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Research article

Multiple solutions for a class of boundary value problems of fractional differential equations with generalized Caputo derivatives

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Abstract: This paper is mainly concerned with the existence of multiple solutions for the following boundary value problems of fractional differential equations with generalized Caputo derivatives:

$$\begin{cases} {}_0^C D_g^\alpha x(t) + f(t,x) = 0, \ 0 < t < 1; \\ x(0) = 0, \ {}_0^C D_g^1 x(0) = 0, \ {}_0^C D_g^\nu x(1) = \int_0^1 h(t) {}_0^C D_g^\nu x(t) g'(t) dt, \end{cases}$$

where $2 < \alpha < 3$, $1 < \nu < 2$, $\alpha - \nu - 1 > 0$, $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$, $g' > 0$, $h \in C([0,1], \mathbb{R}^+)$, $\mathbb{R}^+ = [0, +\infty)$. Applying the fixed point theorem on cone, the existence of multiple solutions for considered system is obtained. The results generalize and improve existing conclusions. Meanwhile, the Ulam stability for considered system is also considered. Finally, three examples are worked out to illustrate the main results.

Keywords: multiple solutions; fixed point theory; fractional differential equation; boundary value problems; generalized Caputo derivatives
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1. Introduction

The fractional calculus is a branch of mathematics, which studies the integration and differentiation of any order in real or complex field. In 1832, the fractional derivative was first formally proposed by Liouville. See [1, 2] for more knowledge on fractional calculus. The fractional order differential equation (FDE, for short) is a generalization of classical integer order differential equation as well, which can describe complex with simple modeling, clear physical meaning of parameters, accurate selection and so on. Hence, it becomes an important tool for mathematical modeling of complex machines, physical processes, fluid dynamics, finance and other areas of applications (see [3, 4] and references therein). In recent decades, more and more researchers pay much attention to the fractional differential equations and have obtained substantial achievements. For example, S. Salahshour and A. Ahmadian et al. researched the heat transfer problem with a approach of fractional modeling [5], successive approximation method for Caputo q-fractional IVPs [6] and M-fractional derivative under interval uncertainty [7]. N. Sene investigated chaotic system involving Caputo fractional derivatives in [8, 9]. [10–12] were concerned with fractional diffusion equation. [13, 14] studied infinitely many solutions for impulsive fractional boundary value problem with p-Laplacian and for fractional schrodinger-maxwell equations, respectively. [15–17] analyzed the Ulam stability of nonlinear FDEs. [18, 19] investigated the controllability for two classes of semilinear fractional evolution systems.

In the last few years, boundary value problems of fractional differential equations (FBVPs, for short) have been extensively studied. Most of them have been considered in the frame of standard fractional derivatives such as Rieman-Liouvile and Caputo derivatives. For instance, [20] is concerned with positive solutions of a two-point boundary value problem for singular fractional differential equations in Banach space. [21–23] developed bifurcation techniques for FBVPs. [24–26] investigated positive solutions for FBVPs. [27, 28] dealt with coupled fractional differential systems with nonlocal boundary conditions. [29–31] studied FBVPs via critical point theory. [32, 33] were concerned with the solvability for multi-order nonlinear fractional systems and periodic boundary value problems of nonlinear fractional hybrid differential equations. [34] investigated positive solutions for nonlinear discrete FBVPs with a p-laplacian operator.

More generally, A. Babakhani in [35] considered

$$\begin{cases} \frac{d}{dt}{}^{C}D_{0+}^{\alpha}u(t) + q(t)f(u(t), u'(t)) = 0, \ 0 \leq t \leq 1, \ 1 < \alpha < 2, \\ u(0) = 0, \ u(1) = v > 0, \ ({}^{C}D_{0+}^{\alpha}u)(1) = \int_0^1 ({}^{C}D_{0+}^{\alpha}u)(s)dg(s), \end{cases}$$

where ${}^{C}D_{0+}^{\alpha}$ is the Caputo fractional derivative of order α, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a given continuous function and $g : [0,1] \to [0,+\infty)$ is nondecreasing function. By constructing a special cone, the existence of at least one positive solution was obtained under some suitable assumptions. In [36], Y. Li studied the following fractional q-difference equations involving q-integral boundary conditions:

$$\begin{cases} D_q^{\alpha}x(t) + f_1(t, y(t)) = 0, \ 0 < t < 1; \\ D_q^{\alpha}y(t) + f_2(t, x(t)) = 0, \ 0 < t < 1; \\ x(0) = 0, \ D_q^1 x(0) = 0, \ D_q^{\nu}x(1) = \int_0^1 h(t)D_q^{\nu}x(t)d_qt; \\ y(0) = 0, \ D_q^1 y(0) = 0, \ D_q^{\nu}y(1) = \int_0^1 h(t)D_q^{\nu}y(t)d_qt, \end{cases}$$

where $2 < \alpha < 3$, $1 < \nu < 2$, D_q^{ν} is α-order Riemann-Liouville's fractional q-derivative. The existence of nontrivial solutions is obtained by using topological degree theory.

With the development of investigation on fractional derivatives, new concepts are constantly being put forward. For instance, F. Jarad et al. proposed a new kind of generalized fractional derivatives and studied their properties in [37, 38]. N. Sene investigates fractional advection-dispersion equation described by the Caputo left generalized fractional derivative in [39]. [40] studied Ulam stabilities of fractional differential equations including generalized Caputo fractional derivative. However, to our best knowledge, there are few studies on the existence of multiple solutions and Ulam-Hyers stability for integral boundary value problems of FDES with generalized Caputo derivatives. The purpose of present paper is to fill this gap.

Motivated by the above discussions, this paper studies multiple solutions and Ulam-Hyers stability for the following FBVPs with generalized Caputo derivatives

$$\begin{cases} {}^C_0D^\alpha_g x(t) + f(t,x) = 0,\ 0 < t < 1; \\ x(0) = 0,\ {}^C_0D^1_g x(0) = 0,\ {}^C_0D^\nu_g x(1) = \int_0^1 h(t)\,{}^C_0D^\nu_g x(t) g'(t)dt, \end{cases} \tag{1.1}$$

where $2 < \alpha < 3$, $1 < \nu < 2$, $\alpha - \nu - 1 > 0$, $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$, $g' > 0$, $h \in C([0,1], \mathbb{R}^+)$, $\mathbb{R}^+ = [0, +\infty)$. The main features of this paper are as follows. Firstly, compared with the above mentioned references, BVP (1.1) is studied in the sense of generalized Caputo fractional derivatives, which is also different from fractional q-difference equations in [36]. Secondly, the fractional boundary value condition we consider here is of integral form, and that makes BVP (1.1) more widely applicable in solving practical problems. Thirdly, the used approach in this paper has certain advantages over some references listed as above. In detail, the distinctive tool used here is the first eigenvalue of corresponding linear operator. At the same time, a suitable cone is established by researching properties of Green's function deeply. So the positive solutions can be obtained by means of the cone expansion and compression fixed point theorem and Leggett-Williams theorem. Finally, the Ulam-Hyers stability and generalized Ulam-Hyers stability for BVP (1.1) are also studied under some suitable assumptions.

The remainder of this paper is organized as follows: Some basic knowledge of fractional calculous and some preliminary results are given in Section 2. The existence results will be given and proved in Section 3. And in Section 4, the Ulam-Hyers stability and generalized Ulam-Hyers stability will be established. Three examples are worked out to illustrate the main results in Section 5. Finally, the conclusion and some future works are given in Section 6.

2. Preliminaries

Definition 2.1. *[38] (1) Let $g \in C^n[a,b]$ such that $g'(t) > 0$ on $[a,b]$. Define*

$$AC^n_g =: \{f : [a,b] \to C \text{ and } f^{[n-1]} \in AC[a,b]\},$$

where $f^{[n-1]} = (\frac{1}{g'(t)}\frac{d}{dt})^{n-1} f$.

(2)

$$C_{\varepsilon,g}[a,b] =: \{f : (a,b] \to \mathbb{R} \text{ such that } (g(t) - g(a))^\varepsilon f(t) \in C[a,b]\},$$

where $C_{0,g} = C[a,b]$.

(3)

$$C^n_{\varepsilon,g}[a,b] =: \{f : (a,b] \to \mathbb{R} \text{ such that } f^{[n-1]} \in C[a,b] \text{ and } f^{[n]} \in C_{\varepsilon,g}[a,b]\},$$

where $C^n_{0,g} = C[a,b]$.

Definition 2.2. *[38] (1) The Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ of a function f on a finite or infinite interval (a,b) is defined as follows:*

$$({}_aI^\alpha y)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} y(s)ds.$$

(2) The Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{R}^+$ of a function f on a finite or infinite interval (a, b) is defined as follows:

$$({}_aD^\alpha y)(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n \int_a^t (t-s)^{n-\alpha-1} y(s)ds,$$

where $n = [\alpha] + 1$, $g^{(i)} \neq 0, i = 1, 2, ..., n$.

Definition 2.3. *[38] The Caputo fractional derivative of order $\alpha \in \mathbb{R}^+$ of a function f on a finite or infinite interval (a, b) is defined as follows:*

$$({}_a^CD^\alpha y)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} y^{(n)}(s)ds.$$

Definition 2.4. *[38] (1) The generalized Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ of a function f with respect to the function g on a finite or infinite interval (a, b) is defined as follows:*

$$({}_aI_g^\alpha y)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (g(t)-g(s))^{\alpha-1} y(s)g'(s)ds.$$

(2) The generalized Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{R}^+$ of a function f with respect to the function g on a finite or infinite interval (a, b) is defined as follows:

$$({}_aD_g^\alpha y)(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{1}{g'(t)}\frac{d}{dt})^n \int_a^t (g(t)-g(s))^{n-\alpha-1} y(s)g'(s)ds,$$

where $n = [\alpha] + 1$, $g^{(i)} \neq 0, i = 2, ..., n$.

Definition 2.5. *[38] The generalized Caputo fractional derivative of order $\alpha \in \mathbb{R}^+$ of a function f with respect to another function g on a finite or infinite interval (a, b) is defined as follows:*

$$({}_a^CD_g^\alpha y)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (g(t)-g(s))^{n-\alpha-1} y^{[n]}(s)g'(s)ds,$$

where $y^{[n]} = (\frac{1}{g'(t)}\frac{d}{dt})^n$, $n = [\alpha] + 1$, $g^{(i)} \neq 0, i = 2, ..., n$.

Remark 2.6. *From the Definition 2.4 (1) and Definition 2.5 above, we can see that*

$$({}_a^CD_g^\alpha y)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (g(t)-g(s))^{n-\alpha-1} y^{[n]}(s)g'(s)ds = {}_aI_g^{n-\alpha} y^{[n]}(t).$$

Lemma 2.7. *[38] Let $g \in C^n[a, b]$ such that $g'(t) > 0$ on $[a, b]$. Then $y \in AC_g^n$ if and only if it can be written as that*

$$y(t) = \frac{1}{(n-1)!} \int_a^t (g(t)-g(s))^{n-1} y^{[n]}(s)g'(s)ds + \sum_{k=0}^{n-1} \frac{y^{[k]}(a)}{k!}(g(t)-g(a))^k. \tag{2.1}$$

Lemma 2.8. *[38] Let $\alpha > 0$, $n = [\alpha] + 1$ and $y \in AC_g^n[a, b]$. Then the fractional derivative of y with respect to g exists almost everywhere and*

$$({}_aD_g^\alpha y)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (g(t)-g(s))^{n-\alpha-1} y^{[n]}(s)g'(s)ds + \sum_{k=0}^{n-1} \frac{y^{[k]}(a^+)}{\Gamma(k-\alpha-1)}(g(t)-g(a))^{k-\alpha}. \tag{2.2}$$

Remark 2.9. *[38] Equation (2.2) can be written as that*

$$({}_aD_g^{\alpha}y)(t) = ({}_aI_g^{n-\alpha})y^{[n]}(t) + \sum_{k=0}^{n-1}\frac{y^{[k]}(a^+)}{\Gamma(k-\alpha-1)}(g(t)-g(a))^{k-\alpha},$$

and thus, one can define the Caputo fractional derivative of a function with respect to another function as

$$\begin{aligned}({}_a^CD_g^{\alpha}y)(t) &= ({}_aD_g^{\alpha}y)(t) - \sum_{k=0}^{n-1}\frac{y^{[k]}(a^+)}{\Gamma(k-\alpha-1)}(g(t)-g(a))^{k-\alpha} \\ &= {}_aD_g^{\alpha}(y(s) - \sum_{k=0}^{n-1}\frac{y^{[k]}(a^+)}{k!}(g(t)-g(a))^{k})(t).\end{aligned}$$

Similar to Caputo fractional derivative, we can easily obtain the following properties.

Lemma 2.10. *Let $\alpha > 0$, ${}_a^CD_g^{\alpha}$ be a generalized Caputo fractional derivative of α, $y(t) \in C[0,1]$. Then,*

$${}_aI_g^{\alpha}\,{}_a^CD_g^{\alpha}y(t) = y(t) - \sum_{k=0}^{n-1}\frac{y^{[k]}(a)}{k!}(g(t)-g(a))^{k},$$

where $g^{(k)} \neq 0$, $k = 0,\ 1,\ 2,\ \cdots,\ n$, $n = [\alpha]+1$.

Proof. The proof is done by using Remark 2.6 and Lemma 2.7.

$${}_aI_g^{\alpha}\,{}_a^CD_g^{\alpha}y(t) = {}_aI_g^{\alpha}\,{}_aI_g^{n-\alpha}y^{[n]}(t) = {}_aI_g^{n}y^{[n]}(t) = y(t) - \sum_{k=0}^{n-1}\frac{y^{[k]}(a)}{k!}(g(t)-g(a))^{k},$$

where $g^{(k)} \neq 0$, $k = 0,\ 1,\ 2,\ \cdots,\ n$, $n = [\alpha]+1$.

The proof is completed. □

Lemma 2.11. *Let $A = 1 - \int_0^1 h(t)(\frac{g(t)-g(0)}{g(1)-g(0)})^{\alpha-\nu-1}g'(t)dt \neq 0$. Then the following boundary value problem*

$$\begin{cases} ({}_0^CD_g^{\alpha}x)(t) + y(t) = 0,\ 0 < t < 1; \\ x(0) = 0,\ {}_0^CD_g^{1}x(0) = 0,\ {}_0^CD_g^{\nu}x(1) = \int_0^1 h(t){}_0^CD_g^{\nu}x(t)g'(t)dt, \end{cases} \tag{2.3}$$

has a unique solution

$$x(t) = \int_0^1 G(t,s)y(s)g'(s)ds := Sy(t),$$

where

$$B = (g(1)-g(0))^{\alpha-\nu-1}A,\ L = \frac{\Gamma(3)}{\Gamma(3-\nu)},$$

$$G(t,s) = G_0(t,s) + \frac{(g(t)-g(0))^2}{BL\Gamma(\alpha-\nu)}\int_0^1 h(t)G_1(t,s)g'(t)dt,$$

$$G_0(t,s)=\begin{cases}\frac{1}{L\Gamma(\alpha-\nu)}(g(t)-g(0))^2\frac{(g(1)-g(s))^{\alpha-\nu-1}}{(g(1)-g(0))^{2-\nu}}-\frac{1}{\Gamma(\alpha)}(g(t)-g(s))^{\alpha-1}, & 0\le s\le t\le 1;\\ \frac{1}{L\Gamma(\alpha-\nu)}(g(t)-g(0))^2\frac{(g(1)-g(s))^{\alpha-\nu-1}}{(g(1)-g(0))^{2-\nu}}, & 0\le t\le s\le 1,\end{cases}$$

$$G_1(t,s)=\frac{1}{L\Gamma(\alpha-\nu)}\begin{cases}(g(t)-g(0))^{2-\nu}\frac{(g(1)-g(s))^{\alpha-\nu-1}}{(g(1)-g(0))^{2-\nu}}-(g(t)-g(s))^{\alpha-\nu-1}, & 0\le s\le t\le 1;\\ (g(t)-g(0))^{2-\nu}\frac{(g(1)-g(s))^{\alpha-\nu-1}}{(g(1)-g(0))^{2-\nu}}, & 0\le t\le s\le 1.\end{cases}$$

Proof. Using Definition 2.4 and Definition 2.5, one can obtain that

$$x(t)=c_0+c_1(g(t)-g(0))+c_2(g(t)-g(0))^2-{}_0I_g^{\alpha}y(t).$$

Noticing that $x(0)=({}_0^CD_g^1x)(0)=0$, one can deduce that $c_0=c_1=0$. Hence,

$$x(t)=c_2(g(t)-g(0))^2-{}_0I_g^{\alpha}y(t). \tag{2.4}$$

By (2.4), one can easily get that

$$\begin{aligned}({}_0^CD_g^{\nu}x)(t)&={}_0^CD_g^{\nu}[c_2(g(t)-g(0))^2-{}_0I_g^{\alpha}y(t)]\\&=c_2\frac{\Gamma(3)}{\Gamma(3-\nu)}(g(t)-g(0))^{2-\nu}-({}_0^CD_g^{\nu})({}_0I_g^{\alpha}y)(t)\\&=c_2\frac{\Gamma(3)}{\Gamma(3-\nu)}(g(t)-g(0))^{2-\nu}-({}_0I_g^{\alpha-\nu}y)(t)\\&=c_2\frac{\Gamma(3)}{\Gamma(3-\nu)}(g(t)-g(0))^{2-\nu}-\frac{1}{\Gamma(\alpha-\nu)}\int_0^t(g(t)-g(s))^{\alpha-\nu-1}y(s)g'(s)ds.\end{aligned}$$

Obviously,

$$({}_0^CD_g^{\nu}x)(1)=c_2\frac{\Gamma(3)}{\Gamma(3-\nu)}(g(1)-g(0))^{2-\nu}-\frac{1}{\Gamma(\alpha-\nu)}\int_0^1(g(1)-g(s))^{\alpha-\nu-1}y(s)g'(s)ds. \tag{2.5}$$

Applying

$$({}_0^CD_g^{\nu}x)(1)=\int_0^1 h(t){}_0^CD_g^{\nu}x(t)g'(t)dt$$

and (2.5), it is immediate to see that

$$\begin{aligned}({}_0^CD_g^{\nu}x)(1)&=\int_0^1 h(t){}_0^CD_g^{\nu}x(t)g'(t)dt\\&=c_2\frac{\Gamma(3)}{\Gamma(3-\nu)}\int_0^1 h(t)(g(t)-g(0))^{2-\nu}g'(t)dt\end{aligned}$$

$$-\frac{1}{\Gamma(\alpha-\nu)}\int_0^1 h(t)[\int_0^t (g(t)-g(s))^{\alpha-\nu-1}y(s)g'(s)ds]g'(t)dt.$$

Hence, we deduce that

$$\begin{aligned}
c_2 &= \frac{1}{BL\Gamma(\alpha-\nu)}\int_0^1 (g(1)-g(s))^{\alpha-\nu-1}y(s)g'(s)ds \\
&\quad -\frac{1}{BL\Gamma(\alpha-\nu)}\int_0^1 h(t)[\int_0^t (g(t)-g(s))^{\alpha-\nu-1}y(s)g'(s)ds]g'(t)dt.
\end{aligned}$$

Moreover, by (2.4), one can get that

$$\begin{aligned}
x(t) &= \frac{1}{BL\Gamma(\alpha-\nu)}\int_0^1 (g(t)-g(0))^2(g(1)-g(s))^{\alpha-\nu-1}y(s)g'(s)ds \\
&\quad -\frac{(g(t)-g(0))^2}{BL\Gamma(\alpha-\nu)}\int_0^1 h(t)[\int_0^t (g(t)-g(s))^{\alpha-\nu-1}y(s)g'(s)ds]g'(t)dt \\
&\quad -\frac{1}{\Gamma(\alpha)}\int_0^t (g(t)-g(s))^{\alpha-1}y(s)g'(s)ds \\
&= \frac{1}{BL\Gamma(\alpha-\nu)}\int_0^1 (g(t)-g(0))^2(g(1)-g(s))^{\alpha-\nu-1}y(s)g'(s)ds \\
&\quad -\frac{(g(t)-g(0))^2}{BL\Gamma(\alpha-\nu)}\int_0^1 h(t)[\int_0^t (g(t)-g(s))^{\alpha-\nu-1}y(s)g'(s)ds]g'(t)dt \\
&\quad -\frac{1}{\Gamma(\alpha)}\int_0^t (g(t)-g(s))^{\alpha-1}y(s)g'(s)ds \\
&\quad +\frac{1}{L(g(1)-g(0))^{2-\nu}\Gamma(\alpha-\nu)}\int_0^1 (g(t)-g(0))^2(g(1)-g(s))^{\alpha-\nu-1}y(s)g'(s)ds \\
&\quad -\frac{1}{L(g(1)-g(0))^{2-\nu}\Gamma(\alpha-\nu)}\int_0^1 (g(t)-g(0))^2(g(1)-g(s))^{\alpha-\nu-1}y(s)g'(s)ds \\
&= \int_0^1 G_0(t,s)y(s)g'(s)ds \\
&\quad +\frac{(g(1)-g(0))^{2-\nu}}{BL\Gamma(\alpha-\nu)}\int_0^1 (g(t)-g(0))^2\frac{(g(1)-g(s))^{\alpha-\nu-1}}{(g(1)-g(0))^{2-\nu}}y(s)g'(s)ds \\
&\quad -\frac{(g(t)-g(0))^2}{BL\Gamma(\alpha-\nu)}\int_0^1 h(t)[\int_0^t (g(t)-g(s))^{\alpha-\nu-1}y(s)g'(s)ds]g'(t)dt \\
&\quad -\frac{B}{BL\Gamma(\alpha-\nu)}\int_0^1 (g(t)-g(0))^2\frac{(g(1)-g(s))^{\alpha-\nu-1}}{(g(1)-g(0))^{2-\nu}}y(s)g'(s)ds \\
&= \int_0^1 G_0(t,s)y(s)g'(s)ds \\
&\quad -\frac{(g(t)-g(0))^2}{BL\Gamma(\alpha-\nu)}\int_0^1 h(t)[\int_0^t (g(t)-g(s))^{\alpha-\nu-1}y(s)g'(s)ds]g'(t)dt \\
&\quad +\frac{(g(1)-g(0))^{2-\nu}(g(t)-g(0))^2}{BL\Gamma(\alpha-\nu)}\int_0^1 \frac{(g(1)-g(s))^{\alpha-\nu-1}}{(g(1)-g(0))^{2-\nu}}y(s)g'(s)ds
\end{aligned}$$

$$
\begin{aligned}
&-\frac{B(g(t)-g(0))^{\alpha-1}}{BL\Gamma(\alpha)}\int_0^1\frac{(g(1)-g(s))^{\alpha-\nu-1}}{(g(1)-g(0))^{2-\nu}}y(s)g'(s)ds]\\
=\;&\int_0^1 G_0(t,s)y(s)g'(s)ds\\
&-\frac{(g(t)-g(0))^2}{BL\Gamma(\alpha-\nu)}\int_0^1 h(t)[\int_0^t (g(t)-g(s))^{\alpha-\nu-1}y(s)g'(s)ds]g'(t)dt\\
&+\frac{(g(t)-g(0))^2}{BL\Gamma(\alpha-\nu)}\int_0^1 h(t)(g(t)-g(0))^{2-\nu}g'(t)dt\int_0^1\frac{(g(1)-g(s))^{\alpha-\nu-1}}{(g(1)-g(0))^{2-\nu}}y(s)g'(s)ds\\
=\;&\int_0^1 G_0(t,s)y(s)g'(s)ds\\
&+\frac{(g(t)-g(0))^2}{BL\Gamma(\alpha-\nu)}\int_0^1[\int_0^1 h(t)(g(t)-g(0))^{2-\nu}\frac{(g(1)-g(s))^{\alpha-\nu-1}}{(g(1)-g(0))^{2-\nu}}g'(t)dt\\
&-\int_s^1 h(t)(g(t)-g(s))^{\alpha-\nu-1}g'(t)dt]y(s)g'(s)ds\\
=\;&\int_0^1 G_0(t,s)y(s)g'(s)ds+\frac{(g(t)-g(0))^2}{BL\Gamma(\alpha-\nu)}\int_0^1[\int_0^1 h(t)G_1(t,s)g'(t)dt]y(s)g'(s)ds\\
=\;&\int_0^1 G(t,s)y(s)g'(s)ds.
\end{aligned}
$$

The proof is completed. □

Suppose that $(2-\nu)(g(t)-g(s)) \leq (\alpha-\nu-1)(g(t)-g(0))$, for $0 \leq s \leq t \leq 1$ in the rest of the paper.

Lemma 2.12. *The functions $G_i (i = 0, 1)$ has the following properties:*
(1) $G_0(t,s) \geq 0$ for $s,\ t \in [0,1]$;
(2) $G_1(t,s) \geq 0$ for $s,\ t \in [0,1]$;
(3) $G_0(t,s) \leq G_0(1,s)$ for $s,\ t \in [0,1]$;
(4) $G_0(t,s) \geq (\frac{g(t)-g(0)}{g(1)-g(0)})^2 G_0(1,s)$ for $s,\ t \in [0,1]$.

Proof. (1) On the one hand, for $0 \leq s \leq t \leq 1$, we know

$$G_0(t,s) = \frac{1}{L\Gamma(\alpha-\nu)}(g(t)-g(0))^2\frac{(g(1)-g(s))^{\alpha-\nu-1}}{(g(1)-g(0))^{2-\nu}} - \frac{1}{\Gamma(\alpha)}(g(t)-g(s))^{\alpha-1}.$$

By careful calculation, we can see

$$G_0^{[3]}(t,s) = (\frac{1}{g'(t)}\frac{d}{dt})^3 G_0(t,s) = -\frac{1}{\Gamma(\alpha)}(\alpha-1)(\alpha-2)(\alpha-3)(g(t)-g(s))^{\alpha-3} \geq 0.$$

With the property of $g(t)$, it implies that

$$G_0^{[2]}(t,s) \geq G_0^{[2]}(s,s) \geq 0.$$

Similarly,

$$G_0^{[1]}(t,s) \geq G_0^{[1]}(s,s) \geq 0.$$

Thus, it is easy to see that

$$G_0(t,s) \geq G_0(s,s) = \frac{1}{L\Gamma(\alpha-\nu)}(g(s)-g(0))^2 \frac{(g(1)-g(s))^{\alpha-\nu-1}}{(g(1)-g(0))^{2-\nu}} \geq 0.$$

On the other hand, for $0 \leq t \leq s \leq 1$, it is easy to see that from Lemma 2.10, $G_0(t,s) \geq 0$ for $s,\ t \in [0,1]$.

(2) For $0 \leq s \leq t \leq 1$, from $(2-\nu)(g(t)-g(s)) \leq (\alpha-\nu-1)(g(t)-g(0))$, $\frac{(g(t)-g(s))^{\alpha-\nu-1}}{(g(t)-g(0))^{2-\nu}}$ is a nondecreasing function for t on $[0,1]$.

Then,

$$\begin{aligned} G_1(t,s) &= \frac{1}{L\Gamma(\alpha-\nu)}[(g(t)-g(0))^{2-\nu}\frac{(g(1)-g(s))^{\alpha-\nu-1}}{(g(1)-g(0))^{2-\nu}} - (g(t)-g(s))^{\alpha-\nu-1}] \\ &= \frac{1}{L\Gamma(\alpha-\nu)}[(g(t)-g(0))^{2-\nu}(\frac{(g(1)-g(s))^{\alpha-\nu-1}}{(g(1)-g(0))^{2-\nu}} - \frac{(g(t)-g(s))^{\alpha-\nu-1}}{(g(t)-g(0))^{2-\nu}})] \\ &\geq 0. \end{aligned}$$

On the other hand, for $0 \leq t \leq s \leq 1$, it is easy to see that from Lemma 2.10 the conclusion is obviously established. Therefore, $G_1(t,s) \geq 0$ for $s, t \in [0,1]$.

(3) For $s, t \in [0,1]$, from (1) and Lemma 2.10, one can easily obtain that $G_0(t,s)$ is an increasing function with respect to t. Then, $G_0(t,s) \leq G_0(1,s)$.

(4) For $0 \leq s \leq t \leq 1$, from $(2-\nu)(g(t)-g(s)) \leq (\alpha-\nu-1)(g(t)-g(0))$, $\frac{(g(t)-g(s))^{\alpha-1}}{(g(t)-g(0))^2}$ is a nondecreasing function with respect to t.

$$\begin{aligned} \frac{G_0(t,s)}{G_0(1,s)} &= \frac{\frac{1}{L\Gamma(\alpha-\nu)}(g(t)-g(0))^2\frac{(g(1)-g(s))^{\alpha-\nu-1}}{(g(1)-g(s))^{\alpha-1}} - \frac{1}{\Gamma(\alpha)}(g(t)-g(s))^{\alpha-1}}{\frac{1}{L\Gamma(\alpha-\nu)}(g(1)-g(0))^2\frac{(g(1)-g(s))^{\alpha-\nu-1}}{(g(1)-g(s))^{\alpha-1}} - \frac{1}{\Gamma(\alpha)}(g(1)-g(s))^{\alpha-1}} \\ &= \frac{(g(t)-g(0))^2[\frac{1}{L\Gamma(\alpha-\nu)}\frac{(g(1)-g(s))^{\alpha-\nu-1}}{(g(1)-g(s))^{\alpha-1}} - \frac{1}{\Gamma(\alpha)}\frac{(g(t)-g(s))^{\alpha-1}}{(g(t)-g(0))^2}]}{(g(1)-g(0))^2[\frac{1}{L\Gamma(\alpha-\nu)}\frac{(g(1)-g(s))^{\alpha-\nu-1}}{(g(1)-g(s))^{\alpha-1}} - \frac{1}{\Gamma(\alpha)}\frac{(g(1)-g(s))^{\alpha-1}}{(g(1)-g(0))^2}]} \\ &\geq (\frac{(g(t)-g(0))}{(g(1)-g(0))})^2. \end{aligned}$$

On the other hand, for $0 \leq t \leq s \leq 1$, it is easy to see that

$$\frac{G_0(t,s)}{G_0(1,s)} = (\frac{(g(t)-g(0))}{(g(1)-g(0))})^2.$$

Therefore, $(\frac{g(t)-g(0)}{g(1)-g(0)})^2 G_0(1,s) \leq G_0(t,s)$ for $s,\ t \in [0,1]$.

The proof is completed. □

By Lemma 2.12, the following conclusion is established obviously.

Lemma 2.13. *Under the assumption in Lemma 2.12, the function G has the following properties:*

$$G(t,s) \geq 0, \ for \ s, \ t \in [0,1];$$

$$(\frac{g(t)-g(0)}{g(1)-g(0)})^2 \varphi(s) \leq G(t,s) \leq \varphi(s), \ for \ s, \ t \in [0,1].$$

where $\varphi(s) = G_0(1,s) + \dfrac{(g(1)-g(0))^2}{BL\Gamma(\alpha-\nu)} \displaystyle\int_0^1 h(t)G_1(t,s)g'(t)dt.$

The following lemmas will be used in the proof of the main results.

Lemma 2.14. *[41] Let $\Omega \subset E$ be a bounded open set and $0 \in \Omega$. $T : P \cap \overline{\Omega} \to P$ be a completely continuous operator. If T satisfies*

$$x \neq \mu T x, \ \forall x \in P \cap \partial\Omega, \ 0 < \mu \leq 1,$$

then $i(T, P \cap \Omega, p) = 1$.

Lemma 2.15. *[41] Let $\Omega \subset E$ be a bounded open set and $0 \in \Omega$. $T : P \cap \overline{\Omega} \to P$ be a completely continuous operator. If there is $\varphi \in P, \varphi \neq 0$ such that T satisfies*

$$x - T x \neq \mu\varphi, \ \forall x \in P \cap \partial\Omega, \ \mu \geq 0,$$

then $i(T, P \cap \Omega, p) = 0$.

Lemma 2.16. *[42] (Leggett-Williams theorem) Let P be a cone in a real Banach space E, $P_c = \{x \in P | \ \|x\| < c\}$, θ be a nonnegative continuous concave functional on P such that $\theta(x) \leq \|x\|$, for $\forall x \in P_c$; and $P(\theta, b, d) = \{x \in P | \ b \leq \theta(x), \ \|x\| \leq d\}$. Suppose that $T : P_c \to P_c$ is completely continuous and there exist constants $0 < a < b < d \leq c$ such that*

(A1) $\{x \in P(\theta, b, d) | \ \theta(x) > b\} \neq \emptyset$ and $\theta(T x) > b$ for $x \in P(\theta, b, d)$;

(A2) $\|T x\| < a$ for $\|x\| \leq a$;

(A3) $\theta(T x) > b$ for $x \in P(\theta, b, c)$ with $\|T x\| > d$.

Then T has at least three fixed points $x_1, \ x_2, \ x_3$ with $\|x_1\| < a$; $b < \theta(x_2)$; $a < \|x_3\|$ and $\theta(x_3) < b$.

3. Existence results

In this section, we establish the existence and multiplicity results for BVP (1.1). Let $E = C[0,1]$, $\|x\| := max_{t\in[0,1]}|x(t)|$ and $P := \{x \in E : x(t) \geq 0, \ \forall t \in [0,1]\}$. Then $(E, \ \|\cdot\|)$ is a real Banach space and P is a cone on E. Hence E is an ordered Banach space and the cone P is normal. Obviously, the normal constant is $N = 1$. Define operator $T : P \to P$ as follows:

$$T x(t) := \int_0^1 G(t,s) f(s, x(s)) g'(s) ds, \ x \in P. \tag{3.1}$$

For any $x \in P$, by the continuity of G, f and g', $T x$ is well defined. Since f is bounded, It is easy to see that T is also bounded. By Lemma (2.11), one can easily see that the existence of solutions for BVP (1.1) is equivalent to the existence of positive fixed point of T. Therefore, we need only to find the positive fixed point of T in the following work.

Subsequently, for simplicity and convenience, set

$$M = (\int_0^1 \varphi(s)g'(s)ds)^{-1},\ N = (\int_0^1 (\frac{g(\frac{1}{2}) - g(0)}{g(1) - g(0)})^2 \varphi(s)g'(s)ds)^{-1}.$$

Let $r(S)$ be the spectral radius of the linear bounded operator S defined by

$$(Sx)(t) = \int_0^1 G(t,s)x(t)g'(s)ds,\ t \in [0,1],\ x \in E.$$

From the Krein-Rutman theorem, we know that $r(S)$ is positive and S has a positive eigenfunction φ_1 corresponding to λ_1 such that $\lambda_1 S(\varphi_1) = \varphi_1$, where λ_1 is the first eigenvalue of S and $\lambda_1 = (r(S))^{-1}$.

Now let's list the following assumptions satisfied throughout the paper.

(H1) $\lim\limits_{x\to 0^+} \sup\limits_{t\in[0,1]} \dfrac{f(t,x)}{x} < \lambda_1$.

(H2) $\lim\limits_{x\to +\infty} \inf\limits_{t\in[0,1]} \dfrac{f(t,x)}{x} > \lambda_1$.

(H3) $\lim\limits_{x\to 0^+} \inf\limits_{t\in[0,1]} \dfrac{f(t,x)}{x} > \lambda_1$.

(H4) $\lim\limits_{x\to +\infty} \sup\limits_{t\in[0,1]} \dfrac{f(t,x)}{x} < \lambda_1$.

By the Arzela-Ascoli theorem, the following conclusion is established obviously.

Lemma 3.1. *The operator $T:\ P \to P$ is completely continuous.*

Now we are in a position to give our main results.

Theorem 3.2. *Under the assumptions (H1) and (H2), BVP (1.1) admits at least one positive solution.*

Proof. For the sake of obtaining the desired result, we need only to prove T has at least a positive fixed point in $P \cap (B_{\overline{R}_1} \backslash \overline{B}_{r_1})$.

First, the assumption (H1) implies that there exist $r_1 > 0$ and $\varepsilon_0 \in (0, \lambda_1)$ such that

$$f(t,x) < (\lambda_1 - \varepsilon_0)x,\ t \in [0,1],\ x \in [0, r_1].$$

We claim that for $\mu \in (0,1]$,

$$x(t) \neq \mu Tx(t),\ \forall x \in P \cap \partial B_{r_1},\ t \in [0,1]. \tag{3.2}$$

Suppose on the contrary that there exist $x_0 \in P \cap \partial B_{r_1}$, $\mu_0 \in (0,1]$ such that

$$x_0(t) = \mu_0 Tx_0(t),\ t \in [0,1].$$

Then,

$$\begin{aligned} x_0(t) &= \mu_0 Tx_0(t) \\ &\le \int_0^1 G(t,s)f(s,x_0(s))g'(s)ds \end{aligned}$$

$$< (\lambda_1 - \varepsilon_0) \int_0^1 G(t,s)x_0(s)g'(s)ds$$
$$= (\lambda_1 - \varepsilon_0) S x_0(t).$$

By nth iteration, we can get that

$$x_0(t) < (\lambda_1 - \varepsilon_0) S x_0(t) < (\lambda_1 - \varepsilon_0)^2 S^2 x_0(t) < \ldots < (\lambda_1 - \varepsilon_0)^n S^n x_0(t).$$

From the definition of the norm $\|\cdot\|$, one can deduce that

$$\|x_0\| < (\lambda_1 - \varepsilon_0)^n \|S^n\| \|x_0\|.$$

It is easy to see that

$$(\lambda_1 - \varepsilon_0)^n \|S^n\| > 1.$$

Hence we have

$$lim_{n\to\infty} \sqrt[n]{\|S^n\|}(\lambda_1 - \varepsilon_0) \geq 1.$$

This is a contradiction with

$$lim_{n\to\infty} \sqrt[n]{\|S^n\|}(\lambda_1 - \varepsilon_0) = (\lambda_1 - \varepsilon_0) r(S) < 1,$$

which means that (3.2) holds. By Lemma 2.14, we get

$$i(T, P \cap \partial B_{r_1}, P) = 1. \tag{3.3}$$

Second, the assumption (H2) implies that there exist $\varepsilon_1 > 0$ and $R_1 > 0$ such that

$$f(t,x) > (\lambda_1 + \varepsilon_1)x, \ t \in [0,1], \ |x| \geq R_1.$$

Let $\widetilde{M} = max_{t\in[0,1], x\in[0,R_1]}[f(t,x) + (\lambda_1 + \varepsilon_1)x]$, we can see

$$f(t,x) > (\lambda_1 + \varepsilon_1)x - \widetilde{M}, \ t \in [0,1], \ \forall x \in [0,+\infty).$$

Choose $\overline{R}_1 > max\{r_1, R_1, \dfrac{\widetilde{M}\|[(\lambda_1 + \varepsilon_1)S - I]^{-1}\|}{M}\}$. We claim that for $\mu \in [0,+\infty)$,

$$x(t) - Tx(t) \neq \mu\varphi_1(t), \ \forall x \in P \cap \partial B_{\overline{R}_1}, \ t \in [0,1]. \tag{3.4}$$

Suppose on the contrary that there exist $x_1 \in P \cap \partial B_{\overline{R}_1}$ and $\mu_1 \geq 0$ such that

$$x_1(t) - Tx_1(t) = \mu_1\varphi_1(t), \ t \in [0,1].$$

Therefore,

$$x_1(t) = Tx_1(t) + \mu_1\varphi_1(t)$$
$$= \int_0^1 G(t,s) f(s, x_1(s)) g'(s)ds + \mu_1\varphi_1(t)$$

$$
\begin{aligned}
&> \int_0^1 G(t,s)[(\lambda_1+\varepsilon_1)x_1(s)-\widetilde{M}]g'(s)ds+\mu_1\varphi_1(t)\\
&= (\lambda_1+\varepsilon_1)\int_0^1 G(t,s)x_1(s)g'(s)ds-\int_0^1 G(t,s)\widetilde{M}g'(s)ds+\mu_1\varphi_1(t)\\
&= (\lambda_1+\varepsilon_1)Sx_1(t)-\int_0^1 G(t,s)\widetilde{M}g'(s)ds+\mu_1\varphi_1(t).
\end{aligned}
$$

Thus, we can see

$$
\begin{aligned}
[(\lambda_1+\varepsilon_1)S-I]x_1(t) &< \int_0^1 G(t,s)\widetilde{M}g'(s)ds-\mu_1\varphi_1(t)\\
&< \int_0^1 G(t,s)\widetilde{M}g'(s)ds.
\end{aligned}
$$

Since $\lambda_1+\varepsilon_1>\lambda_1$, $[(\lambda_1+\varepsilon_1)S-I]$ is a positive linear operator. Hence, it has the inverse operator $[(\lambda_1+\varepsilon_1)S-I]^{-1}$. By normality of cone P, we know

$$
\begin{aligned}
\overline{R}_1=\|x_1\| &< \|[(\lambda_1+\varepsilon_1)S-I]^{-1}\|\|\int_0^1 G(t,s)\widetilde{M}g'(s)ds\|\\
&< \widetilde{M}(\int_0^1 \varphi(s)g'(s)ds)\|[(\lambda_1+\varepsilon_1)S-I]^{-1}\|\\
&= \frac{\widetilde{M}\|[(\lambda_1+\varepsilon_1)S-I]^{-1}\|}{M}\\
&< \overline{R}_1.
\end{aligned}
$$

This is a contradiction, which implies that (3.4) hold. By Lemma 2.15, one can get that

$$
i(T,P\cap B_{\overline{R}_1},P)=0. \tag{3.5}
$$

Together with (3.3) and according to the regional additivity of the fixed point index, we have

$$
i(T,P\cap(B_{\overline{R}_1}\backslash\overline{B}_{r_1}),P)=0-1=-1. \tag{3.6}
$$

The proof is completed. □

Theorem 3.3. *Under the assumptions (H3) and (H4), BVP (1.1) admits at least one positive solution.*

Proof. For the sake of obtaining the desired result, we need only to prove that T has a positive fixed point in $P\cap(B_{\overline{R}_2}\backslash\overline{B}_{r_2})$.

First, the assumption (H3) implies that there exist $\varepsilon_2>0$ and $r_2>0$ such that

$$
f(t,x)>(\lambda_1+\varepsilon_2)x,\ t\in[0,1],\ x\in[0,r_2]. \tag{3.7}
$$

Now we claim that for $\mu\in[0,+\infty)$,

$$
x(t)-Tx(t)\neq\mu\varphi_1(t),\ \forall x\in P\cap\partial B_{r_2},\ t\in[0,1]. \tag{3.8}
$$

Hence, suppose on the contrary that there exist $x_2 \in P \cap \partial B_{r_2}$ and $\mu_2 \geq 0$ such that

$$x_2(t) - Tx_2(t) = \mu_2\varphi_1(t),\ t \in [0,1].$$

Without loss of generality, suppose $\mu_2 > 0$. Then,

$$x_2(t) = Tx_2(t) + \mu_2\varphi_1(t) \geq \mu_2\varphi_1(t).$$

Taking $\mu^* = sup\{\mu \mid x_2 \geq \mu\varphi_1,\ \mu > 0\}$, we have $0 < \mu_2 \leq \mu^* < +\infty$ and $x_2(t) \geq \mu^*\varphi_1(t)$. By the positivity of operator S, we know

$$\lambda_1 Sx_2 \geq \lambda_1 S(\mu^*\varphi_1) = \mu^*\varphi_1.$$

This together with (3.7) guarantees that

$$\begin{aligned}
x_2(t) &= Tx_2(t) + \mu_2\varphi_1(t)\\
&= \int_0^1 G(t,s)f(s,x_2(s))g'(s)ds + \mu_2\varphi_1(t)\\
&> (\lambda_1 + \varepsilon_2)\int_0^1 G(t,s)x_2(s)g'(s)ds + \mu_2\varphi_1(t)\\
&= (\lambda_1 + \varepsilon_2)Sx_2(t) + \mu_2\varphi_1(t)\\
&> (\mu^* + \mu_2)\varphi_1(t),
\end{aligned}$$

which is a contradiction with the definition of μ^*. Therefore, (3.8) is valid. According to Lemma 2.15, we have

$$i(T, P \cap \partial B_{r_2}, P) = 0. \tag{3.9}$$

Second, the assumption (H4) implies that there exist $\varepsilon_3 \in (0, \lambda_1)$ and $R_2 > 0$ such that

$$f(t,x) < (\lambda_1 - \varepsilon_3)x,\ t \in [0,1],\ |x| > R_2.$$

Let $\widetilde{N} = max_{t\in[0,1],x\in[0,R_2]}[f(t,x) + (\lambda_1 - \varepsilon_3)x]$, we can see

$$f(t,x) < (\lambda_1 - \varepsilon_3)x + \widetilde{N},\ t \in [0,1],\ \forall x \in [0,+\infty).$$

Choose $\overline{R}_2 > max\{r_2, R_2, \dfrac{\widetilde{N}[1-(\lambda_1-\varepsilon_3)\|S\|]^{-1}}{M}\}$. We claim that for $\mu \in (0,1]$,

$$x(t) \neq \mu Tx(t),\ \forall x \in P \cap \partial B_{\overline{R}_2},\ t \in [0,1]. \tag{3.10}$$

If it is not true, there exist $x_3 \in P \cap \partial B_{\overline{R}_2}$ and $\mu_3 \in (0,1]$ such that

$$x_3(t) = \mu_3 Tx_3(t),\ t \in [0,1].$$

Then,

$$x_3(t) = \mu_3 Tx_3(t)$$

$$
\begin{aligned}
&\leq \int_0^1 G(t,s)f(s,x_3(s))g'(s)ds \\
&< \int_0^1 G(t,s)[(\lambda_1-\varepsilon_3)x_3(s)+\widetilde{N}]g'(s)ds \\
&= (\lambda_1-\varepsilon_3)Sx_3(t)+\int_0^1 \widetilde{N}G(t,s)g'(s)ds,
\end{aligned}
$$

which means

$$
[I-(\lambda_1-\varepsilon_3)S]x_3(t) < \widetilde{N}\int_0^1 \varphi(s)g'(s)ds.
$$

Because of $0 < \|(\lambda_1-\varepsilon_3)S\| < 1$, $[I-(\lambda_1-\varepsilon_3)S]$ has the bounded and inverse operator $[I-(\lambda_1-\varepsilon_3)S]^{-1}$ and

$$
[I-(\lambda_1-\varepsilon_3)S]^{-1} = \sum_{n=0}^{\infty}[(\lambda_1-\varepsilon_3)S]^n.
$$

By normality of cone P, we have

$$
\begin{aligned}
\overline{R}_2 = \|x_3\| &< \|[I-(\lambda_1-\varepsilon_3)S]^{-1}\int_0^1 \varphi(s)\widetilde{N}g'(s)ds\| \\
&\leq \widetilde{N}\|[I-(\lambda_1-\varepsilon_3)S]^{-1}\|\int_0^1 \varphi(s)g'(s)ds \\
&\leq \frac{\widetilde{N}[1-(\lambda_1-\varepsilon_3)\|S\|]^{-1}}{M} \\
&< \overline{R}_2.
\end{aligned}
$$

This is a contraction, which means that (3.10) is valid. By Lemma 2.14,

$$
i(T, P\cap \partial B_{\overline{R}_2}, P) = 1. \tag{3.11}
$$

It together with (3.9) and the regional additivity of the fixed point index guarantees that

$$
i(T, P\cap (B_{\overline{R}_2}\backslash \overline{B}_{r_2}), P) = 1-0 = 1. \tag{3.12}
$$

The proof is completed. □

Now we are in a position to give the multiple solutions for BVP (1.1).

Theorem 3.4. *Assume that (H2) and (H3) hold. In addition, suppose that there exists $R > 0$ such that*

$$
f(t,x) < MR,\ \forall x\in[0,R],\ t\in[0,1].
$$

Then BVP (1.1) has at least two positive solutions.

Proof. For the sake of obtaining our conclusion, we first claim that for $\mu \in (0, 1]$,

$$x(t) \neq \mu T x(t),\ \forall x \in P \cap \partial B_R,\ t \in [0, 1]. \tag{3.13}$$

Suppose on the contrary that there exist $x_4 \in P \cap \partial B_R$ and $\mu_4 \in (0, 1]$ such that

$$x_4(t) = \mu_4 T x_4(t),\ t \in [0, 1]. \tag{3.14}$$

Then,

$$x_4(t) = \mu_4 T x_4(t) \leq \int_0^1 G(t, s) f(s, x_4(s)) g'(s) ds < \int_0^1 \varphi(s) M R g'(s) ds = R.$$

This is a contradiction, which means that (3.13) is valid. By Lemma 2.14, we have

$$i(T, P \cap B_R, P) = 1. \tag{3.15}$$

Next, similar to the process of proving (3.4) and (3.8), there exist $r \in (0, R)$ and $\widetilde{R} \geq max\{R, \overline{R}_1\}$ such that (3.5) and (3.9) hold.

Together with (3.15), Lemma 2.14 and Lemma 2.15, one can immediately obtain that

$$i(T, P \cap (B_{\widetilde{R}} \backslash \overline{B}_R), P) = i(T, P \cap B_{\widetilde{R}}, P) - i(T, P \cap B_R, P) = 0 - 1 = -1,$$

$$i(T, (P \cap B_R \backslash \overline{B}_r), P) = i(T, P \cap B_R, P) - i(T, P \cap B_r, P) = 1 - 0 = 1.$$

Namely, there exist $x_1 \in P \cap (B_{\widetilde{R}} \backslash \overline{B}_R)$ and $x_2 \in P \cap (B_R \backslash \overline{B}_r)$ satisfying $T x_i = x_i (i = 1, 2)$.

To sum up, Theorem 3.4 is proved. □

Now we are in a position to give at least three solutions for BVP (1.1).

Theorem 3.5. *Assume that there exist positive constants $a,\ b,\ c$ with $0 < a < b < c$ such that*

(H5) $f(t, x) < Ma, (t, x) \in [0, 1] \times [0, a]$;

(H6) $f(t, x) > Nb, (t, x) \in [\frac{1}{2}, 1] \times [b, c]$;

(H7) $f(t, x) \leq Mc, (t, x) \in [0, 1] \times [0, c]$.

Then BVP(1.1) has at least three nonnegative solutions x_1, x_2, x_3 satisfying $\|x_1\| < a$; $b < min_{t\in[\frac{1}{2},1]}|x_2(t)| < \|x_2\| \leq c$, $a < \|x_3\| \leq c$ and $min_{t\in[\frac{1}{2},1]}|x_3(t)| \leq b$.

Proof. We shall prove assumptions of Lemma 2.16 are valid.

Let $\theta(x) = min_{t\in[\frac{1}{2},1]}|x(t)|$. Hence, $\theta(x)$ is a nonnegative continuous concave functional on P.

First, we prove $T : P_c \to P_c$ is completely continuous. In fact, for $x \in \overline{P}_c$, from (H7) and Lemma 2.16, one can deduce that

$$\begin{aligned}
\|Tx\| &= max_{t\in[0,1]}\left|\int_0^1 G(t, s) f(s, x(s)) g'(s) ds\right| \\
&\leq \int_0^1 \varphi(s) f(s, x(s)) g'(s) ds \\
&\leq Mc \int_0^1 \varphi(s) g'(s) ds \\
&= c.
\end{aligned}$$

Thus, $T : \overline{P}_c \to \overline{P}_c$. In addition, by the continuity of G, f and g', we can conclude that $T : P_c \to P_c$ is completely continuous.

Let $x(t) = \dfrac{b+c}{2}$ for $t \in [0,1]$, it is not difficult to see

$$x(t) = \frac{b+c}{2} \in P(\theta, b, c),\ \theta(x) = \theta(\frac{b+c}{2}) > b.$$

This means $\{x \in P(\theta, b, c) |\ \theta(x) > b\} \neq \emptyset$. By condition (H6), for $x \in P(\theta, b, c)$, we have

$$\begin{aligned}
\theta(Tx) &= min_{t\in[\frac{1}{2},1]}|(Tx)(t)| \\
&= min_{t\in[\frac{1}{2},1]}|\int_0^1 G(t,s)f(s,x(s))g'(s)ds| \\
&> \int_0^1 (\frac{g(\frac{1}{2}) - g(0)}{g(1) - g(0)})^2 \varphi(s)Nbg'(s)ds \\
&= b,
\end{aligned}$$

which means that (A1) in Lemma (2.16) is valid.

By similar analysis, by (H5), one can see that

$$\|Tx\| < a,\ \forall x \in \overline{P}_a.$$

That is, (A2) in Lemma (2.16) holds. Taking $c = d$, (A3) is valid obviously.

To sum up, all assumptions of Lemma 2.16 are valid. Therefore, BVP (1.1) has at least three nontrivial solutions $x_1,\ x_2,\ x_3$ satisfying $\|x_1\| < a$; $b < min_{t\in[\frac{1}{2},1]}|x(t)| < \|x_2\| \leq c$, $a < \|x_3\| \leq c$ and $min_{t\in[\frac{1}{2},1]}|x(t)| \leq b$. □

4. The Ulam stability analysis

In this section, we shall give the criteria of Ulam stability for BVP (1.1). First, let us list the following assumption.

(H8) For all $x, y \in \mathbb{R}^+$, there exists a positive constant $0 < L < M$ such that

$$|f(t,x) - f(t,y)| \leq L|x - y|,\ t \in [0,1].$$

Next, for some $\epsilon > 0$, consider the following differential inequalities

$$|{}_0^C D_g^\alpha x(t) - f(t, x(t))| \leq \epsilon,\ t \in [0,1]. \tag{4.1}$$

Definition 4.1. *[17] BVP (1.1) is Ulam-Hyers stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $x \in E$ of the inequality 4.1, there exists a solution $\overline{x} \in E$ of BVP (1.1) with*

$$|x(t) - \overline{x}(t)| \leq C_f \epsilon,\ t \in [0,1].$$

Definition 4.2. *[17] BVP (1.1) is generalized Ulam-Hyers stable if there exist $\Phi_f \in C(\mathbb{R}^+, (0, +\infty))$, $\Phi_f(0) = 0$ such that for each solution $x \in E$ of the inequality 4.1, there exists a solution $\overline{x} \in E$ of BVP (1.1) with*

$$|x(t) - \overline{x}(t)| \leq \Phi_f(\epsilon),\ t \in [0,1].$$

Now, we are in a position to prove the main stable theorem of this section.

Theorem 4.3. *Under the assumptions (H1), (H4) and (H8), BVP (1.1) is Ulam-Hyers stable.*

Proof. Under the assumptions (H1) and (H4), by process similar to proving Theorems 3.2 and 3.3, BVP (1.1) has at least one positive solution.

Let $\overline{x} \in E$ be the solution of BVP (1.1) and $x \in E$ be a solution of

$$\begin{cases} |{}_0^C D_g^\alpha x(t) - f(t,x)| \leq \epsilon, \ 0 < t < 1; \\ x(0) = 0, \ {}_0^C D_g^1 x(0) = 0, \ {}_0^C D_g^\nu x(1) = \int_0^1 h(t) {}_0^C D_g^\nu x(t) g'(t) dt, \end{cases} \tag{4.2}$$

Then, by Lemma 2.10,

$$\overline{x}(t) = \int_0^1 G(t,s) f(s, \overline{x}(s)) g'(s) ds,$$

and

$$x(t) = \int_0^1 G(t,s)(f(s,x(s)) + \mathcal{E}(s)) g'(s) ds,$$

where $\mathcal{E}(t) = {}_0^C D_g^\alpha x(t) - f(t,x)$. By (4.2), it is easy to see $|\mathcal{E}(t)| < \epsilon$.

Then,

$$\begin{aligned} |x(t) - \overline{x}(t)| &= |x(t) - \int_0^1 G(t,s) f(s,\overline{x}(s)) g'(s) ds| \\ &\leq |x(t) - \int_0^1 G(t,s) f(s,x(s)) g'(s) ds| \\ &\quad + |\int_0^1 G(t,s) f(s,x(s)) g'(s) ds - \int_0^1 G(t,s) f(s,\overline{x}(s)) g'(s) ds| \\ &= |\int_0^1 G(t,s) \mathcal{E}(s) g'(s) ds| \\ &\quad + |\int_0^1 G(t,s) f(s,x(s)) g'(s) ds - \int_0^1 G(t,s) f(s,\overline{x}(s)) g'(s) ds| \\ &\leq \frac{1}{M}\epsilon + L(max_{t\in[0,1]} \int_0^1 G(t,s)|x(s) - \overline{x}(s)| g'(s) ds) \\ &= \frac{1}{M}\epsilon + \frac{L}{M}\|x - \overline{x}\|. \end{aligned}$$

Hence,

$$\|x - \overline{x}\| \leq \frac{1}{M - L}\epsilon.$$

Therefore, BVP (1.1) is Ulam-Hyers stable.

In addition, set $\Phi_f(z) = Lz$, then $\Phi_f(0) = 0$. By Definition 4.2, BVP (1.1) is generalized Ulam-Hyers stable.

The proof is completed. □

5. Examples

In this section, three illustrative examples are worked out to show the effectiveness of the obtained results.

Example 5.1. *Consider the following BVP*

$$\begin{cases} {}_0^C D_g^{2.95} + f(t,x) = 0,\ 0 < t < 1; \\ x(0) = 0,\ {}_0^C D_g^1 x(0) = 0,\ {}_0^C D_g^{1.05} x(1) = \int_0^1 h(t) {}_0^C D_g^{1.05} x(t) g'(t)dt, \end{cases} \tag{5.1}$$

where $g(t) = e^{\frac{t}{2}},\ h(t) = 1$ and

$$f(t,x) = \begin{cases} (1+t)(x)^{\frac{1}{2}},\ 0 < x \leq 1,\ 0 \leq t \leq 1; \\ (1+t)(x)^2,\ x > 1,\ 0 \leq t \leq 1. \end{cases}$$

Conclusion: BVP (5.1) has at least two positive solutions.
Proof. BVP (5.1) can be regarded as a BVP of the form (1.1).

By careful calculation and Lemma 2.11, one can obtain that

$$G(t,s) = G_0(t,s) + \frac{(e^{\frac{t}{2}} - 1)^2}{BL\Gamma(1.90)} \int_0^1 G_1(t,s) g'(t)dt,\ L = \frac{\Gamma(3)}{\Gamma(1.95)},$$

$$G_0(t,s) = \begin{cases} \frac{\Gamma(1.95)}{\Gamma(1.9)\Gamma(3)} (e^{\frac{t}{2}} - 1)^2 \frac{(e^{\frac{1}{2}} - e^{\frac{s}{2}})^{0.90}}{(e^{\frac{1}{2}} - 1)^{0.95}} - \frac{1}{\Gamma(2.95)} (e^{\frac{t}{2}} - e^{\frac{s}{2}})^{1.95}, & 0 \leq s \leq t \leq 1; \\ \frac{\Gamma(1.95)}{\Gamma(1.9)\Gamma(3)} (e^{\frac{t}{2}} - 1)^2 \frac{(e^{\frac{1}{2}} - e^{\frac{s}{2}})^{0.90}}{(e^{\frac{1}{2}} - 1)^{0.95}}, & 0 \leq t \leq s \leq 1. \end{cases}$$

$$G_1(t,s) = \frac{\Gamma(1.95)}{\Gamma(1.9)\Gamma(3)} \begin{cases} (e^{\frac{t}{2}} - 1)^{0.95} \frac{(e^{\frac{1}{2}} - e^{\frac{s}{2}})^{0.90}}{(e^{\frac{1}{2}} - 1)^{0.95}} - (e^{\frac{t}{2}} - e^{\frac{s}{2}})^{0.90}, & 0 \leq s \leq t \leq 1; \\ (e^{\frac{t}{2}} - 1)^{0.95} \frac{(e^{\frac{1}{2}} - e^{\frac{s}{2}})^{0.90}}{(e^{\frac{1}{2}} - 1)^{0.95}}, & 0 \leq t \leq s \leq 1. \end{cases}$$

By calculation, we get that

$$\lim_{x \to 0^+} \inf_{t \in [0,1]} \frac{f(t,x)}{x} = \lim_{x \to 0^+} \inf_{t \in [0,1]} x^{-\frac{1}{2}} = +\infty > \lambda_1,$$

$$\lim_{x \to +\infty} \inf_{t \in [0,1]} \frac{f(t,x)}{x} = \lim_{x \to +\infty} \inf_{t \in [0,1]} x = +\infty > \lambda_1.$$

In addition, notice that $M = (\int_0^1 \varphi(s) g'(s) ds)^{-1} \approx 11.42$ and choose $R = 5$.

Thus,

$$0 \leq f(t,x) \leq max_{t \in [0,1]} f(t,x) \leq 2R^2 < MR,\ t \in [0,1],\ x \in [0,R].$$

Consequently, all conditions in Theorem 3.5 hold, which means that our conclusion follows. □

Example 5.2. *Consider the following BVP*

$$\begin{cases} {}_0^C D_g^{2.95} + f(t,x) = 0, \ 0 < t < 1; \\ x(0) = 0, \ {}_0^C D_g^1 x(0) = 0, \ {}_0^C D_g^{1.05} x(1) = \int_0^1 h(t) {}_0^C D_g^{1.05} x(t) g'(t) dt, \end{cases} \tag{5.2}$$

where $g(t) = e^{\frac{t}{2}}, \ h(t) = 1$ and

$$f(t,x) = \begin{cases} \frac{1}{4}t + x^2, \ 0 < x \le 1, \ 0 < t < 1; \\ 500 + \frac{1}{4}t + x, \ x > 1, \ 0 < t < 1. \end{cases}$$

Conclusion: BVP (5.2) has at least three nonnegative solutions.

Proof. BVP (5.2) can be regarded as a BVP of the form (1.1). The function G, G_0, G_1 for BVP (5.2) is the same as that of BVP (5.1) in Example 5.1.

In addition, notice that

$$M = (\int_0^1 \varphi(s) g'(s) ds)^{-1} \approx 11.42,$$

$$N = (\int_0^1 (\frac{g(\frac{1}{2}) - g(0)}{g(1) - g(0)})^2 \varphi(s) g'(s) ds)^{-1} \approx 59.52.$$

Choosing $a = \frac{1}{10}$, $b = 1$, $c = 30$, we have

$$f(t,x) = \frac{1}{4}t + x^2 \le 0.26 < Ma \approx 1.142, \ \forall t \in [0,1], \ x \in [0, \frac{1}{10}];$$

$$f(t,x) = 300 + \frac{1}{4}t + x \ge 301.12 > Nb \approx 59.52, \ \forall t \in [\frac{1}{2}, 1], \ x \in [1, 30];$$

$$f(t,x) = 300 + \frac{1}{4}t + x \le 330.25 < Mc \approx 342.60, \ \forall t \in [0,1], \ x \in [0, 30].$$

By Theorem 3.5, BVP (5.3) has at least three nonnegative solutions $x_1, \ x_2, \ x_3$ with $\|x_1\| < \frac{1}{10}$; $1 < min_{t\in[\frac{1}{2},1]}|x(t)| < \|x_2\| \le 30$; $\frac{1}{10} < \|x_3\| \le 30$ and $min_{t\in[\frac{1}{2},1]}|x(t)| \le 1$.

Example 5.3. *Consider the following BVP*

$$\begin{cases} {}_0^C D_g^{2.95} + f(t,x) = 0, \ 0 < t < 1; \\ x(0) = 0, \ {}_0^C D_g^1 x(0) = 0, \ {}_0^C D_g^{\nu} x(1) = \int_0^1 h(t) {}_0^C D_g^{1.05} x(t) g'(t) dt, \end{cases} \tag{5.3}$$

where $g(t) = e^{\frac{t}{2}}, \ h(t) = 1$ and $f(t,x) = e^t ln(1 + x^2)$.

Conclusion: BVP (5.3) has at least one positive solutions and the solution of BVP (5.3) is Ulam-Hyers stable and generalized Ulam-Hyers stable.

Proof. BVP (5.3) can be regarded as a BVP of the form (1.1). The function G, G_0, G_1 for BVP (5.3) is the same as that of BVP (5.1) in Example 5.1. By calculation, we get that

$$\lim_{x\to 0^+} \sup_{t\in[0,1]} \frac{f(t,x)}{x} = \lim_{x\to 0^+} \sup_{t\in[0,1]} \frac{e^t ln(1 + x^2)}{x} = +\infty > \lambda_1,$$

$$\lim_{x\to+\infty}\sup_{t\in[0,1]}\frac{f(t,x)}{x}=\lim_{x\to+\infty}\sup_{t\in[0,1]}\frac{e^t ln(1+x^2)}{x}=0<\lambda_1,$$

which implies that (H1) and (H4) hold.

In addition, notice that for all $x, y \in \mathbb{R}^+$,

$$|f(t,x)-f(t,y)|=e^t|ln(1+x^2)-ln(1+y^2)|\le e|x-y|.$$

This means that (H8) are satisfied if we set $L = e$.

Consequently, by Theorem 4.3, BVP (5.3) has at least one positive solutions and the solution of BVP (5.3) is Ulam-Hyers stable and generalized Ulam-Hyers stable.

The proof is completed. □

6. Conclusions

The existence of solutions is of the fundamental problems for FDEs. This work studies the existence of positive solutions and multiple positive solutions for a class of FBVPs with generalized Caputo derivatives. Taking full advantage of the properties of Green's function, a suitable cone is established. The positive solutions and multiple positive solutions are obtained by means of the first eigenvalue of corresponding linear operator and the cone expansion and compression fixed point theorem. At the same time, by using Leggett-Williams theorem, we obtain that BVP (1.1) has at least three nonnegative solutions. Moreover, Ulam-Hyers stability and generalized Ulam-Hyers stability are also studied under some suitable assumptions.

For our subsequent work, the following issues will continue to be focused on:
(i) The systems studied on this topic will be more and more extensive and complicated. Therefore, it is valuable to investigate impulsive FDEs with generalized derivatives or hybrid FDEs with delay.
(ii) As an important component of technology and mathematical control theory, controllability has already gained considerable attention. Hence, the controllability for fractional differential system with generalized derivatives may be an interesting issue.
(iii) With the development of the theoretical study on FDEs, application area of FDEs with generalized derivatives in reality needs to be investigated in depth.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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