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Research article

Carathéodory properties of Gaussian hypergeometric function associated with differential inequalities in the complex plane

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Abstract: The results presented in this paper highlight the property of the Gaussian hypergeometric function to be a Carathéodory function and refer to certain differential inequalities interpreted in form of inclusion relations for subsets of the complex plane using the means of the theory of differential superordination and the method of subordination chains also known as Löwner chains.

Keywords: differential superordination; analytic function; convex function; univalent function; subordinant; best subordinant; Gaussian hypergeometric function; Löwner chain
Mathematics Subject Classification: 30C45, 30C80

1. Introduction

Hypergeometric functions have been investigated mainly for their applications in a wide variety of fields as it was nicely presented in the survey-com-expository recent article [1] where the author gives references on scientific literature on the subject highlighting the applications of those functions in the theory of univalent functions. The interest on studying such applications grew once they were surprisingly used by L. de Branges in the proof of Bieberbach's conjecture published in 1985 [2]. Before this event, little research has been done from this point of view. Merkes and Scott investigated the starlikeness of certain Gaussian hypergeometric functions in 1961 [3] and before that the relationship between hypergeometric functions and univalent function theory was taken into consideration in the articles published by E. Kreyszig and J. Todd in 1959, who investigated the univalence of the error function $Erf(z)$ [4] and of the function $exp(z^2) \cdot Erf(z)$ [5].

Once the connection of hypergeometric functions with univalent functions theory was established, they were intensely investigated concerning many aspects. S. Ruscheweyh and V. Singh published results related to the order of starlikeness of certain hypergeometric functions in 1986 [6]. S. S. Miller and P. T. Mocanu have used the method of differential subordinations to investigate univalence, starlikeness and convexity of certain hypergeometric functions in 1990 [7] and their results were different from those previously published by S. Ruscheweyh and V. Singh [6]. After that, many papers appeared presenting interesting results obtained using different types of hypergeometric functions. Convolution operators involving generalized hypergeometric functions with very interesting and numerous applications were introduced such as the celebrated Dziok-Srivastava operator [8] and Srivastava-Wright operator [9] which were intensely used for obtaining elegant new results of which I only mention [10–13] and who's importance is nicely presented in [14]. An extension of the beta function by introducing an extra parameter, which proved to be useful earlier, is applied in [15] to extend the hypergeometric and confluent hypergeometric functions, an entire study on this extension and its relationship with the hypergeometric and confluent hypergeometric functions been conducted. S-generalized Gauss hypergeometric function was introduced in [16] and a new integral transform whose kernel is the S-generalized Gauss hypergeometric function was investigated in [17] and Gauss hypergeometric function transform follows as a simple special case of this integral transform. A new subclass of analytic functions in the open unit disk U, which is defined by the convolution, where the Gaussian hypergeometric function is involved is introduced and studied in [18]. Those are just a few mentions of the use of Gaussian hypergeometric function in obtaining original results related to geometric function theory.

So, it is not a surprise that this function was taken into consideration to obtain original new results presented in this paper. The theory of differential superordination introduced by Miller and Mocanu in 2003 [19] was applied in order to obtain interesting superordination results where Gaussian hypergeometric function is the best subordinant. This theory is relatively new and it provides interesting outcome as it can be seen from the large number of recently published papers of which I only mention [20–22]. It was applied for obtaining results concerning confluent hypergeometric function in [23] and those results inspired the idea of doing a similar study considering Gaussian hypergeometric function. In the same mentioned paper [23], inequalities in the complex plane connected with superordination results were interpreted in terms of inclusion relations between subsets of the complex plane inspired by the way the notion of differential subordination was introduced by Miller and Mocanu in two papers [24, 25]. A short history of the connection between differential subordination and inclusion relations between subsets of the complex plane can be read in [23]. That outcome and the results published very recently using the same idea [26] determined the investigation in the same manner on the superordination results obtained as result of the study on Gausssian hypergeometric function. The outcome was as expected, interesting: The property of Gaussian hypergeometric function to be a Carathéodory function emerged. The class of Carathéodory functions is defined and denoted as $\mathcal{P} = \{p \in H(U) : p(0) = 1,\ Re\, p(z) > 0, z \in U\}$ where $U = \{z \in \mathbb{C} : |z| < 1\}$. This property that certain functions have is of interest nowadays just as it was many years ago as it can be seen in the published results from 1972 [26] which have inspired the ones published just a few years later, in 1975 [27] and continued to push this study forward [28–33] many new results arising in the last years too [34–38], to mention only a small part of the papers.

This study was completed by adding the method of subordination chains or Löwner chains next to the means of the theory of differential superordination. This method was introduced by K. Löwner [39] starting from the idea of embedding the studied domain into domains that continuously "grow" and which can be described by using differential equations. The idea was pick up by many other researchers and led to obtaining important new results in geometric functions theory. This theory combined with the theory of differential superordination gives great outcome as it also can be seen in the next section of this paper which contains the original results.

The definition of the notion of subordination chain as it can be seen in [40] is next reminded:

Definition 1. *[40] A function $L(z,t)$, $z \in U$, $t \geq 0$ is a subordination chain if $L(\cdot,t)$ is analytic and univalent in U for all $t \geq 0$, $L(z,\cdot)$is contiunously differentiable on $\mathbb{R}^+$ for all $z \in U$ and $L(z,s) \prec L(z,t)$ when $0 \leq s \leq t$.*

The classes used throughout the paper are next shown. Let $H(U)$ denote the class of analytic functions in the unit disc. The subclass of functions $f \in H(U)$ which have the serial development $f(z) = a + a_{n+1}z^{n+1} + \ldots$ with $z \in U$, $a \in \mathbb{C}$ and n a positive integer is denoted by $H[a,n]$ with $H_0 = H[0,1]$. The functions $f \in H(U)$ which have the specific form $f(z) = z + a_{n+1}z^{n+1} + \ldots$ with $z \in U$ form the class denoted by A_n with A_1 written simply A. The functions from class A which are univalent in U are contained by the class denoted by S. The univalent functions with the property $\operatorname{Re}\frac{zf'(z)}{f(z)} > 0$ are called starlike functions and the class they form is denoted by S^*. The class of convex functions is denoted by K and consists of the univalent functions having the property $\operatorname{Re}\frac{zf''(z)}{f'(z)} + 1 > 0$.

The next definitions are related to the theory of differential superordination:

Definition 2. *[19] Let f and F be members of $H(U)$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w, analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ and such that $f(z) = F(w(z))$. In such a case we write $f \prec F$ or $f(z) \prec F(z)$. If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U)$ $F(U)$.*

Definition 3. *[19] Let $\varphi(r,s,t;z) : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ and let h be analytic in U. If p and $\varphi(p(z), zp(z), z^2p(z); z)$ are univalent in U and satisfy the (second-order) differential superordination*

$$h(z) \prec \varphi(p(z), zp(z), z^2p(z); z), \tag{1.1}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination or more simply a subordinant, if $q \prec p$ for all p satisfying (1.1). A subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.1) is said to be the best subordinant of (1.1). Note that the best subordinant is unique up to a rotation of U.

Definition 4. *[19] We denote by Q the set of functions f that are analytic and injective on $U \backslash E(f)$ where*

$$E(f) = \{\zeta \in \partial U : f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$, for $\zeta \in \partial U \backslash E(f)$. The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

Definition 5. *[19] Let Ω be a set in $\mathbb{C}$ and $q \in H[a,n]$. The class of admissible functions $\phi_n[\Omega, q]$ consists of those functions $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\varphi(r,s,t;\zeta) \in \Omega, \tag{A}$$

whenever

$$r = q(z), \; s = \frac{zq'(z)}{m}, \; Re\left(\frac{t}{s} + 1\right) \leq \frac{1}{m} Re \left[\frac{zq''(z)}{q(z)} + 1\right], \; \zeta \in \partial U, \; z \in U, \; m \geq n \geq 1.$$

When $n = 1$, $\phi_1[\Omega, q]$ is written as $\phi[\Omega, q]$.

In the special case when h is an analytic mapping of U onto $\Omega \neq \mathbb{C}$ we denote the class $\phi_n\left[h(U), q\right]$ by $\phi_n\left[h, q\right]$.

If $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$ then the admissibility condition (A) reduces to

$$\phi\left(r, s; \zeta\right) \in \Omega \tag{A'}$$

where $\zeta \in \partial U,\ z \in U,\ m \geq n \geq 1.$

A lemma is necessary for proving the original results contained in the next section.

Lemma 1. *(Miller and Mocanu) [19] Let $p \in Q(a)$ and let $q\left(z\right) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$ be analytic in U with $q(z) \not\equiv a$ and $n \geq 1$. If q is not subordinate to p, then there exists points $z_0 = r_0 e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \backslash E(p)$ and an $m \geq n \geq 1$ for which $q(U_{r_o}) \subset p(U)$ and*

(i) $q\left(z_0\right) = p\left(\zeta_0\right)$

(ii) $z_0 q'\left(z_0\right) = m\zeta_0 p'(\zeta_0)$ and

(iii) $Re\frac{z_0 q''(z_0)}{q(z_0)} + 1 \geq mRe\left[\frac{\zeta_0 p(\zeta_0)}{p(\zeta_0)} + 1\right]$.

The definition of Gaussian hypergeometric function studied in [7] is presented below:

Definition 6. *[7] Let a, b and c be complex numbers with $c \neq 0, -1, -2, \ldots$ The function*

$$F\left(a, b, c; z\right) =_2 F_1\left(a, b, c; z\right) = 1 + \frac{ab}{c} \cdot \frac{z}{1!} + \frac{a\left(a+1\right) b\left(b+1\right)}{c\left(c+1\right)} \cdot \frac{z^2}{2!} + ...,\ z \in U, \tag{1.2}$$

is called Gaussian hypergeometric function, is analytic in U and satisfies the hypergeometric equation:

$$z(z-1) \cdot w''\left(z\right) + \left[c - \left(a+b+1\right)z\right] \cdot w'\left(z\right) - ab \cdot w\left(z\right) = 0.$$

If we let

$$(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} = d\left(d+1\right)\left(d+2\right)\ldots\left(d+k-1\right)\ \text{and}\ (d)_0 = 1,$$

then (1.2) can be written in the form

$$F\left(a, b, c; z\right) = \sum_{k=0}^{\infty} \frac{(a)_k \cdot (b)_k}{(c)_k} \cdot \frac{z^k}{k!} = \frac{\Gamma\left(c\right)}{\Gamma\left(a\right)\Gamma\left(b\right)} \cdot \sum_{k=0}^{\infty} \frac{\Gamma\left(a+k\right)\Gamma\left(b+k\right)}{\Gamma\left(c+k\right)} \cdot \frac{z^k}{k!}. \tag{1.3}$$

Remark 1. *The results presented in the next section related to Gaussian hypergeometric function are believed to be new. Analogous methods have been used in earlier publications as well as it can be seen in [23, 26, 41–43].*

2. Results

The first original result presented in this paper is a theorem obtained using the theory of differential superordination which gives necessary and sufficient conditions for the Gaussian hypergeometric function to be the best subordinant for a certain differential superordination. The proof uses Lemma 1 previously given in the introduction.

Theorem 1. *Let h be analytic in U, consider the Gaussian hypergeometric function given by (1.2) and let* $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}, \varphi \in \phi_n [h(U), q]$. *Suppose that the differential equation*

$$\varphi\left(q(z), zq'(z), z^2 q''(z); z\right) = h(z) \tag{2.1}$$

has a univalent solution $q(z) = F(a, b, c; z) \in Q(1)$.

If $p \in Q(1)$ *and* $\varphi(p(z), zp'(z), z^2 p''(z); z)$ *is univalent in U, then*

$$h(z) \prec \varphi\left(q(z), zq'(z), z^2 q''(z); z\right) \text{ or } h(U) \subset \varphi(U) \tag{2.2}$$

implies

$$q(z) = F(z, b, c; z) \prec p(z) \text{ or } F(U) \subset p(U) \tag{2.3}$$

and $q(z) = F(a, b, c; z)$ *is the best subordinant.*

Proof. Suppose that functions $p,\ q,\ h$ satisfy the conditions from Lemma 1 in $\overline{U} = \{z \in \mathbb{C} : \ |z| \leq 1\}$. Let $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$,

$$\varphi(r, s, t; \zeta) \in \phi_n [h(U), q], \ r, s, t \in \mathbb{C}. \tag{2.4}$$

For

$$r = q(z), \ s = \frac{zq'(z)}{m}, \ t = z^2 q''(z),$$

from Definition 5 we know that

$$\varphi\left(q(z), \frac{zq'(z)}{m}, z^2 q''(z); \zeta\right) \in h(U). \tag{2.5}$$

For $z = z_0 \in U$, relation (2.5) becomes

$$\varphi\left(q(z_0), \frac{z_0 q'(z_0)}{m}, z_0^2 q''(z_0); \zeta\right) \in h(U). \tag{2.6}$$

Using relation (2.4) for $z \in U$ the following is obtained:

$$\left\{\varphi\left(p(z), zp'(z), z^2 p''(z); z\right)\right\} \in \varphi(U). \tag{2.7}$$

Let Γ denote the border of the set $\varphi(\overline{U})$.

$$\Gamma = \left\{\varphi\left(p(\zeta), \zeta p'(\zeta), \zeta^2 p''(\zeta); \zeta\right)\right\} \notin \varphi(U). \tag{2.8}$$

For $\zeta = \zeta_0,\ |\zeta| = |\zeta_0| = 1$, (2.8) becomes

$$\varphi\left(p(\zeta_0), \zeta_0 p'(\zeta_0), \zeta_0^2 p''(\zeta_0); \zeta_0\right) \notin \varphi(U). \tag{2.9}$$

Using Definition 2, since $\varphi(p(z), zp'(z), z^2 p''(z); z)$ is univalent in U, from superordination (2.2) we get that

$$h(U) \subset \varphi(U). \tag{2.10}$$

Assume that $q(z) = F(a, b, c; z) \nprec p(z)$ (q is not subordinate to p). By applying Lemma 1, we know that there exist points $z_0 = r_0 e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U$ and an $m \geq n \geq 1$ such that conditions (i)–(iii) from the lemma are satisfied. Using these conditions with $r = q(z_0) = p(\zeta_0)$,

$$s = \frac{zq'(z_0)}{m} = \zeta_0 p'(\zeta_0), \; t = {z_0}^2 q''(z_0) = {\zeta_0}^2 p''(\zeta_0), \; \zeta_0 \in \partial U, \; |\zeta_0| = 1,$$

we get

$$\varphi\left(q(z_0), \frac{z_0 q'(z_0)}{m}, z_0^2 q''(z_0); z_0\right) = \varphi\left(p(\zeta_0), \zeta_0 p'(\zeta_0), \zeta_0^2 p''(\zeta_0); \zeta_0\right), \tag{2.11}$$

and using (2.9) the conclusion is that

$$\varphi\left(q(z_0), \frac{z_0 q'(z_0)}{m}, z_0^2 q''(z_0); z_0\right) \notin \varphi(U). \tag{2.12}$$

Since (2.12) contradicts (2.10), the subordination $q(z) \prec p(z), \; z \in U$ must be satisfied, meaning that $q(U) \subset p(U)$.

Since $q(z) = F(a, b, c; z)$ is the solution of the differential Eq (2.1), q is the best subordinant and $q(U) = F(U) \subset p(U)$.

Having this theorem proved, it is easy to see that the problem of finding the best subordinant of a superordination such as (2.2), reduces to proving that the differential Eq (2.1) has a univalent solution in U.

The conclusion of the theorem can be interpreted as follows:

$$\varphi\left(F(a, b, c; z), zF'(a, b, c; z), z^2 F''(a, b, c; z); z\right) \prec \varphi\left(p(z), zp'(z), z^2 p''(z); z\right) \tag{2.13}$$

implies

$$q(z) = F(a, b, c; z) \prec p(z), \quad z \in U. \tag{2.14}$$

Since $q(z) = F(a, b, c; z)$ is the solution of the differential Eq (2.1), it is the best subordinant. $\square$

In order to use the result proved in Theorem 1 and apply the method of subordination chains, it must be simplified for the case of first-order differential superordiantion. That is possible as it can be seen in the next theorem.

Theorem 2. *Let h be analytic in U, consider the Gaussian hypergeometric function given by (1.2) and let $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$, $\varphi \in \phi_n[h(U), q]$.*

Suppose that the differential equation

$$\varphi(q(z), zq'(z); z) = h(z) \tag{2.15}$$

has a univalent solution $q(z) = F(a, b, c; z) \in Q(1)$ and suppose that

$$\varphi(F(a, b, c; z), tzF'(a, b, c; z); z) \in h(U), \quad t > 0. \tag{2.16}$$

If $p \in Q(1)$ and $\varphi(p(z), zp'(z); z)$ is univalent in U, then

$$h(z) = \varphi(F(a, b, c; z), tzF'(a, b, c; z); z) \prec \varphi(p(z), zp'(z); z) \tag{2.17}$$

implies

$$q(z) = F(a,b,c;z) \prec p(z) \ \ or\ F(U) \subset p(U). \tag{2.18}$$

Furthermore, if $\varphi(F(a,b,c;z), zF'(a,b,c;z); z) = h(z)$ *has a univalent solution* $q(z) = F(a,b,c;z) \in Q(1)$, *then q is the best subordinant.*

Proof. Using Definition 2, superordination (2.17) is equivalent to

$$h(U) \subset \varphi(U). \tag{2.19}$$

For $z = z_0 \in U$, using relation (2.19), we have

$$h(z_0) = \varphi(F(a,b,c;z_0), z_0F'(a,b,c;z_0); z_0) \in \varphi(U). \tag{2.20}$$

Suppose that functions $p,\ q,\ h$ satisfy the conditions from Lemma 1 in $\overline{U}$.

Assume that $q(z) = F(a,b,c;z) \not\prec p(z)$. By applying Lemma 1, we know that there exist points $z_0 \in U$ and $\zeta_0 \in \partial U$ and an $m \geq n \geq 1$ which satisfy conditions (i)–(iii) of the lemma.

Let $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$,

$$\varphi(r,s;z) \in \phi_n[h(U), q], \quad r,s \in \mathbb{C}.$$

Using these conditions with

$$r = F(a,b,c;z_0) = p(\zeta_0),\ s = \frac{z_0F'(a,b,c;z_0)}{m} = \zeta_0 p'(\zeta_0),\ \zeta_0 \in \partial U,\ |\zeta_0| = 1,\ t = 1 > 0$$

we get

$$\varphi\left(F(a,b,c;z_0), \frac{z_0F(a,b,c;z_0)}{m}; z_0\right) = \varphi(p(\zeta_0), \zeta_0 p'(\zeta_0); \zeta_0) \notin \varphi(U).$$

Since this contradicts (2.20), the conclusion must be that $q(z) = F(a,b,c;z) \prec p(z),\ z \in U$. Because q is a solution for Eq (2.15), $q(z) = F(a,b,c;z)$ is the best subordinant. □

The next theorem is proved by using the method of subordination chains and the results stated in Theorem 2. This theorem provides means for obtaining the Carathéodory properties for the Gaussian hypergeometric function.

Theorem 3. *Let* $q(z) = F(a,b,c;z)$ *given by (1.2) be a solution for the equation*

$$h(z) = F(a,b,c;z) + zF'(a,b,c;z), \quad z \in U. \tag{2.21}$$

If

$$L(z,t) = F(a,b,c;z) + tzF'(a,b,c;z),\ t > 0,$$

is a subordination chain, let $p \in H[1,1] \cap Q$ *such that* $p(z) + zp'(z)$ *is univalent in U.*

Then

$$h(z) \prec p(z) + zp'(z)$$

implies

$$q(z) = F(a,b,c;z) \prec p(z) \ \ or\ F(U) \subset p(U),$$

and $q(z) = F(a,b,c;z)$ *is the best subordinant.*

Proof. Suppose that functions $p,\ q,\ h$ satisfy the conditions of this theorem in the closed unit disc $\overline{U}$.
Let $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$,

$$\varphi(r, s; z) = r + ts, \quad r, s \in \mathbb{C}, t > 0. \tag{2.22}$$

For $r = F(a, b, c; z)$, $s = zF'(a, b, c; z)$ relation (2.22) becomes

$$\varphi(F(a, b, c; z), zF'(a, b, c; z); z) = F(a, b, c; z) + tzF'(a, b, c; z) = L(z, t), \ t > 0. \tag{2.23}$$

Since $L(\cdot, t)$ is a subordination chain, from Definition 1, we have

$$L(z, t) \prec L(z, 1), \ 0 < t \leq 1. \tag{2.24}$$

For $t = 1$, relation (2.23) becomes

$$L(z, 1) = F(a, b, c; z) + zF'(a, b, c; z). \tag{2.25}$$

Using relation (2.25), (2.23) gives

$$L(z, 1) = h(z), \ z \in U. \tag{2.26}$$

Using (2.23) and (2.26), relation (2.24) can be written as

$$\varphi(F(a, b, c; z), tzF'(a, b, c; z); z) \prec h(z), \ z \in U. \tag{2.27}$$

Applying Definition 2, superordination (2.27) is equivalent to

$$\varphi(F(a, b, c; z), tzF'(a, b, c; z); z) \in h(U), \ t > 0.$$

By using the results proved in Theorem 2 we have that

$$q(z) = F(a, b, c; z) \prec p(z) \ \text{ or } F(U) \subset p(U). \tag{2.28}$$

Since q is a solution for Eq (2.21), it results that $q(z) = F(a, b, c; z)$ is the best subordinant. □

Remark 2. *If in Theorem 3 we use the convex function in U, $p(z) = \frac{1+z}{1-z}$ with $p(0) = 1$ the following result is obtained which gives a sufficient condition for the Gaussian hypergeometric function $F(a, b, c; z)$ given by (1.2) to be a Carathéodory function.*

Corollary 4. *Let $q(z) = F(a, b, c; z)$ given by (1.2) be a solution for the equation*

$$h(z) = F(a, b, c; z) + zF'(a, b, c; z), \ z \in U. \tag{2.29}$$

If

$$L(z, t) = F(a, b, c; z) + tzF'(a, b, c; z), \ t > 0,$$

is a subordination chain, let $p \in H[1, 1] \cap Q$ such that

$$p(z) + zp'(z) = \frac{1+z}{1-z} + \frac{2z}{(1-z)^2}$$

is univalent in U, then

$$h(z) \prec \frac{1+z}{1-z} + \frac{2z}{(1-z)^2}$$

implies

$$F(a,b,c;z) \prec \frac{1+z}{1-z} \text{ or } Re\, F(a,b,c;z) > 0, \quad z \in U, \tag{2.30}$$

also written as

$$F(U) \subset \{z \in \mathbb{C} : z = x + iy,\ x > 0,\ y \in \mathbb{R}\},$$

which shows that $F \in \mathcal{P}$.

$F(a,b,c;z)$ *being the solution of Eq (2.29),* $q(z) = F(a,b,c;z)$ *is the best subordinant for the superordination in (2.30).*

Proof. Using the relation (2.28) obtained in the proof of Theorem 3, we can write

$$F(a,b,c;z) \prec p(z) = \frac{1+z}{1-z}, \quad z \in U. \tag{2.31}$$

Since this function $p(z)$ is convex in U, differential superordination (2.31) is equivalent to

$$\text{Re}\, F(a,b,c;z) > Re\frac{1+z}{1-z} > 0,\ z \in U,$$

and $F(a,b,c;z)$ is the best subordinant.

Since Re $F(a,b,c;z) > 0$ we conclude that function F is a function with positive real part, hence $F \in \mathcal{P}$. □

Remark 3. *If in Theorem 3 we use the convex function in U,* $p(z) = \frac{1}{1+z}$ *with* $p(0) = 1$, $Re\frac{1}{1+z} > \frac{1}{2}$, *the following result is obtained which gives a sufficient condition for the Gaussian hypergeometric function* $F(a,b,c;z)$ *given by (1.2) to be a Carathéodory function.*

Corollary 5. *Let* $q(z) = F(a,b,c;z)$ *given by (1.2) be a solution for the equation*

$$h(z) = F(a,b,c;z) + zF'(a,b,c;z),\ z \in U. \tag{2.32}$$

If

$$L(z,t) = F(a,b,c;z) + tzF'(a,b,c;z),\ t > 0,$$

is a subordination chain, let $p \in H[1,1] \cap Q$ *such that*

$$p(z) + zp'(z) = \frac{1}{(1+z)^2}$$

is univalent in U, then

$$h(z) \prec \frac{1}{(1+z)^2},\ z \in U,$$

implies

$$F(a,b,c;z) \prec \frac{1}{1+z} \text{ i.e. } Re\, F(a,b,c;z) > \frac{1}{2},\ z \in U, \tag{2.33}$$

also written as

$$F(U) \subset \left\{z \in \mathbb{C} : z = x + iy,\ x > \frac{1}{2},\ y \in \mathbb{R}\right\},$$

which shows that $F \in \mathcal{P}$.

$F(a,b,c;z)$ *being the solution of Eq (2.32),* $q(z) = F(a,b,c;z)$ *is the best subordinant for the superordination in (2.33).*

Proof. Using the relation (2.28) obtained in the proof of Theorem 3, we can write

$$F(a,b,c;z) \prec p(z) = \frac{1}{1+z},\ z \in U. \tag{2.34}$$

Since this function $p(z)$ is convex in U, differential superordination (2.34) is equivalent to

$$\text{Re } F(a,b,c;z)\ > Re\frac{1}{1+z} > \frac{1}{2},\ z \in U,$$

and $F(a,b,c;z)$ is the best subordinant.

Since Re $F(a,b,c;z) > \frac{1}{2}$ we conclude that function F is a Carathéodory function. □

Remark 4. *If in Theorem 3 we use the convex function in U, $p(z) = \frac{1}{2} \cdot \frac{2+z}{1-z}$ with $p(0) = 1$, $Re\ p(z) > \frac{1}{4}$, the following result is obtained which gives a sufficient condition for the Gaussian hypergeometric function $F(a,b,c;z)$ given by (1.2) to be a Carathéodory function.*

Corollary 6. *Let $q(z) = F(a,b,c;z)$ given by (1.2) be a solution for the equation*

$$h(z) = F(a,b,c;z) + zF'(a,b,c;z),\ z \in U. \tag{2.35}$$

If

$$L(z,t) = F(a,b,c;z) + tzF(a,b,c;z),\quad t > 0,$$

is a subordination chain, let $p \in H[1,1] \cap Q$ such that

$$p(z) + zp'(z) = \frac{1}{2} \cdot \frac{2+z}{1-z} + \frac{3}{2} \cdot \frac{1}{(1-z)^2}$$

is univalent in U, then

$$h(z) \prec \frac{1}{2} \cdot \frac{2+z}{1-z} + \frac{3}{2} \cdot \frac{1}{(1-z)^2},\ z \in U,$$

implies

$$F(a,b,c;z) \prec \frac{1}{2} \cdot \frac{2+z}{1-z}\ \text{i.e. } Re\ F(a,b,c;z) > \frac{1}{4},\ z \in U, \tag{2.36}$$

also written as

$$F(U) \subset \left\{ z\in \mathbb{C} : z = x+iy,\ x > \frac{1}{4},\ y\in \mathbb{R} \right\},$$

which shows that $F \in \mathcal{P}$.

$F(a,b,c;z)$ being the solution of Eq (2.35), $q(z) = F(a,b,c;z)$ is the best subordinant for the superordination in (2.36).

Proof. Using the relation (2.28) obtained in the proof of Theorem 3, we can write

$$F(a,b,c;z) \prec p(z) = \frac{1}{2} \cdot \frac{2+z}{1-z}, z \in U. \tag{2.37}$$

Since this function $p(z)$ is convex in U, differential superordination (2.37) is equivalent to

$$\text{Re } F(a,b,c;z)\ > \text{Re}\frac{1}{2} \cdot \frac{2+z}{1-z} > \frac{1}{4},\ z \in U,$$

also written as

$$F(U) \subset \left\{ z \in \mathbb{C} : z = x + iy,\ x > \frac{1}{4},\ y \in \mathbb{R} \right\},$$

which shows that F is a Carathéodory function and is the best subordinant for superordination (2.37). □

Example 1. *For* $a = -1, b = -i, c = i,\ F(-1, -i, i; z) = 1 + z,\ F'(-1, -i, i; z) = 1$ *and*

$$h(z) = F(-1, -i, i; z) + zF'(-1, -i, i; z) = 1 + 2z.$$

If

$$p(z) = \frac{1 + z}{1 - z},$$

then

$$p(z) + zp'(z) = \frac{1 + z}{1 - z} + \frac{2z}{(1 - z)^2}$$

is univalent in U.

Using Corollary 4, we have:

If

$$1 + 2z \prec \frac{1 + z}{1 - z} + \frac{2z}{(1 - z)^2},\ z \in U,$$

then

$$1 + z \prec \frac{1 + z}{1 - z},\ z \in U,$$

or

$$F(U) \subset \{ z \in \mathbb{C} : z = x + iy,\ x > 0,\ y \in \mathbb{R} \}.$$

Indeed, we have

$$Re\ (1 + z) = Re\ (1 + \rho \cos \alpha + i\rho \sin \alpha) = 1 + \rho \cos \alpha > 0,$$

since $0 < \rho < 1,\ -1 \leq \cos \alpha \leq 1$.

3. Conclusions

In this paper, Gaussian hypergeometric function is studied using the means of the theory of differential superordination and the method of subordination chains in order to prove some theorems involving superordinations for which Gaussian hypergeometric function is the best subordinant. Inequalities in the complex plane connected with the superordination results were interpreted in terms of inclusion relations between subsets of the complex plane, corollaries of the theorems giving, for certain functions used, sufficient conditions for the Gaussian hypergeometric function to be a Carathéodory function. An example is also given, making it more obvious how the proven results can be applied and how they are important for the study of this function. The function considered for this investigation could be used applying the dual method of differential subordination .

Conflict of interest

All author declares no conflicts of interest in this paper.

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