$ for all $x \in \Omega$, such that we obtain

$$\Phi(\eta_0, \xi_0) = \Psi(\eta_0, \xi_0) = (0, 0) \tag{4.31}$$

and

$$\begin{aligned}
\Phi(\eta_1, \xi_1) &= \widetilde{M}_1\left(\int_\Omega \frac{|b_1|^{\overline{p}(x)}}{\overline{p}(x)}dx\right) + \widetilde{M}_2\left(\int_\Omega \frac{|b_1|^{\overline{p}(x)}}{\overline{p}(x)}dx\right)\\
&\ge m_1\left(\int_\Omega \frac{|b_1|^{\overline{p}(x)}}{\overline{p}(x)}dx\right) + m_2\left(\int_\Omega \frac{|b_1|^{\overline{p}(x)}}{\overline{p}(x)}dx\right)\\
&\ge |\Omega|\left(\frac{m_1 b_1^{p^-}}{p^+} + \frac{m_2 b_1^{p^-}}{p^+}\right)\\
&\ge \frac{m_1}{p^+} + \frac{m_2}{p^+}\\
&\ge r.
\end{aligned} \tag{4.32}$$

Thus, the combination of (4.31) and (4.32) implies that

$$\Phi(\eta_0, \xi_0) < r < \Phi(\eta_1, \xi_1). \tag{4.33}$$

On the other hand

$$\begin{aligned}
&- \frac{(\Phi(\eta_1, \xi_1) - r)\Psi(\eta_0, \xi_0) + (r - \Phi(\eta_0, \xi_0))\Psi(\eta_1, \xi_1)}{\Phi(\eta_1, \xi_1) - \Psi(\eta_0, \xi_0)}\\
&= r\frac{\int_\Omega F(x, b_1, b_1)dx}{\widetilde{M}_1\left(\int_\Omega \frac{|b_1|^{\overline{p}(x)}}{\overline{p}(x)}dx\right) + \widetilde{M}_2\left(\int_\Omega \frac{|b_1|^{\overline{p}(x)}}{\overline{p}(x)}dx\right)}.
\end{aligned} \tag{4.34}$$

Let $(\eta, \xi) \in X_0$ such that $\Phi(\eta, \xi) \le r$, we get

$$\begin{aligned}\Phi(\eta, v) &= \widetilde{M}_1\left(\delta_{p(\cdot)}(\eta)\right) + \widetilde{M}_2\left(\delta_{p(\cdot)}(\xi)\right)\\ &\geq \min\{m_1, m_2\}\left(\frac{1}{p^+}\rho_{p(\cdot)}^{s(\cdot)}(\eta) + \frac{1}{p^+}\rho_{p(\cdot)}^{s(\cdot)}(\xi)\right),\end{aligned} \tag{4.35}$$

which implies that

$$\rho_{p(\cdot)}^{s(\cdot)}(\eta) \leq \frac{rp^+}{m_1} < 1,\ \rho_{p(\cdot)}^{s(\cdot)}(\xi) \leq \frac{rp^+}{m_2} < 1. \tag{4.36}$$

According to Proposition 2.2, we derive

$$\|\eta\|_{w_0} \leq 1, \quad \|\xi\|_{w_0} \leq 1. \tag{4.37}$$

Combining with (4.36), we deduce

$$\|\eta\|_{w_0} \leq \left(\frac{rp^+}{m_1}\right)^{\frac{1}{p^+}},\ \|\xi\|_{w_0} \leq \left(\frac{rp^+}{m_2}\right)^{\frac{1}{p^+}}. \tag{4.38}$$

Therefore

$$|\eta(x)| \leq C_\phi\left(\frac{rp^+}{m_1}\right)^{\frac{1}{p^+}} \leq a_0,\ |\xi(x)| \leq C_\phi\left(\frac{rp^+}{m_2}\right)^{\frac{1}{p^+}} \leq a_0 \tag{4.39}$$

for all $x \in \Omega$. It follows from (4.34) that

$$\begin{aligned}&-\inf_{(\eta,\xi)\in\Phi^{-1}((-\infty,r])}\Psi(\eta,\xi)\\ &= \sup_{\Phi(\eta,\xi)\leq r}(-\Psi(\eta,\xi))\\ &\leq \sup_{\{(\eta,\xi)\in X_0:|\eta(x)|,|\xi(x)|\leq a_0,\forall x\in\Omega\}}\int_\Omega F(x,\eta,\xi)dx\\ &\leq \int_\Omega \sup_{0\leq s,t\leq a} F(x,s,t)dx\\ &\leq 0\\ &\leq -\frac{(\Phi(\eta_1,\xi_1)-r)\Psi(\eta_0,\xi_0)+(r-\Phi(\eta_0,\xi_0))\Psi(\eta_1,\xi_1)}{\Phi(\eta_1,\xi_1)-\Psi(\eta_0,\xi_0)}.\end{aligned} \tag{4.40}$$

Consequently, the conditions of Theorem 2.3 are satisfied. For every compact interval $\Lambda \subset (0,+\infty)$ and $G \in \mathcal{A}$, we fix $\lambda \in \Lambda$ and put

$$J(\eta,\xi) = -\int_\Omega G(x,\eta,\xi)dx,$$

for all $(\eta,\xi) \in X_0$. Then, J is a compact derivative. Therefore, there exists a positive number δ such that for every $\mu \in [0,\delta]$, $(\eta_i,\xi_i) \in X_0$ are three solutions of the following equation

$$\Phi'(\eta_i,\xi_i) + \lambda\Psi'(\eta_i,\xi_i) + \mu J'(\eta_i,\xi_i) = 0.$$

That is, $(\eta_i,\xi_i) \in X_0$ are three weak solutions of the nonlocal fractional Kirchhoff-type elliptic systems (1.1). □

5. Conclusions

In this article, we consider a kind of $p(\cdot)$-fractional Kirchhoff systems. Under some reasonable assumptions, we obtain two and three solutions based on Bonanno's multiple critical points theorems and Ricceri's three critical points theorem, where the condition of Palais-Smale is not requested. Several recent results of literatures are extended and improved.

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Conflict of interest

The authors declare that they have no competing interests.

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