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Research article

Mapping properties of Janowski-type harmonic functions involving Mittag-Leffler function

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Abstract: In this paper, we examine a connotation between certain subclasses of harmonic univalent functions by applying certain convolution operator regarding Mittag-Leffler function. To be more precise, we confer such influences with Janowski-type harmonic univalent functions in the open unit disc $\mathbb{D}$.

Keywords: analytic functions; univalent functions; harmonic functions; Mittag-Leffler function
Mathematics Subject Classification: 30C45, 30C80, 31A05, 33B15, 33C10, 33E20

1. Introduction and preliminaries

Denote by $\mathcal{A}$, the family of analytic functions in the unit disc

$$\mathbb{D} = \{z : |z| < 1, z \in \mathbb{C}\}$$

whose members are

$$h(z) = z + \sum_{k=2}^{\infty} \mathrm{A}_k z^k, \tag{1.1}$$

and normalized by $h(0) = h'(0) - 1 = 0$. Let continuous complex- valued harmonic function in a complex domain Ω be $f = u + iv$, if both u and v are real and harmonic in Ω. We write $f = h + \overline{g}$, where h and g are analytic in $\mathbf{C} \subset \Omega$, a simply-connected domain. We appeal h the analytic part and g the co-analytic part of f. Clunie and Sheil-Small [1] pointed out that f to be locally univalent and sense preserving in $\mathbf{C}$ if and only if $|h'(z)| > |g'(z)|$ in $\mathbb{D}$.

Denote by H the family of all harmonic functions of the form $f = h + \overline{g}$, is given by,

$$f(z) = z + \sum_{j=2}^{\infty} |\mathrm{A}_j| z^j + \overline{\sum_{j=1}^{\infty} |\mathrm{B}_j| z^j}, \quad (0 \leq \mathrm{B}_1 < 1), \tag{1.2}$$

where

$$h(z) = z + \sum_{j=2}^{\infty} \mathrm{A}_j z^j, \quad g(z) = \sum_{j=1}^{\infty} \mathrm{B}_j z^j, \quad |\mathrm{B}_1| < 1, \quad (z \in \mathbb{D}),$$

are members of $\mathcal{A}$. We let

$$\mathcal{T}_{\mathrm{H}} = \left\{ f(z) = z + \sum_{j=2}^{\infty} |\mathrm{A}_j| z^j - \overline{\sum_{k=1}^{\infty} |\mathrm{B}_j| z^j}, \quad (0 \leq \mathrm{B}_1 < 1) \right\}. \tag{1.3}$$

Symbolize $\mathrm{S_H}$, the subclass of H that are univalent and orientation preserving in $\mathbb{D}$. Note that $\frac{f - \overline{\mathrm{B}_1 f}}{1 - |\mathrm{B}_1|^2} \in S_{\mathrm{H}}$ whenever $f \in \mathrm{S_H}$. Due to Clunie and Sheil-Small [1] we also let

$$S_{\mathrm{H}}^{0} = \{ f = h + \overline{g} \in \mathrm{S_H} : g'(0) = \mathrm{B}_1 = 0 \}.$$

Further, let $\mathrm{K_H^0}$, $\mathrm{ST_H^0}$ and $\mathrm{C_H^0}$ are the subclasses of $\mathrm{S_H^0}$ which are harmonic convex, starlike and close-to-convex in $\mathbb{D}$ respectively. Also $\mathrm{T_H^0}$, denotes typically real harmonic functions for further details refer [1–4].

We recall following lemmas proved in [1, 2].

Lemma 1. *([1, 2]). If $f \in K_{\mathrm{H}}^0$ is assumed as in* (1.2) *with* $\mathrm{B}_1 = 0$*, then*

$$\left|\mathrm{A}_j\right| \leq \frac{j+1}{2} \text{ and } \left|\mathrm{B}_j\right| \leq \frac{j-1}{2}. \tag{1.4}$$

Lemma 2. *[1, 2] If $f \in \mathrm{ST_H^0}$ (or $f \in \mathrm{C_H^0}$) is assumed as in* (1.2) *with* $\mathrm{B}_1 = 0$*, then*

$$\left|\mathrm{A}_j\right| \leq \frac{(2j+1)(j+1)}{6}, \quad \left|\mathrm{B}_j\right| \leq \frac{(2j-1)(j-1)}{6}. \tag{1.5}$$

Lately, for $f \in \mathcal{S}_H$ whose members are given by (1.2), Dziok [5] introduced a new class $\mathcal{S}_{\mathrm{H}}^{*}(F, G)$ $(-G \leq F < G \leq 1)$ by

$$\mathcal{S}_{\mathrm{H}}^{*}(F, G) = \left\{ f = h + \overline{g} \in \mathrm{H} : \frac{\mathfrak{D}_H f(z)}{f(z)} \prec \frac{1 + Fz}{1 + Gz}; \right\}, \tag{1.6}$$

where $\mathfrak{D}_H f(z) = zh'(z) - \overline{zg'(z)}$ and $z \in \mathbb{D}$. Also let

$$\mathcal{K}_H(F, G) := f \in \mathcal{S}_H : \mathfrak{D}_H f \in \mathcal{S}_H^{*}(F, G). \tag{1.7}$$

In particular $\mathcal{R}_{\mathrm{H}}(F, G)$, the class discussed by Dziok [6] is given by

$$\mathcal{R}_{\mathrm{H}}(F, G) = \left\{ f = h + \overline{g} \in \mathrm{H} : \frac{\mathfrak{D}_H f(z)}{z'} \prec \frac{1 + Fz}{1 + Gz} \right\}. \tag{1.8}$$

Remark 1. We note that

1). $\mathcal{S}^*(F,G) := \mathcal{S}^*_H(F,G) \cap \mathcal{A}$ was introduced by Janowski [7].
2). For $0 \le \vartheta < 1$ the classes $\mathcal{S}^*_H(\vartheta) := \mathcal{S}^*_H(2\vartheta - 1, 1)$ and $\mathcal{K}_H(\vartheta) := \mathcal{K}_H(2\vartheta - 1, 1)$ were investigated by Jahangiri [8, 9].
3). Also $\mathcal{S}^*_H := \mathcal{S}^*_H(0)$ and $\mathcal{K}_H := \mathcal{K}_H(0)$ are the classes harmonic starlike and convex functions in $\mathbb{D}$ and $\mathcal{S}^*_H(F,G) \subset \mathcal{S}^*_H$, $\mathcal{K}_H(F,G) \subset \mathcal{K}_H$.

Lately, Dizok [5] gave the following coefficient conditions:

Lemma 3. *[5] Let $f \in \mathrm{H}$ be assumed as in (1.2), then $f \in \mathcal{S}^*_{\mathrm{H}}(F,G)$ if*

$$\sum_{j=2}^{\infty}\left(\frac{j(1+G)-(1+F)}{G-F}|\mathrm{A}_j| + \frac{j(1+G)+(1+F)}{G-F}|\mathrm{B}_j|\right) \le 1, \tag{1.9}$$

where $\mathrm{A}_1 = 1; \mathrm{B}_1 = 0$ *and* $-G \le F < G \le 1$.

Lemma 4. *[5] Let $f \in \mathcal{T}_{\mathrm{H}}$ be assumed as in* (1.3) *and*

$$f \in \mathcal{ST}^*_{\mathrm{H}}(F,G) \Leftrightarrow \sum_{j=2}^{\infty}\left(\frac{j(1+G)-(1+F)}{G-F}|\mathrm{A}_j| + \frac{j(1+G)+(1+F)}{G-F}|\mathrm{B}_j|\right) \le 1, \tag{1.10}$$

where $\mathrm{A}_1 = 1; \mathrm{B}_1 = 0$ *and* $-G \le F < G \le 1$.

Remark 2. If $f \in \mathcal{S}^*_{\mathrm{H}}(F,G)$, then

$$|\mathrm{A}_j| \le \frac{G-F}{j(1+G)-(1+F)} \quad \textit{and} \quad |\mathrm{B}_j| \le \frac{G-F}{j(1+G)+(1+F)}, \quad j \ge 2. \tag{1.11}$$

Lemma 5. *[5] Let $f \in \mathcal{T}_{\mathrm{H}}$ be assumed as in* (1.3) *and*

$$f \in \mathcal{KT}_{\mathrm{H}}(F,G) \Leftrightarrow \sum_{j=2}^{\infty} j\left(\frac{j(1+G)-(1+F)}{G-F}|\mathrm{A}_j| + \frac{j(1+G)+(1+F)}{G-F}|\mathrm{B}_j|\right) \le 1, \tag{1.12}$$

where $\mathrm{A}_1 = 1; \mathrm{B}_1 = 0$ *and* $-G \le F < G \le 1$.

Lemma 6. *[6]Let $f \in \mathrm{H}$ be assumed as in* (1.2) *then $f \in \mathcal{R}_{\mathrm{H}}(F,G)$ if*

$$\sum_{j=2}^{\infty}\left(\frac{j(1+G)}{G-F}|\mathrm{A}_j| + \frac{j(1+G)}{G-F}|\mathrm{B}_j|\right) \le 1, \tag{1.13}$$

where $\mathrm{A}_1 = 1; \mathrm{B}_1 = 0$ *and* $-G \le F < G \le 1$.

Lemma 7. *Let $f \in \mathcal{T}_{\mathrm{H}}$ be assumed as in* (1.3) *and*

$$f \in \mathcal{RT}_{\mathrm{H}}(F,G) \Leftrightarrow \sum_{j=2}^{\infty}\left(\frac{j(1+G)}{G-F}|\mathrm{A}_j| + \frac{j(1+G)}{G-F}|\mathrm{B}_j|\right) \le 1, \tag{1.14}$$

where $\mathrm{A}_1 = 1; \mathrm{B}_1 = 0$ *and* $-G \le F < G \le 1$.

Remark 3. If $f \in \mathcal{R}_{\mathrm{H}}(F,G)$, then $|\mathrm{A}_j| \leq \frac{G-F}{j(1+G)}$ and $|\mathrm{B}_j| \leq \frac{G-F}{j(1+G)}$, $k \geq 2$.

Mittag-Leffler [10] defined a function $\mathbf{E}_\alpha(z)$ given by

$$\mathbf{E}_\alpha(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j+1)}, \quad (z \in \mathbb{C},\ \alpha \in \mathbb{C},\ \text{with } \operatorname{Re}\alpha > 0)$$

was commonly known as the Mittag-Leffler function. Wiman [11] defined a more general function $\mathbf{E}_{\alpha,\beta}$ generalizing $\mathbf{E}_\alpha(z)$ was given by

$$\mathbf{E}_{\alpha,\beta}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j+\beta)}, \quad (z \in \mathbb{C},\ \alpha,\beta \in \mathbb{C},\ \text{with } \operatorname{Re}\alpha > 0,\ \operatorname{Re}\beta > 0.\ ). \tag{1.15}$$

Perceive that the function $\mathbf{E}_{\alpha,\beta}$ comprises many well-known functions as its exceptional case, for example, $\mathbf{E}_{0,0}(z) = \sum_{j=0}^{\infty} z^j$, $\mathbf{E}_{1,1}(z) = e^z$, $\mathbf{E}_{1,2}(z) = \frac{e^z-1}{z}$, $\mathbf{E}_{2,1}(z^2) = \cosh z$, $\mathbf{E}_{2,1}(-z^2) = \cos z$, $\mathbf{E}_{2,2}(z^2) = \frac{\sinh z}{z}$, $\mathbf{E}_{2,2}(-z^2) = \frac{\sin z}{z}$, $\mathbf{E}_4(z) = \frac{1}{2}[\cos z^{1/4} + \cosh z^{1/4}]$ and $\mathbf{E}_3(z) = \frac{1}{2}[e^{z^{1/3}} + 2e^{-\frac{1}{2}z^{1/3}} \cos(\frac{\sqrt{3}}{2} z^{1/3})]$. It is of interest to note that by fixing $\alpha = 1/2$ and $\beta = 1$ we get

$$\mathbf{E}_{\frac{1}{2},1}(z) = e^{z^2} \cdot \operatorname{erfc}(-z) = e^{z^2} \left(1 + \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} z^{2n+1} \right).$$

The Mittag-Leffler function ascends logically in the elucidation of fractional order differential and integral equations, and primarily in the investigations of fractional generalization of kinetic equation, random walks, Lévy flights, super-diffusive transport and in the study of complex systems. Numerous properties of Mittag-Leffler function and generalized Mittag-Leffler function can be found e.g., in [12–15]. We prompt that Mittag-Leffler function $\mathbf{E}_{\alpha,\beta}(z)$ is not a member of $\mathcal{A}$. Consequently, it is probable to cogitate the following normalization of $\mathbf{E}_{\alpha,\beta}(z)$ as below due to Bansal and Prajapat [13] :

$$\mathcal{E}_{\alpha,\beta}(z) = z\Gamma(\beta)\mathbf{E}_{\alpha,\beta}(z) = z + \sum_{j=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)} z^j,$$

it grasps for complex parameters $z \in \mathbb{C},\ \alpha,\beta \in \mathbb{C},$ with $\operatorname{Re}\alpha > 0,\ \operatorname{Re}\beta > 0.$

In our present study, we shall confine our attention to the case of real-valued α,β and $z \in \mathbb{D}$ and hence define a new linear operator based on convolution (or hadamard) product as below:
For real parameters $\alpha,\beta,\gamma,\delta$ $(\alpha,\beta,\gamma,\delta \notin \{0,-1,-2,\dots\})$, we let

$$\mathcal{E}_{\alpha,\beta}(z) = z + \sum_{j=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)} z^j \quad \textit{and} \quad \mathcal{E}_{\gamma,\delta}(z) = \sum_{j=1}^{\infty} \frac{\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)} z^j, \tag{1.16}$$

and define the convolution operator $\Lambda(f)$ by

$$\begin{aligned} \mathcal{F}(z) = \Lambda f(z) &= h(z) * \mathcal{E}_{\alpha,\beta}(z) + \overline{g(z) * \mathcal{E}_{\gamma,\delta}(z)} \\ &= z + \sum_{j=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)} \mathrm{A}_j z^j + \overline{\sum_{j=1}^{\infty} \frac{\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)} \mathrm{B}_j z^j}. \end{aligned} \tag{1.17}$$

Inclusion relations between different subclasses of analytic and univalent functions by using hypergeometric functions (see for example, [16–21]) and by the recent investigations related with distribution series (see for example, [22–24] and references cited there in) were exclusively studied in the literature. Lately, there has been triggering interest to investigate mapping properties and inclusion results for the families of harmonic univalent functions including various linear and nonlinear operators, (see [25–28]) and references cited there in) and also by Bessel functions studied by Porwal [29]. Motivated by aforementioned works and recent study in [30], in our current paper, we launch connections among the classes $\mathrm{K}_{\mathrm{H}}^{0}$, $\mathrm{S}_{\mathrm{H}}^{*,0}$, $\mathrm{C}_{\mathrm{H}}^{0}$, $\mathcal{S}_{\mathrm{H}}(F,G)$, and $\mathcal{R}_{\mathrm{H}}(F,G)$ by applying the convolution operator Λ related with Mittag-Leffler function.

2. Mapping properties of the class H, related with Mittag-Leffler function

In order to establish our main results, throughout this paper we let the following:

$$\mathcal{E}_{\alpha,\beta}(z) = z + \sum_{j=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)} z^{j}; \quad \mathcal{E}_{\alpha,\beta}(1) = 1 + \sum_{j=2}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)} \tag{2.1}$$

and

$$\mathcal{E}'_{\alpha,\beta}(z) = 1 + \sum_{j=2}^{\infty} \frac{j\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)} z^{j-1}; \quad \mathcal{E}'_{\alpha,\beta}(1) - 1 = \sum_{j=2}^{\infty} \frac{j\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)} \tag{2.2}$$

$$\mathcal{E}''_{\alpha,\beta}(1) = \sum_{j=2}^{\infty} \frac{j(j-1)\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)} \tag{2.3}$$

$$\mathcal{E}'''_{\alpha,\beta}(1) = \sum_{j=2}^{\infty} \frac{j(j-1)(j-2)\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)} \tag{2.4}$$

Now by using Lemma 1, we establish connection between harmonic convex functions and $\mathcal{S}_{\mathrm{H}}(F,G)$:

Theorem 2.1. *Let $\alpha,\beta,\gamma,\delta$, $(\alpha,\beta,\gamma,\delta \notin \{0,-1,-2,\dots\})$ are real. If for some $-G \leq F < G \leq 1$ and the inequality*

$$\begin{aligned}(1+G)\mathcal{E}''_{\alpha,\beta}(1) + (1+2G-F)\mathcal{E}'_{\alpha,\beta}(1) - (1+F)\mathcal{E}_{\alpha,\beta}(1)\\ +(1+G)\mathcal{E}''_{\gamma,\delta}(1) + (1+F)\mathcal{E}'_{\gamma,\delta}(1) - (1+F)\mathcal{E}_{\gamma,\delta}(1) \leq 4(G-F).\end{aligned} \tag{2.5}$$

is satisfied then $\Lambda\left(\mathrm{K}_{\mathrm{H}}^{0}\right) \subset \mathcal{S}_{\mathrm{H}}^{*}(F,G)$.

Proof. Let $f \in \mathrm{K}_{\mathrm{H}}^{0}$ be as assumed in (1.2) with $\mathrm{B}_1 = 0$. We requisite to show that $\Lambda(f) = \mathcal{F}(z) \in \mathcal{S}_{\mathrm{H}}^{*}(F,G)$, which is given by (1.17) with $\mathrm{B}_1 = 0$. In view of Lemma 3, we need to assert that

$$\begin{aligned}\Psi_1 \quad = \quad & \sum_{j=2}^{\infty} [j(1+G)-(1+F)] \left|\frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)}\mathrm{A}_j\right| \\ & + \sum_{j=2}^{\infty} [j(1+G)+(1+F)] \left|\frac{\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)}\mathrm{B}_j\right| \leq G-F.\end{aligned}$$

In view of Lemma 1, we have $A_j = \frac{j+1}{2}$ and $B_j = \frac{j-1}{2}$, thus

$$\begin{aligned}\Psi_1 &\leq \frac{1}{2}\left[\sum_{j=2}^{\infty}(j+1)\left[j(1+G)-(1+F)\right]\frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)}\right.\\&\quad\left.+\sum_{j=2}^{\infty}(j-1)\left[j(1+G)+(1+F)\right]\frac{\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)}\right]\\&=\frac{1}{2}\left[\sum_{j=2}^{\infty}\{(j^2+j)(1+G)-(j+1)(1+F)\}\frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)}\right.\\&\quad\left.+\sum_{j=2}^{\infty}\{j(j-1)(1+G)+(j-1)(1+F)\}\frac{\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)}\right].\end{aligned}$$

Expressing $j^2 = j(j-1)+j$, we get

$$\begin{aligned}\Psi_1 &= \frac{1}{2}\left[\sum_{j=2}^{\infty}\{j(j-1)(1+G)+j(1+2G-F)-(1+F)\}\frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)}\right.\\&\quad\left.+\sum_{j=2}^{\infty}\{j(j-1)(1+G)+j(1+F)-(1+F)\}\frac{\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)}\right]\\&=\frac{1}{2}\left[(1+G)\sum_{j=2}^{\infty}\frac{j(j-1)\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)}+(1+2G-F)\sum_{j=2}^{\infty}\frac{j\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)}\right.\\&\quad-(1+F)\sum_{j=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)}+(1+G)\sum_{j=2}^{\infty}\frac{j(j-1)\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)}\\&\quad\left.+(1+F)\sum_{j=2}^{\infty}\frac{j\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)}-(1+F)\sum_{j=2}^{\infty}\frac{\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)}\right].\end{aligned}$$

Now by using (2.1)–(2.3), we get

$$\begin{aligned}\Psi_1 &= \frac{1}{2}\Big[(1+G)\mathcal{E}''_{\alpha,\beta}(1)+(1+2G-F)[\mathcal{E}'_{\alpha,\beta}(1)-1]-(1+F)[\mathcal{E}_{\alpha,\beta}(1)-1]\\&\quad+(1+G)\mathcal{E}''_{\gamma,\delta}(1)+(1+F)[\mathcal{E}'_{\gamma,\delta}(1)-1])-(1+F)[\mathcal{E}_{\gamma,\delta}(1)-1]\Big]\\&=\frac{1}{2}\Big((1+G)\mathcal{E}''_{\alpha,\beta}(1)+(1+2G-F)\mathcal{E}'_{\alpha,\beta}(1)-[1+F]\mathcal{E}_{\alpha,\beta}(1)+2[F-G]\\&\quad+(1+G)\mathcal{E}''_{\gamma,\delta}(1)+(1+F)\mathcal{E}'_{\gamma,\delta}(1)-(1+F)\mathcal{E}_{\gamma,\delta}(1)\Big),\end{aligned}$$

but Ψ_1 is bounded above by $G-F$, if and only if (2.5) holds. □

As in Theorem 2.1, using Lemma 2,we determine the analogous results for the classes $\mathrm{ST}_{\mathrm{H}}^{*,0}$, $\mathrm{C}_{\mathrm{H}}^{0}$ with $\mathcal{S}_{\mathrm{H}}^{*}(F,G)$.

Theorem 2.2. *Let $\alpha,\beta,\gamma,\delta$, $(\alpha,\beta,\gamma,\delta \notin \{0,-1,-2,\dots\})$ are real. If for some $-G \leq F < G \leq 1$ and the inequality*

$$\begin{aligned} & 2(1+G)\mathcal{E}'''_{\alpha,\beta}(1) + [9G-2F+7]\mathcal{E}''_{\alpha,\beta}(1) + [6G-5F+1]\mathcal{E}'_{\alpha,\beta}(1) \\ - & (1+F)\mathcal{E}_{\alpha,\beta}(1) + 6(F-G) + 2(1+G)\mathcal{E}'''_{\gamma,\delta}(1) + [3G+2F+5]\mathcal{E}''_{\gamma,\delta}(1) \\ - & (1+F)\mathcal{E}'_{\gamma,\delta}(1) + (1+F)[\mathcal{E}_{\gamma,\delta}(1) \leq 12[G-F] \end{aligned} \tag{2.6}$$

is satisfied, then $\Lambda(\mathrm{S}_{\mathrm{H}}^{*,0}) \subset \mathcal{S}_{\mathrm{H}}^{*}(F,G)$ and $\Lambda(\mathrm{C}_{\mathrm{H}}^{0}) \subset \mathcal{S}_{\mathrm{H}}^{*}(F,G)$.

Proof. Let $f \in S_{\mathrm{H}}^{*,0}$ *(or C_{H}^{0})* be as assumed in (1.2) with $\mathrm{B}_1 = 0$. It is adequate to verify that $\Lambda(f) = \mathcal{F}(z) \in \mathcal{S}_{\mathrm{H}}^{*}(F,G)$. In sight of Theorem 3, it is profuse to show that

$$\begin{aligned} \Psi_2 &= \sum_{j=2}^{\infty} [j(1+G)-(1+F)] \frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)} \left|\mathrm{A}_j\right| \\ &+ \sum_{j=2}^{\infty} [j(1+G)+(1+F)] \frac{\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)} \left|\mathrm{B}_j\right| \leq G-F. \end{aligned}$$

By Lemma 2, we have $\mathrm{A}_j = \frac{(2j+1)(j+1)}{6}$ and $\mathrm{B}_j = \frac{(2j-1)(j-1)}{6}$, thus

$$\begin{aligned} \Psi_2 \leq & \frac{1}{6}\left[\sum_{j=2}^{\infty}(2j+1)(j+1)[j(1+G)-(1+F)] \frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)} \right. \\ + & \left. \sum_{j=2}^{\infty}(2j-1)(j-1)[j(1+G)+(1+F)] \frac{\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)}\right] \end{aligned}$$

$$\begin{aligned} = & \frac{1}{6}\sum_{j=2}^{\infty}\Big\{2(1+G)j^3 + (3G-2F+1)j^2 \\ + & (G-3F-2)j - (1+F)\Big\} \frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)} \\ + & \frac{1}{6}\sum_{j=2}^{\infty}\Big\{2(1+G)j^3 + (2F-3G-1)j^2 \\ + & (G-3F-2)j + (1+F)\Big\} \frac{\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)}. \end{aligned}$$

Writing $j^3 = j(j-1)(j-2) + 3j(j-1) + j$ and $j^2 = j(j-1) + j$, we have

$$\begin{aligned} \Psi_2 = & \frac{1}{6}\left[2(1+G)\sum_{j=2}^{\infty}\frac{j(j-1)(j-2)\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)} \right. \\ + & (9G-2F+7)\sum_{j=2}^{\infty}\frac{j(j-1)\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)} \end{aligned}$$

$$+\quad (6G-5F+1)\sum_{j=2}^{\infty}\frac{j\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)}-(1+F)\sum_{j=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)}$$
$$+\quad 2(1+G)\sum_{j=2}^{\infty}\frac{j(j-1)(j-2)\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)}+(3G+2F+5)\sum_{j=2}^{\infty}\frac{j(j-1)\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)}$$
$$-\quad (1+F)\sum_{j=2}^{\infty}\frac{j\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)}+(1+F)\sum_{j=2}^{\infty}\frac{\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)}\Bigg].$$

Now by using (2.1)–(2.4), we get

$$\begin{aligned}\Psi_2 &= \frac{1}{6}\Big\{2(1+G)\mathcal{E}'''_{\alpha,\beta}(1)+(9G-2F+7)\mathcal{E}''_{\alpha,\beta}(1)+(6G-5F+1)[\mathcal{E}'_{\alpha,\beta}(1)-1]\\ &- (1+F)[\mathcal{E}_{\alpha,\beta}(1)-1]+2(1+G)\mathcal{E}'''_{\gamma,\delta}(1)+(3G+2F+5)\mathcal{E}''_{\gamma,\delta}(1)\\ &- (1+F)[\mathcal{E}'_{\gamma,\delta}(1)-1]+(1+F)[\mathcal{E}_{\gamma,\delta}(1)-1]\Big\}\\ &= \frac{1}{6}\Big\{2(1+G)\mathcal{E}'''_{\alpha,\beta}(1)+(9G-2F+7)\mathcal{E}''_{\alpha,\beta}(1)+(6G-5F+1)\mathcal{E}'_{\alpha,\beta}(1)\\ &- (1+F)\mathcal{E}_{\alpha,\beta}(1)+6(F-G)+2(1+G)\mathcal{E}'''_{\gamma,\delta}(1)+(3G+2F+5)\mathcal{E}''_{\gamma,\delta}(1)\\ &- (1+F)\mathcal{E}'_{\gamma,\delta}(1)+(1+F)\mathcal{E}_{\gamma,\delta}(1)\Big\}.\end{aligned}$$

But Ψ_2 confined above by $G-F$, if and only if (2.6) holds. □

Theorem 2.3. *Let $\alpha,\beta,\gamma,\delta$, $(\alpha,\beta,\gamma,\delta\notin\{0,-1,-2,\dots\})$ are real. If for some $-\mathbf{G}\le\mathbf{F}<\mathbf{G}\le 1$ and $-G\le F<G\le 1$ the inequality*

$$\begin{aligned}(\mathbf{G}-\mathbf{F})\Big\{\Big[\mathcal{E}_{\alpha,\beta}(1)-1\Big]-\tfrac{1+F}{G-F}\Big(\int_0^1\tfrac{\mathcal{E}_{\alpha,\beta}(t)}{t}dt-1\Big)\\ +\mathcal{E}_{\gamma,\delta}(1)+\tfrac{1+F}{G-F}\Big(\int_0^1\mathcal{E}_{\gamma,\delta}(t)dt\Big)\Big\}\le G-F,\end{aligned}\tag{2.7}$$

is satisfied then $\Lambda\left(\mathcal{RT}_{\mathrm{H}}(\mathbf{G},\mathbf{F})\right)\subset\mathcal{S}^*_{\mathrm{H}}(F,G)$.

Proof. Let $f\in\mathcal{RT}_{\mathrm{H}}(\mathbf{G},\mathbf{F})$ be as assumed in (1.3). By virtue of Lemma 3, it is enough to show that $\Psi_3\le G-F$, where

$$\begin{aligned}\Psi_3 &= \sum_{j=2}^{\infty}\left[j(1+G)-(1+F)\right]\frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)}\left|\mathrm{A}_j\right|\\ &+ \sum_{j=1}^{\infty}\left[j(1+G)+(1+F)\right]\frac{\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)}\left|\mathrm{B}_j\right|.\end{aligned}$$

By Remark 3, we have

$$\begin{aligned}\Psi_3\le(\mathbf{G}-\mathbf{F})\Bigg[\sum_{j=2}^{\infty}\left(1-\frac{1+F}{G-F}\right)\frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)}\\ +\sum_{j=1}^{\infty}\left(1+\frac{1+F}{G-F}\right)\frac{\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)}\Bigg]\end{aligned}$$

$$
\begin{aligned}
&= (\mathbf{G}-\mathbf{F})\left[\sum_{j=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)}-\frac{1+F}{G-F}\sum_{j=2}^{\infty}\frac{\Gamma(\beta)}{j\Gamma(\alpha(j-1)+\beta)}\right.\\
&\quad\left.+\sum_{j=1}^{\infty}\frac{\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)}+\frac{1+F}{G-F}\sum_{j=1}^{\infty}\frac{\Gamma(\delta)}{k\Gamma(\gamma(j-1)+\delta)}\right]\\
&= (\mathbf{G}-\mathbf{F})\left[\left\{\mathcal{E}_{\alpha,\beta}(1)-1\right\}-\frac{1+F}{G-F}\left\{\int_0^1\frac{\mathcal{E}_{\alpha,\beta}(t)}{t}dt-1\right\}\right.\\
&\quad\left.+\mathcal{E}_{\gamma,\delta}(1)+\frac{1+F}{G-F}\left\{\int_0^1\mathcal{E}_{\gamma,\delta}(t)dt\right\}\right],
\end{aligned}
$$

and Ψ_3 is bounded above by $G-F$, if and only if (2.7) holds. □

In next theorem, we establish association between $\Lambda(\mathcal{S}^*_{\mathrm{H}}(F,G))$ and $\mathcal{S}^*_{\mathrm{H}}(F,G)$.

Theorem 2.4. *Let $\alpha,\beta,\gamma,\delta$, $(\alpha,\beta,\gamma,\delta\notin\{0,-1,-2,\dots\})$ are real. If for some $-G\le F<G\le 1$ the inequality*

$$\mathcal{E}_{\alpha,\beta}(1)+\mathcal{E}_{\gamma,\delta}(1)\le 2 \tag{2.8}$$

is satisfied, then $\Lambda\left(\mathcal{S}^*_{\mathrm{H}}(F,G)\right)\subset\mathcal{S}^*_{\mathrm{H}}(F,G)$.

Proof. In sight of Lemma 3,it is profuse to show that

$$
\begin{aligned}
\Psi_4 \quad = \quad &\sum_{j=2}^{\infty}\left[j(1+G)-(1+F)\right]\frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)}\left|\mathrm{A}_j\right|\\
+ \quad &\sum_{j=1}^{\infty}\left[j(1+G)+(1+F)\right]\frac{\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)}\left|\mathrm{B}_j\right|\le G-F.
\end{aligned}\tag{2.9}
$$

By Remark 2, it follows that

$$
\begin{aligned}
\Psi_4 &\le (G-F)\left[\sum_{j=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)}+\sum_{j=1}^{\infty}\frac{\Gamma(\delta)}{\Gamma(\gamma(j-1)+\delta)}\right]\\
&= (G-F)\left[\sum_{j=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(j-1)+\beta)}+\sum_{j=0}^{\infty}\frac{\Gamma(\delta)}{\Gamma(\gamma j+\delta)}\right]\\
&= (G-F)\left[\mathcal{E}_{\alpha,\beta}(1)-1+\mathcal{E}_{\gamma,\delta}(1)\right].
\end{aligned}
$$

But Ψ_4 is bounded above by $G-F$, if and only if (2.8) holds. □

Now we attain a depiction for Λ which maps $\mathcal{ST}_{\mathrm{H}}(F,G)$ on to itself.

Theorem 2.5. *Let $\alpha,\beta,\gamma,\delta$, $(\alpha,\beta,\gamma,\delta\notin\{0,-1,-2,\dots\})$ are real and $-G\le F<G\le 1$. Then*

$$\Lambda\left(\mathcal{ST}^*_{\mathrm{H}}(F,G)\right)\subset\mathcal{ST}^*_{\mathrm{H}}(F,G),$$

if and only if,

$$\mathcal{E}_{\alpha,\beta}(1)+\mathcal{E}_{\gamma,\delta}(1)\le 2.$$

Proof. The proof follows in lines similar to the proof of Theorem 2.4, so we overlook the details. □

3. Conclusions

In this investigation we obtained sufficient conditions and inclusion results for functions $f \in \mathcal{A}$ to be in the classes $\mathcal{S}_{\mathrm{H}}^{*}(F, G)$ and information regarding the images of functions by applying convolution operator with Mittag-Leffler function. By specializing the parameter $G = 1$ and $F = 2\varrho - 1 (0 \leq \varrho < 1)$ we can easily derive the inclusion results and mapping properties for the function classes $f \in \mathcal{S}_{\mathrm{H}}(\varrho)$ and $f \in \mathcal{R}_{\mathrm{H}}(\varrho)$ in association with Mittag-Leffler function. By using the Alexander theorem $f \in \mathcal{K}_H \Leftrightarrow zf' \in \mathcal{S}_H$ we can define $\mathcal{K}_{\mathrm{H}}(F, G)$ whose members are given by (1.2), satisfying the condition

$$\mathcal{K}_{\mathrm{H}}(F, G) = \left\{ f = h + \overline{g} \in \mathrm{H} : \frac{\mathfrak{D}_H(z \mathfrak{D}_H f(z)}{\mathfrak{D}_H f(z)} \prec \frac{1 + Fz}{1 + Gz} \right\}. \tag{3.1}$$

By using the result given in Lemma 5,one can easily discuss the above results for $f \in \mathcal{K}_{\mathrm{H}}(F, G)$ on lines similar to above theorems, we left this as exercise for interested readers.

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Conflict of interest

The authors declares no conflicts of interest in this paper.

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