Mathematics

Research article

Pullback random attractors of stochastic strongly damped wave equations with variable delays on unbounded domains

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Abstract: In this paper, we consider the asymptotic behavior of solutions to stochastic strongly damped wave equations with variable delays on unbounded domains, which is driven by both additive noise and deterministic non-autonomous forcing. We first establish a continuous cocycle for the equations. Then we prove asymptotic compactness of the cocycle by tail-estimates and a decomposition technique of solutions. Finally, we obtain the existence of a tempered pullback random attractor.

Keywords: strongly damped wave equation; variable delays; random attractor; asymptotical compactness; tail-estimates
Mathematics Subject Classification: 37L55, 35B41, 35B40

1. Introduction

The aim of this paper is to establish the existence of pullback random attractors of the following stochastic non-autonomous strongly damped wave equation with variable delays and with additive noise in $\mathbb{R}^d$:

$$u_{tt} - \alpha\Delta u_t - \Delta u + u_t + \lambda u = f(x, u(t - \rho(t))) + g(t, x) + \sum_{j=1}^{m} h_j(x)\frac{dW_j}{dt}, \qquad (1.1)$$

with initial conditions

$$u(s + \tau, x) = \phi(s, x), \quad u_t(s + \tau, x) = \psi(s, x), \quad s \in [-h, 0] \qquad (1.2)$$

where $x \in \mathbb{R}^d$, $t \geqslant \tau, \tau \in \mathbb{R}$; $\alpha > 0$ is the strong damping coefficient, λ is a positive constant; $g \in L_{loc}^2(\mathbb{R}, L^2(\mathbb{R}^d))$ and $h_j \in H^2(\mathbb{R}^d)$; f is a nonlinear function satisfying some conditions, ρ is a given delay

function; $\{W_j\}_{j=1}^m$ are independent real-valued two-sided Wiener process on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, which will be specified later.

As we know, the concept of random attractor, as an extension of the global attractor for the deterministic systems, was first introduced in [8], which has been studied in many papers, see [2, 3, 9–12, 20, 21, 23, 28, 30, 31, 39] and references therein.

Time delay differential equations arise from some evolution phenomena in physics, biology and life science, which depend not only on the current states but also on their past history. There have been many works on the asymptotic behavior of delay differential equations, see [6, 13, 19] in the deterministic case and [4, 7, 16, 17, 29, 35, 36] in the stochastic case and references therein.

The asymptotic behavior of solutions of stochastic wave equation have been studied extensively in [15, 22, 25, 37] in the autonomous case. For the non-autonomous stochastic wave equation, the existence of random attractors was obtained in [33] on bounded domains and in [5, 18, 27, 32, 34] on unbounded domains. Wave equations with delays are widely used in biology, physics, engineering and chemistry. Therefore, it is important for us to study the asymptotic behavior of stochastic delay wave equation. In [14, 38], stochastic wave equations with delays on bounded domains are considered. However, the results for the stochastic delay wave equation on unbounded domains are very few.

In this work, we study the pullback random attractors of stochastic non-autonomous strongly damped wave equations with variable delays on unbounded domains. To prove the existence of pullback random attractors, we need to derive some kind compactness. The main difficulty in this paper is to establish the asymptotic compactness since the Sobolev embedding is no longer compact on unbounded domains. We here overcome the difficulty by showing that the uniform tail-estimates of solutions are sufficiently small (see Lemma 2.3). On bounded domains, we decompose the solutions into a sum of two parts. One part decays exponentially and the other part has higher regularity. For the higher regularity part, we first give some uniform estimates (see Lemma 2.4) and obtain the Hölder continuity of solutions in time (see Lemma 2.5). Then we apply Arzela-Ascoli theorem to prove the precompactness (see Lemma 3.2) and hence establish our main result (see Theorem 3.3). In addition, the strongly damped term $\alpha \Delta u_t$ and the delay term $f(x, u(t - \rho(t)))$ introduce an additional difficulty in deriving the uniform estimates, which needs some nontrivial arguments.

The paper is organized as follows. In the next section, we establish a continuous cocycle for problem (1.1) and (1.2) and some uniform estimates of solutions are derived. Then we prove the existence and uniqueness of the tempered pullback attractors for (1.1) and (1.2) in Section 3. In Section 4, we make conclusion as well as some comments on our results.

Notations: Let $(\Omega, \mathscr{F}, \mathbb{P})$ be the standard probability space with $\Omega = \{\omega = (\omega_1, \omega_2, \cdots, \omega_m) \in C(\mathbb{R}, \mathbb{R}^m) : \omega(0) = 0\}$, $\mathscr{F}$ is the Bore σ−algebra generated by the compact open topology of Ω and $\mathbb{P}$ is the Wiener measure on $(\Omega, \mathscr{F})$. Define the shift operator $\{\theta_t\}_{t \in \mathbb{R}}$ by $\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), t \in \mathbb{R}, \omega \in \Omega$. Then $(\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$ is a metric dynamical system.

Throughout this paper, we use $(\cdot)$ and $\| \cdot \|$ to denote the inner product and norm of $L^2(\mathbb{R}^d)$, respectively, and use $\| \cdot \|_X$ to denote the norm of a general Banach space X. For $h > 0$, let C_h be the Banach space $C([-h, 0]; L^2(\mathbb{R}^d))$ endowed with the norm $\|\varphi\|_{C_h} = \sup_{s \in [-h, 0]} \|\varphi(s)\|$ and u^t be the function defined by $u^t = u(t + s), s \in [-h, 0]$. Let C be a positive constant whose value may be different from line to line or even in the same line.

Let $E = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, endowed with the following norm

$$\|(u, v)\|_E^2 = (\sigma^2 + \lambda - \sigma)\|u\|^2 + (1 - \alpha\sigma)\|\nabla u\|^2 + \|v\|^2, \quad (u, v) \in E \tag{1.3}$$

where $\sigma > 0$ is a fixed constant such that

$$1 - \sigma > 0,$$

$$1 - \alpha\sigma > 0,$$

$$\sigma^2 + \lambda - \sigma > 0.$$

Let $\mathscr{E} = \{(u, v) : u \in C_h, \|\nabla u\| \in C_h, v \in C_h\}$, with the norm

$$\|(u, v)\|_{\mathscr{E}}^2 = (\sigma^2 + \lambda - \sigma)\|u\|_{C_h}^2 + (1 - \alpha\sigma)\|\nabla u\|_{C_h}^2 + \|v\|_{C_h}^2. \tag{1.4}$$

2. Materials and methods

2.1. Preliminaries and cocycles

In this subsection, we first show that the system (1.1) and (1.2) generates a continuous cocycle. Then, we recall some results for the existence of pullback random attractors for non-autonomous random dynamical systems.

For our purpose, we transform the system (1.1) and (1.2) into a deterministic system with random parameters but without white noise, and then show that it generates a continuous cocycle on $\mathscr{E}$ over $\mathbb{R}$ and $(\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})$.

For $j = 1, 2, \cdots, m$, consider the one-dimensional Ornstein-Uhlenbeck equation:

$$dz_j + z_j dt = dW_j, \tag{2.1}$$

whose solution is given by

$$z_j(t) = z_j(\theta_t \omega_j) \equiv - \int_{-\infty}^0 e^s(\theta_t \omega_j)(s) ds, \quad t \in \mathbb{R}. \tag{2.2}$$

It is known that the random variable $|z_j(\omega_j)|$ is tempered and there exists a θ_t−invariant subset $\widetilde{\Omega} \subset \Omega$ of full measure such that $z_j(\theta_t \omega_j)$ is continuous in t for each $\omega \in \widetilde{\Omega}$. From now on, we will not distinguish $\widetilde{\Omega}$ and Ω, and write the space $\widetilde{\Omega}$ as Ω.

Set

$$z(\theta_t \omega) = \sum_{j=1}^m h_j(x) z_j(\theta_t \omega_j),$$

then from (2.1) we have that

$$dz + z dt = \sum_{j=1}^m h_j dW_j. \tag{2.3}$$

It follows from [1, Proposition 4.3.3] that, there exists a tempered function $r(\omega) > 0$ such that

$$\sum_{j=1}^{m} |z_j(\omega_j)|^2 \leqslant r(\omega), \tag{2.4}$$

where $r(\omega) > 0$ satisfies for each $\omega \in \Omega$,

$$r(\theta_t\omega) \leqslant e^{\sigma'\frac{|t|}{2}} r(\omega), \quad t \in \mathbb{R}, \tag{2.5}$$

here σ' is a positive constant which will be fixed later. Then by (2.4) and (2.5), we obtain, for each $\omega \in \Omega$,

$$\sum_{j=1}^{m} |z_j(\theta_t\omega_j)|^2 \leqslant e^{\sigma'\frac{|t|}{2}} r(\omega), \quad t \in \mathbb{R}. \tag{2.6}$$

In the rest of this subsection, we show that there is a continuous cocycle generated by the system (1.1) and (1.2). Firstly we give the following assumptions on f and g:

(A1) There exist a function $k_1(x) \in L^2(\mathbb{R}^d)$ and a positive constant k_2 such that the functions $f \in C(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}), \rho \in C(\mathbb{R}, [0, h])$ satisfy

$$|f(x, u)|^2 \leqslant |k_1(x)|^2 + k_2^2 |u|^2, \quad \forall x \in \mathbb{R}^d, u \in \mathbb{R};$$

and

$$|\rho'(t)| \leqslant \rho^* < 1, \quad \forall t \in \mathbb{R};$$

(A2) There exists a constant $L > 0$, such that

$$|f(x, u) - f(x, v)| \leqslant L|u - v|, \quad \forall x \in \mathbb{R}^d, u, v, \in \mathbb{R};$$

(A3) The deterministic forcing $g(t, x) \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^d))$, and

$$\int_{-\infty}^{\tau} e^{\lambda r} \|g(r, \cdot)\|^2 dr < \infty, \quad \forall \tau \in \mathbb{R},$$

which implies

$$\lim_{k \to \infty} \int_{-\infty}^{\tau} \int_{|x| \geqslant k} e^{\lambda r} |g(r, \cdot)|^2 dx dr = 0, \quad \forall \tau \in \mathbb{R}.$$

Let $v = u_t + \sigma u - z(\theta_t\omega)$, where σ is given in (1.3), then (1.1) and (1.2) can be rewritten as the following equivalent form:

$$\begin{cases} \dfrac{du}{dt} = v - \sigma u + z(\theta_t\omega), \\ \dfrac{dv}{dt} = \alpha \Delta v - (1 - \sigma)v + (1 - \alpha\sigma)\Delta u - (\sigma^2 + \lambda - \sigma)u \\ \qquad + f(x, u(t - \rho(t))) + g(t, x) + \sigma z(\theta_t\omega) + \alpha \Delta z(\theta_t\omega), \end{cases} \tag{2.7}$$

with the initial conditions

$$u(\tau + s, x) = u^\tau(x) \equiv \phi(s, x), v(\tau + s, x) = v^\tau(x), \quad s \in [-h, 0], x \in \mathbb{R}^d, \tag{2.8}$$

where $v^\tau(x) \equiv \psi(s, x) + \sigma\phi(s, x) - z(\theta_{\tau+s}\omega)$. Put $\varphi(\tau + t, \tau, \theta_\tau, \varphi^\tau) = (u(\tau + t, \tau, \theta_\tau, u^\tau), v(\tau + t, \tau, \theta_\tau, v^\tau))^\top$, where $\varphi^\tau = (u^\tau, v^\tau)^\top$. By the classical theory in [24], we may show the following existence results of solutions of (2.7) and (2.8).

Assume that $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}^d))$ and the assumption (A1) holds. Then for each $\omega \in \Omega, \tau \in \mathbb{R}, \varphi^\tau \in \mathscr{E}$, there exists a solution $\varphi(\cdot, \tau, \omega, \varphi^\tau)$ to the problem (2.7) and (2.8), which satisfies $\varphi(\cdot, \tau, \omega, \varphi^\tau) \in C([\tau - h, T]; E)$, for any $T > \tau$, and for any $t \in [\tau, T]$, $\varphi^t(\cdot, \tau, \omega, \varphi^\tau) \in C([-h, T]; \mathscr{E})$.

Assume moreover (A2) holds. Then the solutions to the problem (2.7) and (2.8) are unique, and the solutions depend continuously on the initial data $\varphi^\tau \in \mathscr{E}$, for any $\omega \in \Omega, t \geqslant \tau$.

Now, we define a mapping: $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \mathscr{E} \to \mathscr{E}$ by

$$\Phi(t, \tau, \omega, \varphi^\tau) = \varphi^{t+\tau}(\cdot, \tau, \theta_{-\tau}\omega, \varphi^\tau),$$

where $\varphi^{t+\tau}(s, \tau, \theta_{-\tau}\omega, \varphi^\tau) = \varphi(t + \tau + s, \tau, \theta_{-\tau}\omega, \varphi^\tau)$ for $s \in [-h, 0]$. Then Φ is a continuous cocycle on $\mathscr{E}$ over $\mathbb{R}$ and $(\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$.

In the following, let $\mathcal{D}(X)$ be the collection of all tempered families of nonempty bounded subsets of X. Recall that $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(X)$ is said to be tempered in X if for each $\gamma > 0$,

$$\lim_{t\to-\infty} e^{\gamma t}\|D(\tau + t, \theta_t\omega)\|_X = 0, \tag{2.9}$$

where $\|D\|_X = \sup_{x\in D} \|x\|_X$. The cocycle Φ is said to be $\mathcal{D}(X)$-pullback asymptotically compact in X if for all $\tau \in \mathbb{R}, \omega \in \Omega$, the sequence

$$\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^\infty \text{ has a convergent subsequence in } X, \tag{2.10}$$

whenever $t_n \to \infty$, and $x_n \in D(\tau - t_n, \theta_{-t_n}\omega)$ with $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}(X)$.

Next, we provide the following result for non-autonomous random dynamical systems from [26].

Proposition 2.1. *Let Φ be a continuous cocycle on X over $\mathbb{R}$ and $(\Omega, \mathscr{F}, \mathbb{P}, \{\theta_t\}_{t\in\mathbb{R}})$. Suppose Φ is $\mathcal{D}(X)$-pullback asymptotically compact in X and has a closed measurable $\mathcal{D}(X)$-pullback absorbing set K in $\mathcal{D}(X)$. Then Φ has an unique $\mathcal{D}(X)$ pullback attractor $\mathcal{A}$ in $\mathcal{D}(X)$. For each $\tau \in \mathbb{R}$ and $\omega \in \Omega$, $\mathcal{A}$ is given by,*

$$\mathcal{A}(\tau, \omega) = \bigcap_{\tau\geqslant 0} \overline{\bigcup_{t\geqslant\tau} \Phi(t, \tau - t, \theta_{-t}\omega, K(\tau - t, \theta_t\omega))}.$$

In the rest of this paper, we will use Proposition 2.1 to prove the existence and uniqueness of a pullback random attractor for the continuous cocycle Φ in $\mathscr{E}$.

2.2. *Uniform tail-estimates*

In this subsection, we derive some uniform tail-estimates of solutions of problem (2.7) and (2.8). Hereafter we suppose that $\mathcal{D}$ is the collection of all tempered families of nonempty bounded subsets of X.

Lemma 2.2. *In addition to the assumptions (A1)–(A3), suppose that there exists $\sigma' > 0$ such that*

$$1 - \sigma > \sigma', \tag{2.11}$$

$$\sigma > \sigma', \tag{2.12}$$

and

$$(2\sigma - \sigma')(\sigma^2 + \lambda - \sigma) - \frac{3k_2^2 e^{\sigma' h}}{(1 - \sigma)(1 - \rho^*)} > 0. \tag{2.13}$$

Then for each $\tau \in \mathbb{R}, \omega \in \Omega$, and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(\tau, \omega, D) > 0$, such that for all $t \geqslant T, -h \leqslant s \leqslant 0$, the solution φ of (2.7) and (2.8) satisfies

$$\|\varphi^\tau(s, \tau - t, \theta_{-\tau}\omega, \varphi^{\tau-t})\|_{\mathscr{E}}^2 + C \int_{\tau-t}^{\tau} e^{\sigma' r} \|\varphi^r(s, \tau - t, \theta_{-\tau}\omega, \varphi^{\tau-t})\|_{\mathscr{E}}^2 dr$$
$$+ 2\alpha \int_{\tau-t}^{\tau} e^{\sigma'(r-\tau-s)} \|\nabla v(r, \tau - t, \theta_{-\tau}\omega, v^{\tau-t})\|^2 dr \leqslant r_1(\tau, \omega), \tag{2.14}$$

where

$$r_1(\tau, \omega) = C + Cr(\omega) + Ce^{-\sigma' \tau} \int_{-\infty}^{\tau} e^{\sigma' r} \|g(r, x)\|^2 dr, \tag{2.15}$$

$\varphi^{\tau-t} = (u^{\tau-t}, v^{\tau-t}) \in D(\tau - t, \theta_{-t}\omega), r(\omega)$ is the tempered function satisfying (2.4), C is a constant independent of τ, ω and D.

Proof. Taking the inner product of the second equation of (2.7) by v in $L^2(\mathbb{R}^d)$, we have

$$\frac{d}{dt}\|v\|^2 = -2\alpha\|\nabla v\|^2 - 2(1 - \sigma)\|v\|^2 + 2(1 - \alpha\sigma)(\Delta u, v) - 2(\sigma^2 + \lambda - \sigma)(u, v)$$
$$+ 2(f(x, u(t - \rho(t))), v) + 2(g(t, x), v) + 2\sigma(z(\theta_t\omega), v) + 2\alpha(\Delta z(\theta_t\omega), v). \tag{2.16}$$

Note that

$$\frac{du}{dt} = v - \sigma u + z(\theta_t\omega). \tag{2.17}$$

Then by (2.17), we derive that

$$(\Delta u, v) = (\Delta u, \frac{du}{dt} + \sigma u - z(\theta_t\omega)) = -\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 - \sigma\|\nabla u\|^2 + (\Delta z(\theta_t\omega), u) \tag{2.18}$$

and

$$(u, v) = (u, \frac{du}{dt} + \sigma u - z(\theta_t\omega)) = \frac{1}{2}\frac{d}{dt}\|u\|^2 + \sigma\|u\|^2 - (z(\theta_t\omega), u). \tag{2.19}$$

It follows from (2.16)–(2.19) that

$$\frac{d}{dt}(\|v\|^2 + (1 - \alpha\sigma)\|\nabla u\|^2 + (\sigma^2 + \lambda - \sigma)\|u\|^2) + 2\alpha\|\nabla v\|^2$$

$$\begin{aligned}
&= -2(1 - \sigma)\|v\|^2 - 2\sigma(1 - \alpha\sigma)\|\nabla u\|^2 - 2\sigma(\sigma^2 + \lambda - \sigma)\|u\|^2 \\
&\quad + [2(\sigma^2 + \lambda - \sigma)(z(\theta_t\omega), u) + 2(1 - \alpha\sigma)(\Delta z(\theta_t\omega), u)] \\
&\quad + [2\sigma(z(\theta_t\omega), v) + 2\alpha(\Delta z(\theta_t\omega), v)] \\
&\quad + 2(f(x, u(t - \rho(t))), v) + 2(g(t, x), v) \\
&=: -2(1 - \sigma)\|v\|^2 - 2\sigma(1 - \alpha\sigma)\|\nabla u\|^2 - 2\sigma(\sigma^2 + \lambda - \sigma)\|u\|^2 + \sum_{i=1}^{4} \mathscr{I}_i.
\end{aligned} \quad (2.20)$$

Now we estimate the terms on the right-hand of (2.20). By Young's inequality, Cauchy-Schwarz inequality and (A1), we have

$$\mathscr{I}_1 \leqslant \frac{(\sigma^2 + \lambda - \sigma)^2}{\varepsilon_1}\|z(\theta_t\omega)\|^2 + \frac{(1 - \alpha\sigma)^2}{\varepsilon_1}\|\Delta z(\theta_t\omega)\|^2 + 2\varepsilon_1\|u\|^2,$$

$$\mathscr{I}_2 \leqslant \frac{\sigma^2}{\varepsilon_2}\|z(\theta_t\omega)\|^2 + \frac{\alpha^2}{\varepsilon_2}\|\Delta z(\theta_t\omega)\|^2 + 2\varepsilon_2\|v\|^2,$$

$$\mathscr{I}_3 \leqslant \frac{1}{2\varepsilon_3}\|f(x, u(t - \rho(t)))\|^2 + 2\varepsilon_3\|v\|^2 \leqslant \frac{1}{2\varepsilon_3}\|k_1\|^2 + \frac{k_2^2}{2\varepsilon_3}\|u(t - \rho(t))\|^2 + 2\varepsilon_3\|v\|^2$$

and

$$\mathscr{I}_4 \leqslant 2\|g(t, x)\|^2\|v\|^2 \leqslant \frac{1}{2\varepsilon_4}\|g(t, x)\|^2 + 2\varepsilon_4\|v\|^2,$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are fixed positive constants which will be chosen later. Combining these estimates with (2.20), we have

$$\begin{aligned}
&\frac{d}{dt}(\|v\|^2 + (1 - \alpha\sigma)\|\nabla u\|^2 + (\sigma^2 + \lambda - \sigma)\|u\|^2) + 2\alpha\|\nabla v\|^2 \\
&\leqslant -2((1 - \sigma) - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\|v\|^2 - 2\sigma(1 - \alpha\sigma)\|\nabla u\|^2 - 2(\sigma(\sigma^2 + \lambda - \sigma) - \varepsilon_1)\|u\|^2 \\
&\quad + \frac{1}{2\varepsilon_3}\|k_1\|^2 + \frac{k_2^2}{2\varepsilon_3}\|u(t - \rho(t))\|^2 + \frac{1}{2\varepsilon_4}\|g(t, x)\|^2 \\
&\quad + \left(\frac{(\sigma^2 + \lambda - \sigma)^2}{\varepsilon_1} + \frac{\sigma^2}{\varepsilon_2}\right)\|z(\theta_t\omega)\|^2 + \left(\frac{(1 - \alpha\sigma)^2}{\varepsilon_1} + \frac{\alpha^2}{\varepsilon_2}\right)\|\Delta z(\theta_t\omega)\|^2.
\end{aligned} \quad (2.21)$$

Recalling the definition of norm $\|\cdot\|_E$, from (2.21), we get

$$\begin{aligned}
&\frac{d}{dt}e^{\sigma' t}\|(u, v)\|_E^2 + 2\alpha e^{\sigma' t}\|\nabla v\|^2 \\
&\leqslant -(2((1 - \sigma) - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) - \sigma')\|v\|^2 e^{\sigma' t} \\
&\quad - (2\sigma - \sigma')(1 - \alpha\sigma)\|\nabla u\|^2 e^{\sigma' t} \\
&\quad - ((2\sigma - \sigma')(\sigma^2 + \lambda - \sigma) - 2\varepsilon_1)\|u\|^2 e^{\sigma' t} \\
&\quad + \frac{1}{2\varepsilon_3}\|k_1\|^2 e^{\sigma' t} + \frac{k_2^2}{2\varepsilon_3}\|u(t - \rho(t))\|^2 e^{\sigma' t} + \frac{1}{2\varepsilon_4}\|g(t, x)\|^2 e^{\sigma' t}
\end{aligned}$$

$$+ \left(\frac{(\sigma^2 + \lambda - \sigma)^2}{\varepsilon_1} + \frac{\sigma^2}{\varepsilon_2} \right) \|z(\theta_t \omega)\|^2 e^{\sigma' t} + \left(\frac{(1 - \alpha\sigma)^2}{\varepsilon_1} + \frac{\alpha^2}{\varepsilon_2} \right) \|\Delta z(\theta_t \omega)\|^2 e^{\sigma' t}. \tag{2.22}$$

Integrating (2.22) from $\tau - t$ to $\tau + s$, where $s \in [-h, 0]$, we obtain

$$e^{\sigma'(\tau+s)} \|\varphi(\tau + s, \tau - t, \omega, \varphi^{\tau-t})\|_E^2 + 2\alpha \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|\nabla v(r, \tau - t, \omega, v^{\tau-t})\|^2 dr$$

$$\leqslant e^{\sigma'(\tau-t)} \|\varphi(\tau - t, \tau - t, \omega, \varphi^{\tau-t})\|_E^2$$

$$- (2((1 - \sigma) - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) - \sigma') \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|v(r, \tau - t, \omega, v^{\tau-t})\|^2 dr$$

$$- (2\sigma - \sigma')(1 - \alpha\sigma) \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|\nabla u(r, \tau - t, \omega, u^{\tau-t})\|^2 dr$$

$$- ((2\sigma - \sigma')(\sigma^2 + \lambda - \sigma) - 2\varepsilon_1) \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|u(r, \tau - t, \omega, u^{\tau-t})\|^2 dr$$

$$+ \frac{\|k_1\|^2}{2\varepsilon_3 \sigma'} e^{\sigma'(\tau+s)} + \frac{k_2^2}{2\varepsilon_3} \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|u(r - \rho(r), \tau - t, \omega, u^{\tau-t})\|^2 dr$$

$$+ \frac{1}{2\varepsilon_4} \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|g(r, x)\|^2 dr$$

$$+ \left(\frac{(\sigma^2 + \lambda - \sigma)^2}{\varepsilon_1} + \frac{\sigma^2}{\varepsilon_2} \right) \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|z(\theta_r \omega)\|^2 dr$$

$$+ \left(\frac{(1 - \alpha\sigma)^2}{\varepsilon_1} + \frac{\alpha^2}{\varepsilon_2} \right) \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|\Delta z(\theta_r \omega)\|^2 dr. \tag{2.23}$$

Set $r' = r - \rho(r)$. Combining $\rho(r) \in [0, h]$ and the fact $\frac{1}{1-\rho'(r)} \leqslant \frac{1}{1-\rho^*}$ for all $r \in \mathbb{R}$, we infer that

$$\frac{k_2^2}{2\varepsilon_3} \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|u(r - \rho(r), \tau - t, \omega, u^{\tau-t})\|^2 dr$$

$$\leqslant \frac{k_2^2 e^{\sigma' h}}{2\varepsilon_3 (1 - \rho^*)} \int_{\tau-t-h}^{\tau+s} e^{\sigma' r} \|u(r, \tau - t, \omega, u^{\tau-t}\|^2 dr$$

$$= \frac{k_2^2 e^{\sigma' h}}{2\varepsilon_3 (1 - \rho^*)} \left(\int_{\tau-t-h}^{\tau-t} e^{\sigma' r} \|u(r, \tau - t, \omega, u^{\tau-t}\|^2 dr + \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|u(r, \tau - t, \omega, u^{\tau-t}\|^2 dr \right)$$

$$\leqslant \frac{k_2^2 h e^{\sigma' h} e^{\sigma'(\tau-t)}}{2\varepsilon_3 (1 - \rho^*)} \|u^{\tau-t}\|_{C_h}^2 + \frac{k_2^2 e^{\sigma' h}}{2\varepsilon_3 (1 - \rho^*)} \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|u(r, \tau - t, \omega, u^{\tau-t}\|^2 dr. \tag{2.24}$$

It follows from (2.23) and (2.24) that

$$e^{\sigma'(\tau+s)} \|\varphi(\tau + s, \tau - t, \omega, \varphi^{\tau-t})\|_E^2 + 2\alpha \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|\nabla v(r, \tau - t, \omega, v^{\tau-t})\|^2 dr$$

$$\leqslant e^{\sigma'(\tau-t)} \|\varphi(\tau - t, \tau - t, \omega, \varphi^{\tau-t})\|_E^2$$

$$- (2((1 - \sigma) - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) - \sigma') \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|v(r, \tau - t, \omega, v^{\tau-t})\|^2 dr$$

$$- (2\sigma - \sigma')(1 - \alpha\sigma) \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|\nabla u(r, \tau - t, \omega, u^{\tau-t})\|^2 dr$$

$$- \left((2\sigma - \sigma')(\sigma^2 + \lambda - \sigma) - \frac{k_2^2 e^{\sigma' h}}{2\varepsilon_3(1 - \rho^*)} - 2\varepsilon_1\right) \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|u(r, \tau - t, \omega, u^{\tau-t}\|^2 dr$$

$$+ \frac{\|k_1\|^2}{2\varepsilon_3 \sigma'} e^{\sigma'(\tau+s)} + \frac{k_2^2 h e^{\sigma' h} e^{\sigma'(\tau-t)}}{2\varepsilon_3(1 - \rho^*)} \|u^{\tau-t}\|_{C_h}^2 + \frac{1}{2\varepsilon_4} \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|g(r, x)\|^2 dr$$

$$+ \left(\frac{(\sigma^2 + \lambda - \sigma)^2}{\varepsilon_1} + \frac{\sigma^2}{\varepsilon_2}\right) \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|z(\theta_r \omega)\|^2 dr$$

$$+ \left(\frac{(1 - \alpha\sigma)^2}{\varepsilon_1} + \frac{\alpha^2}{\varepsilon_2}\right) \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|\Delta z(\theta_r \omega)\|^2 dr. \tag{2.25}$$

Let $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \frac{1-\sigma}{6}$. By (2.11)–(2.13), we can choose ε_1 small enough such that

$$1 - \sigma - \sigma' > 0, \tag{2.26}$$

$$2\sigma - \sigma' > 0 \tag{2.27}$$

and

$$(2\sigma - \sigma')(\sigma^2 + \lambda - \sigma) - \frac{3k_2^2 e^{\sigma' h}}{(1 - \sigma)(1 - \rho^*)} - 2\varepsilon_1 > 0 \tag{2.28}$$

Replacing ω with $\theta_{-\tau}\omega$ and by (2.25)–(2.28), we get

$$\|\varphi(\tau + s, \tau - t, \theta_{-\tau}\omega, \varphi^{\tau-t})\|_E^2 + Ce^{-\sigma'(\tau+s)} \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|\varphi(r, \tau - t, \theta_{-\tau}\omega, \varphi^{\tau-t})\|_E^2 dr$$

$$+ 2\alpha e^{-\sigma'(\tau+s)} \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|\nabla v(r, \tau - t, \theta_{-\tau}\omega, v^{\tau-t})\|^2 dr$$

$$\leqslant e^{\sigma' h} e^{-\sigma' t} \|\varphi^{\tau-t}\|_E^2 + C\|u^{\tau-t}\|_{C_h}^2 + C + Ce^{-\sigma' \tau} \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|g(r, x)\|^2 dr$$

$$+ Ce^{-\sigma' \tau} \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|z(\theta_{r-\tau}\omega)\|^2 dr + Ce^{-\sigma' \tau} \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|\Delta z(\theta_{r-\tau}\omega)\|^2 dr$$

$$\leqslant Ce^{-\sigma' t} \|\varphi^{\tau-t}\|_{\mathscr{E}}^2 + C + Ce^{-\sigma' \tau} \int_{-\infty}^{\tau} e^{\sigma' r} \|g(r, x)\|^2 dr$$

$$+ Ce^{-\sigma' \tau} \int_{\tau-t}^{\tau} e^{\sigma' r} \|z(\theta_{r-\tau}\omega)\|^2 dr + Ce^{-\sigma' \tau} \int_{\tau-t}^{\tau} e^{\sigma' r} \|\Delta z(\theta_{r-\tau}\omega)\|^2 dr. \tag{2.29}$$

Note that $z(\theta_t \omega) = \sum_{j=1}^{m} h_j z(\theta_t \omega_j)$ and $h_j \in H^2(\mathbb{R}^d)$, we deduce that for each $\omega \in \Omega$,

$$e^{-\sigma' \tau} \int_{\tau-t}^{\tau} e^{\sigma' r} (\|z(\theta_{r-\tau}\omega)\|^2 + \|\Delta z(\theta_{r-\tau}\omega)\|^2) dr$$

$$= e^{-\sigma' \tau} \int_{-t}^{0} e^{\sigma' r} (\|z(\theta_r \omega)\|^2 + \|\Delta z(\theta_r \omega)\|^2) dr$$

$$\leqslant e^{-\sigma' \tau} \int_{-t}^{0} e^{\sigma' r} \left(\sum_{j=1}^{m} |h_j z(\theta_r \omega_j)|^2 + \sum_{j=1}^{m} |\Delta h_j z(\theta_r \omega_j)|^2\right) dr$$

$$\leqslant e^{-\sigma'\tau} \int_{-\infty}^{0} e^{\sigma'r} \sum_{j=1}^{m} |z(\theta_r\omega_j)|^2 dr \leqslant Cr(\omega). \tag{2.30}$$

Since $\varphi^{\tau-t} = (u^{\tau-t}, v^{\tau-t}) \in D(\tau - t, \theta_{-t}\omega)$, then

$$\limsup_{t\to\infty} Ce^{-\sigma't}\|\varphi^{\tau-t}\|_{\mathscr{E}}^2 \leqslant \limsup_{t\to\infty} Ce^{-\sigma't}\|D(\tau - t, \theta_{-t}\omega)\|_{\mathscr{E}}^2 = 0. \tag{2.31}$$

By (2.31), there exists $T = T(\tau, \omega, D) > 0$, such that for all $t \geqslant T$,

$$Ce^{-\sigma't}\|\varphi^{\tau-t}\|_{\mathscr{E}}^2 \leqslant 1. \tag{2.32}$$

Combining (2.32), (2.30) and (2.29), we obtain for all $t \geqslant T$

$$\|\varphi(\tau + s, \tau - t, \theta_{-\tau}\omega, \varphi^{\tau-t})\|_{\mathscr{E}}^2 + C \int_{\tau-t}^{\tau} e^{\sigma'r}\|\varphi^r(s, \tau - t, \theta_{-\tau}\omega, \varphi^{\tau-t})\|_{\mathscr{E}}^2 dr$$

$$+ 2\alpha \int_{\tau-t}^{\tau} e^{\sigma'(r-\tau-s)}\|\nabla v(r, \tau - t, \theta_{-\tau}\omega, v^{\tau-t})\|^2 dr$$

$$\leqslant C + Cr(\omega) + Ce^{-\sigma'\tau} \int_{-\infty}^{\tau} e^{\sigma'r}\|g(r, x)\|^2 dr \equiv r_1(\tau, \omega). \tag{2.33}$$

Thus, we complete the proof of Lemma 2.2. $\qquad\qquad\square$

Now we establish the following estimates on the exterior of a ball.

Lemma 2.3. *Suppose the hypotheses in Lemma 2.2 hold, and let $\tau \in \mathbb{R}, \omega \in \Omega$ and $D \in \mathcal{D}$. Then for each $\epsilon > 0$. there exists $T = T(\tau, \omega, \epsilon, D) > 0$, and $K = K(\tau, \omega, \epsilon) \geqslant 1$, such that for all $t \geqslant T, k \geqslant K$ and $-h \leqslant s \leqslant 0$,*

$$\|\varphi^\tau(s, \tau - t, \theta_{-t}\omega, \varphi^{\tau-t})\|_{\mathscr{E}(\mathbb{R}^d\backslash\Omega_k)}^2 \leqslant \epsilon,$$

where $\Omega_k = \{x \in \mathbb{R}^d : |x| \leqslant k\}$ and $\varphi^{\tau-t} = (u^{\tau-t}, v^{\tau-t}) \in D(\tau - t, \theta_{-t}\omega)$.

Proof. Choose a smooth function $\xi(\cdot)$ as follows:

$$\xi(s) = \begin{cases} 0, & 0 \leqslant s \leqslant 1, \\ 1, & s \geqslant 2, \end{cases} \tag{2.34}$$

where $0 \leqslant \xi(s) \leqslant 1, s \in \mathbb{R}^+$, and with the constants μ_1, μ_2 satisfying $|\xi'(s)| \leqslant \mu_1, |\xi''(s)| \leqslant \mu_2$ for $s \in \mathbb{R}^+$. Taking the inner product of the second equation of (2.7) with $\xi^2(\frac{|x|^2}{k^2})v$ in $L^2(\mathbb{R}^d)$, we have

$$\frac{d}{dt} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|v|^2 dx$$

$$= -2(1-\sigma) \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|v|^2 dx + 2\alpha \int_{\mathbb{R}^d} (\Delta v)\xi^2(\frac{|x|^2}{k^2})v dx$$

$$+ 2(1-\alpha\sigma) \int_{\mathbb{R}^d} (\Delta u)\xi^2(\frac{|x|^2}{k^2})v dx - 2(\sigma^2 + \lambda - \sigma) \int_{\mathbb{R}^d} u\xi^2(\frac{|x|^2}{k^2})v dx$$

$$+ 2 \int_{\mathbb{R}^d} f(x, u(t - \rho(t)))\xi^2(\frac{|x|^2}{k^2})v dx + 2 \int_{\mathbb{R}^d} g(t, x)\xi^2(\frac{|x|^2}{k^2})v dx$$

$$+ 2\int_{\mathbb{R}^d}(\sigma z(\theta_t\omega) + \alpha\Delta z(\theta_t\omega))\xi^2(\frac{|x|^2}{k^2})vdx =: -2(1-\sigma)\int_{\mathbb{R}^d}\xi^2(\frac{|x|^2}{k^2})|v|^2dx + \sum_{i=1}^{6}\mathscr{J}_i. \qquad (2.35)$$

Now we estimate each term on the right hand of (2.35). By Young's inequality, Cauchy-Schwarz inequality and (A1), we infer that

$$\begin{aligned}
\mathscr{J}_1 &= -\frac{8\alpha}{k^2}\int_{\mathbb{R}^d}(\nabla v)\xi(\frac{|x|^2}{k^2})\xi'(\frac{|x|^2}{k^2})xvdx - 2\alpha\int_{\mathbb{R}^d}\xi^2(\frac{|x|^2}{k^2})|\nabla v|^2dx \\
&\leqslant \frac{8\sqrt{2}\alpha\mu_1}{k}\int_{k\leqslant|x|\leqslant\sqrt{2}k}|\nabla v||v|dx - 2\alpha\int_{\mathbb{R}^d}\xi^2(\frac{|x|^2}{k^2})|\nabla v|^2dx \\
&\leqslant \frac{4\sqrt{2}\alpha\mu_1}{k}(\|\nabla v\|^2 + \|v\|^2) - 2\alpha\int_{\mathbb{R}^d}\xi^2(\frac{|x|^2}{k^2})|\nabla v|^2dx.
\end{aligned} \qquad (2.36)$$

$$\mathscr{J}_2 = -\frac{8(1-\alpha\sigma)}{k^2}\int_{\mathbb{R}^d}(\nabla u)\xi(\frac{|x|^2}{k^2})\xi'(\frac{|x|^2}{k^2})xvdx - 2(1-\alpha\sigma)\int_{\mathbb{R}^d}(\nabla u)\xi^2(\frac{|x|^2}{k^2})\nabla vdx, \qquad (2.37)$$

where

$$\begin{aligned}
-\frac{8(1-\alpha\sigma)}{k^2}\int_{\mathbb{R}^d}(\nabla u)\xi(\frac{|x|^2}{k^2})\xi'(\frac{|x|^2}{k^2})xvdx &\leqslant \frac{8\sqrt{2}(1-\alpha\sigma)\mu_1}{k}\int_{k\leqslant|x|\leqslant\sqrt{2}k}|\nabla u||v|dx \\
&\leqslant \frac{4\sqrt{2}(1-\alpha\sigma)\mu_1}{k}(\|\nabla u\|^2 + \|v\|^2),
\end{aligned} \qquad (2.38)$$

and

$$\begin{aligned}
&-2(1-\alpha\sigma)\int_{\mathbb{R}^d}(\nabla u)\xi^2(\frac{|x|^2}{k^2})\nabla vdx \\
&= -2(1-\alpha\sigma)\int_{\mathbb{R}^d}(\nabla u)\xi^2(\frac{|x|^2}{k^2})\nabla(\frac{du}{dt} + \sigma u - z(\theta_t\omega))dx \\
&= -(1-\alpha\sigma)\frac{d}{dt}\int_{\mathbb{R}^d}\xi^2(\frac{|x|^2}{k^2})|\nabla u|^2dx - 2\sigma(1-\alpha\sigma)\int_{\mathbb{R}^d}\xi^2(\frac{|x|^2}{k^2})|\nabla u|^2dx \\
&\quad - 2(1-\alpha\sigma)\int_{\mathbb{R}^d}\xi^2(\frac{|x|^2}{k^2})|u||\Delta z(\theta_t\omega)|dx \\
&\leqslant -(1-\alpha\sigma)\frac{d}{dt}\int_{\mathbb{R}^d}\xi^2(\frac{|x|^2}{k^2})|\nabla u|^2dx - 2\sigma(1-\alpha\sigma)\int_{\mathbb{R}^d}\xi^2(\frac{|x|^2}{k^2})|\nabla u|^2dx \\
&\quad + \frac{(1-\alpha\sigma)^2}{\varepsilon_1}\int_{\mathbb{R}^d}\xi^2(\frac{|x|^2}{k^2})|\Delta z(\theta_t\omega)|^2dx + \varepsilon_1\int_{\mathbb{R}^d}\xi^2(\frac{|x|^2}{k^2})|u|^2dx.
\end{aligned} \qquad (2.39)$$

$$\begin{aligned}
\mathscr{J}_3 &= -2(\sigma^2 + \lambda - \sigma)\int_{\mathbb{R}^d}u\xi^2(\frac{|x|^2}{k^2})(\frac{du}{dt} + \sigma u - z(\theta_t\omega))dx \\
&= -(\sigma^2 + \lambda - \sigma)\frac{d}{dt}\int_{\mathbb{R}^d}\xi^2(\frac{|x|^2}{k^2})|u|^2dx - 2\sigma(\sigma^2 + \lambda - \sigma)\int_{\mathbb{R}^d}\xi^2(\frac{|x|^2}{k^2})|u|^2dx
\end{aligned}$$

$$+ 2(\sigma^2 + \lambda - \sigma) \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|u||z(\theta_t\omega)|dx$$

$$\leqslant -(\sigma^2 + \lambda - \sigma)\frac{d}{dt} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|u|^2 dx - 2\sigma(\sigma^2 + \lambda - \sigma) \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|u|^2 dx$$

$$+ \frac{(\sigma^2 + \lambda - \sigma)^2}{\varepsilon_1} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|z(\theta_t\omega)|^2 dx + \varepsilon_1 \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|u|^2 dx, \tag{2.40}$$

$$\mathscr{J}_4 \leqslant \frac{1}{2\varepsilon_3} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|k_1(x)|^2 dx + \frac{k_2^2}{2\varepsilon_3} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|u(t - \rho(t))|^2 dx$$

$$+ 2\varepsilon_3 \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|v|^2 dx, \tag{2.41}$$

$$\mathscr{J}_5 \leqslant \frac{1}{2\varepsilon_4} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|g(t, x)|^2 dx + 2\varepsilon_4 \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|v|^2 dx, \tag{2.42}$$

$$\mathscr{J}_6 \leqslant \frac{\sigma^2}{\varepsilon_2} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|z(\theta_t\omega)|^2 dx + \frac{\alpha^2}{\varepsilon_2} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|\Delta z(\theta_t\omega)|^2 dx + 2\varepsilon_2 \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|v|^2 dx, \tag{2.43}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ are positive constants which will be given later. It follows from (2.35)–(2.42) that

$$\frac{d}{dt} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})(|v|^2 + (1 - \alpha\sigma)|\nabla u|^2 + (\sigma^2 + \lambda - \sigma)|u|^2) dx$$

$$+ 2\alpha \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|\nabla v|^2 dx$$

$$\leqslant -2((1 - \sigma) - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|v|^2 dx$$

$$- (2\sigma(\sigma^2 + \lambda - \sigma) - 2\varepsilon_1) \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|u|^2 dx$$

$$- 2\sigma(1 - \alpha\sigma) \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|\nabla u|^2 dx$$

$$+ \frac{4\sqrt{2}\alpha\mu_1}{k}(\|\nabla v\|^2 + \|v\|^2) + \frac{4\sqrt{2}(1 - \alpha\sigma)\mu_1}{k}(\|\nabla u\|^2 + \|v\|^2)$$

$$+ \frac{1}{2\varepsilon_3} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|k_1(x)|^2 dx + \frac{k_2^2}{2\varepsilon_3} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|u(t - \rho(t))|^2 dx$$

$$+ \frac{1}{2\varepsilon_4} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|g(t, x)|^2 dx + \left(\frac{(1 - \alpha\sigma)^2}{\varepsilon_1} + \frac{\alpha^2}{\varepsilon_2}\right) \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|\Delta z(\theta_t\omega)|^2 dx$$

$$+ \left(\frac{(\sigma^2 + \lambda - \sigma)^2}{\varepsilon_1} + \frac{\sigma^2}{\varepsilon_2}\right) \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|z(\theta_t\omega)|^2 dx. \tag{2.44}$$

Let $Y = |v|^2 + (1 - \alpha\sigma)|\nabla u|^2 + (\sigma^2 + \lambda - \sigma)|u|^2$. Multiplying (2.44) by $e^{\sigma' t}$ and integrating over $(\tau - t, \tau + s)$, where $s \in [-h, 0]$, σ' is a fixed constant choosen as in Lemma 2.2, we obtain for each $\omega \in \Omega$

$$e^{\sigma'(\tau+s)} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})Y(\tau + s, \tau - t, \omega, Y^{\tau-t}) dx$$

$$+ 2\alpha \int_{\tau-t}^{\tau+s} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|\nabla v(\tau+s, \tau-t, \omega, v^{\tau-t})|^2 dx dr$$

$$\leqslant e^{\sigma'(\tau-t)} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2}) Y(\tau-t, \tau-t, \omega, Y^{\tau-t}) dx$$

$$- (2((1-\sigma) - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) - \sigma') \int_{\tau-t}^{\tau+s} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|v(r, \tau-t, \omega, v^{\tau-t})|^2 dx dr$$

$$- ((2\sigma - \sigma')(\sigma^2 + \lambda - \sigma) - 2\varepsilon_1) \int_{\tau-t}^{\tau+s} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|u(r, \tau-t, \omega, u^{\tau-t})|^2 dx dr$$

$$- (2\sigma - \sigma')(1 - \alpha\sigma) \int_{\tau-t}^{\tau+s} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|\nabla u(r, \tau-t, \omega, u^{\tau-t})|^2 dx dr$$

$$+ \frac{C}{k} \int_{\tau-t}^{\tau+s} e^{\sigma' r}(\|\nabla u(r, \tau-t, \omega, u^{\tau-t})\|^2 + \|v(r, \tau-t, \omega, v^{\tau-t})\|^2) dr$$

$$+ \frac{C}{k} \int_{\tau-t}^{\tau+s} e^{\sigma' r} \|\nabla v(r, \tau-t, \omega, v^{\tau-t})\|^2 dr$$

$$+ \frac{1}{2\varepsilon_3} \int_{\tau-t}^{\tau+s} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|k_1(x)|^2 dx dr$$

$$+ \frac{k_2^2}{2\varepsilon_3} \int_{\tau-t}^{\tau+s} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|u(r-\rho(r), \tau-t, \omega, u^{\tau-t})|^2 dx dr$$

$$+ \frac{1}{2\varepsilon_4} \int_{\tau-t}^{\tau+s} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|g(r, x)|^2 dx dr$$

$$+ \left(\frac{(1-\alpha\sigma)^2}{\varepsilon_1} + \frac{\alpha^2}{\varepsilon_2}\right) \int_{\tau-t}^{\tau+s} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|\Delta z(\theta_r \omega)|^2 dx dr$$

$$+ \left(\frac{(\sigma^2 + \lambda - \sigma)^2}{\varepsilon_1} + \frac{\sigma^2}{\varepsilon_2}\right) \int_{\tau-t}^{\tau+s} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|z(\theta_r \omega)|^2 dx dr. \tag{2.45}$$

Set $r' = r - \rho(r)$. By using the similar arguments in (2.24), we obtain

$$\frac{k_2^2}{2\varepsilon_3} \int_{\tau-t}^{\tau+s} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})\|u(r-\rho(r), \tau-t, \omega, u^{\tau-t})\|^2 dx dr$$

$$\leqslant \frac{k_2^2 e^{\sigma' h}}{2\varepsilon_3(1-\rho^*)} \int_{\tau-t-h}^{\tau+s} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})\|u(r, \tau-t, \omega, u^{\tau-t}\|^2 dx dr$$

$$= \frac{k_2^2 e^{\sigma' h}}{2\varepsilon_3(1-\rho^*)} \int_{\tau-t-h}^{\tau-t} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})\|u(r, \tau-t, \omega, u^{\tau-t}\|^2 dx dr$$

$$+ \frac{k_2^2 e^{\sigma' h}}{2\varepsilon_3(1-\rho^*)} \int_{\tau-t}^{\tau+s} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})\|u(r, \tau-t, \omega, u^{\tau-t}\|^2 dx dr$$

$$\leqslant \frac{k_2^2 h e^{\sigma' h} e^{\sigma'(\tau-t)}}{2\varepsilon_3(1-\rho^*)} \|u^{\tau-t}\|_{C_h}^2 + \frac{k_2^2 e^{\sigma' h}}{2\varepsilon_3(1-\rho^*)} \int_{\tau-t}^{\tau+s} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})\|u(r, \tau-t, \omega, u^{\tau-t}\|^2 dx dr. \tag{2.46}$$

Combining (2.45) and (2.46), we have for $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \dfrac{1-\sigma}{6}$

$$\int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2}) Y(\tau+s, \tau-t, \omega, Y^{\tau-t}) dx$$

$$+ 2\alpha e^{-\sigma'(\tau+s)} \int_{\tau-t}^{\tau+s} e^{\sigma'r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|\nabla v(\tau+s, \tau-t, \omega, v^{\tau-t})|^2 dxdr$$

$$\leqslant e^{\sigma'h} e^{-\sigma't} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})Y(\tau-t, \tau-t, \omega, Y^{\tau-t})dx + \frac{3k_2^2 h e^{2\sigma'h} e^{-\sigma't}}{(1-\sigma)(1-\rho^*)}\|u^{\tau-t}\|_{C_h}^2$$

$$- ((1-\sigma) - \sigma')e^{-\sigma'(\tau+s)} \int_{\tau-t}^{\tau+s} e^{\sigma'r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|v(r, \tau-t, \omega, v^{\tau-t})|^2 dxdr$$

$$- ((2\sigma-\sigma')(\sigma^2+\lambda-\sigma) - \frac{3k_2^2 e^{\sigma'h}}{(1-\sigma)(1-\rho^*)} - 2\varepsilon_1)$$

$$e^{-\sigma'(\tau+s)} \int_{\tau-t}^{\tau+s} e^{\sigma'r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|u(r, \tau-t, \omega, u^{\tau-t})|^2 dxdr$$

$$- (2\sigma-\sigma')(1-\alpha\sigma)e^{-\sigma'(\tau+s)} \int_{\tau-t}^{\tau+s} e^{\sigma'r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|\nabla u(r, \tau-t, \omega, u^{\tau-t})|^2 dxdr$$

$$+ \frac{C}{k} e^{-\sigma'(\tau+s)} \int_{\tau-t}^{\tau+s} e^{\sigma'r}(\|\nabla u(r, \tau-t, \omega, u^{\tau-t})\|^2 + \|v(r, \tau-t, \omega, v^{\tau-t})\|^2)dr$$

$$+ \frac{C}{k} e^{-\sigma'(\tau+s)} \int_{\tau-t}^{\tau+s} e^{\sigma'r}\|\nabla v(r, \tau-t, \omega, v^{\tau-t})\|^2 dr$$

$$+ \frac{3}{1-\sigma} e^{-\sigma'(\tau+s)} \int_{\tau-t}^{\tau+s} e^{\sigma'r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|k_1(x)|^2 dxdr$$

$$+ \frac{3}{1-\sigma} e^{-\sigma'(\tau+s)} \int_{\tau-t}^{\tau+s} e^{\sigma'r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|g(r, x)|^2 dxdr$$

$$+ \left(\frac{(1-\alpha\sigma)^2}{\varepsilon_1} + \frac{3\alpha^2}{1-\sigma}\right) e^{-\sigma'(\tau+s)} \int_{\tau-t}^{\tau+s} e^{\sigma'r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|\Delta z(\theta_r\omega)|^2 dxdr$$

$$+ \left(\frac{(\sigma^2+\lambda-\sigma)^2}{\varepsilon_1} + \frac{3\sigma^2}{1-\sigma}\right) e^{-\sigma'(\tau+s)} \int_{\tau-t}^{\tau+s} e^{\sigma'r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|z(\theta_r\omega)|^2 dxdr. \tag{2.47}$$

Choosing ε_1 sufficiently small, together with (2.26)–(2.28), and replacing ω by $\theta_{-\tau}\omega$, we have

$$\int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})Y(\tau+s, \tau-t, \theta_{-\tau}\omega, Y^{\tau-t})dx$$

$$+ 2\alpha e^{-\sigma'(\tau+s)} \int_{\tau-t}^{\tau+s} e^{\sigma'r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|\nabla v(\tau+s, \tau-t, \theta_{-\tau}\omega, v^{\tau-t})|^2 dxdr$$

$$\leqslant C e^{-\sigma't} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})Y^{\tau-t}dx + C e^{-\sigma't}\|u^{\tau-t}\|_{C_h}^2$$

$$+ \frac{C}{k} e^{-\sigma'\tau} \int_{\tau-t}^{\tau} e^{\sigma'r}(\|\nabla u(r, \tau-t, \theta_{-\tau}\omega, u^{\tau-t})\|^2 + \|v(r, \tau-t, \theta_{-\tau}\omega, v^{\tau-t})\|^2)dr$$

$$+ \frac{C}{k} e^{-\sigma'\tau} \int_{\tau-t}^{\tau} e^{\sigma'r}\|\nabla v(r, \tau-t, \theta_{-\tau}\omega, v^{\tau-t})\|^2 dr$$

$$+ C e^{-\sigma'\tau} \int_{\tau-t}^{\tau} e^{\sigma'r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|k_1(x)|^2 dxdr$$

$$+ C e^{-\sigma'\tau} \int_{\tau-t}^{\tau} e^{\sigma'r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|g(r, x)|^2 dxdr$$

$$+ Ce^{-\sigma' \tau} \int_{\tau-t}^{\tau} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|\Delta z(\theta_{r-\tau}\omega)|^2 dxdr$$

$$+ Ce^{-\sigma' \tau} \int_{\tau-t}^{\tau} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|z(\theta_{r-\tau}\omega)|^2 dxdr. \tag{2.48}$$

We now estimate the terms on the right-hand side of (2.48). Since $\varphi^{\tau-t} = (u^{\tau-t}, v^{\tau-t}) \in D(\tau - t, \theta_{-t}\omega)$, given $\epsilon > 0$, there exists $T_1 = T_1(\tau, \omega, D, \epsilon) > 0$, $K_1 = K_1(\tau, \omega, \epsilon) \geqslant 1$, such that for all $t > T_1, k > K_1$,

$$Ce^{-\sigma' t} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})Y^{\tau-t}dx + Ce^{-\sigma' t}\|u^{\tau-t}\|_{C_h}^2 \leqslant Ce^{-\sigma' t}\|\varphi^{\tau-t}\|_{\mathcal{E}}^2 \leqslant C\epsilon. \tag{2.49}$$

Due to (A3) and $k_1 \in L^2(\mathbb{R}^d)$, there exists $K_2 = K_2(\tau, \omega, \epsilon) \geqslant 1$, such that for all $k > K_2$

$$Ce^{-\sigma' \tau} \int_{\tau-t}^{\tau} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|k_1(x)|^2 dxdr \leqslant Ce^{-\sigma' \tau} \int_{\tau-t}^{\tau} e^{\sigma' r} \int_{|x| \geqslant k} |k_1(x)|^2 dxdr \leqslant C\epsilon, \tag{2.50}$$

and

$$Ce^{-\sigma' \tau} \int_{\tau-t}^{\tau} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})|g(r, x)|^2 dxdr \leqslant Ce^{-\sigma' \tau} \int_{-\infty}^{\tau} e^{\sigma' r} \int_{|x| \geqslant k} |g(r, x)|^2 dxdr$$

$$\leqslant Ce^{-\sigma' \tau} \int_{-\infty}^{\tau} e^{\sigma' r}\|g(r, x)\|^2 dxdr \leqslant C\epsilon. \tag{2.51}$$

Note that $z(\theta_t\omega) = \sum_{j=1}^{m} h_j z(\theta_t\omega_j)$ and $h_j \in H^2(\mathbb{R}^d)$, there exists $K_3 = K_3(\omega, \epsilon)$ such that for all $k \geqslant K_3$ and $j = 1, 2, \cdots, m$,

$$\int_{|x| \geqslant k} (|h_j(x)|^2 + |\nabla h_j(x)|^2 + |\Delta h_j(x)|^2)dx \leqslant \frac{\epsilon}{r(\omega)}, \tag{2.52}$$

where $r(\omega)$ is tempered satisfying (2.4) and (2.5).
By (2.52), we have

$$Ce^{-\sigma' \tau} \int_{\tau-t}^{\tau} e^{\sigma' r} \int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})(|\Delta z(\theta_{r-\tau}\omega)|^2 + |z(\theta_{r-\tau}\omega)|^2)dxdr$$

$$\leqslant C \int_{\tau-t}^{\tau} e^{\sigma'(r-\tau)} \sum_{j=1}^{m} \int_{|x| \geqslant k} (|\Delta h_i|^2|z_j(\theta_{r-\tau}\omega_j)|^2 + |h_j|^2|z_j(\theta_{r-\tau}\omega_j)|^2)dxdr$$

$$\leqslant C \int_{-\infty}^{0} e^{\sigma' r} \sum_{j=1}^{m} \int_{|x| \geqslant k} (|\Delta h_i|^2|z_j(\theta_r\omega_j)|^2 + |h_j|^2|z_j(\theta_r\omega_j)|^2)dxdr$$

$$\leqslant \frac{\epsilon}{r(\omega)} \int_{-\infty}^{0} e^{\sigma' r} \sum_{j=1}^{m} |z_j(\theta_r\omega_j)|^2 dr \leqslant C\epsilon. \tag{2.53}$$

In view of Lemma 2.2, (2.4) and (A3), there exists $T_2 = T_2(\tau, \omega, D, \epsilon)$, $K_4 = K_4(\epsilon) \geqslant 1$ such that

$$\frac{C}{k}e^{-\sigma' \tau} \int_{\tau-t}^{\tau} e^{\sigma' r}(\|\nabla u(r, \tau - t, \theta_{-\tau}\omega, u^{\tau-t})\|^2 + \|v(r, \tau - t, \theta_{-\tau}\omega, v^{\tau-t})\|^2)dr$$

$$+ \frac{C}{k}e^{-\sigma'\tau}\int_{\tau-t}^{\tau} e^{\sigma'r}\|\nabla v(r, \tau - t, \theta_{-\tau}\omega, v^{\tau-t})\|^2 dr$$

$$\leqslant \frac{C}{k}e^{-\sigma'\tau}\int_{-\infty}^{\tau} e^{\sigma'r}\|g(r, x)\|^2 dr + \frac{C}{k}\int_{-\infty}^{0} e^{\sigma'r}\sum_{j=1}^{m}|z_j(\theta_r\omega_j)|^2 dr + \frac{C}{k} \leqslant C\epsilon. \qquad (2.54)$$

Let $T = \max\{T_1, T_2\} > 0, K = \max\{K_1, K_2, K_3, K_4\} \geqslant 1$. From (2.48)–(2.54) we have for all $t > T, k > K$,

$$\int_{\mathbb{R}^d} \xi^2(\frac{|x|^2}{k^2})Y(\tau + s, \tau - t, \theta_{-\tau}\omega, Y^{\tau-t})dx \leqslant C\epsilon,$$

which implies

$$\|\varphi^\tau(s, \tau - t, \theta_{-t}\omega, \varphi^{\tau-t})\|^2_{\mathscr{E}(\mathbb{R}^d\backslash\Omega_k)} \leqslant C\epsilon.$$

Then the proof of Lemma 2.3 is finished. □

2.3. Estimates on bounded domain

We now decompose the solutions of (2.7) and (2.8) in bounded domains and derive some uniform estimates.

Let $\Omega_k = \{x \in \mathbb{R}^d : |x| \leqslant k\}$, given $k \geqslant 1$ and set

$$\begin{cases} \tilde{u}(t, \tau, \omega, \tilde{u}^\tau) = (1 - \xi^2(\frac{|x|^2}{k^2}))u(t, \tau, \omega, u^\tau), \\ \tilde{v}(t, \tau, \omega, \tilde{v}^\tau) = (1 - \xi^2(\frac{|x|^2}{k^2}))v(t, \tau, \omega, v^\tau), \end{cases} \qquad (2.55)$$

where ξ is the cutoff function defined in (2.34).
Multiplying (2.7) by $1 - \xi^2(\frac{|x|^2}{k^2})$, we have

$$\begin{cases} \dfrac{d\tilde{u}}{dt} = \tilde{v} - \sigma\tilde{u} + (1 - \xi^2(\frac{|x|^2}{k^2}))z(\theta_t\omega), \\ \dfrac{d\tilde{v}}{dt} = \alpha\Delta\tilde{v} + 2\alpha\nabla v\nabla\xi^2(\frac{|x|^2}{k^2}) + \alpha v\Delta\xi^2(\frac{|x|^2}{k^2}) - (1 - \sigma)\tilde{v} + (1 - \alpha\sigma)\Delta\tilde{u} \\ \qquad + 2(1 - \alpha\sigma)\nabla u\nabla\xi^2(\frac{|x|^2}{k^2}) + (1 - \alpha\sigma)\Delta\xi^2(\frac{|x|^2}{k^2}) - (\sigma^2 + \lambda - \sigma)\tilde{u} \\ \qquad + (1 - \xi^2(\frac{|x|^2}{k^2}))(f(x, u(t - \rho(t))) + g(t, x) + \sigma z(\theta_t\omega) + \alpha\Delta z(\theta_t\omega)), \end{cases} \qquad (2.56)$$

with the initial conditions

$$\tilde{u}(\tau + s, x) = \tilde{u}^\tau(x) \equiv (1 - \xi^2(\frac{|x|^2}{k^2}))\phi(s, x), \tilde{v}(\tau + s, x) = \tilde{v}^\tau(x), \quad s \in [-h, 0], x \in \mathbb{R}^d, \qquad (2.57)$$

where $\tilde{v}^\tau(x) \equiv (1-\xi^2(\frac{|x|^2}{k^2}))\psi(s, x)+\sigma(1-\xi^2(\frac{|x|^2}{k^2}))\phi(s, x)-(1-\xi^2(\frac{|x|^2}{k^2}))z(\theta_{\tau+s}\omega)$. Now we decompose (2.56) into two parts. Set $\tilde{u} = u_1 + u_2, \tilde{v} = v_1 + v_2$, we have two new systems:

$$\begin{cases} \dfrac{du_2}{dt} = v_2 - \sigma u_2, \\ \dfrac{dv_2}{dt} = \alpha\Delta v_2 - (1 - \sigma)v_2 + (1 - \alpha\sigma)\Delta u_2 - (\sigma^2 + \lambda - \sigma)u_2, \end{cases} \qquad (2.58)$$

with the initial conditions

$$u_2^\tau(x) = \tilde{u}(\tau + s, x), v_2^\tau(x) = \tilde{v}(\tau + s, x), \quad s \in [-h, 0], x \in \Omega_{2k}, \tag{2.59}$$

and boundary conditions

$$u_2(t, x) = 0, v_2(t, x) = 0, \quad t \in [-h, +\infty), |x| = 2k, \tag{2.60}$$

and

$$\begin{cases} \dfrac{du_1}{dt} = v_1 - \sigma u_1 + (1 - \xi^2(\dfrac{|x|^2}{k^2}))z(\theta_t\omega), \\[2mm] \dfrac{dv_1}{dt} = \alpha\Delta v_1 + 2\alpha\nabla v\nabla\xi^2(\dfrac{|x|^2}{k^2}) + \alpha v\Delta\xi^2(\dfrac{|x|^2}{k^2}) - (1 - \sigma)v_1 + (1 - \alpha\sigma)\Delta u_1 \\[2mm] \qquad + 2(1 - \alpha\sigma)\nabla u\nabla\xi^2(\dfrac{|x|^2}{k^2}) + (1 - \alpha\sigma)u\Delta\xi^2(\dfrac{|x|^2}{k^2}) - (\sigma^2 + \lambda - \sigma)u_1 \\[2mm] \qquad + (1 - \xi^2(\dfrac{|x|^2}{k^2}))(f(x, u(t - \rho(t))) + g(t, x) + \sigma z(\theta_t\omega) + \alpha\Delta z(\theta_t\omega)), \end{cases} \tag{2.61}$$

with the initial conditions

$$u_1^\tau(x) = 0, v_1^\tau(x) = 0, \quad s \in [-h, 0], x \in \Omega_{2k}, \tag{2.62}$$

and boundary conditions

$$u_1(t, x) = 0, v_1(t, x) = 0, \quad t \in [-h, +\infty), |x| = 2k. \tag{2.63}$$

Using the similar arguments in Lemma 2.2, we derive the following estimate of solutions of (2.56) and (2.57),

$$\|(\tilde{u}(\tau + s, \tau - t, \theta_{-\tau}\omega, \tilde{u}^{\tau-t}), \tilde{v}(\tau + s, \tau - t, \theta_{-\tau}\omega, \tilde{v}^{\tau-t}))\|^2_{\mathscr{E}(\Omega_{2k})}$$

$$+ C \int_{\tau-t}^{\tau} e^{\sigma'r}\|(\tilde{u}(r + s, \tau - t, \theta_{-\tau}\omega, \tilde{u}^{\tau-t}), \tilde{v}(r + s, \tau - t, \theta_{-\tau}\omega, \tilde{v}^{\tau-t}))\|^2_{\mathscr{E}(\Omega_{2k})}dr$$

$$+ 2\alpha \int_{\tau-t}^{\tau} e^{\sigma'(r-\tau-s)}\|\nabla\tilde{v}(r, \tau - t, \theta_{-\tau}\omega, \tilde{v}^{\tau-t})\|^2dr$$

$$\leqslant Ce^{-\sigma't}\|\tilde{\varphi}^{\tau-t}\|^2_{\mathscr{E}(\Omega_{2k})} + Ce^{-\sigma'\tau}\int_{-\infty}^{\tau} e^{\sigma'r}\|g(r, x)\|^2dr + Cr(\omega) + C. \tag{2.64}$$

It follows from (2.64), we obtain the estimate of (u_2, v_2)

$$\|(u_2(\tau + s, \tau - t, \theta_{-\tau}\omega, u_2^{\tau-t}), v_2(\tau + s, \tau - t, \theta_{-\tau}\omega, v_2^{\tau-t}))\|^2_{\mathscr{E}(\Omega_{2k})}$$

$$+ C \int_{\tau-t}^{\tau} e^{\sigma'r}\|(u_2(r + s, \tau - t, \theta_{-\tau}\omega, u_2^{\tau-t}), v_2(r + s, \tau - t, \theta_{-\tau}\omega, v_2^{\tau-t}))\|^2_{\mathscr{E}(\Omega_{2k})}dr$$

$$+ 2\alpha \int_{\tau-t}^{\tau} e^{\sigma'(r-\tau-s)}\|\nabla v_2(r, \tau - t, \theta_{-\tau}\omega, v_2^{\tau-t})\|^2dr$$

$$\leqslant Ce^{-\sigma't}\|\varphi_2^{\tau-t}\|^2_{\mathscr{E}(\Omega_{2k})} = Ce^{-\sigma't}\|\tilde{\varphi}^{\tau-t}\|^2_{\mathscr{E}(\Omega_{2k})}. \tag{2.65}$$

It follows from (2.64) and (2.65) that

$$\|(u_1(\tau+s,\tau-t,\theta_{-\tau}\omega,0),v_1(\tau+s,\tau-t,\theta_{-\tau}\omega,0)\|^2_{\mathscr{E}(\Omega_{2k})}$$

$$+ C\int_{\tau-t}^{\tau} e^{\sigma' r}\|(u_1(r+s,\tau-t,\theta_{-\tau}\omega,0),v_1(r+s,\tau-t,\theta_{-\tau}\omega,0))\|^2_{\mathscr{E}(\Omega_{2k})}dr$$

$$+ 2\alpha\int_{\tau-t}^{\tau} e^{\sigma'(r-\tau-s)}\|\nabla v_1(r,\tau-t,\theta_{-\tau}\omega,0)\|^2 dr$$

$$\leqslant 2\|(\tilde{u}(\tau+s,\tau-t,\theta_{-\tau}\omega,\tilde{u}^{\tau-t}),\tilde{v}(\tau+s,\tau-t,\theta_{-\tau}\omega,\tilde{v}^{\tau-t}))\|^2_{\mathscr{E}(\Omega_{2k})}$$

$$+ 2\|(u_2(\tau+s,\tau-t,\theta_{-\tau}\omega,u_2^{\tau-t}),v_2(\tau+s,\tau-t,\theta_{-\tau}\omega,v_2^{\tau-t}))\|^2_{\mathscr{E}(\Omega_{2k})}$$

$$+ 2C\int_{\tau-t}^{\tau} e^{\sigma' r}\|(\tilde{u}(r+s,\tau-t,\theta_{-\tau}\omega,\tilde{u}^{\tau-t}),\tilde{v}(r+s,\tau-t,\theta_{-\tau}\omega,\tilde{v}^{\tau-t}))\|^2_{\mathscr{E}(\Omega_{2k})}dr$$

$$+ 2C\int_{\tau-t}^{\tau} e^{\sigma' r}\|(u_2(r+s,\tau-t,\theta_{-\tau}\omega,u_2^{\tau-t}),v_2(r+s,\tau-t,\theta_{-\tau}\omega,v_2^{\tau-t}))\|^2_{\mathscr{E}(\Omega_{2k})}dr$$

$$+ 4\alpha\int_{\tau-t}^{\tau} e^{\sigma'(r-\tau-s)}\|\nabla\tilde{v}(r,\tau-t,\theta_{-\tau}\omega,\tilde{v}^{\tau-t})\|^2 dr$$

$$+ 4\alpha\int_{\tau-t}^{\tau} e^{\sigma'(r-\tau-s)}\|\nabla v_2(r,\tau-t,\theta_{-\tau}\omega,v_2^{\tau-t})\|^2 dr$$

$$\leqslant Ce^{-\sigma' t}\|\tilde{\varphi}^{\tau-t}\|^2_{\mathscr{E}(\Omega_{2k})} + Ce^{-\sigma'\tau}\int_{-\infty}^{\tau} e^{\sigma' r}\|g(r,x)\|^2 dr + Cr(\omega) + C,$$

Since $\tilde{\varphi}^{\tau-t} = (\tilde{u}^{\tau-t},\tilde{v}^{\tau-t}) \in D(\tau-t,\theta_{-t}\omega) \in \mathcal{D}(\mathscr{E}(\Omega_{2k}))$, we have

$$\limsup_{t\to\infty} Ce^{-\sigma' t}\|\tilde{\varphi}^{\tau-t}\|^2_{\mathscr{E}(\Omega_{2k})} \leqslant \limsup_{t\to\infty} Ce^{-\sigma' t}\|D(\tau-t,\theta_{-t}\omega)\|^2_{\mathscr{E}(\Omega_{2k})} = 0,$$

Therefore, there exists $T = T(\tau,\omega,D) > 0$ such that for all $t \geqslant T$,

$$Ce^{-\sigma' t}\|\tilde{\varphi}^{\tau-t}\|^2_{\mathscr{E}(\Omega_{2k})} \leqslant 1.$$

Thus, for all $t \geqslant T$, we have

$$\|(u_1(\tau+s,\tau-t,\theta_{-\tau}\omega,0),v_1(\tau+s,\tau-t,\theta_{-\tau}\omega,0)\|^2_{\mathscr{E}(\Omega_{2k})}$$

$$+ C\int_{\tau-t}^{\tau} e^{\sigma' r}\|(u_1(r+s,\tau-t,\theta_{-\tau}\omega,0),v_1(r+s,\tau-t,\theta_{-\tau}\omega,0))\|^2_{\mathscr{E}(\Omega_{2k})}dr$$

$$+ 2\alpha\int_{\tau-t}^{\tau} e^{\sigma'(r-\tau-s)}\|\nabla v_1(r,\tau-t,\theta_{-\tau}\omega,0)\|^2 dr$$

$$\leqslant C + + Cr(\omega) + Ce^{-\sigma'\tau}\int_{-\infty}^{\tau} e^{\sigma' r}\|g(r,x)\|^2 dr \leqslant Cr_1(\tau,\omega), \tag{2.66}$$

where $r_1(\tau,\omega)$ is defined in Lemma 2.2.

Furthermore, we can give the uniform estimates of $(\Delta u_1, \Delta v_1)$, which imply the higher regularity of (u_1, v_1).

Lemma 2.4. *Assume the hypotheses in Lemma 2.2 hold, and let $\tau \in \mathbb{R}, \omega \in \Omega$ and $D \in \mathcal{D}(\mathscr{E}(\Omega_{2k}))$. Then for given $\epsilon > 0$, there exist random variables $r_2(\tau,\omega)$, and $T = T(\tau,\omega,D) > 0$, such that for all $t \geqslant T, k \geqslant 1$ and $-h \leqslant s \leqslant 0$,*

$$(1 - \alpha\sigma)\|\Delta u_1(\tau+s,\tau-t,\theta_{-\tau}\omega,0)\|^2 + \|\nabla v_1(\tau+s,\tau-t,\theta_{-\tau}\omega,0)\|^2$$

$$+ \alpha \int_{\tau-t}^{\tau} e^{\sigma'(r-\tau-s)} \|\Delta v_1(r, \tau - t, \theta_{-\tau}\omega, 0)\|^2 dr \leqslant r_2(\tau, \omega),$$

where $r_2(\tau, \omega) = Cr_1(\tau, \omega), r_1(\tau, \omega)$ is defined as in Lemma 2.2.

Proof. From the first equation in (2.61), we have

$$\frac{d}{dt}\Delta u_1 = \Delta v_1 - \sigma\Delta u_1 + (1 - \xi^2(\frac{|x|^2}{k^2}))\Delta z(\theta_t\omega). \tag{2.67}$$

Taking the inner product of the second equation in (2.61) and (2.67) with $-\Delta v_1$ and Δu_1 respectively, we have

$$\frac{d}{dt}\|v_1\|^2$$
$$= -2\alpha\|v_1\|^2 - 2(1 - \sigma)\|\nabla v_1\|^2 - 2(1 - \alpha\sigma)(\Delta u_1, \Delta v_1) + 2(\sigma^2 + \lambda - \sigma)(u_1, \Delta v_1)$$
$$- 4\alpha(\nabla v\nabla\xi^2(\frac{|x|^2}{k^2}), \Delta v_1) - 2\alpha(v\Delta\xi^2(\frac{|x|^2}{k^2}), \Delta v_1)$$
$$- 4(1 - \alpha\sigma)(\nabla u\nabla\xi^2(\frac{|x|^2}{k^2}), \Delta v_1) - 2(1 - \alpha\sigma)(u\Delta\xi^2(\frac{|x|^2}{k^2}), \Delta v_1)$$
$$- 2(1 - \xi^2(\frac{|x|^2}{k^2}))(f(x, u(t - \rho(t))) + g(t, x) + \sigma z(\theta_t\omega) + \alpha\Delta z(\theta_t\omega), \Delta v_1), \tag{2.68}$$

and

$$\frac{d}{dt}\|\Delta u_1\|^2 = 2(\Delta v_1, \Delta u_1) - 2\sigma\|\Delta u_1\|^2 + 2(1 - \xi^2(\frac{|x|^2}{k^2}))(\Delta z(\theta_t\omega), \Delta u_1). \tag{2.69}$$

Adding up (2.69)×$(1 - \alpha\sigma)$ and (2.68), we obtain

$$\frac{d}{dt}((1 - \alpha\sigma)\|\Delta u_1\|^2 + \|\nabla v_1\|^2)$$
$$= -2\sigma(1 - \alpha\sigma)\|\Delta u_1\|^2 - 2\alpha\|\Delta v_1\|^2 - 2(1 - \sigma)\|\nabla v_1\|^2 + 2(\sigma^2 + \lambda - \sigma)(u_1, \Delta v_1)$$
$$- 2(1 - \xi^2(\frac{|x|^2}{k^2}))(f(x, u(t - \rho(t))) + g(t, x), \Delta v_1)$$
$$- 2(1 - \xi^2(\frac{|x|^2}{k^2}))(\sigma z(\theta_t\omega) + \alpha\Delta z(\theta_t\omega), \Delta v_1) + 2(1 - \alpha\sigma)(1 - \xi^2(\frac{|x|^2}{k^2}))(\Delta z(\theta_t\omega), \Delta u_1)$$
$$- [4\alpha(\nabla v\nabla\xi^2(\frac{|x|^2}{k^2}), \Delta v_1) + 2\alpha(v\Delta\xi^2(\frac{|x|^2}{k^2}), \Delta v_1)]$$
$$- [4(1 - \alpha\sigma)(\nabla u\nabla\xi^2(\frac{|x|^2}{k^2}), \Delta v_1) + 2(1 - \alpha\sigma)(u\Delta\xi^2(\frac{|x|^2}{k^2}), \Delta v_1)]$$
$$= -2\sigma(1 - \alpha\sigma)\|\Delta u_1\|^2 - 2\alpha\|\Delta v_1\|^2 - 2(1 - \sigma)\|\nabla v_1\|^2 + \sum_{i=1}^{6} \mathscr{T}_i.. \tag{2.70}$$

Next, we give the estimates of the terms on the right-hand of (2.70). It follows from Cauchy-Schwarz inequality and Young's inequality that

$$\mathscr{T}_1 \leqslant \frac{\alpha}{9}\|\Delta v_1\|^2 + \frac{9(\sigma^2 + \lambda - \sigma)^2}{\alpha}\|u_1\|^2,$$

$$\mathscr{T}_2 \leqslant \frac{2\alpha}{9}\|\Delta v_1\|^2 + \frac{9}{\alpha}\|k_1\|^2 + +\frac{9k_2^2}{\alpha}\|u^t\|_{C_h}^2 + \frac{9}{\alpha}\|g(t,x)\|^2,$$

$$\mathscr{T}_3 \leqslant \frac{2\alpha}{9}\|\Delta v_1\|^2 + \frac{9\sigma^2}{\alpha}\|z(\theta_t\omega)\|^2 + \frac{9}{\alpha}\|\Delta z(\theta_t\omega)\|^2,$$

and

$$\mathscr{T}_4 \leqslant \sigma(1 - \alpha\sigma)\|\Delta u_1\|^2 + \sigma(1 - \alpha\sigma)\|\Delta z(\theta_t\omega)\|^2.$$

Using the properties of the cutoff function ξ, we infer that

$$\mathscr{T}_5 = -4\alpha(\frac{4x}{k^2}\nabla v\xi(\frac{|x|^2}{k^2})\xi'(\frac{|x|^2}{k^2}), \Delta v_1)$$
$$- 2\alpha(v(\frac{4}{k^2}\xi(\frac{|x|^2}{k^2})\xi'(\frac{|x|^2}{k^2}) + \frac{8x^2}{k^4}(\xi'(\frac{|x|^2}{k^2}))^2 + \frac{8x^2}{k^4}\xi(\frac{|x|^2}{k^2})\xi''(\frac{|x|^2}{k^2})), \Delta v_1),$$

where

$$-4\alpha(\frac{4x}{k^2}\nabla v\xi(\frac{|x|^2}{k^2})\xi'(\frac{|x|^2}{k^2}), \Delta v_1) = -\frac{16\alpha}{k^2}\int_{\Omega_{2k}} x\nabla v\xi(\frac{|x|^2}{k^2})\xi'(\frac{|x|^2}{k^2})\Delta v_1 dx$$

$$\leqslant \frac{16\sqrt{2}\alpha\mu_1}{k}\int_{k\leqslant|x|\leqslant\sqrt{2}k} |\nabla v||\Delta v_1|dx$$

$$\leqslant \frac{16\sqrt{2}\alpha\mu_1}{k}\int_{\Omega_{2k}} |\nabla v||\Delta v_1|dx$$

$$\leqslant \frac{\alpha}{9}\|\Delta v_1\|^2 + \frac{2^7 3^2 \mu_1^2 \alpha}{k^2}\|\nabla v\|^2,$$

and similarly,

$$- 2\alpha(v(\frac{4}{k^2}\xi(\frac{|x|^2}{k^2})\xi'(\frac{|x|^2}{k^2}) + \frac{8x^2}{k^4}(\xi'(\frac{|x|^2}{k^2}))^2 + \frac{8x^2}{k^4}\xi(\frac{|x|^2}{k^2})\xi''(\frac{|x|^2}{k^2})), \Delta v_1)$$

$$\leqslant \frac{\alpha}{9}\|\Delta v_1\|^2 + (\frac{2^4 3^3 \mu_1^2 \alpha}{k^4} + \frac{2^8 3^3 \mu_1^4 \alpha}{k^4} + \frac{2^8 3^3 \mu_2^2 \alpha}{k^4})\|v\|^2.$$

Thus we have

$$\mathscr{T}_5 \leqslant \frac{2\alpha}{9}\|\Delta v_1\|^2 + \frac{2^7 3^2 \mu_1^2 \alpha}{k^2}\|\nabla v\|^2 + (\frac{2^4 3^3 \mu_1^2 \alpha}{k^4} + \frac{2^8 3^3 \mu_1^4 \alpha}{k^4} + \frac{2^8 3^3 \mu_2^2 \alpha}{k^4})\|v\|^2.$$

In the same way we obtain

$$\mathscr{T}_6 \leqslant \frac{2\alpha}{9}\|\Delta v_1\|^2 + \frac{2^7 3^2 \mu_1^2 \alpha}{k^2}\|\nabla u\|^2$$
$$+ (\frac{2^4 3^3 \mu_1^2 (1 - \alpha\sigma)}{k^4\alpha} + \frac{2^8 3^3 \mu_1^4 (1 - \alpha\sigma)}{k^4\alpha} + \frac{2^8 3^3 \mu_2^2 (1 - \alpha\sigma)}{k^4\alpha})\|u\|^2.$$

Combining these estimates with (2.70), we get

$$\frac{d}{dt}((1 - \alpha\sigma)\|\Delta u_1\|^2 + \|\nabla v_1\|^2) + \alpha\|\Delta v_1\|^2$$

$$\leqslant -\sigma(1 - \alpha\sigma)\|\Delta u_1\|^2 - 2(1 - \sigma)\|\nabla v_1\|^2 + \frac{9(\sigma^2 + \lambda - \sigma)^2}{\alpha}\|u_1\|^2$$

$$+ \frac{C}{k^2}(\|\nabla u\|^2 + \|\nabla v\|^2) + \frac{C}{k^4}(\|u\|^2 + \|v\|^2)$$

$$+ \frac{9}{\alpha}\|k_1\|^2 + +\frac{9k_2^2}{\alpha}\|u^t\|_{C_h}^2 + \frac{9}{\alpha}\|g(t, x)\|^2$$

$$+ \frac{9\sigma^2}{\alpha}\|z(\theta_t\omega)\|^2 + (\sigma(1 - \alpha\sigma) + \frac{9}{\alpha})\|\Delta z(\theta_t\omega)\|^2. \tag{2.71}$$

Let $\sigma' > 0$ be a constant satisfying (2.11)–(2.13). Multiplying $e^{\sigma't}$ on both sides of (2.71) and integrating over $(\tau - t, \tau + s)$, where $s \in [-h, 0]$, we obtain

$$(1 - \alpha\sigma)\|\Delta u_1(\tau + s, \tau - t, \omega, 0)\|^2 + \|\nabla v_1(\tau + s, \tau - t, \omega, 0)\|^2$$

$$+ \alpha e^{-\sigma'(\tau+s)} \int_{\tau-t}^{\tau+s} e^{\sigma'r}\|\Delta v_1(r, \tau - t, \omega, 0)\|^2 dr$$

$$\leqslant -(\sigma - \sigma')(1 - \alpha\sigma)e^{-\sigma'(\tau+s)} \int_{\tau-t}^{\tau+s} e^{\sigma'r}\|\Delta u_1(r, \tau - t, \omega, 0)\|^2 dr$$

$$- (2(1 - \sigma) - \sigma')e^{-\sigma'(\tau+s)} \int_{\tau-t}^{\tau+s} e^{\sigma'r}\|\nabla v_1(r, \tau - t, \omega, 0)\|^2 dr$$

$$+ \frac{9(\sigma^2 + \lambda - \sigma)^2}{\alpha} e^{-\sigma'\tau} \int_{\tau-t}^{\tau} e^{\sigma'r}\|u_1(r, \tau - t, \omega, 0)\|^2 dr$$

$$+ \frac{C}{k^2} e^{-\sigma'\tau} \int_{\tau-t}^{\tau} e^{\sigma'r}(\|\nabla u(r, \tau - t, \omega, u^{\tau-t})\|^2 + \|\nabla v(r, \tau - t, \omega, v^{\tau-t})\|^2) dr$$

$$+ \frac{C}{k^4} e^{-\sigma'\tau} \int_{\tau-t}^{\tau} e^{\sigma'r}(\|u(r, \tau - t, \omega, u^{\tau-t})\|^2 + \|v(r, \tau - t, \omega, v^{\tau-t})\|^2) dr$$

$$+ C + \frac{9k_2^2}{\alpha} e^{-\sigma'\tau} \int_{\tau-t}^{\tau} e^{\sigma'r}\|u^r(s, \tau - t, \omega, u^{\tau-t})\|_{C_h}^2 dr$$

$$+ Ce^{-\sigma'\tau} \int_{-\infty}^{\tau} e^{\sigma'r}\|g(r, x)\|^2 dr + Ce^{-\sigma'\tau} \int_{-\infty}^{0} e^{\sigma'r}(\|\Delta z(\theta_r\omega)\|^2 + \|z(\theta_r\omega)\|^2) dr. \tag{2.72}$$

Now we replace ω by $\theta_{-\tau}\omega$ and estimate the terms on the right-hand of (2.72).

By Lemma 2.2, there exists $T_1 = T_1(\tau, \omega, D) > 0$, such that for all $t \geqslant T_1$,

$$\frac{C}{k^2} e^{-\sigma'\tau} \int_{\tau-t}^{\tau} e^{\sigma'r}(\|\nabla u(r, \tau - t, \theta_{-\tau}\omega, u^{\tau-t})\|^2 + \|\nabla v(r, \tau - t, \theta_{-\tau}\omega, v^{\tau-t})\|^2) dr$$

$$+ \frac{C}{k^4} e^{-\sigma'\tau} \int_{\tau-t}^{\tau} e^{\sigma'r}(\|u(r, \tau - t, \theta_{-\tau}\omega, u^{\tau-t})\|^2 + \|v(r, \tau - t, \theta_{-\tau}\omega, v^{\tau-t})\|^2) dr$$

$$\leqslant Cr_1(\tau, \omega), \tag{2.73}$$

and

$$\frac{9k_2^2}{\alpha} e^{-\sigma'\tau} \int_{\tau-t}^{\tau} e^{\sigma'r}\|u^r(s, \tau - t, \theta_{-\tau}\omega, u^{\tau-t})\|_{C_h}^2 dr \leqslant Cr_1(\tau, \omega). \tag{2.74}$$

From (2.66), there exists $T_2 = T_2(\tau, \omega, D) > 0$, such that for all $t \geqslant T_2$,

$$\frac{9(\sigma^2 + \lambda - \sigma)^2}{\alpha} e^{-\sigma'\tau} \int_{\tau-t}^{\tau} e^{\sigma'r} \|u_1(r, \tau - t, \theta_{-\tau}\omega, 0)\|^2 dr \leqslant Cr_1(\tau, \omega). \tag{2.75}$$

By (2.30), we obtain

$$Ce^{-\sigma'\tau} \int_{-\infty}^{\tau} e^{\sigma'r} \|g(r, x)\|^2 dr + Ce^{-\sigma'\tau} \int_{\tau-t}^{\tau} e^{\sigma'r} (\|\Delta z(\theta_{r-\tau}\omega)\|^2 + \|z(\theta_{r-\tau}\omega)\|^2) dr \leqslant Cr_1(\tau, \omega). \tag{2.76}$$

Let $T = \max\{T_1, T_2\}$. It follows from (2.72)–(2.76) that for all $t \geqslant T$,

$$(1 - \alpha\sigma)\|\Delta u_1(\tau + s, \tau - t, \theta_{-\tau}\omega, 0)\|^2 + \|\nabla v_1(\tau + s, \tau - t, \theta_{-\tau}\theta_{-\tau}\omega, 0)\|^2$$
$$+ \alpha \int_{\tau-t}^{\tau} e^{\sigma'(r-\tau-s)} \|\Delta v_1(r, \tau - t, \theta_{-\tau}\omega, 0)\|^2 dr \leqslant Cr_1(\tau, \omega) \equiv r_2(\tau, \omega).$$

Thus we complete the proof of Lemma 2.4. □

Next, we establish the Hölder continuity of φ_1 in time, which will be useful to show the equicontinuity of solutions in $C([-h, 0], E(\Omega_{2k}))$ based on the Arzela-Ascoli theorem.

Lemma 2.5. *Assume the hypotheses in Lemma 2.2 hold, and let $\tau \in \mathbb{R}, \omega \in \Omega$ and $D \in \mathcal{D}(\mathcal{E}(\Omega_{2k}))$. Then there exist random variables $r_3(\tau, \omega)$ and $T = T(\tau, \omega, D) > 0$, such that for all $t \geqslant T, k \geqslant 1$ and $-h \leqslant \eta_1 < \eta_2 \leqslant 0$,*

$$\|\varphi_1^\tau(\eta_2, \tau - t, \theta_{-\tau}\omega, 0) - \varphi_1^\tau(\eta_1, \tau - t, \theta_{-\tau}\omega, 0)\|_{E(\Omega_{2k})}^2 \leqslant r_3(\tau, \omega)|\eta_2 - \eta_1|^{\frac{1}{2}}$$

where $\varphi_1^\tau(\eta_i, \tau - t, \theta_{-\tau}\omega, 0) = (u_1^\tau(\eta_i, \tau - t, \theta_{-\tau}\omega, 0), v_1^\tau(\eta_i, \tau - t, \theta_{-\tau}\omega, 0)), i = 1, 2.$

Proof. By the definition of the norm $\|\cdot\|_E$, we have

$$\|\varphi_1^\tau(\eta_2, \tau - t, \theta_{-\tau}\omega, 0) - \varphi_1^\tau(\eta_1, \tau - t, \theta_{-\tau}\omega, 0)\|_{E(\Omega_{2k})}^2$$
$$= (\sigma^2 + \lambda - \sigma)\|u_1^\tau(\eta_2, \tau - t, \theta_{-\tau}\omega, 0) - u_1^\tau(\eta_1, \tau - t, \theta_{-\tau}\omega, 0)\|^2$$
$$+ (1 - \alpha\sigma)\|\nabla u_1^\tau(\eta_2, \tau - t, \theta_{-\tau}\omega, 0) - \nabla u_1^\tau(\eta_1, \tau - t, \theta_{-\tau}\omega, 0)\|^2$$
$$+ \|v_1^\tau(\eta_2, \tau - t, \theta_{-\tau}\omega, 0) - v_1^\tau(\eta_1, \tau - t, \theta_{-\tau}\omega, 0)\|^2)^{\frac{1}{2}}$$
$$\leqslant (\sigma^2 + \lambda - \sigma)\|u_1^\tau(\eta_2, \tau - t, \theta_{-\tau}\omega, 0) - u_1^\tau(\eta_1, \tau - t, \theta_{-\tau}\omega, 0)\|$$
$$+ (1 - \alpha\sigma)\|\nabla u_1^\tau(\eta_2, \tau - t, \theta_{-\tau}\omega, 0) - \nabla u_1^\tau(\eta_1, \tau - t, \theta_{-\tau}\omega, 0)\|$$
$$+ \|v_1^\tau(\eta_2, \tau - t, \theta_{-\tau}\omega, 0) - v_1^\tau(\eta_1, \tau - t, \theta_{-\tau}\omega, 0)\|$$
$$\leqslant (\sigma^2 + \lambda - \sigma) \int_{\tau+\eta_1}^{\tau+\eta_2} \|\frac{d}{dr}u_1(r, \tau - t, \theta_{-\tau}\omega, 0)\| dr$$
$$+ (1 - \alpha\sigma) \int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla \frac{d}{dr}u_1(r, \tau - t, \theta_{-\tau}\omega, 0)\| dr$$
$$+ \int_{\tau+\eta_1}^{\tau+\eta_2} \|\frac{d}{dr}v_1(r, \tau - t, \theta_{-\tau}\omega, 0)\| dr. \tag{2.77}$$

From (2.61), we have

$$\frac{du_1}{dt} = v_1 - \sigma u_1 + (1 - \xi^2(\frac{|x|^2}{k^2}))z(\theta_t\omega),$$

together with (2.66), we know that

there exists $T_1 = T_1(\tau, \omega, D) > 0$ such that for all $t \geqslant T_1$,

$$\int_{\tau+\eta_1}^{\tau+\eta_2} \|\frac{d}{dr}u_1(r, \tau - t, \theta_{-\tau}\omega, 0)\|dr$$

$$\leqslant \int_{\tau+\eta_1}^{\tau+\eta_2} \|v_1(r, \tau - t, \theta_{-\tau}\omega, 0)\|dr + \sigma \int_{\tau+\eta_1}^{\tau+\eta_2} \|u_1(r, \tau - t, \theta_{-\tau}\omega, 0)\|dr$$

$$+ \int_{\tau+\eta_1}^{\tau+\eta_2} \|z(\theta_{r-\tau}\omega)\|dr$$

$$\leqslant \left\{ \left(\int_{\tau+\eta_1}^{\tau+\eta_2} \|v_1(r, \tau - t, \theta_{-\tau}\omega, 0)\|^2 dr\right)^{\frac{1}{2}} + \sigma \left(\int_{\tau+\eta_1}^{\tau+\eta_2} \|u_1(r, \tau - t, \theta_{-\tau}\omega, 0)\|^2 dr\right)^{\frac{1}{2}} \right.$$

$$\left. + \left(\int_{\tau+\eta_1}^{\tau+\eta_2} \|z(\theta_{r-\tau}\omega)\|^2 dr\right)^{\frac{1}{2}} \right\} |\eta_2 - \eta_1|^{\frac{1}{2}}$$

$$\leqslant \left\{ \left(1 + \frac{\sigma}{\sqrt{\sigma^2 + \lambda} - \sigma}\right) \left(\int_{\tau+\eta_1}^{\tau+\eta_2} r_1(\tau, \theta_{r-\tau}\omega)dr\right)^{\frac{1}{2}} \right.$$

$$\left. + \left(\int_{\tau+\eta_1}^{\tau+\eta_2} \|z(\theta_{r-\tau}\omega)\|^2 dr\right)^{\frac{1}{2}} \right\} |\eta_2 - \eta_1|^{\frac{1}{2}}$$

$$= \left\{ \left(1 + \frac{\sigma}{\sqrt{\sigma^2 + \lambda} - \sigma}\right) \left(\int_{\eta_1}^{\eta_2} r_1(\tau, \theta_r\omega)dr\right)^{\frac{1}{2}} + \left(\int_{\eta_1}^{\eta_2} \|z(\theta_r\omega)\|^2 dr\right)^{\frac{1}{2}} \right\} |\eta_2 - \eta_1|^{\frac{1}{2}}$$

$$\leqslant \left\{ \left(1 + \frac{\sigma}{\sqrt{\sigma^2 + \lambda} - \sigma}\right) h^{\frac{1}{2}} \left(\sup_{r\in[-h,0]} r_1(\tau, \theta_r\omega)\right)^{\frac{1}{2}} + e^{\frac{\sigma'h}{4}} h^{\frac{1}{2}} r_2^{\frac{1}{2}}(\omega) \right\} |\eta_2 - \eta_1|^{\frac{1}{2}} \quad (2.78)$$

Since

$$e^{-\sigma'h} \int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla v_1(r, \tau - t, \theta_{-\tau}\omega, 0)\|^2 dr \leqslant \int_{\tau+\eta_1}^{\tau+\eta_2} e^{\sigma'(r-\tau-\eta_2)} \|\nabla v_1(r, \tau - t, \theta_{-\tau}\omega, 0)\|^2 dr$$

$$\leqslant \int_{\tau-t}^{\tau+\eta_2} e^{\sigma'(r-\tau-\eta_2)} \|\nabla v_1(r, \tau - t, \theta_{-\tau}\omega, 0)\|^2 dr, \quad (2.79)$$

by (2.66) and (2.79), we get for $t \geqslant T_1$,

$$\int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla v_1(r, \tau - t, \theta_{-\tau}\omega, 0)\|^2 dr \leqslant Ce^{\sigma'h}r_1(\tau, \omega). \quad (2.80)$$

Using the similar computation in (2.78), we obtain for $t \geqslant T_1$,

$$\int_{\tau+\eta_1}^{\tau+\eta_2} \|\frac{d}{dr}\nabla u_1(r, \tau - t, \theta_{-\tau}\omega, 0)\|dr$$

$$\leqslant \int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla v_1(r, \tau - t, \theta_{-\tau}\omega, 0)\| dr + \sigma \int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla u_1(r, \tau - t, \theta_{-\tau}\omega, 0)\| dr$$

$$+ \int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla z(\theta_{r-\tau}\omega)\| dr$$

$$\leqslant \left\{ \left(\int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla v_1(r, \tau - t, \theta_{-\tau}\omega, 0)\|^2 dr \right)^{\frac{1}{2}} + \sigma \left(\int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla u_1(r, \tau - t, \theta_{-\tau}\omega, 0)\|^2 dr \right)^{\frac{1}{2}} \right.$$

$$\left. + \left(\int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla z(\theta_{r-\tau}\omega)\|^2 dr \right)^{\frac{1}{2}} \right\} |\eta_2 - \eta_1|^{\frac{1}{2}}$$

$$\leqslant \left\{ C^{\frac{1}{2}} \sqrt{r_1(\tau, \omega)} + \frac{C^{\frac{1}{2}}\sigma}{\sqrt{1 - \alpha\sigma}} \left(\int_{\tau+\eta_1}^{\tau+\eta_2} r_1(\tau, \theta_{r-\tau}\omega) dr \right)^{\frac{1}{2}} \right.$$

$$\left. + \left(\int_{\tau+\eta_1}^{\tau+\eta_2} \|z(\theta_{r-\tau}\omega)\|^2 dr \right)^{\frac{1}{2}} \right\} |\eta_2 - \eta_1|^{\frac{1}{2}}$$

$$= \left\{ C^{\frac{1}{2}} \sqrt{r_1(\tau, \omega)} + \frac{C^{\frac{1}{2}}\sigma}{\sqrt{1 - \alpha\sigma}} \left(\int_{\eta_1}^{\eta_2} r_1(\tau, \theta_r\omega) dr \right)^{\frac{1}{2}} + \left(\int_{\eta_1}^{\eta_2} \|z(\theta_r\omega)\|^2 dr \right)^{\frac{1}{2}} \right\} |\eta_2 - \eta_1|^{\frac{1}{2}}$$

$$\leqslant \left\{ C^{\frac{1}{2}} \sqrt{r_1(\tau, \omega)} + \frac{C^{\frac{1}{2}}\sigma}{\sqrt{1 - \alpha\sigma}} h^{\frac{1}{2}} \left(\sup_{r\in[-h,0]} r_1(\tau, \theta_r\omega) \right)^{\frac{1}{2}} + e^{\frac{\sigma' h}{4}} h^{\frac{1}{2}} r_2^{\frac{1}{2}}(\omega) \right\} |\eta_2 - \eta_1|^{\frac{1}{2}} \tag{2.81}$$

By (2.61), we get

$$\int_{\tau+\eta_1}^{\tau+\eta_2} \|\frac{d}{dr} v_1(r, \tau - t, \theta_{-\tau}\omega, 0)\| dr$$

$$\leqslant \alpha \int_{\tau+\eta_1}^{\tau+\eta_2} \|\Delta v_1(r, \tau - t, \theta_{-\tau}\omega, 0)\| dr + (1 - \alpha\sigma) \int_{\tau+\eta_1}^{\tau+\eta_2} \|\Delta u_1(r, \tau - t, \theta_{-\tau}\omega, 0)\| dr$$

$$+ (1 - \sigma) \int_{\tau+\eta_1}^{\tau+\eta_2} \|v_1(r, \tau - t, \theta_{-\tau}\omega, 0)\| dr + (\sigma^2 + \lambda - \sigma) \int_{\tau+\eta_1}^{\tau+\eta_2} \|u_1(r, \tau - t, \theta_{-\tau}\omega, 0)\| dr$$

$$+ 2\alpha \int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla v(r, \tau - t, \theta_{-\tau}\omega, v^{\tau-t}) \nabla \xi^2(\frac{|x|^2}{k^2})\| dr$$

$$+ \alpha \int_{\tau+\eta_1}^{\tau+\eta_2} \|v(r, \tau - t, \theta_{-\tau}\omega, v^{\tau-t}) \Delta \xi^2(\frac{|x|^2}{k^2})\| dr$$

$$+ 2(1 - \alpha\sigma) \int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla u(r, \tau - t, \theta_{-\tau}\omega, u^{\tau-t}) \nabla \xi^2(\frac{|x|^2}{k^2})\| dr$$

$$+ (1 - \alpha\sigma) \int_{\tau+\eta_1}^{\tau+\eta_2} \|u(r, \tau - t, \theta_{-\tau}\omega, u^{\tau-t}) \Delta \xi^2(\frac{|x|^2}{k^2})\| dr$$

$$+ \int_{\tau+\eta_1}^{\tau+\eta_2} (\|f(x, u(r - \rho(r)))\| + \|g(r, x)\|) dr + \int_{\tau+\eta_1}^{\tau+\eta_2} (\sigma\|z(\theta_r\omega)\| + \alpha\|\Delta z(\theta_r\omega)\|) dr. \tag{2.82}$$

By (2.66), we have for $t \geqslant T_1$,

$$(1 - \sigma) \int_{\tau+\eta_1}^{\tau+\eta_2} \|v_1(r, \tau - t, \theta_{-\tau}\omega, 0)\| dr + (\sigma^2 + \lambda - \sigma) \int_{\tau+\eta_1}^{\tau+\eta_2} \|u_1(r, \tau - t, \theta_{-\tau}\omega, 0)\| dr$$

$$\leqslant C \left(\int_{\tau+\eta_1}^{\tau+\eta_2} \|(u_1(r+s, \tau-t, \theta_{-\tau}\omega, 0), v_1(r+s, \tau-t, \theta_{-\tau}\omega, 0))\|^2_{\mathscr{E}(\Omega_{2k})} dr \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}$$

$$\leqslant C \left(\int_{\tau+\eta_1}^{\tau+\eta_2} r_1(\tau, \theta_{r-\tau}\omega) dr \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}$$

$$\leqslant C h^{\frac{1}{2}} \left(\sup_{r \in [-h, 0]} r_1(\tau, \theta_r\omega) \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}. \tag{2.83}$$

By Lemma 2.4, we know that there exists $T_2 = T_2(\tau, \omega, D)$ such that for $t > T_2$,

$$\alpha \int_{\tau+\eta_1}^{\tau+\eta_2} \|\Delta v_1(r, \tau-t, \theta_{-\tau}\omega, 0)\| dr + (1 - \alpha\sigma) \int_{\tau+\eta_1}^{\tau+\eta_2} \|\Delta u_1(r, \tau-t, \theta_{-\tau}\omega, 0)\| dr$$

$$\leqslant \left\{ \alpha \left(\int_{\tau+\eta_1}^{\tau+\eta_2} \|\Delta v_1(r, \tau-t, \theta_{-\tau}\omega, 0)\|^2 dr \right)^{\frac{1}{2}} \right.$$

$$\left. + (1 - \alpha\sigma) \left(\int_{\tau+\eta_1}^{\tau+\eta_2} \|\Delta u_1(r, \tau-t, \theta_{-\tau}\omega, 0)\|^2 dr \right)^{\frac{1}{2}} \right\} |\eta_2 - \eta_1|^{\frac{1}{2}}$$

$$\leqslant \left\{ \alpha \sqrt{r_2(\tau, \omega)} + (1 - \alpha\sigma) \left(\sup_{r \in [-h, 0]} r_2(\tau, \theta_r\omega) \right)^{\frac{1}{2}} \right\} |\eta_2 - \eta_1|^{\frac{1}{2}}. \tag{2.84}$$

From Lemma 2.2, we find that there exists $T_3 = T_3(\tau, \omega, D)$ such that for $t > T_3$,

$$2\alpha \int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla v(r, \tau-t, \theta_{-\tau}\omega, v^{\tau-t}) \nabla \xi^2(\frac{|x|^2}{k^2})\| dr$$

$$= \frac{8\alpha}{k^2} \int_{\tau+\eta_1}^{\tau+\eta_2} \|x\xi(\frac{|x|^2}{k^2})\xi'(\frac{|x|^2}{k^2})\nabla v(r, \tau-t, \theta_{-\tau}\omega, v^{\tau-t})\| dr$$

$$\leqslant \frac{8\sqrt{2}\alpha\mu_1}{k} \left(\int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla v(r, \tau-t, \theta_{-\tau}\omega, v^{\tau-t})\|^2 dr \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}$$

$$\leqslant \frac{8\sqrt{\alpha}\mu_1 e^{\frac{\sigma' h}{2}}}{k} \left(\int_{\tau-t}^{\tau+\eta_2} e^{r-\tau-\eta_2} \|\nabla v(r, \tau-t, \theta_{-\tau}\omega, v^{\tau-t})\|^2 dr \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}$$

$$\leqslant \frac{8\sqrt{\alpha}\mu_1 e^{\frac{\sigma' h}{2}}}{k} \sqrt{r_1(\tau, \omega)} |\eta_2 - \eta_1|^{\frac{1}{2}}, \tag{2.85}$$

and

$$\alpha \int_{\tau+\eta_1}^{\tau+\eta_2} \|v(r, \tau-t, \theta_{-\tau}\omega, v^{\tau-t}) \Delta \xi^2(\frac{|x|^2}{k^2})\| dr$$

$$= \frac{4\alpha}{k^2} \int_{\tau+\eta_1}^{\tau+\eta_2} \|xv(r, \tau-t, \theta_{-\tau}\omega, v^{\tau-t}) \left((\xi'(\frac{|x|^2}{k^2}))^2 + \xi(\frac{|x|^2}{k^2})\xi''(\frac{|x|^2}{k^2}) \right) \| dr$$

$$+ \frac{4\alpha}{k^2} \int_{\tau+\eta_1}^{\tau+\eta_2} \|v(r, \tau-t, \theta_{-\tau}\omega, v^{\tau-t})\xi(\frac{|x|^2}{k^2})\xi'(\frac{|x|^2}{k^2})\| dr$$

$$\leqslant \frac{4\sqrt{2}k\alpha(\mu_1^2 + \mu_2) + 4\alpha\mu_1}{k^2} \left(\int_{\tau+\eta_1}^{\tau+\eta_2} \|v(r, \tau-t, \theta_{-\tau}\omega, v^{\tau-t})\|^2 dr \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}$$

$$\leqslant \frac{4\sqrt{2}k\alpha(\mu_1^2 + \mu_2) + 4\alpha\mu_1}{k^2} h^{\frac{1}{2}} \left(\sup_{r\in[-h,0]} r_1(\tau, \theta_r\omega) \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}. \tag{2.86}$$

Similarly, we have for $t > T_3$,

$$(1 - \alpha\sigma) \int_{\tau+\eta_1}^{\tau+\eta_2} \|u(r, \tau - t, \theta_{-\tau}\omega, u^{\tau-t})\Delta\xi^2(\frac{|x|^2}{k^2})\|dr$$

$$\leqslant \frac{4\sqrt{2}k(1 - \alpha\sigma)(\mu_1^2 + \mu_2) + 4(1 - \alpha\sigma)\mu_1}{k^2} h^{\frac{1}{2}} \left(\sup_{r\in[-h,0]} r_1(\tau, \theta_r\omega) \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}, \tag{2.87}$$

and

$$2(1 - \alpha\sigma) \int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla u(r, \tau - t, \theta_{-\tau}\omega, u^{\tau-t})\nabla\xi^2(\frac{|x|^2}{k^2})\|dr$$

$$= \frac{8(1 - \alpha\sigma)}{k^2} \int_{\tau+\eta_1}^{\tau+\eta_2} \|x\xi(\frac{|x|^2}{k^2})\xi'(\frac{|x|^2}{k^2})\nabla u(r, \tau - t, \theta_{-\tau}\omega, u^{\tau-t})\|dr$$

$$\leqslant \frac{8\sqrt{2}(1 - \alpha\sigma)\mu_1}{k} \left(\int_{\tau+\eta_1}^{\tau+\eta_2} \|\nabla u(r, \tau - t, \theta_{-\tau}\omega, u^{\tau-t})\|^2 dr \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}$$

$$\leqslant \frac{8\sqrt{2}(1 - \alpha\sigma)\mu_1}{k} h^{\frac{1}{2}} \left(\sup_{r\in[-h,0]} r_1(\tau, \theta_r\omega) \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}. \tag{2.88}$$

By (A1) and Lemma 2.2, we obtain for $t > T_3$,

$$\int_{\tau+\eta_1}^{\tau+\eta_2} \|f(x, u(r - \rho(r)))\|dr$$

$$\leqslant \left(\int_{\tau+\eta_1}^{\tau+\eta_2} \|f(x, u(r - \rho(r)))\|^2 dr \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}$$

$$\leqslant \left(\int_{\tau+\eta_1}^{\tau+\eta_2} \|k_1\|^2 + k_2^2 \|u(r - \rho(r))\|dr \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}$$

$$\leqslant \left(h\|k_1\|^2 + k_2^2 \int_{\tau+\eta_1}^{\tau+\eta_2} \|u^r\|_{C_h} dr \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}$$

$$\leqslant \left(h\|k_1\|^2 + k_2^2 h^{\frac{1}{2}} \sup_{r\in[-h,0]} r_1(\tau, \theta_r\omega) \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}, \tag{2.89}$$

and

$$\int_{\tau+\eta_1}^{\tau+\eta_2} \|g(r, x)\|dr \leqslant \left(\int_{\tau+\eta_1}^{\tau} \|g(r, x)\|^2 dr \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}$$

$$\leqslant \left(\int_{-\infty}^{\tau} e^{\sigma'(r-\tau)} \|g(r, x)\|^2 dr \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}}$$

$$\leqslant \left(\int_{-\infty}^{\tau} e^{\sigma'(r-\tau)} \|g(r, x)\|^2 dr \right)^{\frac{1}{2}} |\eta_2 - \eta_1|^{\frac{1}{2}} \leqslant C^{\frac{1}{2}} \sqrt{r_1(\tau, \omega)} |\eta_2 - \eta_1|^{\frac{1}{2}}. \tag{2.90}$$

Noting that $h_j \in H^2(\mathbb{R}^d)$ and by (2.6), we have

$$\int_{\tau+\eta_1}^{\tau+\eta_2} (\sigma\|z(\theta_r\omega)\| + \alpha\|\Delta z(\theta_r\omega)\|)dr \leqslant (\sigma + \alpha)C^{\frac{1}{2}}h^{\frac{1}{2}}e^{\frac{\sigma'h}{4}}r^{\frac{1}{2}}(\omega)|\eta_2 - \eta_1|^{\frac{1}{2}}. \tag{2.91}$$

Set $T = \max\{T_1, T_2, T_3\}$,

$$
\begin{aligned}
r_3(\tau, \omega) &\equiv (\sigma^2 + \lambda - \sigma)\left\{\left(1 + \frac{\sigma}{\sqrt{\sigma^2 + \lambda - \sigma}}\right)h^{\frac{1}{2}}\left(\sup_{r\in[-h,0]} r_1(\tau, \theta_r\omega)\right)^{\frac{1}{2}} + e^{\frac{\sigma'h}{4}}h^{\frac{1}{2}}r^{\frac{1}{2}}(\omega)\right\} \\
&\quad + (1 - \alpha\sigma)\left\{C^{\frac{1}{2}}\sqrt{r_1(\tau, \omega)} + \frac{C^{\frac{1}{2}}\sigma}{\sqrt{1 - \alpha\sigma}}h^{\frac{1}{2}}\left(\sup_{r\in[-h,0]} r_1(\tau, \theta_r\omega)\right)^{\frac{1}{2}} + e^{\frac{\sigma'h}{4}}h^{\frac{1}{2}}r^{\frac{1}{2}}(\omega)\right\} \\
&\quad + Ch^{\frac{1}{2}}\left(\sup_{r\in[-h,0]} r_1(\tau, \theta_r\omega)\right)^{\frac{1}{2}} + \left\{\alpha\sqrt{r_2(\tau, \omega)} + (1 - \alpha\sigma)\left(\sup_{r\in[-h,0]} r_2(\tau, \theta_r\omega)\right)^{\frac{1}{2}}\right\} \\
&\quad + \frac{8\sqrt{\alpha}\mu_1 e^{\frac{\sigma'h}{2}}}{k}\sqrt{r_1(\tau, \omega)} + \frac{4\sqrt{2}k\alpha(\mu_1^2 + \mu_2) + 4\alpha\mu_1}{k^2}h^{\frac{1}{2}}\left(\sup_{r\in[-h,0]} r_1(\tau, \theta_r\omega)\right)^{\frac{1}{2}} \\
&\quad + \frac{4\sqrt{2}k(1 - \alpha\sigma)(\mu_1^2 + \mu_2) + 4(1 - \alpha\sigma)\mu_1}{k^2}h^{\frac{1}{2}}\left(\sup_{r\in[-h,0]} r_1(\tau, \theta_r\omega)\right)^{\frac{1}{2}} \\
&\quad + \frac{8\sqrt{2}(1 - \alpha\sigma)\mu_1}{k}h^{\frac{1}{2}}\left(\sup_{r\in[-h,0]} r_1(\tau, \theta_r\omega)\right)^{\frac{1}{2}} + C^{\frac{1}{2}}\sqrt{r_1(\tau, \omega)} \\
&\quad + \left(h\|k_1\|^2 + k_2^2 h^{\frac{1}{2}}\sup_{r\in[-h,0]} r_1(\tau, \theta_r\omega)\right)^{\frac{1}{2}} + (\sigma + \alpha)C^{\frac{1}{2}}h^{\frac{1}{2}}e^{\frac{\sigma'h}{4}}r^{\frac{1}{2}}(\omega). \tag{2.92}
\end{aligned}
$$

It follows from (2.77)–(2.92) that, for $t > T$,

$$\|\varphi_1^\tau(\eta_2, \tau - t, \theta_{-\tau}\omega, 0) - \varphi_1^\tau(\eta_1, \tau - t, \theta_{-\tau}\omega, 0))\|_{E(\Omega_{2k})}^2 \leqslant r_3(\tau, \omega)|\eta_2 - \eta_1|^{\frac{1}{2}}.$$

This completes the proof of Lemma 2.5. $\qquad\square$

3. Results and discussion

In this section, we aim to prove the existence of tempered pullback random attractors for the system (1.1) and (1.2) in $\mathscr{E}$. Firstly we show the existence of the pullback absorbing set as follows.

Lemma 3.1. *Suppose the hypotheses in Lemma 2.2 hold. Then the continuous cocycle Φ has a closed measurable $\mathcal{D}$-pullback absorbing set $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$.*

Proof. Set

$$K(\tau, \omega) = \{\varphi \in \mathscr{E} : \|\varphi\|_{\mathscr{E}}^2 \leqslant r_1(\tau, \omega)\},$$

where $r_1(\tau, \omega)$ is given by (2.15). It is evident that, for each $\tau \in \mathbb{R}$, $r_1(\tau, \cdot) : \Omega \to \mathbb{R}$ is $(\mathscr{F}, \mathscr{B}(\mathbb{R}))$-measurable.

Note that

$$r_1(\tau, \omega) = C + Cr(\omega) + Ce^{-\sigma'\tau}\int_{-\infty}^{\tau} e^{\sigma'r}\|g(r, x)\|^2 dr.$$

By simple calculations, we have for each $\gamma > 0$,

$$\lim_{t \to -\infty} e^{\gamma t} \|K(\tau + t, \theta_t \omega)\|_{\mathscr{E}}^2 = \lim_{t \to -\infty} e^{\gamma t} r_1(\tau + t, \theta_t \omega) = 0.$$

In addition, for each $\tau \in \mathbb{R}, \omega \in \Omega$, and $D \in \mathcal{D}$, by Lemma 2.2, there exists $T = T(\tau, \omega, D) > 0$, such that for all $t \geqslant T$,

$$\Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega)) \subseteq K(\tau, \omega),$$

that is, $K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}$ is a closed measurable $\mathcal{D}$-pullback absorbing set for Φ. $\qquad \square$

Next, we will prove that the continuous cocycle Φ is asymptotically compact in $\mathscr{E}$.

Lemma 3.2. *Suppose the hypotheses in Lemma 2.2 hold. Then the continuous cocycle Φ is $\mathcal{D}$-pullback asymptotically compact in $\mathscr{E}$. That is, if for all $\tau \in \mathbb{R}, \omega \in \Omega$, the sequence $\{\Phi(t_m, \tau - t_m, \theta_{-t_m}\omega, x_m)\}_{m=1}^{\infty}$ has a convergent subsequence in $\mathscr{E}$.*

Proof. Let $t_m \to \infty, D \in \mathcal{D}(\mathscr{E}(\Omega_{2k}))$ and $\tilde{\varphi}^{\tau - t_m} = (\tilde{u}^{\tau - t_m}, \tilde{v}^{\tau - t_m}) \in D(\tau - t_m, \theta_{-t_m}\omega)$. We firstly show that $\tilde{\varphi}^{\tau}(\cdot, \tau - t_m, \theta_{-\tau}\omega, \tilde{\varphi}^{\tau - t_m})$ is precompact in $\mathscr{E}(\Omega_{2k})$, where $\tilde{\varphi}^{\tau}(\cdot, \tau - t_m, \theta_{-\tau}\omega, \tilde{\varphi}^{\tau - t_m}) = (\tilde{u}^{\tau}(\cdot, \tau - t_m, \theta_{-\tau}\omega, \tilde{u}^{\tau - t_m}), \tilde{v}^{\tau}(\cdot, \tau - t_m, \theta_{-\tau}\omega, \tilde{v}^{\tau - t_m}))$. It follows from Lemma 2.4 that, for $M_1 = M_1(\tau, \omega, D)$ large enough, $m > M_1$ and $r \in [-h, 0]$,

$$\|u_1(r, \tau - t_m, \theta_{-\tau}\omega, 0)\|_{H^2(\Omega_{2k})}^2 + \|v_1(r, \tau - t_m, \theta_{-\tau}\omega, 0)\|_{H^1(\Omega_{2k})}^2 \leqslant C r_1(\tau, \omega).$$

We know that $H^1(\Omega_{2k}) \hookrightarrow L^2(\Omega_{2k})$ and $H^2(\Omega_{2k}) \hookrightarrow H^1(\Omega_{2k})$ are compact. Therefore, for $m > M_1$ and $r \in [-h, 0]$, $\{u_1(r, \tau - t_m, \theta_{-\tau}\omega, 0), v_1(r, \tau - t_m, \theta_{-\tau}\omega, 0)\}$ is precompact in $E(\Omega_{2k})$. From Lemma 2.5, for $m > M_1$, $\{u_1(\cdot, \tau - t_m, \theta_{-\tau}\omega, 0), v_1(\cdot, \tau - t_m, \theta_{-\tau}\omega, 0)\}$ is equi-continuous in $C([-h, 0], E(\Omega_{2k}))$. Then by Arzela-Ascoli theorem, $\{u_1(r, \tau - t_m, \theta_{-\tau}\omega, 0), v_1(r, \tau - t_m, \theta_{-\tau}\omega, 0)\}$ is precompact in $C([-h, 0], E(\Omega_{2k}))$. Hence, there exists a subsequence t_{m_k}, still denote as t_m, such that

$$(u_1^{\tau}(\cdot, \tau - t_m, \theta_{-\tau}\omega, 0), v_1^{\tau}(\cdot, \tau - t_m, \theta_{-\tau}\omega, 0)) \to (\zeta(\cdot), \xi(\cdot)), \quad in \ C([-h, 0], E(\Omega_{2k})).$$

In other words, for any $\epsilon > 0$, there exists $M_2 = M_2(\tau, \omega, \epsilon, D)$, such that for $m > M_2$ and $r \in [-h, 0]$,

$$\|(u_1^{\tau}(\cdot, \tau - t_m, \theta_{-\tau}\omega, 0) - \zeta(\cdot), v_1^{\tau}(\cdot, \tau - t_m, \theta_{-\tau}\omega, 0) - \xi(\cdot))\|_{\mathscr{E}(\Omega_{2k})} < \epsilon. \tag{3.1}$$

For any $(u_2^{\tau - t_m}, v_2^{\tau - t_m}) = (\tilde{u}^{\tau - t_m}, \tilde{v}^{\tau - t_m}) \in D, D \in \mathcal{D}(\mathscr{E}(\Omega_{2k}))$, by (2.65), there exists $M_3 = M_3(\tau, \omega, \epsilon, D)$, such that for $m > M_3$,

$$\|(u_2^{\tau}(\cdot, \tau - t_m, \theta_{-\tau}\omega, u_2^{\tau - t_m}), v_2^{\tau}(\cdot, \tau - t_m, \theta_{-\tau}\omega, v_2^{\tau - t_m}))\|_{\mathscr{E}(\Omega_{2k})} < \epsilon. \tag{3.2}$$

By (3.1) and (3.2), we derive for $m > \tilde{M} = \max\{M_1, M_2, M_3\}$,

$$\|(\tilde{u}^{\tau}(\cdot, \tau - t_m, \theta_{-\tau}\omega, \tilde{u}^{\tau - t_m}) - \zeta(\cdot), \tilde{v}^{\tau}(\cdot, \tau - t_m, \theta_{-\tau}\omega, \tilde{v}^{\tau - t_m}) - \xi(\cdot))\|_{\mathscr{E}(\Omega_{2k})}$$
$$\leqslant 2\|(u_1^{\tau}(\cdot, \tau - t_m, \theta_{-\tau}\omega, 0) - \zeta(\cdot), v_1^{\tau}(\cdot, \tau - t_m, \theta_{-\tau}\omega, 0) - \xi(\cdot))\|_{\mathscr{E}(\Omega_{2k})}$$
$$+ 2\|(u_2^{\tau}(\cdot, \tau - t_m, \theta_{-\tau}\omega, u_2^{\tau - t_m}), v_2^{\tau}(\cdot, \tau - t_m, \theta_{-\tau}\omega, v_2^{\tau - t_m}))\|_{\mathscr{E}(\Omega_{2k})} \leqslant 4\epsilon. \tag{3.3}$$

Thus, $\tilde{\varphi}^\tau(\cdot, \tau - t_m, \theta_{-\tau}\omega, \tilde{\varphi}^{\tau-t_m}) = (\tilde{u}^\tau(\cdot, \tau - t_m, \theta_{-\tau}\omega, \tilde{u}^{\tau-t_m}), \tilde{v}^\tau(\cdot, \tau - t_m, \theta_{-\tau}\omega, \tilde{v}^{\tau-t_m}))$ is precompact in $\mathscr{E}(\Omega_{2k})$. By Lemma 2.3, there exist $k_1 = k_1(\tau, \omega, \epsilon)$ and $M_4 = M_4(\tau, \omega, \epsilon, D)$, such that for each $m > M_4$,

$$\|\varphi^\tau(\cdot, \tau - t_m, \theta_{-\tau}\omega, \varphi^{\tau-t_m})\|_{\mathscr{E}(\mathbb{R}^d\backslash\Omega_{k_1})} \leqslant \epsilon. \tag{3.4}$$

By (3.3), there exists $k_2 = k_2(\tau, \omega, \epsilon) \geqslant k_1$ such that $\tilde{\varphi}^\tau(\cdot, \tau - t_m, \theta_{-\tau}\omega, \tilde{\varphi}^{\tau-t_m})$ is precompact in $E(\Omega_{2k_2})$. Recalling (2.55) and the fact $1 - \xi^2(\frac{|x|^2}{k_2^2}) = 1$ for $|x| \leqslant k_2$, we know that $\varphi^\tau(\cdot, \tau - t_m, \theta_{-\tau}\omega, \varphi^{\tau-t_m})$ is precompact in $\mathscr{E}(\Omega_{k_2})$. Along with (3.4), we have that the continuous cocycle Φ is asymptotically compact in $\mathscr{E}$. $\square$

We are now to give our main result.

Theorem 3.3. *Suppose the hypotheses in Lemma 2.2 hold. Then the continuous cocycle Φ has a unique $\mathcal{D}$-pullback random attractor in $\mathscr{E}$.*

Proof. By Proposition 2.1, Lemma 3.1 and Lemma 3.2, we can obtain the existence and uniqueness of $\mathcal{D}$-pullback random attractor of Φ in $\mathscr{E}$ immediately. $\square$

4. Conclusions

Since the Sobolev embedding is no longer compact on unbounded domains, we obtained the existence of random attractor for the problem (1.1) and (1.2) by using the uniform tail-estimates of solutions and the decomposition technique as well as the compactness argument. In addition, to derive the uniform estimates, we make some nontrivial arguments due to the presence of strongly damped term $\alpha\Delta u_t$ and the delay term $f(x, u(t - \rho(t)))$ in (1.1).

Conflict of interest

The author declares that there is no conflict of interest.

References

1. L. Arnold, *Random Dynamical Systems*, Berlin: Springer-Verlag, 1998.

2. P. W. Bates, K. Lu, B. Wang, Random attractors for stochastic reaction-diffusion equations on unbounded domains, *J. Differ. Equations*, **246** (2009), 845–869.

3. Z. Brzézniak, Y. Li, Asymptotic compactness and absorbing sets for 2D stochastic navier-stokes equations on some unbounded domains, *Trans. Amer. Math. Soc.*, **358** (2006), 5587–5629.

4. T. Caraballo, M. J. Garrido-Atienza, B. Schmalfuss, J. Valero, Non-autonomous and random attractors for delay random semilinear equations without uniqueness, *Discrete Cont. Dyn-A.*, **21** (2008), 415.

5. T. Caraballo, B. Guo, N. H. Tuan, R. Wang, Asymptotically autonomous robustness of random attractors for a class of weakly dissipative stochastic wave equations on unbounded domains, *Proc. Roy. Soc. Edinb. Sect. A*, (2020), 1–31.

6. T. Caraballo, J. Real, Attractors for 2D-Navier-Stokes models with delays, *J. Differ. Equations*, **205** (2004), 271–297.

7. P. Chen, X. Zhang, Upper semi-continuity of attractors for non-autonomous fractional stochastic parabolic equations with delay, *Discrete Cont. Dyn-B.*, **26** (2021), 4325.

8. H. Crauel, F. Flandoli, Attractors for random dynamical systems, *Probab. Theory Rel.*, **100** (1994), 365–393.

9. F. Flandoli, B. Gess, M. Scheutzow, Synchronization by noise, *Probab. Theory Rel.*, **168** (2017), 511–556.

10. B. Gess, Random attractors for stochastic porous media equations perturbed by space-time linear multiplicative noise. *Ann. Probab.*, **42** (2014), 818–864.

11. B. Gess, W. Liu, A. Schenke, Random attractors for locally monotone stochastic partial differential equations, *J. Differ. Equations*, **269** (2020), 3414–3455.

12. A. Gu, B. Wang, Asymptotic behavior of random Fitzhugh-Nagumo systems driven by colored noise, *Discrete Cont. Dyn-B.*, **23** (2018), 1689–1720.

13. J. U. Jeong, J. Park, Pullback attractors for a 2D-non-autonomous incompressible non-Newtonian fluid with variable delays, *Discrete Cont. Dyn-B.*, **21** (2016), 2687–2702.

14. X. Jia, X. Ding, Random attractors for stochastic retarded strongly damped wave equations with additive noise on bounded domains, *Open Math.*, **17** (2019), 472–486.

15. R. Jones, B. Wang, Asymptotic behavior of a class of stochastic nonlinear wave equations with dispersive and dissipative terms, *Nonlinear Anal-Real.*, **14** (2013), 1308–1322.

16. D. Li, K. Lu, B. Wang, X. Wang, Limiting dynamics for non-autonomous stochastic retarded reaction-diffusion equations on thin domains, *Discrete Cont. Dyn-A.*, **39** (2019), 3717–3747.

17. D. Li, L. Shi, Upper semicontinuity of attractors of stochastic delay reaction-diffusion equations in the delay, *J. Math. Phys.*, **59** (2018), 032703.

18. H. Li, Y. You, J. Tu, Random attractors and averaging for non-autonomous stochastic wave equations with nonlinear damping, *J. Differ. Equations*, **258** (2015), 148–190.

19. L. Liu, T. Caraballo, P. Marín-Rubio, Stability results for 2D Navier-Stokes equations with unbounded delay, *J. Differ. Equations*, **265** (2018), 5685–5708.

20. H. Lu, J. Qi, B. Wang, M. Zhang, Random attractors for non-autonomous fractional stochastic parabolic equations on unbounded domains, *Discrete Cont. Dyn-A.*, **39** (2019), 683–706.

21. K. Lu, B. Wang, Wong-Zakai approximations and long term behavior of stochastic partial differential equations, *J. Dyn. Differ. Equ.*, **31** (2019), 1341–1371.

22. Y. Lv, W. Wang, Limiting dynamics for stochastic wave equations, *J. Differ. Equations*, **244** (2008), 1–23.

23. L. Shi, R. Wang, K. Lu, B. Wang, Asymptotic behavior of stochastic FitzHugh-Nagumo systems on unbounded thin domains, *J. Differ. Equations*, **267** (2019), 4373–4409.

24. R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, New York: Springer-Verlag, 1997.

25. B. Wang, Asymptotic behavior of stochastic wave equations with critical exponents on $\mathbb{R}^3$, *T. Am. Math. Soc.*, **363** (2011), 3639–3663.

26. B. Wang, Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, *J. Differ. Equations*, **253** (2012), 1544–1583.

27. B. Wang, Random attractors for non-autonomous stochastic wave equations with multiplicative noise, *Discrete Cont. Dyn-A.*, **34** (2014), 269–300.

28. B. Wang, Dynamics of fractional stochastic reaction-diffusion equations on unbounded domains driven by nonlinear noise, *J. Differ. Equations*, **268** (2019), 1–59.

29. X. Wang, K. Lu, B. Wang, Random attractors for delay parabolic equations with additive noise and deterministic nonautonomous forcing, *SIAM J. Appl. Dyn. Syst.*, **14** (2015), 1018–1047.

30. R. Wang, L. Shi, B. Wang, Asymptotic behavior of fractional nonclassical diffusion equations driven by nonlinear colored noise on $\mathbb{R}^N$, *Nonlinearity*, **32** (2019), 4524–4556.

31. R. Wang, B. Wang, Asymptotic behavior of non-autonomous fractional stochastic p-Laplacian equations, *Comput. Math. Appl.*, **78** (2019), 3527–3543.

32. Z. Wang, S. Zhou, Random attractor for non-autonomous stochastic strongly damped wave equation on unbounded domains, *J. Appl. Anal. Comput.*, **5** (2015), 363–387.

33. Z. Wang, S. Zhou, Random attractor for stochastic non-autonomous damped wave equation with critical exponent, *Discrete Cont. Dyn-A.*, **37** (2017), 545–573.

34. Z. Wang, S. Zhou, A. Gu, Random attractor for a stochastic damped wave equation with multiplicative noise on unbounded domains, *Nonlinear Anal-Real.*, **12** (2011), 3468–3482.

35. F. Wu, P. E. Kloeden, Mean-square random attractors of stochastic delay differential equations with random delay, *Discrete Cont. Dyn-B.*, **18** (2013), 1715.

36. Y. Yan, X. Li, Long time behavior for the stochastic parabolic-wave systems with delay on infinite lattice, *Nonlinear Anal.*, **197** (2020), 111866.

37. M. Yang, J. Duan, P. E. Kloeden, Asymptotic behavior of solutions for random wave equations with nonlinear damping and white noise, *Nonlinear Anal-Real.*, **12** (2011), 464–478.

38. S. Zhou, M. Zhao, Random attractors for damped non-autonomous wave equations with memory and white noise, *Nonlinear Anal.*, **120** (2015), 202–226.

39. R. Zhu, X. Zhu, Random attractor associated with the quasi-geostrophic equation, *J. Dyn. Differ. Equ.*, **29** (2017), 289–322.