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Research article

Asymptotic formulas for generalized gcd-sum and lcm-sum functions over r-regular integers $(\bmod\ n^r)$

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Abstract: In this paper we perform a further investigation for r-gcd-sum function over r-regular integers $(\bmod\ n^r)$, and we derive two kinds of asymptotic formulas by making use of Dirichlet product, Euler product and some techniques. Moreover, we also establish estimates for the generalized r-lcm-sum function over r-regular integers $(\bmod\ n)$.

Keywords: regular integers; r-gcd-sum function; r-lcm-sum function; asymptotic formula
Mathematics Subject Classification: 11A25, 11N37

1. Introduction and main results

For any integer $n \geq 1$, the Pillai's arithmetical function, which is also known as the gcd-sum function is defined by

$$P(n) = \sum_{k=1}^{n}(k, n) = \sum_{d|n} d\varphi(n/d), \tag{1.1}$$

where φ is the Euler's totient function, and (k, n) denotes the greatest common divisor of k and n. Over the years, a great deal of mathematical effort in number theory has been devoted to the study of classical gcd-sum function. Its distribution properties, numerous important arithmetical and algebraic information have been investigated by many mathematicians; see, for example [1–3]. In addition, some natural generalizations of the usual gcd-sum function also have been considered; see [4]. Let $r \geq 1$ be a fixed integer, the greatest r^{th} power common divisor of positive integers a and b is defined to be the largest positive integers d^r such that $d^r|a$ and $d^r|b$, which is denoted by $(a, b)_r$ and called the r-gcd of a and b. Note that $(a, b)_1 = (a, b)$. With this definition, Prasad, Reddy and Rao defined the r-gcd-sum function as

$$P_r(n^r) = \sum_{k=1}^{n^r}(k, n^r)_r, \tag{1.2}$$

and for every $\varepsilon > 0$, they derived an asymptotic formula of its summatory function as follow:

$$\sum_{n^r \leq x} P_r(n^r) = \frac{x^{1+\frac{1}{r}}}{(r+1)\zeta(r+1)} \left(\frac{\log x}{r} + 2\gamma - \frac{1}{r+1} - \frac{\zeta'(r+1)}{\zeta(r+1)} \right) + O_\varepsilon \left(x^{1+\frac{\theta+\varepsilon}{r}} \right),$$

where $\zeta(n)$ is the Riemann zeta function and θ is the positive real number appearing in the Dirichlet divisor problem, for which one seeks the smallest positive real number θ. That is, for all real $x \geq 1$ and any $\varepsilon > 0$, the asymptotic formula given below holds:

$$\sum_{n \leq x} \tau(n) = x(\log x + 2\gamma - 1) + O_\varepsilon \left(x^{\theta+\varepsilon} \right),$$

where $\tau(n)$ is the number of positive divisors of n and γ is Euler's constant.

Let r be a fixed positive integer. A positive integer k is said to be r-regular (mod n^r) if there exists an integer x such that $k^{r+1}x \equiv k^r \pmod{n^r}$. Also, it was observed in [5] that k is r-regular (mod n^r) if and only if $(k, n^r)_r$ is a unitary divisor of n^r. We recall that d is said to be a unitary divisor of n if $d|n$ and $(d, n/d) = 1$, written as $d \parallel n$. When $r = 1$, it gives usual regular integers (mod n), and a detailed study was initiated by L. Tóth [6]. Let $Reg_r(n)$ denotes the set of all r-regular integers modulo n and $Reg(n)$ denote the set of all regular integers modulo n. That is, $Reg_r(n^r)$=$\{k : 1 \leq k \leq n^r$, k is r-regular mod $n^r\}$, $Reg(n)$=$\{k : 1 \leq k \leq n$, k is regular mod $n\}$. L. Tóth [7] introduced another generalized gcd-sum function over regular integers modulo n as

$$\tilde{P}(n) = \sum_{k \epsilon Reg(n)} (k, n). \tag{1.3}$$

He also proved that $\tilde{P}(n)$ is multiplicative and gave the following asymptotic formula

$$\sum_{n \leq x} \tilde{P}(n) = \frac{x^2}{2\zeta(2)} (K_1 \log x + K_2) + O\left(x^{3/2} \delta(x) \right), \tag{1.4}$$

where $\delta(x) = \exp\left(-C(\log x)^{3/5} (\log\log x)^{-1/5} \right)$, and K_1,K_2 are given by

$$K_1 = \prod_p \left(1 - \frac{1}{p(p+1)} \right),$$

$$K_2 = K_1 \left(2\gamma - \frac{1}{2} - \frac{2\zeta'(2)}{\zeta(2)} \right) - \sum_{n=1}^{\infty} \frac{\mu(n)(\log n - \alpha(n) + 2\beta(n))}{n\psi(n)},$$

where $\psi(n)$ is Dedekind function defined by $\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p} \right)$ and

$$\alpha(n) = \prod_{p|n} \frac{\log p}{p-1}, \; \beta(n) = \prod_{p|n} \frac{\log p}{p^2 - 1}.$$

Recently, under the Riemann hypothesis, the error term $R(x)$ in (1.4) has been improved by Zhang and Zhai [8] to

$$R(x) = O\left(x^{15/11+\varepsilon} \right),$$

where $\varepsilon > 0$ is any sufficiently small positive number.

Afterwards, Prasad, Reddy and Rao [9] introduced the r-gcd-sum function over r-regular integers (mod n^r), which is defined as

$$\tilde{P}_r(n^r) := \sum_{k \epsilon Reg_r(n^r)} (k, n^r)_r.$$

Based on the properties of a generalization of Euler's φ-function [10], they obtain some arithmetic properties of $\tilde{P}_r(n^r)$ and an asymptotic formula for its summatory function. Motivated and inspired by the work of L.Tóth, Prasad, Reddy and Rao [7, 9], in this paper we offer a different proof for Prasad, Reddy and Rao's result [9] firstly, then we perform a further investigation for r-gcd-sum function over r-regular integers (mod n^r) and derive two kinds of asymptotic formulas. Furthermore, we also establish estimates for the generalized r-lcm-sum function over r-regular integers (mod n).

Firstly, we state the following results.

Theorem 1. *For any real number $x > e$ sufficiently large, we have the following estimate*

$$\sum_{n^r \le x} \tilde{P}_r(n^r) = \frac{x^{1+\frac{1}{r}} C_1}{(r+1)\zeta(2)} \left(\frac{\log x}{r} - \frac{2\zeta'(2)}{\zeta(2)} + 2\gamma - \frac{1}{r+1} + \frac{C_2}{C_1} \right) + O\left(x^{1+\frac{1}{2r}} \delta(x) \right),$$

where

$$C_1 = \prod_p \left\{ 1 - \frac{1}{p^r(p+1)} \right\},$$

$$C_2 = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^r \psi(d)} \left(\alpha(d) - 2\beta(d) - \log d \right).$$

Note. Taking $r = 1$, L. Tóth's result ([7], Theorem 2) can be deduced from Theorem 1.

Theorem 2. *For a positive integer $s > 1$, define the generalized r-gcd-sum function over r-regular integers* (mod n^r) *as*

$$\tilde{P}_{r,s}(n^r) = \sum_{k \epsilon Reg_r(n^r)} ((k, n^r)_r)^s.$$

Then for any real number $x > e$ sufficiently large and $r > 1$, we have

$$\sum_{n^r \le x} \tilde{P}_{r,s}(n^r) = \frac{x^{s+\frac{1}{r}}}{rs+1} \prod_p \left(1 + \frac{(p-1)(p^r-1)}{p^{rs-r+1} - p^{r+1}} \right) + O\left(x^s\right).$$

On the other hand, let $[k, n]$ denotes the least common multiple of k and n. Ikeda and Matsuoka [11] studied the functions

$$L_a(n) = \sum_{k=1}^{n} ([k, n])^a,$$

$$T_a(x) = \sum_{n \le x} L_a(n),$$

and obtained

$$\sum_{n \le x} L_a(n) = \frac{\zeta(a+2)}{2(a+1)^2 \zeta(2)} x^{2a+2} + O\left(x^{2a+1} (\log x)^{2/3} (\log\log x)^{4/3} \right)$$

for $a \in \mathbb{N}$. So naturally, we're thinking about generalized r-lcm-sum function here. Define the least r^{th} power common multiple as

$$[k,n]_r = \frac{kn}{(k,n)_r},$$

and for a positive integer $a > 1$, and generalized r-lcm-sum function over r-regular integers (mod n) as

$$\tilde{L}_{r,a}(n) = \sum_{k \epsilon Reg_r(n)} ([k,n]_r)^a .$$

Then we can deduce,

Theorem 3. *For any real number $x > e$ sufficiently large, define the r-lcm-sum function as $L_{r,a}(n) = \sum_{k=1}^{n} ([k,n]_r)^a$, then*

$$\sum_{n \le x} L_{r,a}(n) = \frac{x^{2a+2}\zeta(ar+2r)}{2(a+1)^2\zeta(2r)} + O\left(x^{2a+1}\right).$$

Theorem 4. *For any real number $x > e$ sufficiently large, we obtain*

$$\sum_{n \le x} \tilde{L}_{r,a}(n) = \frac{x^{2a+2}}{2\zeta(2r)(a+1)^2} \prod_{p} \left(1 + \frac{p^{2r-1}(p-1)}{(p^{2r}-1)(p^{a+2}-1)}\right) + O\left(x^{2a+1}\right).$$

Taking $r = 1$ in Theorem 3, we can get,

Corollary 1. *For any real number $x > e$ sufficiently large,*

$$\sum_{n \le x} \tilde{L}_{1,a}(n) = \sum_{k \epsilon Reg(n)} ([k,n])^a = \frac{3x^{2a+2}}{(a+1)^2\pi^2} \prod_{p} \left(1 + \frac{p}{(p+1)(p^{a+2}-1)}\right) + O\left(x^{2a+1}\right).$$

2. Several lemmas

In this section, we give several lemmas which are necessary in the proof of our theorems. First of all, in the proofs of these lemmas, we need some knowledge of elementary and analytic number theory. And we know that an integer is rth-power-free if it is not divisible by the rth power of any integer > 1. Then let $\phi_r(n)$ denotes the number of integers k in the set $1, \cdots, n$, for which the greatest common divisor (k,n) is rth-power-free, and $\mu^r(n)$ is defined as follows:

$$\mu^r(1) = 1;$$

if $n > 1$ and $n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ is the canonical factorization of n, then

$$\mu^r(n) = \begin{cases} (-1)^t, & \text{if } a_1 = a_2 = \cdots = a_t = r, \\ 0, & \text{if for some } i,\ a_i \ne r; \end{cases}$$

and let $\tau^*(n;m)$ denotes the number of unitary divisors of n which are relatively prime to m, where $m \ge 1$ be an integer. In symbols,

$$\tau^*(n;m) = \sum_{\substack{d \| n \\ (d,m)=1}} 1.$$

By the notations above, we have the following:

Lemma 1. *For every $\varepsilon > 0$, we get*

$$\begin{aligned} &\sum_{n\le x} n^r\tau^*(n;m) \\ = &\ \frac{mx^{r+1}}{(r+1)\zeta(2)\psi(m)}\left(\log x+\alpha(m)-2\beta(m)-\frac{2\zeta'(2)}{\zeta(2)}+2\gamma-\frac{1}{r+1}\right) \\ &+O\left(\sigma^*_{-1+\varepsilon}(m)\sigma^*_{-\theta}(m)x^{r+\frac{1}{2}}\delta(x)\right), \end{aligned}$$

where $\sigma^*_a(n)$ is the sum of the a-th power of square free divisors of n and $\alpha(m)$, $\beta(m)$, $\psi(m)$, $\delta(m)$ are the same as given in the introduction.

Proof. For any integer $m \ge 1$ and every $\varepsilon > 0$, one knows that ([12], Theorem 4.3),

$$\begin{aligned} \sum_{n\le x}\tau^*(n;m) &= \frac{mx}{\zeta(2)\psi(m)}\left(\log x+\alpha(m)-2\beta(m)-\frac{2\zeta'(2)}{\zeta(2)}+2\gamma-1\right) \\ &\quad +O\left(\sigma^*_{-1+\varepsilon}(m)\sigma^*_{-\theta}(m)x^{\frac{1}{2}}\delta(x)\right). \end{aligned}$$

Then by using partial summation formula, we immediately have

$$\begin{aligned} \sum_{n\le x} n^r\tau^*(n;m) &= x^r\sum_{n\le x}\tau^*(n;m)-x^r\int_1^x t^{r-1}\sum_{n\le t}\tau^*(n;m)dt \\ &= \frac{mx^{r+1}}{(r+1)\zeta(2)\psi(m)}\left(\log x+\alpha(m)-2\beta(m)-\frac{2\zeta'(2)}{\zeta(2)}+2\gamma-\frac{1}{r+1}\right) \\ &\quad +O\left(\sigma^*_{-1+\varepsilon}(m)\sigma^*_{-\theta}(m)x^{r+\frac{1}{2}}\delta(x)\right). \end{aligned}$$

□

Lemma 2. *([13], Corollary 2.1) For any integer $k \ge 1$ and $s \ge 0$*

$$\sum_{\substack{n\le x\\(n,k)=1}} n^s = \frac{x^{s+1}\varphi(k)}{(s+1)k}+O(x^s\tau(k)),$$

where $\tau(k)$ denotes the number of divisors of k.

Lemma 3. *For every $n, a, r \in \mathbb{N}$, we have the identity*

$$\sum_{\substack{k\le n\\(k,n)_r=1}} k^a = \frac{n^a}{2}g_0(n)+\frac{n^a\phi_r(n)}{a+1}+\frac{n^a}{a+1}\sum_{m=1}^{\lfloor a/2\rfloor}\binom{a+1}{2m}B_{2m}\sum_{d|n}\mu^r(d)\frac{n^{1-2m}}{d^{1-2m}},$$

where $g_0(n) = 1$, if n is rth-power-free; 0, otherwise. And B_n are the Bernoulli numbers defined by exponential generating function $\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}$.

Proof. Firstly, we know the function μ^r has following property [10]:

$$\sum_{d|n}\mu^r(d) = \begin{cases} 1, & \text{if } n \text{ is } r\text{th-power-free}, \\ 0, & \text{if } n \text{ is not } r\text{th-power-free}. \end{cases}$$

In addition, it is well known that for every $n, a \in \mathbb{N}$,

$$\begin{aligned}\sum_{k=1}^{n} k^a &= \frac{1}{a+1}\sum_{m=0}^{a}(-1)^m\binom{a+1}{m}B_m n^{a+1-m}\\ &= \frac{n^a}{2}+\frac{1}{a+1}\sum_{m=0}^{\lfloor a/2\rfloor}\binom{a+1}{2m}B_{2m}n^{a+1-2m}.\end{aligned}$$

So we can deduce

$$\begin{aligned}&\sum_{\substack{k\le n\\(k,n)_r=1}} k^a\\ =\ &\sum_{k\le n}k^a\sum_{d|(k,n)_r}\mu^r(d)=\sum_{d|n}d^a\mu^r(d)\sum_{m\le n/d}m^a\\ =\ &\sum_{d|n}d^a\mu^r(d)\left[\frac{1}{2}\left(\frac{n}{d}\right)^a+\frac{1}{a+1}\sum_{m=0}^{\lfloor a/2\rfloor}\binom{a+1}{2m}B_{2m}\left(\frac{n}{d}\right)^{a+1-2m}\right]\\ =\ &\frac{n^a}{2}\sum_{d|n}\mu^r(d)+\frac{n^a}{a+1}\sum_{m=0}^{\lfloor a/2\rfloor}\binom{a+1}{2m}B_{2m}\sum_{d|n}\mu^r(d)\left(\frac{n}{d}\right)^{1-2m}\\ =\ &\frac{n^a}{2}\sum_{d|n}\mu^r(d)+\frac{n^a}{a+1}\sum_{d|n}\mu^r(d)\frac{n}{d}+\frac{n^a}{a+1}\sum_{m=1}^{\lfloor a/2\rfloor}\binom{a+1}{2m}B_{2m}\sum_{d|n}\mu^r(d)\left(\frac{n}{d}\right)^{1-2m}\\ =\ &\frac{n^a}{2}g_0(n)+\frac{n^a\phi_r(n)}{a+1}+\frac{n^a}{a+1}\sum_{m=1}^{\lfloor a/2\rfloor}\binom{a+1}{2m}B_{2m}\sum_{d|n}\mu^r(d)\frac{n^{1-2m}}{d^{1-2m}}.\end{aligned}$$

This completes the proof of Lemma 3. □

Lemma 4. *For any integer $k \geq 1$ and $a \in \mathbb{N}$, we have*

$$\sum_{\substack{n\le x\\(n,k)=1}} n^{2a}\phi_r(n)=\frac{k^{2r-1}\varphi(k)x^{2a+2}}{(2a+2)\zeta(2r)\varphi_{2r}(k)}+O\left(x^{2a+1}\log x\tau(k)\right), \tag{a}$$

$$\sum_{n\le x} n^{2a}\phi_r(n)=\frac{x^{2a+2}}{(2a+2)\zeta(2r)}+O\left(x^{2a+1}\log x\right). \tag{b}$$

where φ_r is Jordan totient function, defined by

$$\varphi_r(n)=n^r\prod_{p|n}(1-p^{-r}).$$

Proof. It follows from Lemma 2 that

$$
\begin{aligned}
&\sum_{\substack{n\le x\\(n,k)=1}} n^{2a}\phi_r(n)\\
=\ &\sum_{\substack{n\le x\\(n,k)=1}} n^{2a}\sum_{de=n}\mu^r(d)e = \sum_{\substack{d\le x\\(d,k)=1}}\mu^r(d)d^{2a}\sum_{\substack{e\le x/d\\(e,k)=1}} e^{2a+1}\\
=\ &\sum_{\substack{d\le x\\(d,k)=1}}\mu^r(d)d^{2a}\left(\frac{\varphi(k)}{(2a+2)k}\left(\frac{x}{d}\right)^{2a+2}+O\left(\left(\frac{x}{d}\right)^{2a+1}\tau(k)\right)\right)\\
=\ &\frac{\varphi(k)x^{2a+2}}{(2a+2)k}\sum_{\substack{d\le x\\(d,k)=1}}\frac{\mu^r(d)}{d^2}+O\left(x^{2a+1}\tau(k)\sum_{d\le x}\frac{1}{d}\right)\\
=\ &\frac{\varphi(k)x^{2a+2}}{(2a+2)k}\sum_{\substack{d=1\\(d,k)=1}}^{\infty}\frac{\mu^r(d)}{d^2}+O\left(x^{2a+2}\sum_{d>x}\frac{1}{d^2}\right)+O\left(x^{2a+1}\tau(k)\sum_{d\le x}\frac{1}{d}\right)\\
=\ &\frac{\varphi(k)x^{2a+2}}{(2a+2)k}\sum_{\substack{d=1\\(d,k)=1}}^{\infty}\frac{\mu^r(d)}{d^2}+O\left(x^{2a+1}\log x\tau(k)\right),
\end{aligned}
$$

where we use the familiar estimate $\sum_{n\le x}\frac{1}{n} = O(\log x)$ and $\sum_{n>x}\frac{1}{n^s} = O\left(x^{1-s}\right)$, when $s > 1$. Furthermore, it can be deduced

$$
\begin{aligned}
\sum_{\substack{d\le x\\(d,k)=1}}\frac{\mu^r(d)}{d^2} &= \sum_{\substack{d=1\\(d,k)=1}}^{\infty}\frac{\mu^r(d)}{d^2}-\sum_{\substack{d>x\\(d,k)=1}}\frac{\mu^r(d)}{d^2}\\
&= \prod_{p\nmid k}\left(1-\frac{1}{p^{2r}}\right)+O\left(\sum_{d>x}\frac{|\mu^r(d)|}{d^2}\right)\\
&= \frac{\prod_p\left(1-\frac{1}{p^{2r}}\right)}{\prod_{p|k}\left(1-\frac{1}{p^{2r}}\right)}+O\left(\sum_{d>x}\frac{|\mu^r(d)|}{d^2}\right)\\
&= \frac{1/\zeta(2r)}{\varphi_{2r}(k)/k^{2r}}+O\left(x^{-1}\right).
\end{aligned}
$$

So we have

$$
\sum_{\substack{n\le x\\(n,k)=1}} n^{2a}\phi_r(n) = \frac{k^{2r-1}\varphi(k)x^{2a+2}}{(2a+2)\zeta(2r)\varphi_{2r}(k)}+O\left(x^{2a+1}\log x\tau(k)\right).
$$

By using similar methods and elementary asymptotic formula

$$
\sum_{n\le x} n^a = \frac{x^{a+1}}{a+1}+O(x^a),
$$

(b) can be easily derived. □

3. The proof of Theorem 1

Let $R_{n,r}$=$\{k : 1 \le k \le n^r, (k,n^r)_r = 1\}$ and $\phi_r(n^r)$ denote the number of elements in $R_{n,r}$. Since $\phi_r(n^r) = n^r \sum_{d|n} \mu(d)/d^r$ (see, [14]), we have

$$\begin{aligned}
\tilde{P}_r(n^r) &= \sum_{k \epsilon Reg_r(n^r)} (k,n^r)_r = \sum_{d^r \| n^r} d^r \phi_r\left(\frac{n^r}{d^r}\right) = \sum_{\substack{t^r u^r = n^r \\ (t^r,u^r)=1}} t^r \sum_{dv=u} \mu(d) v^r \\
&= \sum_{\substack{tdv=n \\ (t,dv)=1}} t^r v^r \mu(d) = \sum_{ld=n} l^r \mu(d) \sum_{\substack{tv=l \\ (t,v)=1 \\ (t,d)=1}} 1 \\
&= \sum_{ld=n} l^r \mu(d) \tau^*(l;d).
\end{aligned}$$

Taking $y = x^{1/r}$, it follows from Lemma 1 and the above identity that

$$\begin{aligned}
&\sum_{n^r \le x} \tilde{P}_r(n^r) \\
=\ & \sum_{n^r \le x} \sum_{ld=n} l^r \mu(d) \tau^*(l;d) = \sum_{d \le y} \mu(d) \sum_{l \le y/d} l^r \tau^*(l;d) \\
=\ & \frac{y^{r+1}}{(r+1)\zeta(2)} \sum_{d \le y} \frac{\mu(d)}{d^r \psi(d)} \left(\log y - \log d + \alpha(d) - 2\beta(d) - \frac{2\zeta'(2)}{\zeta(2)} + 2\gamma - \frac{1}{r+1}\right) \\
& + O\left(\sum_{d \le y} |\mu(d)| \sigma^*_{-1+\varepsilon}(d) \sigma^*_{-\theta}(d) (y/d)^{r+\frac{1}{2}} \delta(y/d)\right) \\
=\ & \frac{y^{r+1}}{(r+1)\zeta(2)} \sum_{d \le y} \frac{\mu(d)}{d^r \psi(d)} \left(\log y - \frac{2\zeta'(2)}{\zeta(2)} + 2\gamma - \frac{1}{r+1}\right) \\
& + \frac{y^{r+1}}{(r+1)\zeta(2)} \sum_{d \le y} \frac{\mu(d)}{d^r \psi(d)} \left(\alpha(d) - 2\beta(d) - \log d\right) \\
& + O\left(\sum_{d \le y} \sigma^*_{-1+\varepsilon}(d) \sigma^*_{-\theta}(d) (y/d)^{r+\frac{1}{2}} \delta(y/d)\right).
\end{aligned}$$

Notice that for any integer $n \ge 1$,

$$\alpha(n) = \prod_{p|n} \frac{\log p}{p-1} \le \prod_{p|n} \log p \le \log n,$$

$$\beta(n) = \prod_{p|n} \frac{\log p}{p^2 - 1} \le \prod_{p|n} \frac{\log p}{p^2},$$

which implies that $\alpha(n) = O(\log(n)), \beta(n) = O(1)$.

Then by the definition of ψ, it is clear that for any integer $r \ge 0$, the series

$$\sum_{d=1}^{\infty} \frac{\mu(d)}{d^r \psi(d)}$$

and
$$\sum_{d=1}^{\infty}\frac{\mu(d)}{d^r\psi(d)}\left(\alpha(d)-2\beta(d)-\log d\right)$$
are both absolutely convergent. Applying Euler product,we can obtain
$$\sum_{d=1}^{\infty}\frac{\mu(d)}{d^r\psi(d)}=\prod_{p}\left(1-\frac{1}{p^r(p+1)}\right),$$
Moreover, since $\sigma^*_{-\theta}(n)\le\tau(n)$, $\sigma^*_{-1+\varepsilon}(n)\le\tau(n)$ for $\varepsilon\le 1$, and $y^r\delta(y)$ is increasing. so the error term can be simplified as
$$\begin{aligned}&O\left(\sum_{d\le y}\sigma^*_{-1+\varepsilon}(d)\sigma^*_{-\theta}(d)(y/d)^{r+\frac{1}{2}}\delta(y/d)\right)\\ =\ &O\left(y^{r+\frac{1}{2}}\delta(y)\sum_{d\le y}\frac{\tau^2(d)}{d^{r+\frac{1}{2}-\varepsilon}}\right)\\ =\ &O\left(y^{r+\frac{1}{2}}\delta(y)\right).\end{aligned}$$
Now, by a simple calculation, we get
$$\sum_{n^r\le x}\tilde{P}_r(n^r)=\frac{x^{1+\frac{1}{r}}C_1}{(r+1)\zeta(2)}\left(\frac{\log x}{r}-\frac{2\zeta'(2)}{\zeta(2)}+2\gamma-\frac{1}{r+1}+\frac{C_2}{C_1}\right)+O\left(x^{1+\frac{1}{2r}}\delta(x)\right).$$
This completes the proof of Theorem 1.

4. The proof of Theorem 2

For any integer $r>1$, $s>1$, by taking $y=x^{1/r}$ and using lemma 2, we have
$$\begin{aligned}\sum_{n^r\le x}\tilde{P}_{r,s}(n^r)&=\sum_{n^r\le x}\sum_{k\epsilon Reg_r(n^r)}((k,n^r)_r)^s=\sum_{n^r\le x}\sum_{(k,n^r)_r\|n^r}((k,n^r)_r)^s\\ &=\sum_{n^r\le x}\sum_{a^r\|n^r}a^{rs}\sum_{\substack{k=1\\(k,n^r/a^r)_r=1}}^{n^r/a^r}1=\sum_{\substack{ab\le y\\(a,b)=1}}a^{rs}\phi_r(b^r)\\ &=\sum_{b\le y}\phi_r(b^r)\sum_{\substack{a\le y/b\\(a,b)=1}}a^{rs}\\ &=\frac{y^{rs+1}}{rs+1}\sum_{b\le y}\frac{\phi_r(b^r)\varphi(b)}{b^{rs+2}}+O\left(y^{rs}\sum_{b\le y}\frac{\phi_r(b^r)\tau(b)}{b^{rs}}\right)\\ &=\frac{y^{rs+1}}{rs+1}\sum_{b=1}^{\infty}\frac{\phi_r(b^r)\varphi(b)}{b^{rs+2}}+O\left(y^{rs+1}\sum_{b>y}\frac{1}{b^{rs-r}}\right)\\ &\quad+O\left(y^{rs}\sum_{b\le y}\frac{\phi_r(b^r)\tau(b)}{b^{rs}}\right).\end{aligned}$$

Since $r > 1, s > 1$, the series

$$\sum_{n=1}^{\infty} \frac{\phi_r(n^r)\varphi(n)}{n^{rs+2}} \leq \frac{1}{n^{rs-r}}$$

is absolutely convergent and the general term is a multiplicative function, thus the series can be expended into an infinite product of Euler type

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{\phi_r(n^r)\varphi(n)}{n^{rs-r+1}} \\
= & \prod_p \left(1 + \frac{(p-1)(p^r-1)}{p^{rs-r+1}} + \frac{(p-1)p(p^r-1)p^r}{p^{2(rs-r+1)}} + \frac{(p-1)p^2(p^r-1)p^{2r}}{p^{3(rs-r+1)}} \cdots \right) \\
= & \prod_p \left(1 + \frac{(p-1)(p^r-1)}{p^{rs-r+1} - p^{r+1}}\right).
\end{aligned}$$

For the error term, we can deduce that

$$O\left(y^{rs+1} \sum_{b>y} \frac{1}{b^{rs-r}}\right) = O\left(y^{r+2}\right),$$

$$O\left(y^{rs} \sum_{b \leq y} \frac{\tau(b)}{b^{rs-r}}\right) = O\left(y^{rs}\right).$$

To sum up, we can obtain

$$\sum_{n^r \leq x} \tilde{P}_{r,s}(n^r) = \frac{x^{s+\frac{1}{r}}}{rs+1} \prod_p \left(1 + \frac{(p-1)(p^r-1)}{p^{rs-r+1} - p^{r+1}}\right) + O\left(x^s\right).$$

5. The proof of Theorems 3 and 4

Firstly, we prove Theorem 4. By using Lemma 3, it is easily seen that

$$\begin{aligned}
& \tilde{L}_{r,a}(n) = \sum_{k \epsilon Reg_r(n)} ([k,n]_r)^a \\
= & \sum_{k \epsilon Reg_r(n)} \left(\frac{kn}{(k,n)_r}\right)^a = \sum_{d^r \| n} \frac{n^a}{d^{ra}} \sum_{\substack{k \leq n \\ gcd(k,n)_r = d^r}} k^a \\
= & \sum_{d^r \| n} \frac{n^a}{d^{ra}} \sum_{\substack{j \leq n/d^r \\ gcd(j, n/d^r)_r = 1}} (jd^r)^a = n^a \sum_{d^r \| n} \sum_{\substack{j \leq n/d^r \\ gcd(j, n/d^r)_r = 1}} j^a \\
= & \sum_{d^r \| n} d^{ra} \left(\frac{n}{d^r}\right)^{2a} \cdot \left(\frac{1}{2} g_0\left(\frac{n}{d^r}\right) + \frac{1}{a+1} \phi_r\left(\frac{n}{d^r}\right)\right) \\
& + \sum_{d^r \| n} d^{ra} \left(\frac{n}{d^r}\right)^{2a} \cdot \left(\frac{1}{a+1} \sum_{m=1}^{\lfloor a/2 \rfloor} \binom{a+1}{2m} B_{2m} \sum_{ab = \frac{n}{d^r}} \mu^r(a) b^{1-2m}\right).
\end{aligned}$$

Then

$$
\begin{aligned}
&\sum_{n\le x}\tilde{L}_{r,a}(n)\\
=\;&\sum_{d^r\le x} d^{ra}\sum_{\substack{e\le x/d^r\\ gcd(e,d^r)=1}} e^{2a}\cdot\left(\frac{1}{2}g_0(e)+\frac{1}{a+1}\phi_r(e)\right)\\
&+\sum_{d^r\le x} d^{ra}\sum_{\substack{e\le x/d^r\\ gcd(e,d^r)=1}} e^{2a}\cdot\left(\frac{1}{a+1}\sum_{m=1}^{\lfloor a/2\rfloor}\binom{a+1}{2m}B_{2m}\sum_{ab=e}\mu^r(a)b^{1-2m}\right)\\
=\;&\frac{1}{a+1}\sum_{m=1}^{\lfloor a/2\rfloor}\binom{a+1}{2m}B_{2m}\sum_{d^r\le x} d^{ra}\sum_{\substack{e\le x/d^r\\ (d^r,e)=1}} e^{2a}\sum_{ab=e}\mu^r(a)b^{1-2m}\\
&+\frac{1}{a+1}\sum_{d^r\le x} d^{ra}\sum_{\substack{e\le x/d^r\\ (d^r,e)=1}} e^{2a}\phi_r(e)+O\left(x^{2a}\right)\\
=\;&A_0+A_1+O\left(x^{2a}\right).
\end{aligned}
$$

By using Lemma 4(a), we get

$$
\begin{aligned}
A_1 =\;&\frac{1}{a+1}\sum_{d^r\le x} d^{ra}\sum_{\substack{e\le x/d^r\\ (d^r,e)=1}} e^{2a}\phi_r(e)\\
=\;&\frac{1}{a+1}\sum_{d^r\le x} d^{ra}\cdot\frac{\varphi(d^r)d^{2r\cdot r}}{(2a+2)\zeta(2r)\varphi_{2r}(d^r)d^r}\left(\frac{x}{d^r}\right)^{2a+2}\\
&+\frac{1}{a+1}\sum_{d^r\le x} d^{ra}\cdot O\left(\left(\frac{x}{d^r}\right)^{2a+1}\log\left(\frac{x}{d^r}\right)\tau(d^r)\right)\\
=\;&\frac{x^{2a+2}}{2(a+1)^2\zeta(2r)}\sum_{d^r\le x}\frac{\varphi(d^r)d^{2r\cdot r}}{\varphi_{2r}(d^r)d^{ra+3r}}+O\left(x^{2a+1}\log x\sum_{d^r\le x}\frac{\tau(d^r)}{d^{ar+r}}\right)\\
=\;&\frac{x^{2a+2}}{2\zeta(2r)(a+1)^2}\sum_{d=1}^{\infty}\frac{\varphi(d^r)d^{2r\cdot r}}{\varphi_{2r}(d^r)d^{ra+3r}}\\
&+O\left(x^{2a+2}\sum_{d>x^{1/r}}\frac{\varphi(d^r)d^{2r\cdot r}}{\varphi_{2r}(d^r)d^{ra+3r}}\right)+O\left(x^{2a+1}\log x\sum_{d\le x^{1/r}}\frac{\tau(d^r)}{d^{ar+r}}\right).
\end{aligned}
$$

And it can be calculated by Euler product that

$$
\sum_{d=1}^{\infty}\frac{\varphi(d^r)d^{2r\cdot r}}{\varphi_{2r}(d^r)d^{ra+3r}}=\prod_p\left(1+\frac{p^{2r-1}(p-1)}{(p^{2r}-1)(p^{a+2}-1)}\right).
$$

So

$$
A_1=\frac{x^{2a+2}}{2\zeta(2r)(a+1)^2}\prod_p\left(1+\frac{p^{2r-1}(p-1)}{(p^{2r}-1)(p^{a+2}-1)}\right)+O\left(x^{a+2}\log^2 x\right),
$$

where we have used

$$\sum_{d>x}\frac{\varphi(d^r)}{d^{ra+3r}} \le \sum_{d>x}\frac{1}{d^{ra+2r}} = O(x^{1-ar-2r}),$$

$$\sum_{n\le x}\frac{\tau(n)}{n^a} = O(x^{1-a}\log x).$$

And for $m, a \in \mathbb{N}$ with $a > m$,

$$\sum_{\substack{n\le x/d^r\\(n,k)=1}} n^{2a}\sum_{ab=n}\mu^r(a)b^{1-2m} = \sum_{\substack{d\le x\\(d,k)=1}}\mu^r(d)d^{2a}\sum_{\substack{e\le x/d\\(e,k)=1}} e^{2a-2m+1} \ll x^{2a+1},$$

so

$$A_0 = O\left(\sum_{d^r\le x} d^{ra}\left(\frac{x}{d^r}\right)^{2a+1}\right) = O\left(x^{2a+1}\right).$$

To sum up, we can obtain that

$$\begin{aligned}\sum_{n\le x}\tilde{L}_{r,a}(n) &= A_0 + A_1 + O\left(x^{2a}\right)\\ &= \frac{x^{2a+2}}{2\zeta(2r)(a+1)^2}\prod_p\left(1+\frac{p^{2r-1}(p-1)}{(p^{2r}-1)(p^{a+2}-1)}\right) + O\left(x^{2a+1}\right).\end{aligned}$$

This completes the proof of Theorem 4.

Theorem 3 can be deduced by Lemma 3 and Lemma 4(b), and the proof method is similar to Theorem 4.

6. Conclusions

In this article, we firstly introduced the basic concepts and properties of r-regular integers (mod n^r), and the r-gcd-sum function and r-lcm-sum function over those integers. Let $\tilde{P}_r(n^r)$ denote the r-gcd-sum function over r-regular integers (mod n^r), Theorem 1 offered a different proof of Prasad, Reddy and Rao's result, and L. Tóth's result ([7], Theorem 2) can be deduced from our result. Theorem 2 gave an asymptotic formula for generalized r-gcd-sum function over r-regular integers (mod n^r). In addition, we defined least rth power common multiple $[k,n]_r$ based on the relationship between the common lcm and gcd, we also defined r-lcm-sum function $L_{r,a}(n)$ and generalized r-lcm-sum function $\tilde{L}_{r,a}(n)$ over r-regular integers (mod n). By making use of Euler product and some knowledge of elementary and analytic number theory, we obtained asymptotic formulas for their summatory function. As a supplement, we can also got an asymptotic formula for lcm-sum function over regular integers (mod n) easily.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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