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Research article

Quantum Hermite-Hadamard type inequalities for generalized strongly preinvex functions

Humaira Kalsoom[1], Muhammad Amer Latif[2], Muhammad Idrees[3], Muhammad Arif [4] and Zabidin Salleh[5,*]

[1] Department of Mathematical, Zhejiang Normal University, Jinhua 321004, China

[2] Department of Basic Sciences, Deanship of Preparatory Year, King Faisal University, Hofuf 31982, Al-Hasa, Saudi Arabia

[3] Zhejiang Province Key Laboratory of Quantum Technology and Device, Department of Physics, Zhejiang University, Hangzhou 310027, China

[4] Department of Mathematics, Abdul Wali Khan University Mardan, Mardan 23200, Pakistan

[5] Department of Mathematics, Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terengganu, 21030 Kuala Nerus, Terengganu, Malaysia

* **Correspondence:** Email: zabidin@umt.edu.my; Tel: +6096683312.

Abstract: In accordance with the quantum calculus, the quantum Hermite-Hadamard type inequalities shown in recent findings provide improvements to quantum Hermite-Hadamard type inequalities. We acquire a new q_{κ_1}-integral and q^{κ_2}-integral identities, then employing these identities, we establish new quantum Hermite-Hadamard q_{κ_1}-integral and q^{κ_2}-integral type inequalities through generalized higher-order strongly preinvex and quasi-preinvex functions. The claim of our study has been graphically supported, and some special cases are provided as well. Finally, we present a comprehensive application of the newly obtained key results. Our outcomes from these new generalizations can be applied to evaluate several mathematical problems relating to applications in the real world. These new results are significant for improving integrated symmetrical function approximations or functions of some symmetry degree.

Keywords: quantum calculus; quantum Hermite-Hadamard inequality; higher order generalized preinvex mapping; $q_{\kappa_1}, q^{\kappa_2}$-derivatives; $q_{\kappa_1}, q^{\kappa_2}$-integrals
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1. Introduction

In mathematics, the quantum calculus is equivalent to usual infinitesimal calculus without the concept of limits or the investigation of calculus without limits (quantum is from the Latin word "quantus" and literally it means how much, in Swedish "Kvant"). The renowned mathematician Euler was the genius who introduced the analysis q-calculus in the eighteenth century by integrating the parameter q into Newton's work of infinite series. At the beginning of the twentieth century, Jackson [1] has started a study of q-calculus and described quantum-definite integrals. The topic of quantum calculus has very long origins in the past. But to keep up with times, it has undergone rapid growth over the past few decades. I believe this strongly because it is a bridge between mathematics and physics, which is useful when dealing with physics. To get more information, please check the application and results of Ernst [2], Gauchman [3], and Kac and Cheung [4] in the theory of quantum calculus and theory of inequalities in quantum calculus. In previous papers, the authors Ntouyas and Tariboon [5, 6] investigated how quantum-derivatives and quantum-integrals are solved over the intervals of the form $[\kappa_1, \kappa_2] \subset \mathbb{R}$ and set several quantum analogs. Our investigation here is motivated essentially by the fact that basic (or q-)Hölder inequality, Hermite-Hadamard inequality and Ostrowski inequality, Cauchy-Bunyakovsky-Schwarz, Gruss, Gruss-Cebysev and other integral inequalities that use classical convexity. Also, Noor et al. [7], Sudsutad et al. [8], and Zhuang et al. [9], played an active role in the study, and some integral inequalities have been established which give quantum analog for the right part of Hermite-Hadamard inequality by using q-differentiable convex and quasi-convex functions. Many mathematicians have done studies in q-calculus analysis, the interested reader can check [10–18].

Srivastava [19] presented (or q-)calculus and fractional q-calculus and their applications in geometric function theory of complex analysis. There is also a clear connection between the classical q-analysis, which we used here, and the so-called (p, q)-analysis. We emphasize that the results for the q-analogues, which we discussed in this article for $0 < q < 1$, can be easily (and probably trivially) converted into the corresponding results for the (p, q)-analogues (with $0 < q < p \leq 1$) by making a few obvious parametric and argument changes, with the additional parameter p being superfluous. Inspired and motivated by some of the above applications in the field of q-calculus. However, new q-Hermite-Hadamard-type inequalities for quantum integrals on finite intervals has not been studied yet. This gap is the motivation and inspiration for this research.

The discussion and application of convex functions have become a prosperous source of motivational material in pure and applied science. This vision promoted new and profound results in many branches of mathematical and engineering sciences and provided a comprehensive framework for the study of many problems. This discovery produced new and profound results in many mathematical and engineering sciences branches and provided a systematic structure for analyzing many issues in many fields. Many scholars have studied the various classes of convex sets and convex functions. A mapping $\mathcal{K} : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ is considered convex if the mapping $\mathcal{K}$ satisfies the following inequality:

$$\mathcal{K}((1-\tau)\kappa_1 + \tau\kappa_2) \leq (1-\tau)\mathcal{K}(\kappa_1) + \tau\mathcal{K}(\kappa_2)$$

for all $\kappa_1, \kappa_2 \in \mathcal{I}$ and $\tau \in [0, 1]$.

One of the most famous inequalities in the theory of Convex Functional Analysis, Hermite-Hadamard established by Hermite and Hadamard in [20]. It has a very fascinating geometric representation with many significant applications. The extraordinary inequality states that, if $\mathcal{K}:\mathcal{I}\subseteq\mathbb{R}\rightarrow\mathbb{R}$ is a convex mapping on the interval $\mathcal{I}$ of real numbers and $\kappa_1,\kappa_2\in\mathcal{I}$ with $\kappa_1<\kappa_2$. Then,

$$\mathcal{K}\Big(\frac{\kappa_1+\kappa_2}{2}\Big)\leq\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{K}(\tau)d\tau\leq\frac{\mathcal{K}(\kappa_1)+\mathcal{K}(\kappa_2)}{2}. \tag{1.1}$$

For $\mathcal{K}$ to be concave, both inequalities hold in the inverted direction. Many mathematicians have paid considerable attention to the Hermite-Hadamard inequality due to its quality and integrity in mathematical inequality. For significant developments, modifications, and consequences regarding the Hermite-Hadamard uniqueness property and general convex function definitions, the interested reader would like to refer to [21–28] and references therein.

It is noted that quasi-convex functions are a generalization of the convex function class since there are quasi-convex functions that are not convex. Weir et al. [29] introduced the concept of preinvex functions, which were then used in non-linear programming to describe appropriate optimal conditions and duality. Polyak [30] considered and studied the idea of strongly convex functions, which makes an essential contribution to the adaptation of most machine learning models that require the resolution of some form of optimization problem and areas concerned. Zu et al. [31] researched convergence by using iterative techniques based on the strong convex functional theory to resolve variational inequalities and equilibrium issues. Nikodem et al. discovered the new and innovative implementation of the inner product area's characterization with strongly convex functions in [32].

Throughout this paper, we are using continuous bifunctions $\mu(.,.):\mathbb{R}^n\times\mathbb{R}^n\rightarrow\mathbb{R}^n$ and $\mathcal{I}_\mu=[\kappa_1,\kappa_1+\mu(\kappa_2,\kappa_1)]$. Weir and Mond [29], has been introduced the definition of invex sets and preinvex mapping.

Definition 1.1. If $\mathcal{I}_\mu\subset\mathbb{R}^n$, then $\mathcal{I}_\mu\subset\mathbb{R}^n$ is said to be invex set

$$\kappa_1+\tau\mu(\kappa_2,\kappa_1)\in\mathcal{I}_\mu,$$

for all $\kappa_1,\kappa_2\in\mathcal{I}_\mu,\ \tau\in[0,1]$.

Note that, the invex set $\mathcal{I}_\mu$ is also called μ-connected set. If $\mu(\kappa_2,\kappa_1)=\kappa_2-\kappa_1$, then the invex set $\mathcal{I}_\mu$ is a convex set, but the reverse is not true.

Definition 1.2. Let a mapping $\mathcal{K}:\mathcal{I}_\mu\subset\mathbb{R}^n\rightarrow\mathbb{R}$ is called preinvex, if

$$\mathcal{K}\left(\kappa_1+\tau\mu(\kappa_2,\kappa_1)\right)\leq(1-\tau)\mathcal{K}(\kappa_1)+\tau\mathcal{K}(\kappa_2),$$

for all $\kappa_1,\kappa_2\in\mathcal{I}_\mu,\ \tau\in[0,1]$.

Here, we would like to point out that Humaira et al. [11] has introduced and studied generalized higher-order strong preinvex functions, which play a crucial role in studying the theory of optimization and related fields.

Definition 1.3. A function $\mathcal{K}:\mathcal{I}_\mu\subset\mathbb{R}^n\rightarrow\mathbb{R}$ is considered generalized higher-order strong preinvex function of order $\theta>0$ with modulus $\chi\geq0$, if

$$\mathcal{K}\left(\kappa_1+\tau\mu\left(\kappa_2,\kappa_1\right)\right)\leq\left(1-\tau\right)\mathcal{K}\left(\kappa_1\right)+\tau\mathcal{K}\left(\kappa_2\right)-\chi\tau\left(1-\tau\right)\|\mu\left(\kappa_2,\kappa_1\right)\|^\theta,$$

for all $\kappa_1,\kappa_2\in\mathcal{I}_\mu$ and all $\tau\in[0,1]$.

Properties that belong to generalized higher-order strongly preinvex functions are more robust versions of well-known properties of preinvex functions. Let us note the definition of the following generalized higher-order strongly quasi-preinvex functions.

Definition 1.4. [11] A function $\mathcal{K} : \mathcal{I}_\mu \subset \mathbb{R}^n \to \mathbb{R}$ is considered generalized higher-order strong preinvex function of order $\theta > 0$ with modulus $\chi \geq 0$, if

$$\mathcal{K}(\kappa_1 + \tau\mu(\kappa_2, \kappa_1)) \leq \max\{\mathcal{K}(\kappa_1), \mathcal{K}(\kappa_2)\} - \chi\tau(1-\tau)\|\mu(\kappa_2, \kappa_1)\|^\theta,$$

for all $\kappa_1, \kappa_2 \in \mathcal{I}$ and all $\tau \in [0, 1]$.

Remark 1. The notion of generalized higher-order strongly quasi-preinvexity strengthens the concept of quasi-preinvexity.

Several fundamental inequalities that are well known in classical analysis, like Hölder inequality, Ostrowski inequality, Cauchy-Schwarz inequality, Grüess-Chebyshev inequality, Grüess inequality. Using classical convexity, other fundamental inequalities have been proven and applied to q-calculus. Our objective is to develop new Hermite-Hadamard type inequalities by using quantum calculus and to support this claim graphically.

2. Preliminaries of q-calculus and some inequalities

In this section, we discuss some required definitions of quantum calculus and important left and right sides bonds of quantum Hermite-Hadamard integral type inequalities.

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + ... + q^{n-1}, \quad q \in (0, 1).$$

Jackson derived the q-Jackson integral in [1] from 0 to κ_2 for $q \in (0, 1)$ as follows:

$$\int_0^{\kappa_2} \mathcal{K}(\kappa)\ d_q\kappa = (1-q)\kappa_2 \sum_{n=0}^{\infty} q^n \mathcal{K}(\kappa_2 q^n) \tag{2.1}$$

provided the sum converge absolutely.

The q-Jackson integral in a generic interval $[\kappa_1, \kappa_2]$ was given by in [1] and defined as follows:

$$\int_{\kappa_1}^{\kappa_2} \mathcal{K}(\kappa)\ d_q\kappa = \int_0^{\kappa_2} \mathcal{K}(\kappa)\ d_q\kappa - \int_0^{\kappa_1} \mathcal{K}(\kappa)\ d_q\kappa .$$

Definition 2.1. [5] We suppose that $\mathcal{K} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is an arbitrary function. Then q_{κ_1}-derivative of $\mathcal{K}$ at $\kappa \in [\kappa_1, \kappa_2]$ is defined as follows:

$${}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa) = \frac{\mathcal{K}(\kappa) - \mathcal{K}(q\kappa + (1-q)\kappa_1)}{(1-q)(\kappa - \kappa_1)}, \ \kappa \neq \kappa_1. \tag{2.2}$$

Since $\mathcal{K}$ is a arbitrary function from $[\kappa_1, \kappa_2]$ to $\mathbb{R}$, then ${}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1) = \lim_{\kappa \to \kappa_1} {}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa)$. The function $\mathcal{K}$ is called q_{κ_1}- differentiable on $[\kappa_1, \kappa_2]$, if ${}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\tau)$ exists for all $\kappa \in [\kappa_1, \kappa_2]$. If $\kappa_1 = 0$ in (2.2), then ${}_0\mathcal{D}_q\mathcal{K}(\kappa) = \mathcal{D}_q\mathcal{K}(\kappa)$, where $\mathcal{D}_q\mathcal{K}(\kappa)$ is familiar q_{κ_1}-derivative of $\mathcal{K}$ at $\kappa \in [\kappa_1, \kappa_2]$ defined by the expression (see [4])

$$\mathcal{D}_q\mathcal{K}(\kappa) = \frac{\mathcal{K}(\kappa) - \mathcal{K}(q\kappa)}{(1-q)\kappa},\ \kappa \neq 0.$$

The lemma below is play key part to calculate q_{κ_1}-derivatives.

Lemma 2.2. *[5] Taking $\xi \in \mathbb{R}$ and $q \in (0,1)$, we have*

$${}_{\kappa_1}\mathcal{D}_q (x-\kappa_1)^{\xi} = \left(\frac{1-q^{\xi}}{1-q}\right)(x-\kappa_1)^{\xi-1}.$$

Definition 2.3. [5] We suppose that $\mathcal{K} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is an arbitrary function, then the q_{κ_1}-definite integral on $[\kappa_1, \kappa_2]$ is described as below:

$$\int_{\kappa_1}^{\kappa_2} \mathcal{K}(\kappa)\ {}_{\kappa_1}d_q\kappa = (1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty} q^n\mathcal{K}(q^n\kappa_2 + (1-q^n)\kappa_1) = (\kappa_2-\kappa_1)\int_0^1 \mathcal{K}((1-\tau)\kappa_1 + \tau\kappa_2)\ d_q\tau\ . \tag{2.3}$$

for $x \in \mathcal{I}$. If $\chi \in (\kappa_1, x)$, then the definite q_{κ_1}-integral on $\mathcal{I}$ is described as:

$$\begin{aligned}\int_{\kappa_1}^{x} \mathcal{K}(x)\ {}_{\kappa_1}d_qx &= \int_{\kappa_1}^{x} \mathcal{K}(x)\ {}_{\kappa_1}d_qx - \int_{\kappa_1}^{\chi} \mathcal{K}(x)\ {}_{\kappa_1}d_qx \\ &= (x-\kappa_1)(1-q)\sum_{n=0}^{\infty} q^n\mathcal{K}(q^nx + (1-q^n)\kappa_1) \\ &+ (\chi-\kappa_1)(1-q)\sum_{n=0}^{\infty} q^n\mathcal{K}(q^n\chi + (1-q^n)\kappa_1).\end{aligned}$$

If $\kappa_1 = 0$ in (refA3), then we obtain the classical definite q_{κ_1}-integral which is proved in (see [6])

$$\int_0^x \mathcal{K}(x)\ d_qx = (1-q)\,x\sum_{n=0}^{\infty} q^n\mathcal{K}(q^nx),\ x \in [0,\infty).$$

The following properties are very important in quantum calculus.

Theorem 2.4. *[5] We suppose that $\mathcal{K} : \mathcal{I} \to \mathbb{R}$ be a arbitrary function. Then*

1. ${}_{\kappa_1}\mathcal{D}_q \int_{\kappa_1}^{x} \mathcal{K}(\tau)\ {}_{\kappa_1}d_q\tau = \mathcal{K}(x) - \mathcal{K}(\kappa_1)$;
2. $\int_{\chi}^{x} {}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\tau)\ {}_{\kappa_1}d_q\tau = \mathcal{K}(x) - \mathcal{K}(\chi),\ \chi \in (\kappa_1, x)$.

The following is useful results for evaluating such $q_{\kappa 1}$-integrals.

Lemma 2.5. *[5] The following formula holds for $\zeta \in \mathbb{R}\backslash\{-1\}$ with $q \in (0,1)$, then*

$$\int_{\kappa_1}^{\sigma} (\tau-\kappa_1)^{\zeta}\ {}_{\kappa_1}d_q\tau = \left(\frac{1-q}{1-q^{\zeta+1}}\right)(\sigma-\kappa_1)^{\zeta+1}.$$

In [10], Alp et al. established the q_{κ_1}-Hermite-Hadamard inequalities for convexity, which is defined as follows:

Theorem 2.6. *We suppose that $\mathcal{K} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a convex differentiable function on $[\kappa_1, \kappa_2]$ and $q \in (0, 1)$. Then q_{κ_1}-Hermite-Hadamard inequalities are as follows:*

$$\mathcal{K}\left(\frac{q\kappa_1 + \kappa_2}{[2]_q}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{K}(\kappa) \ {}_{\kappa_1}d_q\kappa \ \leq \frac{q\mathcal{K}(\kappa_1) + \mathcal{K}(\kappa_2)}{[2]_q}. \tag{2.4}$$

In [18], Bermudo et al. established the q^{κ_2}-derivative, q^{κ_2}-integration and q^{κ_2}-Hermite-Hadamard inequalities for convexity, which is defined as follows:

Definition 2.7. [18] We suppose that $\mathcal{K} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is an arbitrary function, then q^{κ_2}-derivative of $\mathcal{K}$ at $\kappa \in [\kappa_1, \kappa_2]$ is defined as follows:

$${}^{\kappa_2}\mathcal{D}_q\mathcal{K}(\kappa) \ = \frac{\mathcal{K}(q\kappa + (1-q)\kappa_2) - \mathcal{K}(\kappa)}{(1-q)(\kappa_2 - \kappa)}, \ \kappa \neq \kappa_2. \tag{2.5}$$

Since $\mathcal{K}$ is a arbitrary function from $[\kappa_1, \kappa_2]$ to $\mathbb{R}$, then ${}^{\kappa_2}\mathcal{D}_q\mathcal{K}(\kappa_2) \ = \lim_{\kappa \to \kappa_2} {}^{\kappa_2}\mathcal{D}_q\mathcal{K}(\kappa)$. The function $\mathcal{K}$ is called q^{κ_2}- differentiable on $[\kappa_1, \kappa_2]$, if ${}^{\kappa_2}\mathcal{D}_q\mathcal{K}(\tau)$ exists for all $\kappa \in [\kappa_1, \kappa_2]$. If $\kappa_2 = 0$ in (2.5), then ${}^{0}\mathcal{D}_q\mathcal{K}(\kappa) = \mathcal{D}_q\mathcal{K}(\kappa)$, where $\mathcal{D}_q\mathcal{K}(\kappa)$ is familiar q^{κ_2}-derivative of $\mathcal{K}$ at $\kappa \in [\kappa_1, \kappa_2]$ defined by the expression (see [1])

$$\mathcal{D}_q\mathcal{K}(\kappa) \ = \frac{\mathcal{K}(q\kappa) - \mathcal{K}(\kappa)}{(1-q)\kappa}, \ \kappa \neq 0.$$

Definition 2.8. We suppose that $\mathcal{K} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is an arbitrary function. Then, the q^{κ_2}-definite integral on $[\kappa_1, \kappa_2]$ is defined as:

$$\int_{\kappa_1}^{\kappa_2} \mathcal{K}(\kappa) \ {}^{\kappa_2}d_q\kappa \ = (1-q)(\kappa_2 - \kappa_1) \sum_{n=0}^{\infty} q^n \mathcal{K}(q^n\kappa_1 + (1-q^n)\kappa_2) = (\kappa_2 - \kappa_1) \int_0^1 \mathcal{K}(t\kappa_1 + (1-\tau)\kappa_2) \ d_q\tau \ .$$

Theorem 2.9. *[18] We suppose that $\mathcal{K} : \mathcal{I} \to \mathbb{R}$ is a continuous function. Then*

1. ${}^{\kappa_2}\mathcal{D}_q \int_x^{\kappa_2} \mathcal{K}(\tau) \ {}^{\kappa_2}d_q\tau = \mathcal{K}(\kappa_2) - \mathcal{K}(x)$;
2. $\int_x^{\chi} \ {}^{\kappa_2}\mathcal{D}_q\mathcal{K}(\tau) \ {}^{\kappa_2}d_q\tau = \mathcal{K}(\chi) - \mathcal{K}(x), \chi \in (x, \kappa_2)$.

Theorem 2.10. *[18] We suppose that $\mathcal{K} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a convex function on $[\kappa_1, \kappa_2]$ and $q \in (0, 1)$. Then, q^{κ_2}-Hermite-Hadamard inequalities are as follows:*

$$\mathcal{K}\left(\frac{\kappa_1 + q\kappa_2}{[2]_q}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{K}(\kappa) \ {}^{\kappa_2}d_q\kappa \ \leq \frac{\mathcal{K}(\kappa_1) + q\mathcal{K}(\kappa_2)}{[2]_q}. \tag{2.6}$$

From Theorem 2.6 and Theorem 2.10, one can the following inequalities:

Corollary 1. *[18] For any convex function $\mathcal{K} : [\kappa_1, \kappa_2] \to \mathbb{R}$ and $q \in (0, 1)$, we have*

$$\mathcal{K}\left(\frac{q\kappa_1 + \kappa_2}{[2]_q}\right) + \mathcal{K}\left(\frac{\kappa_1 + q\kappa_2}{[2]_q}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \left\{ \int_{\kappa_1}^{\kappa_2} \mathcal{K}(\kappa) \ {}_{\kappa_1}d_q\kappa \ + \int_{\kappa_1}^{\kappa_2} \mathcal{K}(\kappa) \ {}^{\kappa_2}d_q\kappa \right\} \leq \mathcal{K}(\kappa_1) + \mathcal{K}(\kappa_2) \tag{2.7}$$

and

$$\mathcal{K}\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{1}{2(\kappa_2-\kappa_1)}\left\{\int_{\kappa_1}^{\kappa_2} \mathcal{K}(\kappa)\ {}_{\kappa_1}d_q\kappa + \int_{\kappa_1}^{\kappa_2} \mathcal{K}(\kappa)\ {}^{\kappa_2}d_q\kappa\right\} \leq \frac{\mathcal{K}(\kappa_1)+\mathcal{K}(\kappa_2)}{2}. \tag{2.8}$$

Theorem 2.11. *[7] Suppose that $\mathcal{K} : [\kappa_1, \kappa_1+\mu(\kappa_2,\kappa_1)] \subset \mathbb{R} \to \mathbb{R}$ is a q_{κ_1}-differentiable function on $(\kappa_1, \kappa_1+\mu(\kappa_2,\kappa_1))$ such that ${}_{\kappa_1}\mathcal{D}_q\mathcal{K}$ being continuous and q_{κ_1}-integrable on $[\kappa_1, \kappa_1+\mu(\kappa_2,\kappa_1)]$ and $q \in (0,1)$. If $\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}\right|^{\sigma}$ is preinvex function for $\sigma \geq 1$, then*

$$\left|\frac{1}{\mu(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\mu(\kappa_2,\kappa_1)} \mathcal{K}(x)\ {}_{\kappa_1}d_qx - \frac{q\mathcal{K}(\kappa_1)+\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))}{[2]_q}\right| \leq \frac{q\mu(\kappa_2,\kappa_1)}{[2]_q}\left(\frac{q\left(2+q+q^3\right)}{[2]_q^3}\right)^{1-\frac{1}{\sigma}}$$
$$\times\left[\frac{q\left(1+4q+q^2\right)}{[3]_q[2]_q^3}\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1)\right|^{\sigma} + \frac{q\left(1+3q^2+2q^3\right)}{[3]_q[2]_q^3}\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_2)\right|^{\sigma}\right]^{\frac{1}{\sigma}}. \tag{2.9}$$

Theorem 2.12. *[7] Suppose that $\mathcal{K} : [\kappa_1, \kappa_1+\mu(\kappa_2,\kappa_1)] \subset \mathbb{R} \to \mathbb{R}$ is a q_{κ_1}-differentiable function on $(\kappa_1, \kappa_1+\mu(\kappa_2,\kappa_1))$ such that ${}_{\kappa_1}\mathcal{D}_q\mathcal{K}$ being continuous and q_{κ_1}-integrable on $[\kappa_1, \kappa_1+\mu(\kappa_2,\kappa_1)]$ and $q \in (0,1)$. If $\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}\right|^{\sigma}$ is quasi-preinvex function for $\sigma \geq 1$, then*

$$\left|\frac{1}{\mu(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\mu(\kappa_2,\kappa_1)} \mathcal{K}(x)\ {}_{\kappa_1}d_qx - \frac{q\mathcal{K}(\kappa_1)+\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))}{[2]_q}\right|$$
$$\leq \frac{q^2\mu(\kappa_2,\kappa_1)\left(2+q+q^3\right)}{[2]_q^4}\left(\max\left\{\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1)\right|^{\sigma}, \left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_2)\right|^{\sigma}\right\}\right)^{\frac{1}{\sigma}}. \tag{2.10}$$

3. Hermite-Hadamard-type inequalities for quantum integrals

We are now providing some new Hermite-Hadamard-type inequalities for functions whose absolute value of first q_{κ_1}-, q^{κ_2}-derivatives are generalized higher-order strongly preinvex functions. To prove our main results, we will initially suggest the following useful lemmas.

Lemma 3.1. *Suppose that $\mathcal{K} : [\kappa_1, \kappa_1+\mu(\kappa_2,\kappa_1)] \subset \mathbb{R} \to \mathbb{R}$ is a q_{κ_1}-differentiable function on $(\kappa_1, \kappa_1+\mu(\kappa_2,\kappa_1))$ such that ${}_{\kappa_1}\mathcal{D}_q\mathcal{K}$ being continuous and q_{κ_1}-integrable on $[\kappa_1, \kappa_1+\mu(\kappa_2,\kappa_1)]$ and $q \in (0,1)$, then the following identity holds*

$$\frac{1}{\mu(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\mu(\kappa_2,\kappa_1)} \mathcal{K}(x)\ {}_{\kappa_1}d_qx - \frac{q\mathcal{K}(\kappa_1)+\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))}{[2]_q} = \frac{q\mu(\kappa_2,\kappa_1)}{2}$$
$$\times\int_0^1\int_0^1 (\epsilon-\tau)\ \left[{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1+\tau\mu(\kappa_2,\kappa_1)) - {}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1+\epsilon\mu(\kappa_2,\kappa_1))\right]\ d_q\tau\ {}_0d_q\epsilon. \tag{3.1}$$

Proof. By using Definition 2.1 and Definition 2.3, we have

$$
\begin{aligned}
&\int_0^1 \int_0^1 (\epsilon - \tau)) \left[{}_{\kappa_1}\mathcal{D}_q \mathcal{K}(\kappa_1 + \tau\mu(\kappa_2, \kappa_1)) - {}_{\kappa_1}\mathcal{D}_q \mathcal{K}(\kappa_1 + \epsilon\mu(\kappa_2, \kappa_1)) \right] d_q\tau \, d_q\epsilon \\
&\quad = \int_0^1 \int_0^1 (\epsilon - \tau) \left[\frac{\mathcal{K}(\kappa_1 + \tau\mu(\kappa_2, \kappa_1)) - \mathcal{K}(\kappa_1 + q\tau\mu(\kappa_2, \kappa_1))}{(1-q)\mu(\kappa_2, \kappa_1)\tau} \right. \\
&\qquad \left. - \frac{\mathcal{K}(\kappa_1 + \epsilon\mu(\kappa_2, \kappa_1)) - \mathcal{K}(\kappa_1 + q\epsilon\mu(\kappa_2, \kappa_1))}{(1-q)\mu(\kappa_2, \kappa_1)\epsilon} \right] d_q\tau \, d_q\epsilon \\
&\quad = \int_0^1 \int_0^1 \frac{\epsilon \left[\mathcal{K}(\kappa_1 + \tau\mu(\kappa_2, \kappa_1)) - \mathcal{K}(\kappa_1 + q\tau\mu(\kappa_2, \kappa_1)) \right]}{(1-q)\mu(\kappa_2, \kappa_1)\tau} d_q\tau \, d_q\epsilon \\
&\qquad - \int_0^1 \int_0^1 \frac{\mathcal{K}(\kappa_1 + \epsilon\mu(\kappa_2, \kappa_1)) - \mathcal{K}(\kappa_1 + q\epsilon\mu(\kappa_2, \kappa_1))}{(1-q)\mu(\kappa_2, \kappa_1)} d_q\tau \, d_q\epsilon \\
&\qquad - \int_0^1 \int_0^1 \frac{\mathcal{K}(\kappa_1 + \tau\mu(\kappa_2, \kappa_1)) - \mathcal{K}(\kappa_1 + q\tau\mu(\kappa_2, \kappa_1))}{(1-q)\mu(\kappa_2, \kappa_1)} d_q\tau \, d_q\epsilon \\
&\qquad + \int_0^1 \int_0^1 \frac{\tau \left[\mathcal{K}(\kappa_1 + \epsilon\mu(\kappa_2, \kappa_1)) - \mathcal{K}(\kappa_1 + q\epsilon\mu(\kappa_2, \kappa_1)) \right]}{(1-q)\mu(\kappa_2, \kappa_1)\epsilon} d_q\tau \, d_q\epsilon. \qquad (3.2)
\end{aligned}
$$

We observe that

$$
\begin{aligned}
&\int_0^1 \int_0^1 \frac{\epsilon \left[\mathcal{K}(\kappa_1 + \tau\mu(\kappa_2, \kappa_1)) - \mathcal{K}(\kappa_1 + q\tau\mu(\kappa_2, \kappa_1)) \right]}{(1-q)\mu(\kappa_2, \kappa_1)\tau} d_q\tau \; {}_0d_q\epsilon \\
&\quad = \int_0^1 \epsilon \, d_q\epsilon \int_0^1 \frac{\mathcal{K}(\kappa_1 + \tau\mu(\kappa_2, \kappa_1))}{(1-q)\mu(\kappa_2, \kappa_1)\tau} \; {}_0d_q\tau - \int_0^1 \epsilon \, d_q\epsilon \int_0^1 \frac{\mathcal{K}(\kappa_1 + q\tau\mu(\kappa_2, \kappa_1))}{(1-q)\mu(\kappa_2, \kappa_1)\tau} d_q\tau \\
&\quad = \frac{(1-q)}{\mu(\kappa_2, \kappa_1)} \sum_{n=0}^{\infty} q^{2n} \left[\sum_{n=0}^{\infty} \mathcal{K}(\kappa_1 + q^n\mu(\kappa_2, \kappa_1)) - \sum_{n=0}^{\infty} \mathcal{K}\left(\kappa_1 + q^{n+1}\mu(\kappa_2, \kappa_1)\right) \right] \\
&\quad = \frac{1}{[2]_q\mu(\kappa_2, \kappa_1)} \left[\sum_{n=0}^{\infty} \mathcal{K}(\kappa_1 + q^n\mu(\kappa_2, \kappa_1)) - \sum_{n=1}^{\infty} \mathcal{K}(\kappa_1 + q^n\mu(\kappa_2, \kappa_1)) \right] \\
&\quad = \frac{\mathcal{K}(\kappa_1 + \mu(\kappa_2, \kappa_1)) - \mathcal{K}(\kappa_1)}{[2]_q\mu(\kappa_2, \kappa_1)}, \qquad (3.3)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 \int_0^1 \frac{\mathcal{K}(\kappa_1 + \epsilon\mu(\kappa_2, \kappa_1)) - \mathcal{K}(\kappa_1 + q\epsilon\mu(\kappa_2, \kappa_1))}{(1-q)\mu(\kappa_2, \kappa_1)} d_q\tau \, d_q\epsilon \\
&\quad = \int_0^1 d_q\tau \int_0^1 \frac{\mathcal{K}(\kappa_1 + \epsilon\mu(\kappa_2, \kappa_1))}{(1-q)\mu(\kappa_2, \kappa_1)} \; {}_0d_q\epsilon - \int_0^1 d_q\tau \int_0^1 \frac{\mathcal{K}(\kappa_1 + q\epsilon\mu(\kappa_2, \kappa_1))}{(1-q)\mu(\kappa_2, \kappa_1)} d_q\epsilon \\
&\quad = \frac{(1-q)}{\mu(\kappa_2, \kappa_1)} \sum_{n=0}^{\infty} q^n \left[\sum_{n=0}^{\infty} q^n\mathcal{K}(\kappa_1 + q^n\mu(\kappa_2, \kappa_1)) - \sum_{n=0}^{\infty} q^n\mathcal{K}\left(\kappa_1 + q^{n+1}\mu(\kappa_2, \kappa_1)\right) \right] \\
&\quad = \frac{1}{\mu(\kappa_2, \kappa_1)} \sum_{n=0}^{\infty} q^n\mathcal{K}(\kappa_1 + q^n\mu(\kappa_2, \kappa_1)) - \frac{1}{\mu(\kappa_2, \kappa_1)} \sum_{n=0}^{\infty} q^n\mathcal{K}\left(\kappa_1 + q^{n+1}\mu(\kappa_2, \kappa_1)\right) \\
&\quad = \frac{1}{\mu(\kappa_2, \kappa_1)} \left[\sum_{n=0}^{\infty} q^n\mathcal{K}(\kappa_1 + q^n\mu(\kappa_2, \kappa_1)) - \frac{1}{q} \sum_{n=0}^{\infty} q^{n+1}\mathcal{K}\left(\kappa_1 + q^{n+1}\mu(\kappa_2, \kappa_1)\right) \right]
\end{aligned}
$$

$$= -\frac{1}{q\mu(\kappa_2,\kappa_1)^2}\int_{\kappa_1}^{\kappa_1+\mu(\kappa_2,\kappa_1)}\mathcal{K}(x)\ {}_{\kappa_1}d_qx + \frac{\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))}{q\mu(\kappa_2,\kappa_1)}. \tag{3.4}$$

Similarly

$$\begin{aligned}&\int_0^1\int_0^1 \frac{\mathcal{K}(\kappa_1+\tau\mu(\kappa_2,\kappa_1))-\mathcal{K}(\kappa_1+q\tau\mu(\kappa_2,\kappa_1))}{(1-q)\mu(\kappa_2,\kappa_1)}\,d_q\tau\,d_q\epsilon\\ &= \int_0^1 d_q\epsilon\int_0^1 \frac{\mathcal{K}(\kappa_1+\tau\mu(\kappa_2,\kappa_1))-\mathcal{K}(\kappa_1+q\tau\mu(\kappa_2,\kappa_1))}{(1-q)\mu(\kappa_2,\kappa_1)}\,d_q\tau\\ &= -\frac{1}{q\mu(\kappa_2,\kappa_1)^2}\int_{\kappa_1}^{\kappa_1+\mu(\kappa_2,\kappa_1)}\mathcal{K}(x)\ {}_{\kappa_1}d_qx + \frac{\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))}{q\mu(\kappa_2,\kappa_1)},\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}&\int_0^1\int_0^1 \frac{\tau\left[\mathcal{K}(\kappa_1+\epsilon\mu(\kappa_2,\kappa_1))-\mathcal{K}(\kappa_1+q\epsilon\mu(\kappa_2,\kappa_1))\right]}{(1-q)\mu(\kappa_2,\kappa_1)\epsilon}\,d_q\tau\ {}_0d_q\epsilon\\ &= \int_0^1 \tau\, d_q\tau\int_0^1 \frac{\mathcal{K}(\kappa_1+\epsilon\mu(\kappa_2,\kappa_1))-\mathcal{K}(\kappa_1+q\epsilon\mu(\kappa_2,\kappa_1))}{(1-q)\mu(\kappa_2,\kappa_1)\epsilon}\,d_q\epsilon\\ &= \frac{\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))-\mathcal{K}(\kappa_1)}{[2]_q\mu(\kappa_2,\kappa_1)}.\end{aligned} \tag{3.6}$$

The equalities (3.3)–(3.6), give

$$\begin{aligned}&\int_0^1\int_0^1(\epsilon-\tau)\left[{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1+\tau\mu(\kappa_2,\kappa_1)) -{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1+\epsilon\mu(\kappa_2,\kappa_1))\right]d_q\tau\,d_q\epsilon\\ &= \frac{2}{q\mu(\kappa_2,\kappa_1)^2}\int_{\kappa_1}^{\kappa_1+\mu(\kappa_2,\kappa_1)}\mathcal{K}(x)\ {}_{\kappa_1}d_qx - 2\frac{\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))}{q\mu(\kappa_2,\kappa_1)} + 2\frac{\left[\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))-\mathcal{K}(\kappa_1)\right]}{[2]_q\mu(\kappa_2,\kappa_1)}.\end{aligned} \tag{3.7}$$

Multiplying both sides of (3.7) by $\frac{q\mu(\kappa_2,\kappa_1)}{2}$, we get (3.1). □

Lemma 3.2. *Suppose that $\mathcal{K} : [\kappa_2+\mu(\kappa_1,\kappa_2),\kappa_2] \subset \mathbb{R} \to \mathbb{R}$ is a q^{κ_2}-differentiable function on $(\kappa_2+\mu(\kappa_1,\kappa_2),\kappa_2)$ such that ${}^{\kappa_2}\mathcal{D}_q\mathcal{K}$ being continuous and q^{κ_2}-integrable on $[\kappa_2+\mu(\kappa_1,\kappa_2),\kappa_2]$ with $q\in(0,1)$ and $\mu(\kappa_2,\kappa_1) = -\mu(\kappa_1,\kappa_2) > 0$, then the following identity holds*

$$\begin{aligned}&\frac{1}{\mu(\kappa_1,\kappa_2)}\int_{\kappa_2+\mu(\kappa_1,\kappa_2)}^{\kappa_2}\mathcal{K}(x)\ {}^{\kappa_2}d_qx - \frac{\mathcal{K}(\kappa_2+\mu(\kappa_1,\kappa_2))+q\mathcal{K}(\kappa_2)}{[2]_q} = \frac{q\mu(\kappa_1,\kappa_2)}{2}\\ &\times\int_0^1\int_0^1(\epsilon-\tau)\left[{}^{\kappa_2}\mathcal{D}_q\mathcal{K}(\kappa_2+\tau\mu(\kappa_1,\kappa_2)) - {}^{\kappa_2}\mathcal{D}_q\mathcal{K}(\kappa_2+\epsilon\mu(\kappa_1,\kappa_2))\right]d_q\tau\ {}_0d_q\epsilon.\end{aligned} \tag{3.8}$$

Proof. The proof is directly followed by Definition 2.7 and Definition 2.8. We omit the details. □

Theorem 3.3. *If we assume all the conditions of Lemma 3.1, then the following inequality, shows that $\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}\right|^{\sigma}$ is generalized higher-order strongly preinvex functions of order $\theta > 0$ with modulus $\chi \geq 0$ on $[\kappa_1,\kappa_1+\mu(\kappa_2,\kappa_1)]$ for $\sigma \geq 1$, then*

$$\left|\frac{1}{\mu(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\mu(\kappa_2,\kappa_1)}\mathcal{K}(x)\ {}_{\kappa_1}d_qx-\frac{q\mathcal{K}(\kappa_1)+\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))}{[2]_q}\right|\leq q\mu(\kappa_2,\kappa_1)\left[\rho_3(q)\right]^{1-\frac{1}{\sigma}}$$
$$\times\left[\rho_1(q)\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1)\right|^{\sigma}+\rho_2(q)\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_2)\right|^{\sigma}-\chi\mu(\kappa_2,\kappa_1)^{\theta}\rho_4(q)\right]^{\frac{1}{\sigma}},\quad(3.9)$$

where

$$\rho_1(q)=\frac{q\left(2q^2-q+1\right)}{q^5+2q^4+3q^3+3q^2+2q+1},$$
$$\rho_2(q)=\frac{q}{q^4+q^3+2q^2+q+1},$$
$$\rho_3(q)=\frac{2q}{q^3+2q^2+2q+1},$$

and

$$\rho_4(q)=\frac{q^2\left(q^4+q^3+q^2-q+1\right)}{q^9+3q^8+6q^7+9q^6+11q^5+11q^4+9q^3+6q^2+3q+1}.$$

Proof. Taking modulus on Eq (3.1) and using the power-mean inequality, we have

$$\left|\frac{1}{\mu(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\mu(\kappa_2,\kappa_1)}\mathcal{K}(x)\ {}_{\kappa_1}d_qx-\frac{q\mathcal{K}(\kappa_1)+\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))}{[2]_q}\right|$$
$$\leq\frac{q\mu(\kappa_2,\kappa_1)}{2}\left(\int_0^1\int_0^1|\epsilon-\tau|\ d_q\tau\ d_q\epsilon\right)^{1-\frac{1}{\sigma}}$$
$$\times\left\{\left(\int_0^1\int_0^1|\epsilon-\tau|\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1+\tau\mu(\kappa_2,\kappa_1))\right|^{\sigma}\ d_q\tau\ d_q\epsilon\right)^{\frac{1}{\sigma}}\right.$$
$$\left.+\left(\int_0^1\int_0^1|\epsilon-\tau|\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1+\epsilon\mu(\kappa_2,\kappa_1))\right|^{\sigma}\ d_q\tau\ d_q\epsilon\right)^{\frac{1}{\sigma}}\right\}.\quad(3.10)$$

Since $\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}\right|^{\sigma}$ is generalized higher-order strongly preinvex function for $\sigma\geq 1$, we have

$$\int_0^1\int_0^1|\epsilon-\tau|\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1+\tau\mu(\kappa_2,\kappa_1))\right|^{\sigma}\ d_q\tau\ {}_0d_q\epsilon$$
$$\leq\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1)\right|^{\sigma}\int_0^1\int_0^1|\epsilon-\tau|(1-\tau)\ {}_0d_q\tau\ d_q\epsilon+\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_2)\right|^{\sigma}$$
$$\times\int_0^1\int_0^1|\epsilon-\tau|\,\tau\ d_q\tau\ d_q\epsilon-\chi\mu(\kappa_2,\kappa_1)^{\theta}\int_0^1\int_0^1|\epsilon-\tau|\,\tau(1-\tau)\ d_q\tau\ d_q\epsilon,\quad(3.11)$$

by using Definition 2.1 and Definition 2.3

$$\rho_1(q)=\int_0^1|\epsilon-\tau|(1-\tau)\ {}_0d_q\tau\ d_q\epsilon=\int_0^1\int_0^1\left[-\frac{2q^2\epsilon^3}{[2]_q[3]_q}+\frac{2q\epsilon^2}{[2]_q}-\frac{q\epsilon}{[2]_q}\frac{q^2}{[2]_q[3]_q}\right]d_q\epsilon$$
$$=\frac{q\left(2q^2-q+1\right)}{q^5+2q^4+3q^3+3q^2+2q+1},\quad(3.12)$$

$$\rho_2(q) = \int_0^1 \int_0^1 |\epsilon - \tau| \, \tau \; {}_0d_q\tau \; d_q\epsilon = \int_0^1 \left[\frac{2q^2\epsilon^3}{[2]_q[3]_q} - \frac{\epsilon}{[2]_q} + \frac{1}{[3]_q} \right] d_q\epsilon$$
$$= \frac{q}{q^4 + q^3 + 2q^2 + q + 1}, \tag{3.13}$$

and

$$\rho_4(q) \int_0^1 \int_0^1 |\epsilon - \tau| \, \tau(1-\tau) \; d_q\tau \; {}_0d_q\epsilon$$
$$= \int_0^1 \left(-2 \int_0^\epsilon (\tau - \epsilon) \tau(1-\tau) \; d_q\tau + \int_0^1 (\tau - \epsilon) \tau(1-\tau) \; d_q\tau \right) {}_0d_q\epsilon$$
$$= \frac{q^2 \left(q^4 + q^3 + q^2 - q + 1 \right)}{q^9 + 3q^8 + 6q^7 + 9q^6 + 11q^5 + 11q^4 + 9q^3 + 6q^2 + 3q + 1}. \tag{3.14}$$

Using (3.12)–(3.14) in (3.11) and we get the resulting inequality

$$\int_0^1 \int_0^1 |\epsilon - \tau| \left| {}_{\kappa_1}\mathcal{D}_q \mathcal{K} (\kappa_1 + \tau \mu (\kappa_2, \kappa_1)) \right|^\sigma \; d_q\tau \; {}_0d_q\epsilon$$
$$\leq \left| {}_{\kappa_1}\mathcal{D}_q \mathcal{K} (\kappa_1) \right|^\sigma \rho_1(q) + \left| {}_{\kappa_1}\mathcal{D}_q \mathcal{K} (\kappa_2) \right|^\sigma \rho_2(q) - \chi \mu (\kappa_2, \kappa_1)^\theta \rho_4(q). \tag{3.15}$$

Similarly, we also observe that

$$\int_0^1 \int_0^1 |\epsilon - \tau| \left| {}_{\kappa_1}\mathcal{D}_q \mathcal{K} (\kappa_1 + \epsilon \mu (\kappa_2, \kappa_1)) \right|^\sigma \; d_q\tau \; {}_0d_q\epsilon$$
$$\leq \left| {}_{\kappa_1}\mathcal{D}_q \mathcal{K} (\kappa_1) \right|^\sigma \int_0^1 \int_0^1 |\epsilon - \tau| (1-\epsilon) \; {}_0d_q\tau \; d_q\epsilon + \left| {}_{\kappa_1}\mathcal{D}_q \mathcal{K} (\kappa_2) \right|^\sigma \int_0^1 \int_0^1 |\epsilon - \tau| \, \epsilon \; d_q\tau \; d_q\epsilon$$
$$- \chi \mu (\kappa_2, \kappa_1)^\theta \int_0^1 \int_0^1 |\epsilon - \tau| \, \tau(1-\tau) \; d_q\tau \; {}_0d_q\epsilon$$
$$= \rho_1(q) \left| {}_{\kappa_1}\mathcal{D}_q \mathcal{K} (\kappa_1) \right|^\sigma + \rho_2(q) \left| {}_{\kappa_1}\mathcal{D}_q \mathcal{K} (\kappa_2) \right|^\sigma - \chi \mu (\kappa_2, \kappa_1)^\theta \rho_4(q). \tag{3.16}$$

We also have

$$\rho_3(q) = \int_0^1 \int_0^1 |\epsilon - \tau| \; d_q\tau \; {}_0d_q\epsilon = \int_0^1 \left(-2 \int_0^\epsilon (\tau - \epsilon) \; d_q\tau + \int_0^1 (\tau - \epsilon) \; d_q\tau \right) {}_0d_q\epsilon$$
$$= \int_0^1 \left(\frac{2q\epsilon^2}{[2]_q} - \epsilon + \frac{1}{[2]_q} \right) d_q\epsilon = \frac{2q}{[2]_q[3]_q}, \tag{3.17}$$

Applying (3.15)–(3.17) in (3.10), we obtain the desired inequality. □

Corollary 2. *If $\sigma = 1$ together with the assumptions of Theorem 3.3, we obtain*

$$\left| \frac{1}{\mu(\kappa_2, \kappa_1)} \int_{\kappa_1}^{\kappa_1 + \mu(\kappa_2, \kappa_1)} \mathcal{K}(x) \; {}_{\kappa_1}d_qx - \frac{q\mathcal{K}(\kappa_1) + \mathcal{K}(\kappa_1 + \mu(\kappa_2, \kappa_1))}{[2]_q} \right|$$
$$\leq q\mu(\kappa_2, \kappa_1) \left[\rho_1(q) \left| {}_{\kappa_1}\mathcal{D}_q \mathcal{K} (\kappa_1) \right| + \rho_2(q) \left| {}_{\kappa_1}\mathcal{D}_q \mathcal{K} (\kappa_2) \right| - \chi \mu (\kappa_2, \kappa_1)^\theta \rho_4(q) \right], \tag{3.18}$$

where $\rho_1(q), \rho_2(q)$ and $\rho_4(q)$ are defined in Theorem 3.3.

Corollary 3. *As $q \to 1^-$ in Theorem 3.3, we get the inequality*

$$\left|\frac{1}{\mu(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\mu(\kappa_2,\kappa_1)}\mathcal{K}(x)\,dx-\frac{\mathcal{K}(\kappa_1)+\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))}{2}\right|$$
$$\leq \mu(\kappa_2,\kappa_1)\left(\frac{1}{3}\right)^{1-\frac{1}{\sigma}}\left[\frac{|\mathcal{K}'(\kappa_1)|^{\sigma}+|\mathcal{K}'(\kappa_2)|^{\sigma}}{6}-\frac{\chi\mu(\kappa_2,\kappa_1)^{\theta}}{20}\right]^{\frac{1}{\sigma}}. \quad (3.19)$$

Corollary 4. *Suppose that the assumptions of Theorem 3.3 with $\sigma = 1$ and letting $q \to 1^-$, we obtain the inequality*

$$\left|\frac{1}{\mu(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\mu(\kappa_2,\kappa_1)}\mathcal{K}(x)\,dx-\frac{\mathcal{K}(\kappa_1)+\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))}{2}\right|$$
$$\leq \mu(\kappa_2,\kappa_1)\left[\frac{|\mathcal{K}'(\kappa_1)|+|\mathcal{K}'(\kappa_2)|}{6}-\frac{\chi\mu(\kappa_2,\kappa_1)^{\theta}}{20}\right]. \quad (3.20)$$

Theorem 3.4. *If we assume all the conditions of Lemma 3.1, then the following inequality, shows that $\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}\right|^{\sigma}$ is generalized higher-order strongly preinvex function of order $\theta > 0$ with modulus $\chi \geq 0$ on $[\kappa_1, \kappa_1 + \mu(\kappa_2,\kappa_1)]$ for $\frac{1}{p}+\frac{1}{\sigma}=1$, then*

$$\left|\frac{1}{\mu(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\mu(\kappa_2,\kappa_1)}\mathcal{K}(x)\ {}_{\kappa_1}d_qx-\frac{q\mathcal{K}(\kappa_1)+\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))}{[2]_q}\right| \leq q\mu(\kappa_2,\kappa_1)\left[\rho_3(p,q)\right]^{1-\frac{1}{p}}$$
$$\times\left(\frac{q\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1)\right|^{\sigma}+\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_2)\right|^{\sigma}}{[2]_q}-\frac{\chi\mu(\kappa_2,\kappa_1)^{\theta}q^2}{[2]_q[3]_q}\right)^{\frac{1}{\sigma}}, \quad (3.21)$$

where

$$\rho_3(p,q)=\frac{(q-1)^2}{(q^{p+1}-1)}\sum_{m=0}^{\infty}(-1)^{m-1}\frac{\left(3+q^{p-m+1}-q^{m+1}-2q^{p+1}-q^{p+2}\right)p(p-1)\cdots(p-m+1)}{m!\,(q^{p-m+1}+1)(q^{m+1}-1)}.$$

Proof. Taking modulus on Eq (3.1) and using Hölder's inequality, we have

$$\left|\frac{1}{\mu(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\mu(\kappa_2,\kappa_1)}\mathcal{K}(x)\ {}_{\kappa_1}d_qx-\frac{q\mathcal{K}(\kappa_1)+\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))}{[2]_q}\right|$$
$$\leq \frac{q\mu(\kappa_2,\kappa_1)}{2}\left(\int_0^1\int_0^1|\epsilon-\tau|^p\ d_q\tau\ d_q\epsilon\right)^{1-\frac{1}{p}} \qquad q$$
$$\times\left\{\left(\int_0^1\int_0^1\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1+\tau\mu(\kappa_2,\kappa_1))\right|^{\sigma}\ d_q\tau\ {}_0d_q\epsilon\right)^{\frac{1}{\sigma}}\right.$$
$$\left.+\left(\int_0^1\int_0^1\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1+\epsilon\mu(\kappa_2,\kappa_1))\right|^{\sigma}\ d_q\tau\ d_q\epsilon\right)^{\frac{1}{\sigma}}\right\}. \quad (3.22)$$

We now evaluate the integrals involved in (3.22). We observe that

$$\int_0^1 \int_0^1 |\epsilon - \tau|^p \; d_q\tau \, d_q\epsilon = \int_0^1 \left(\int_0^\epsilon (\epsilon - \tau)^p \; d_q\tau \right) d_q\epsilon$$
$$+ \int_0^1 \left(\int_\epsilon^1 (\tau - \epsilon)^p \; {}_0d_q\tau \right) d_q\epsilon = \int_0^1 \left(\int_0^\epsilon (\epsilon - \tau)^p \; d_q\tau \right) d_q\epsilon$$
$$+ \int_0^1 \left(\int_0^\epsilon (\tau - \epsilon)^p \; {}_0d_q\tau \right) d_q\epsilon + \int_0^1 \left(\int_0^1 (\tau - \epsilon)^p \; d_q\tau \right) d_q\epsilon, \quad (3.23)$$

$$\int_0^1 \left(\int_0^\epsilon (\epsilon - \tau)^p \; {}_0d_q\tau \right) d_q\epsilon$$
$$= \frac{1-q}{1-q^{p+1}} \left[1 - p\frac{1}{[2]_q} + \frac{p(p-1)}{2!} \frac{1}{[3]_q} - \cdots \right]$$
$$= \frac{(1-q)^2}{1-q^{p+1}} \sum_{m=0}^{\infty} (-1)^{m-1} \frac{p(p-1)\cdots(p-m+1)}{m!\,(1-q^{m+1})}, \quad (3.24)$$

$$\int_0^1 \left(\int_0^\epsilon (\tau - \epsilon)^p \; {}_0d_q\tau \right) d_q\epsilon = \int_0^1 \int_{q\tau}^1 (\tau - \epsilon)^p \; d_q\epsilon \, d_q\tau$$
$$= \int_0^1 \int_0^1 (\tau - \epsilon)^p \; d_q\epsilon \; {}_0d_q\tau - \int_0^1 \int_0^{q\tau} (\tau - \epsilon)^p \; d_q\epsilon \, d_q\tau$$
$$= (1-q)^2 \sum_{m=0}^{\infty} (-1)^{m-1} \frac{p(p-1)\cdots(p-m+1)}{m!\left([2]_q^{p-m+1}\right)(1-q^{m+1})}$$
$$- \frac{q(1-q)^2}{1-q^{p+1}} \sum_{m=0}^{\infty} (-1)^{m-1} \frac{q^m p(p-1)\cdots(p-m+1)}{m!\,(1-q^{m+1})}, \quad (3.25)$$

and

$$\int_0^1 \left(\int_0^1 (\tau - \epsilon)^p \; d_q\tau \right) d_q\epsilon = \int_0^1 \left(\int_0^1 (\tau - \epsilon)^p \; d_q\epsilon \right) d_q\tau$$
$$= (1-q)^2 \sum_{m=0}^{\infty} (-1)^{m-1} \frac{p(p-1)\cdots(p-m+1)}{m!\left([2]_q^{p-m+1}\right)(1-q^{m+1})}. \quad (3.26)$$

Using the generalized higher-order strongly preinvexity of $\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}\right|^\sigma$ on $[kappa_1, \kappa_1 + \mu(\kappa_2, \kappa_1)]$, we obtain

$$\int_0^1 \int_0^1 \left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1 + \tau\mu(\kappa_2, \kappa_1))\right|^\sigma \; d_q\tau \, d_q\epsilon$$
$$\leq \left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1)\right|^\sigma \int_0^1 (1-\tau) \; d_q\tau + \left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_2)\right|^\sigma \int_0^1 \tau \, d_q\tau$$
$$- \chi\mu(\kappa_2, \kappa_1)^\theta \int_0^1 \int_0^1 (1-\tau)\,\tau d_q\tau \, d_q\epsilon$$

$$= \frac{q\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1)\right|^{\sigma} + \left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_2)\right|^{\sigma}}{[2]_q} - \frac{\chi\mu(\kappa_2,\kappa_1)^{\theta} q^2}{[2]_q[3]_q}. \quad (3.27)$$

and similarly, we get

$$\int_0^1\int_0^1 \left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1+\epsilon\mu(\kappa_2,\kappa_1))\right|^{\sigma}\, d_q\tau\, d_q\epsilon$$
$$= \frac{q\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1)\right|^{\sigma} + \left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_2)\right|^{\sigma}}{[2]_q} - \frac{\chi\mu(\kappa_2,\kappa_1)^{\theta} q^2}{[2]_q[3]_q}. \quad (3.28)$$

Making use of (3.23) and (3.28) in (3.22), we get the required result. □

Theorem 3.5. *If we assume all the conditions of lemma 3.1, then the following inequality, shows that* $\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}\right|^{\sigma}$ *is generalized higher-order strongly quasi-preinvex function of order* $\theta > 0$ *with modulus* $\chi \geq 0$ *on* $[\kappa_1, \kappa_1+\mu(\kappa_2,\kappa_1)]$ *for* $\sigma \geq 1$*, then*

$$\left|\frac{1}{\mu(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\mu(\kappa_2,\kappa_1)}\mathcal{K}(x)\ {}_{\kappa_1}d_qx - \frac{q\mathcal{K}(\kappa_1)+\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))}{[2]_q}\right|$$
$$\leq q\mu(\kappa_2,\kappa_1)(\rho_3(q))^{1-\frac{1}{\sigma}}\left(\rho_3(q)\rho_5(q) - \chi\mu(\kappa_2,\kappa_1)^{\theta}\rho_4(q)\right)^{\frac{1}{\sigma}}, \quad (3.29)$$

where

$$\rho_5(q) = \max\left\{\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1)\right|^{\sigma}, \quad \left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_2)\right|^{\sigma}\right\},$$

$\rho_3(q)$ *and* $\rho_4(q)$ *are defined in Theorem 3.3.*

Proof. Taking modulus on equation (3.1) and using the power-mean inequality, we have

$$\left|\frac{1}{\mu(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\mu(\kappa_2,\kappa_1)}\mathcal{K}(x)\ {}_{\kappa_1}d_qx - \frac{q\mathcal{K}(\kappa_1)+\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))}{[2]_q}\right|$$
$$\leq \frac{q\mu(\kappa_2,\kappa_1)}{2}\left(\int_0^1\int_0^1 |\epsilon-\tau|\ d_q\tau\, d_q\epsilon\right)^{1-\frac{1}{\sigma}}$$
$$\times\left\{\left(\int_0^1\int_0^1 |\epsilon-\tau|\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1+\tau\mu(\kappa_2,\kappa_1))\right|^{\sigma}\ d_q\tau\, d_q\epsilon\right)^{\frac{1}{\sigma}}\right.$$
$$\left.+\left(\int_0^1\int_0^1 |\epsilon-\tau|\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1+\epsilon\mu(\kappa_2,\kappa_1))\right|^{\sigma} d_q\tau\, d_q\epsilon\right)^{\frac{1}{\sigma}}\right\}. \quad (3.30)$$

By using the generalized higher-order strongly quasi-preinvexity of $\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}\right|^{\sigma}$ on $\sigma \geq 1$, we obtain

$$\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1+\tau\mu(\kappa_2,\kappa_1))\right|^{\sigma} \leq \max\left\{\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1)\right|^{\sigma}, \left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_2)\right|^{\sigma}\right\} - \chi\mu(\kappa_2,\kappa_1)^{\theta}\tau(1-\tau) \quad (3.31)$$

and

$$\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1+\epsilon\mu(\kappa_2,\kappa_1))\right|^{\sigma} \leq \max\left\{\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1)\right|^{\sigma}, \left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_2)\right|^{\sigma}\right\} - \chi\mu(\kappa_2,\kappa_1)^{\theta}\epsilon(1-\epsilon), \quad (3.32)$$

for all $0 \leq \tau, \epsilon \leq 1$.

Applying (3.14), (3.17), (3.31) and (3.32) in (3.30), we get the desired result. □

Corollary 5. *Letting $\sigma = 1$ in Theorem 3.5, we obtain*

$$\left|\frac{1}{\mu(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\mu(\kappa_2,\kappa_1)}\mathcal{K}(x)\ {}_{\kappa_1}d_qx-\frac{q\mathcal{K}(\kappa_1)+\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))}{[2]_q}\right| \\ \leq q\mu(\kappa_2,\kappa_1)\left(\rho_3(q)\rho_6(q)-\chi\mu(\kappa_2,\kappa_1)^{\theta}\rho_4(q)\right), \quad (3.33)$$

where $\rho_6(q)=\max\left\{\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_1)\right|,\left|{}_{\kappa_1}\mathcal{D}_q\mathcal{K}(\kappa_2)\right|\right\}$.

Corollary 6. *Letting $q\to 1^-$ in Theorem 3.5, we obtain*

$$\left|\frac{1}{\mu(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\mu(\kappa_2,\kappa_1)}\mathcal{K}(x)\ {}_{\kappa_1}dx-\frac{\mathcal{K}(\kappa_1)+\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))}{2}\right| \\ \leq \mu(\kappa_2,\kappa_1)\left(\frac{1}{3}\right)^{1-\frac{1}{\sigma}}\left(\frac{\rho_7(1)}{3}-\frac{\chi\mu(\kappa_2,\kappa_1)^{\theta}}{20}\right)^{\frac{1}{\sigma}}, \quad (3.34)$$

where $\rho_7(1)=\max\left\{\left|{}_{\kappa_1}\mathcal{D}\mathcal{K}(\kappa_1)\right|^{\sigma},\left|{}_{\kappa_1}\mathcal{D}\mathcal{K}(\kappa_2)\right|^{\sigma}\right\}$.

Corollary 7. *Letting $q\to 1^-$ in Theorem 3.5 together with $\sigma = 1$, we obtain*

$$\left|\frac{1}{\mu(\kappa_2,\kappa_1)}\int_{\kappa_1}^{\kappa_1+\mu(\kappa_2,\kappa_1)}\mathcal{K}(x)\ {}_{\kappa_1}dx-\frac{\mathcal{K}(\kappa_1)+\mathcal{K}(\kappa_1+\mu(\kappa_2,\kappa_1))}{2}\right| \\ \leq \mu(\kappa_2,\kappa_1)\left(\frac{\rho_8(1)}{3}-\frac{\chi\mu(\kappa_2,\kappa_1)^{\theta}}{20}\right), \quad (3.35)$$

where $\rho_8(1)=\max\left\{\left|{}_{\kappa_1}\mathcal{D}\mathcal{K}(\kappa_1)\right|,\left|{}_{\kappa_1}\mathcal{D}\mathcal{K}(\kappa_2)\right|\right\}$.

Theorem 3.6. *If we assume all the conditions of Lemma 3.2, then the following inequality, shows that $\left|{}^{\kappa_2}\mathcal{D}_q\mathcal{K}\right|^{\sigma}$ is generalized higher-order strongly preinvex function of order $\theta>0$ with modulus $\chi\geq 0$ on $[\kappa_2+\mu(\kappa_1,\kappa_2),\kappa_2]$ for $\sigma\geq 1$, then*

$$\left|\frac{1}{\mu(\kappa_1,\kappa_2)}\int_{\kappa_2+\mu(\kappa_1,\kappa_2)}^{\kappa_2}\mathcal{K}(x)\ {}^{\kappa_2}d_qx-\frac{\mathcal{K}(\kappa_2+\mu(\kappa_1,\kappa_2))+q\mathcal{K}(\kappa_2)}{[2]_q}\right| \leq q\mu(\kappa_1,\kappa_2)\left[\rho_3(q)\right]^{1-\frac{1}{\sigma}} \\ \times\left[\rho_2(q)\left|{}^{\kappa_2}\mathcal{D}_q\mathcal{K}(\kappa_1)\right|^{\sigma}+\rho_1(q)\left|{}^{\kappa_2}\mathcal{D}_q\mathcal{K}(\kappa_2)\right|^{\sigma}-\chi\mu(\kappa_1,\kappa_2)^{\theta}\rho_4(q)\right]^{\frac{1}{\sigma}}, \quad (3.36)$$

where $\rho_1(q)$,$\rho_2(q)$,$\rho_3(q)$ and $\rho_4(q)$ are defined in Theorem 3.3.

Proof. The desired inequality (3.36) can be obtained by applying the strategy used in the proof of Theorem 3.3 and taking into account the Lemma 3.8. □

Corollary 8. *If $\sigma = 1$ together with the assumptions of Theorem 3.6, we obtain*

$$\left|\frac{1}{\mu(\kappa_1,\kappa_2)}\int_{\kappa_2+\mu(\kappa_1,\kappa_2)}^{\kappa_2}\mathcal{K}(x)\ {}^{\kappa_2}d_qx-\frac{\mathcal{K}(\kappa_2+\mu(\kappa_1,\kappa_2))+q\mathcal{K}(\kappa_2)}{[2]_q}\right| \\ \leq q\mu(\kappa_1,\kappa_2)\left[\rho_2(q)\left|{}^{\kappa_2}\mathcal{D}_q\mathcal{K}(\kappa_1)\right|+\rho_1(q)\left|{}^{\kappa_2}\mathcal{D}_q\mathcal{K}(\kappa_2)\right|-\chi\mu(\kappa_1,\kappa_2)^{\theta}\rho_4(q)\right], \quad (3.37)$$

where $\rho_1(q)$, $\rho_2(q)$ and $\rho_4(q)$ are defined in Theorem 3.3.

Theorem 3.7. *If we assume all the conditions of Lemma 3.2, then the following inequality, shows that* $\left|{}^{\kappa_2}\mathcal{D}_q\mathcal{K}\right|^{\sigma}$ *is generalized higher-order strongly preinvex function of order* $\theta > 0$ *with modulus* $\chi \geq 0$ *on* $[\kappa_2 + \mu(\kappa_1, \kappa_2), \kappa_2]$ *for* $\frac{1}{p} + \frac{1}{\sigma} = 1$, *then*

$$\left|\frac{1}{\mu(\kappa_1,\kappa_2)}\int_{\kappa_2+\mu(\kappa_1,\kappa_2)}^{\kappa_2}\mathcal{K}(x)\ {}^{\kappa_2}d_qx - \frac{\mathcal{K}(\kappa_2+\mu(\kappa_1,\kappa_2)) + q\mathcal{K}(\kappa_2)}{[2]_q}\right| \leq q\mu(\kappa_1,\kappa_2)\left[\rho_3(p,q)\right]^{\frac{1}{p}}$$
$$\times\left(\frac{\left|{}^{\kappa_2}\mathcal{D}_q\mathcal{K}(\kappa_1)\right|^{\sigma} + q\left|{}^{\kappa_2}\mathcal{D}_q\mathcal{K}(\kappa_2)\right|^{\sigma}}{[2]_q} - \frac{\chi\mu(\kappa_1,\kappa_2)^{\theta}q^2}{[2]_q[3]_q}\right)^{\frac{1}{\sigma}}, \quad (3.38)$$

where $\rho_3(p,q)$ *is defined in Theorem 3.4.*

Proof. The desired inequality (3.38) can be obtained by applying the strategy used in the proof of Theorem 3.4 and taking into account the Lemma 3.8. □

Theorem 3.8. *If we assume all the conditions of Lemma 3.2, then the following inequality, shows that* $\left|{}^{\kappa_2}\mathcal{D}_q\mathcal{K}\right|^{\sigma}$ *is generalized higher-order strongly quasi-preinvex function of order* $\theta > 0$ *with modulus* $\chi \geq 0$ *on* $[\kappa_2 + \mu(\kappa_1, \kappa_2), \kappa_2]$ *for* $\sigma \geq 1$, *then*

$$\left|\frac{1}{\mu(\kappa_1,\kappa_2)}\int_{\kappa_2+\mu(\kappa_1,\kappa_2)}^{\kappa_2}\mathcal{K}(x)\ {}^{\kappa_2}d_qx - \frac{\mathcal{K}(\kappa_2+\mu(\kappa_1,\kappa_2)) + q\mathcal{K}(\kappa_2)}{[2]_q}\right|$$
$$\leq q\mu(\kappa_2,\kappa_1)\left(\rho_3(q)\right)^{1-\frac{1}{\sigma}}\left(\rho_3(q)\rho_9(q) - \chi\mu(\kappa_1,\kappa_2)^{\theta}\rho_4(q)\right)^{\frac{1}{\sigma}}, \quad (3.39)$$

where $\rho_9(q) = \max\left\{\left|{}^{\kappa_2}\mathcal{D}_q\mathcal{K}(\kappa_1)\right|^{\sigma}, \quad \left|{}^{\kappa_2}\mathcal{D}_q\mathcal{K}(\kappa_2)\right|^{\sigma}\right\}$.

Proof. The desired inequality (3.39) can be obtained by applying the strategy used in the proof of Theorem 3.5 and taking into account the Lemma 3.8. □

4. Comparison of results

In this section we compare our results with the existing results graphically.

Consider the function $\mathcal{K} : [0,3] \to \mathbb{R}$ defined by $\mathcal{K}(\omega) = \omega^2$. Then $\mathcal{K}$ is a continuous function on $[0,3] \subset \mathbb{R}$ and is q_{κ_1}-differentiable on $[0,3]$. Its q_{κ_1}-derivative at ω is given by

$${}_0\mathcal{D}_q\mathcal{K}(\omega) = [2]_q\omega, \quad \omega \neq 0$$

which is continuous and q_{κ_1}-integrable on $[0,3]$ for $q \in (0,1)$.

Let $\sigma = 4, \chi = 2 = \theta$, and $\mu(3,0) = 3$, hence

$$\left|{}_0\mathcal{D}_q\mathcal{K}(\omega)\right|^4 = [2]_q^4\omega^4, \quad \omega \neq 0.$$

We observe that

$$\left|{}_0\mathcal{D}_q\mathcal{K}(0)\right|^4 = \lim_{x\to 0^+}[2]_q^4\omega^4 = 0$$

and

$$\left|{}_0\mathcal{D}_q\mathcal{K}(3)\right|^4 = 81[2]_q^4.$$

The RHS of the inequality (2.9) becomes

$$\alpha(q) = \frac{9q}{[2]_q}\left(\frac{q\left(2+q+q^3\right)}{[2]_q^3}\right)^{\frac{3}{4}}\left(\frac{q[2]_q\left(1+3q^2+2q^3\right)}{[3]_q}\right)^{\frac{1}{4}} \tag{4.1}$$

and the RHS of the inequality (3.9) takes the form

$$\beta(q) = 3q\left(\frac{2q}{[2]_q\left(q^2+q+1\right)}\right)^{\frac{3}{4}}\left(\frac{q81[2]_q^4}{q^4+q^3+2q^2+q+1} - \frac{18q^2\left(q^4+q^3+q^2-q+1\right)}{q^9+3q^8+6q^7+9q^6+11q^5+11q^4+9q^3+6q^2+3q+1}\right)^{\frac{1}{4}}. \tag{4.2}$$

From Figure 1, it can be seen that $\beta(q) > \alpha(q)$, i.e. the inequality (2.9) provides better estimate than that of the inequality (3.9).

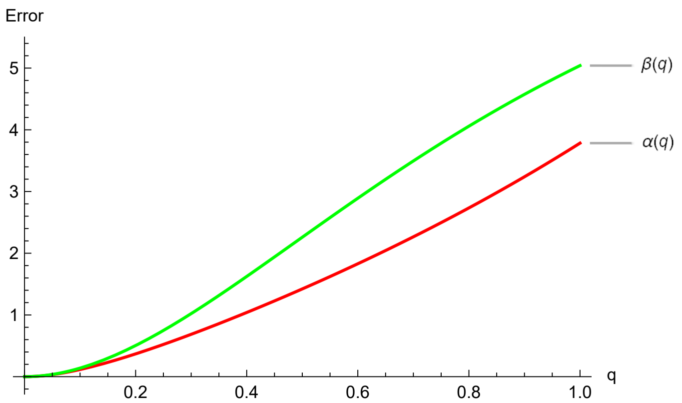

Figure 1. comparison of $\alpha(q)$ and $\beta(q)$.

Now consider the RHS of the inequality (2.10)

$$\gamma(q) = \frac{9q^2\left(2+q+q^3\right)}{[2]_q^3} \tag{4.3}$$

and the RHS of the inequality (3.29)

$$\delta(q) = 3q\left(\frac{2q}{q^3+2q^2+2q+1}\right)^{\frac{3}{4}}\left(\frac{162q[2]_q^4}{q^3+2q^2+2q+1} - \frac{18q^2\left(q^4+q^3+q^2-q+1\right)}{q^9+3q^8+6q^7+9q^6+11q^5+11q^4+9q^3+6q^2+3q+1}\right)^{\frac{1}{4}}. \tag{4.4}$$

From Figure 2, it can be seen that $\delta(q) > \gamma(q)$, i.e. the inequality (2.10) provides better estimate than that of the inequality (3.29).

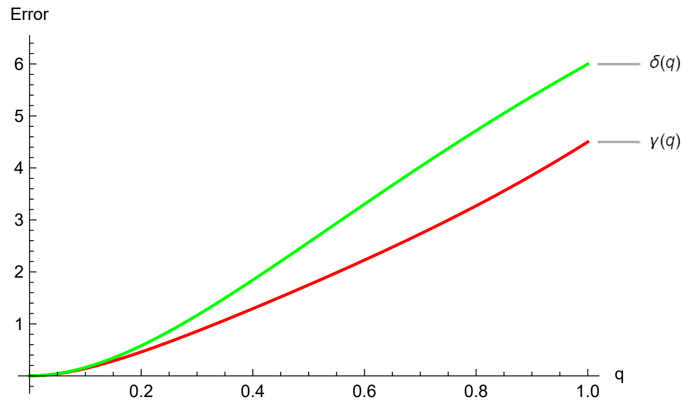

Figure 2. comparison of $\gamma(q)$ and $\delta(q)$.

5. Conclusions

In this research, the generalized class of preinvex functions has been considered. We also obtained attractive quantum analogs of new Hermite-Hadamard type inequalities for generalized higher-order strongly preinvex and quasi-preinvex functions. New integral identities for q_{κ_1}- and q^{κ_2}-differentiable functions were proven, which played an important part in obtaining quantum estimates of Hermite-Hadamard type inequalities for q_{κ_1}- and q^{κ_2}-differentiable generalized higher-order strongly preinvex and quasi-preinvex functions. Our study's claim has been graphically supported. Finally, the innovative definition of generalized higher-order strongly preinvex functions has potential applications in parallelogram law of L_p-spaces in functional analysis and opening new avenues for future study. Moreover, Srivastava [19] we presented (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis. There is also a clear connection between the classical q-analysis, which we used here, and the so-called (p, q)-analysis. We emphasize that the results for the q-analogues, which we discussed in this article for $0 < q < 1$, can be easily (and probably trivially) converted into the corresponding results for the (p, q)-analogues (with $0 < q < p \leq 1$) by making a few obvious parametric and argument changes, with the additional parameter p being superfluous.

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Conflict of interest

The authors declare that they have no competing interests.

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