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Research article

A generalized alternating harmonic series

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Abstract: This paper introduces a generalization of the alternating harmonic series, expresses the sum in two closed forms, and examines the relationship between these sums and the harmonic numbers.

Keywords: alternating harmonic series; sum of infinite series; rearrangement of series; harmonic numbers
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1. Introduction

Riemann's Rearrangement Theorem says that if an infinite series of real numbers is conditionally convergent, then its terms can be rearranged in in such a way that the resulting series converges to any real sum [3]. It is well known that the alternating harmonic series converges to log 2, that is,

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2.$$

As the prototypical conditionally convergent series, rearrangements of the alternating harmonic series have been studied extensively [1]. However, Riemann's Rearrangement Theorem is non-constructive; there is no general method to find the sum of a re-arrangement. It was shown in [8] that assigning plus or minus signs randomly produces sums that converge almost surely.

In this article, we consider a generalized version of the alternating harmonic series, one with the assignment of plus or minus signs, as follows. For each positive integer k, we consider the series

$$\begin{aligned} S_k = &\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right) - \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k}\right) \\ &+ \left(\frac{1}{2k+1} + \frac{1}{2k+2} + \cdots + \frac{1}{3k}\right) - \left(\frac{1}{3k+1} + \frac{1}{3k+2} + \cdots + \frac{1}{4k}\right) + \cdots \end{aligned}$$

We will find the infinite sum, S_k, of the infinite series in two different formats and examine the interesting relationship between this sum and the harmonic number,

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} = \sum_{n=1}^{k} \frac{1}{n},$$

which appears often throughout mathematics [2].

Definition 1.1. *We take $H_0 = 0$ and define the terms of S_k by*

$$a_n(k) = H_{nk} - H_{(n-1)k},$$

for each positive integer n and k. Thus,

$$S_k = \sum_{n=1}^{\infty} (-1)^{n+1} a_n(k).$$

Clearly, for a fixed k, $a_n(k) > 0$ and decreases to 0 as n increases. Hence, by the alternating series test, S_k is convergent for all positive integers k.

2. Sum of generalized alternating harmonic series

Our first summation formula is given in integral form, as follows.

Theorem 2.1. *If k is a positive integer, then*

$$S_k = \sum_{m=0}^{k-1} \int_0^1 \frac{x^m}{1 + x^k} dx. \tag{2.1}$$

Proof. Since

$$\int_0^1 \frac{1}{1 + x} dx = \log 2,$$

Equation (2.1) holds for $k = 1$. For $k \geq 2$, we first note that the harmonic number has an integral expression [7] as

$$\begin{aligned} H_n &= 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \int_0^1 dx + \int_0^1 x dx + \cdots + \int_0^1 x^{n-1} dx \\ &= \int_0^1 (1 + x + \cdots + x^{n-1}) dx = \int_0^1 \frac{1 - x^n}{1 - x} dx. \end{aligned}$$

Hence when $k \geq 2$,

$$a_n(k) = H_{nk} - H_{(n-1)k} = \int_0^1 \frac{1 - x^{nk}}{1 - x} dx - \int_0^1 \frac{1 - x^{(n-1)k}}{1 - x} dx$$

$$= \int_0^1 \frac{x^{(n-1)k} - x^{nk}}{1-x} dx = \int_0^1 x^{(n-1)k} \frac{1-x^k}{1-x} dx. \tag{2.2}$$

It follows that for $k \geq 2$,

$$\begin{aligned} S_k &= \sum_{n=1}^{\infty} (-1)^{n+1} a_n(k) = \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^1 x^{(n-1)k} \frac{1-x^k}{1-x} dx \\ &= \int_0^1 \frac{1-x^k}{1-x} \sum_{n=1}^{\infty} \left(-x^k\right)^{n-1} dx = \int_0^1 \frac{1-x^k}{1-x} \frac{1}{1+x^k} dx \\ &= \int_0^1 \frac{1+x+\cdots+x^{k-1}}{1+x^k} dx = \sum_{m=0}^{k-1} \int_0^1 \frac{x^m}{1+x^k} dx, \end{aligned}$$

proving the theorem. □

As an application of Theorem 2.1, we have

Example 2.2.

$$S_2 = \int_0^1 \frac{1+x}{1+x^2} dx = \arctan x\Big|_0^1 + \frac{1}{2} \log(1+x^2)\Big|_0^1 = \frac{\pi}{4} + \frac{1}{2} \log 2.$$

Our second summation formula is given in terms of trigonometric functions, as follows.

Corollary 2.3. *If If k is a positive integer, then*

$$S_k = \frac{\pi}{2k} \sum_{m=1}^{k-1} \csc \frac{m\pi}{k} + \frac{1}{k} \log 2.$$

Proof. By Theorem 2.1,

$$\begin{aligned} S_k &= \sum_{m=0}^{k-1} \int_0^1 \frac{x^m}{1+x^k} dx = \sum_{m=0}^{k-2} \int_0^1 \frac{x^m}{1+x^k} dx + \int_0^1 \frac{x^{k-1}}{1+x^k} dx \\ &= \sum_{m=1}^{k-1} \int_0^1 \frac{x^{m-1}}{1+x^k} dx + \frac{1}{k} \log 2 \\ &= \frac{1}{2} \left(\sum_{m=1}^{k-1} \int_0^1 \frac{x^{m-1}}{1+x^k} dx + \sum_{m=1}^{k-1} \int_0^1 \frac{x^{m-1}}{1+x^k} dx \right) + \frac{1}{k} \log 2 \\ &= \frac{1}{2} \left(\sum_{m=1}^{k-1} \int_0^1 \frac{x^{m-1}}{1+x^k} dx + \sum_{m'=1}^{k-1} \int_0^1 \frac{x^{k-m'-1}}{1+x^k} dx \right) + \frac{1}{k} \log 2 \end{aligned} \tag{2.3}$$

$$= \frac{1}{2}\sum_{m=1}^{k-1}\int_0^1 \frac{x^{m-1}+x^{k-m-1}}{1+x^k}dx + \frac{1}{k}\log 2, \tag{2.4}$$

where in Eq (2.3), for the second summation in the parenthesis, we changed the index from m to $m' = k - m$. We simplify Eq (2.4) using the following well known result, see page 323 of [5].

$$\text{For } k > m > 0, \quad \int_0^1 \frac{x^{m-1}+x^{k-m-1}}{1+x^k}dx = \frac{\pi}{k}\csc\frac{m\pi}{k}.$$

This yields the desired result,

$$S_k = \frac{\pi}{2k}\sum_{m=1}^{k-1}\csc\frac{m\pi}{k} + \frac{1}{k}\log 2,$$

and this proves the Theorem. □

Example 2.4.

$$S_3 = \frac{\pi}{6}\sum_{m=1}^{2}\csc\frac{m\pi}{3} + \frac{1}{3}\log 2 = \frac{2\pi\sqrt{3}}{9} + \frac{1}{3}\log 2.$$

3. The relationship between S_k and H_k

We now set about finding the relationship between S_k and H_k. Since on $[0,1]$, $1 \leq 1 + x^k \leq 2$ for all integer $k \geq 1$, we have that,

$$\sum_{m=0}^{k-1}\int_0^1 \frac{x^m}{2}dx \leq \sum_{m=0}^{k-1}\int_0^1 \frac{x^m}{1+x^k}dx \leq \sum_{m=0}^{k-1}\int_0^1 x^m dx.$$

Hence by Theorem 2.1, we have

$$\frac{1}{2}H_k \leq S_k \leq H_k, \quad \text{for all} \quad k \geq 1.$$

We will now calculate and simplify the difference between H_k and S_k. Notice that

$$H_k - S_k = a_1(k) - \sum_{n=1}^{\infty}(-1)^{n+1}a_n(k) = -\sum_{n=2}^{\infty}(-1)^{n+1}a_n(k) = \sum_{n=1}^{\infty}(-1)^{n+1}a_{n+1}(k).$$

We make the following definition.

Definition 3.1. *Let*

$$D_k = H_k - S_k = \sum_{n=1}^{\infty}(-1)^{n+1}a_{n+1}(k).$$

Note that for each integer $k \geq 1$, D_k is a convergent alternating series, which easily follows from the properties of $a_n(k)$. More interestingly, D_k itself forms a convergent sequence, as given in the following.

Theorem 3.2.
$$\lim_{k\to\infty} D_k = \lim_{k\to\infty}(H_k - S_k) = \log\frac{\pi}{2}.$$

Proof. Let ε be an arbitrarily fixed positive real number. By the Wallis product formula [6], we have

$$\prod_{n=1}^{\infty}\frac{(2n)^2}{(2n-1)(2n+1)} = \frac{\pi}{2}.$$

Since the natural logarithm is a continuous function, there exist a positive integer N_1 such that when $l \geq N_1$,

$$\left|\log\prod_{n=1}^{l}\frac{(2n)^2}{(2n-1)(2n+1)} - \log\frac{\pi}{2}\right| < \frac{\varepsilon}{3}. \tag{3.1}$$

Let m be any positive integer and let $k \geq 1$. By (2.2), we have

$$\begin{aligned}
\left|\sum_{n=m}^{\infty}(-1)^{n+1}a_{n+1}(k)\right| &= \left|\sum_{n=m}^{\infty}(-1)^{n+1}\int_0^1 x^{nk}\frac{1-x^k}{1-x}dx\right| = \left|-\sum_{n=m}^{\infty}\int_0^1(-x^k)^n\frac{1-x^k}{1-x}dx\right| \\
&\leq \int_0^1\frac{1-x^k}{1-x}\left|\sum_{n=m}^{\infty}(-x^k)^n\right|dx \leq \int_0^1\frac{1-x^k}{1-x}\frac{x^{mk}}{1+x^k}dx \\
&= \int_0^1(1+x+\cdots+x^{k-1})\frac{x^{mk}}{1+x^k}dx \leq \int_0^1 kx^{mk}dx = \frac{k}{mk+1} \\
&< \frac{1}{m}.
\end{aligned}$$

Hence, there exists a positive integer N_2 such than when $m \geq N_2$,

$$\left|\sum_{n=m}^{\infty}(-1)^{n+1}a_{n+1}(k)\right| < \frac{\varepsilon}{3}, \tag{3.2}$$

for each and every $k \geq 1$.

Let $N = \max\{N_1, N_2\}$. Recall the Euler-Mascheroni constant [4] is defined as

$$\gamma = \lim_{n\to\infty}\left(H_n - \log n\right).$$

Since $\{H_n - \log n\}$ converges, it is a Cauchy sequence. We therefore may choose a positive integer K such that when $j, m \geq K$,

$$\left|(H_j - H_m) - \log\frac{j}{m}\right| = |(H_j - \log j) - (H_m - \log m)| < \frac{\varepsilon}{6N}. \tag{3.3}$$

For convenience, we denote

$$D^{2N}(k) = \sum_{n=1}^{2N}(-1)^{n+1}a_{n+1}(k) \qquad \text{and} \qquad T_{2N}(k) = \sum_{n=2N+1}^{\infty}(-1)^{n+1}a_{n+1}(k).$$

Clearly, $D_k = D^{2N}(k) + T_{2N}(k)$ and by (3.2), for any $k \geq 1$, there is

$$|T_{2N}(k)| < \frac{\varepsilon}{3}. \tag{3.4}$$

Also notice that, for any integer $k \geq 1$, we have

$$\begin{aligned} D^{2N}(k) &= \sum_{n=1}^{2N} (-1)^{n+1} a_{n+1}(k) \\ &= a_2(k) - a_3(k) + a_4(k) - a_5(k) + \cdots + a_{2N}(k) - a_{2N+1}(k) \\ &= \sum_{n=1}^{N} \left[a_{2n}(k) - a_{2n+1}(k)\right] \\ &= \sum_{n=1}^{N} \left[(H_{2nk} - H_{(2n-1)k}) - (H_{(2n+1)k} - H_{2nk})\right]. \end{aligned} \tag{3.5}$$

Let

$$W_N = \log \prod_{n=1}^{N} \frac{(2n)^2}{(2n-1)(2n+1)}.$$

Henceforth, let $k \geq K$, where K is defined by Eq (3.3). Using the representation of $D^{2N}(k)$ in Eq (3.5), we have

$$\begin{aligned} &\left|D^{2N}(k) - W_N\right| \\ &= \left|\sum_{n=1}^{N} \left[(H_{2nk} - H_{(2n-1)k}) - (H_{(2n+1)k} - H_{2nk})\right] - \log \prod_{n=1}^{N} \frac{(2n)^2}{(2n+1)(2n-1)}\right| \\ &= \left|\sum_{n=1}^{N} \left\{\left[(H_{2nk} - H_{(2n-1)k}) - (H_{(2n+1)k} - H_{2nk})\right] - \log \frac{(2n)^2}{(2n+1)(2n-1)}\right\}\right| \\ &\leq \sum_{n=1}^{N} \left|\left[(H_{2nk} - H_{(2n-1)k}) - (H_{(2n+1)k} - H_{2nk})\right] - \log \frac{(2n)^2}{(2n-1)(2n+1)}\right| \\ &= \sum_{n=1}^{N} \left|\left[(H_{2nk} - H_{(2n-1)k}) - \log \frac{2n}{2n-1}\right] - \left[(H_{(2n+1)k} - H_{2nk}) - \log \frac{2n+1}{2n}\right]\right| \\ &\leq \sum_{n=1}^{N} \left\{\left|(H_{2nk} - H_{(2n-1)k}) - \log \frac{2nk}{(2n-1)k}\right| + \left|(H_{(2n+1)k} - H_{2nk}) - \log \frac{(2n+1)k}{2nk}\right|\right\} \\ &\leq N\left(\frac{\varepsilon}{6N} + \frac{\varepsilon}{6N}\right) = \frac{\varepsilon}{3}, \end{aligned} \tag{3.6}$$

where "$\leq$" in (3.6) follows from (3.3) because $k \geq K$, which makes all of $2nk$, $(2n-1)k$, and $(2n+1)k$ greater than K.

Finally, for $k \geq K$, we have

$$\begin{aligned}
\left|H_k - S_k - \log\frac{\pi}{2}\right| &= \left|D_k - \log\frac{\pi}{2}\right| \\
&= \left|D^{2N}(k) + T_{2N}(k) - W_N + W_N - \log\frac{\pi}{2}\right| \\
&\leq \left|D^{2N}(k) - W_N\right| + |T_{2N}(k)| + \left|W_N - \log\frac{\pi}{2}\right| \\
&< \varepsilon,
\end{aligned}$$

where the last step follows from (3.1), (3.4), and (3.6). Therefore, by the arbitrariness of ε, we have

$$\lim_{k\to\infty} D_k = \lim_{k\to\infty}(H_k - S_k) = \log\frac{\pi}{2},$$

and this completes the proof. □

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Conflict of interest

The authors declare that there is no conflicts of interest.

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