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Research article

Switched coupled system of nonlinear impulsive Langevin equations with mixed derivatives

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Abstract: In this paper, we consider switched coupled system of nonlinear impulsive Langevin equations with mixed derivatives. Some sufficient conditions are constructed to observe the existence, uniqueness and generalized Ulam-Hyers-Rassias stability of our proposed model, with the help of generalized Diaz-Margolis's fixed point approach, over generalized complete metric space. We give an example which supports our main result.

Keywords: Langevin equation; Caputo derivative; impulse; Ulam-Hyers-Rassias stability
Mathematics Subject Classification: 26A33, 34A08, 34B27

1. Introduction

At Wisconsin university, Ulam raised a question about the stability of functional equations in the year 1940. The question of Ulam was: Under what conditions does there exist an additive mapping near an approximately additive mapping [28]? In 1941, Hyers was the first mathematician who gave partial answer to Ulam's question [11], over Banach space. Afterwards, stability of such form is known as Ulam-Hyers stability. In 1978, Rassias [18], provided a remarkable generalization of the Ulam-Hyers stability of mappings by considering variables. For more information about the topic, we refer the reader to [21, 25, 26, 29, 39, 40, 42].

Fractional Langevin differential equations have been one of the important subject in physics, chemistry and electrical engineering. The Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments. For instance, Brownian motion is well described by the Langevin equation when the random fluctuation force is assumed to be white noise. In case the random fluctuation force is not white noise, the motion of the particle is described by the generalized Langevin equation. For systems in complex media, ordinary Langevin equation does not provide the correct description of the dynamics. Various generalizations of Langevin equations have been proposed to describe dynamical processes in a fractal medium. One such generalization is the generalized Langevin equation which incorporates the fractal and memory properties with a dissipative memory kernel into the Langevin equation. Another possible extension requires the replacement of ordinary derivative by a fractional derivative in the Langevin equation to give the fractional Langevin equation. For more details, see [2, 10, 15, 16, 19, 33, 34, 35].

Fractional order differential equations are the generalizations of the classical integer order differential equations. Fractional calculus has become a speedily developing area and its applications can be found in diverse fields ranging from physical sciences, porous media, electrochemistry, economics, electromagnetics, medicine and engineering to biological sciences. Progressively, fractional differential equations play a very important role in fields such as thermodynamics, statistical physics, nonlinear oscillation of earthquakes, viscoelasticity, defence, optics, control, signal processing, electrical circuits, astronomy etc. There are some outstanding articles which provide the main theoretical tools for the qualitative analysis of this research field, and at the same time, shows the interconnection as well as the distinction between integral models, classical and fractional differential equations, see [1, 12, 17, 20, 23, 27, 32].

Impulsive fractional differential equations are used to describe both physical, social sciences and many dynamical systems such as evolution processes pharmacotherpy. There are two types of impulsive fractional differential equations the first one is instantaneous impulsive fractional differential equations while the other one is non-instantaneous impulsive fractional differential equations. In last few decades, the theory of impulsive fractional differential equations are well utilized in medicine, mechanical engineering, ecology, biology and astronomy etc. see [3, 8, 13, 24, 30, 36, 38, 41].

Recently, many mathematicians received a considerable attention to the existence, uniqueness and different types of Hyres-Ulam stability of the solutions of nonlinear implicit fractional differential equations with Caputo fractional derivative, see [5, 7, 22, 26].

Wang et al. [31], studied generalized Ulam-Hyers-Rassias stability of the following fractional differential equation:

$$\begin{cases} {}^{c}\mathcal{D}^{\nu}_{0,w}z(w) = f(w, z(w)), \quad w \in (w_k, s_k], \quad k = 0, 1, \dots, m, \quad 0 < \nu < 1, \\ z(w) = g_k(w, z(w)), \quad w \in (s_{k-1}, w_k], \quad k = 1, 2, \dots, m. \end{cases}$$

Zada et al. [37], studied existence and uniqueness of solutions by using Diaz Margolis's fixed point theorem and presented different types of Ulam-Hyers stability for a class of nonlinear implicit fractional differential equation with non-instantaneous integral impulses and nonlinear integral boundary conditions:

$$\begin{cases} {}^{c}\mathcal{D}^{\nu}_{0,w}z(w) = f(w, z(w), {}^{c}\mathcal{D}^{\nu}_{0,w}z(w)), \ w \in (w_k, s_k], \ k = 0, 1, \dots, m, \ 0 < \nu < 1, \ w \in (0, 1], \\ z(w) = I^{\nu}_{s_{k-1},w_k}(\xi_k(w, z(w))), \ w \in (s_{k-1}, w_k], k = 1, 2, \dots, m, \\ z(0) = \dfrac{1}{\Gamma\nu}\displaystyle\int_0^T (T-s)^{\nu-1}\eta(s, z(s))ds. \end{cases}$$

In recent years, many researchers paid much attention to the coupled system of fractional differential equations due to its applications in different fields [3, 6, 32].

Ali et al. [4], studied the existence, uniqueness of solutions by using using the classical fixed point theorems such as Banach contraction principle and Leray-Schauder of cone type and presented various kinds of Ulam stability including Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability of the solutions to a nonlinear coupled systems of implicit fractional differential equations involving Caputo derivative of the form:

$$\begin{cases} {}^{c}\mathcal{D}^{\nu}u(w) - f(w, v(w), {}^{c}\mathcal{D}^{\nu}u(w)) = 0, & \nu \in (2,3],\ w \in J, \\ {}^{c}\mathcal{D}^{\mu}v(w) - f(w, u(w), {}^{c}\mathcal{D}^{\nu}v(w)) = 0, & \mu \in (2,3],\ w \in J, \\ \acute{u}(w)|_{w=0} = \acute{\acute{u}}(w)|_{w=0}, & u(w)|_{w=1} = \lambda u(\eta),\ \lambda, \eta \in (0,1), \\ \acute{v}(w)|_{w=0} = \acute{\acute{v}}(w)|_{w=0}, & v(w)|_{w=1} = \lambda v(\eta),\ \lambda, \eta \in (0,1) \end{cases}$$

In this paper, we study switched coupled system of nonlinear impulsive Langevin equations with mixed derivatives of the form:

$$\begin{cases} \begin{cases} {}^{c}\mathcal{D}^{\nu}_{0,w}(\mathcal{D} + \lambda_1)u(w) = f_1(w, v(w), u(w)), & w \in (w_k, s_k], \quad k = 0, 1, \ldots, m, \\ u(w) = g_k(w, u(w)), & w \in (s_{k-1}, w_k], \quad k = 1, 2, \ldots, m, \\ u(0) = u_0, \quad u(T) = \int_0^{\eta_1} \frac{1}{\Gamma p_1}(\eta_1 - s)^{p1-1}u(s)ds, & 0 < \eta_1 < T, \end{cases} \\ \begin{cases} {}^{c}\mathcal{D}^{\mu}_{0,w}(\mathcal{D} + \lambda_2)v(w) = f_2(w, u(w), v(w)), & w \in (w_k, s_k], \quad k = 0, 1, \ldots, m, \\ v(w) = g_k(w, v(w)), & w \in (s_{k-1}, w_k], \quad k = 1, 2, \ldots, m, \\ v(0) = v_0, \quad v(T) = \int_0^{\eta_2} \frac{1}{\Gamma p_2}(\eta_2 - s)^{p_2-1}v(s)ds, & 0 < \eta_2 < T, \end{cases} \end{cases} \tag{1.1}$$

where ${}^{c}\mathcal{D}^{\nu}_{0,w}$ and ${}^{c}\mathcal{D}^{\mu}_{0,w}$ represents classical Caputo derivative [12], of order ν and μ respectively with the lower bound zero, $0 = w_0 < s_0 < w_1 < s_1 < \cdots < w_m < s_m = T$, T is the pre–fixed number and $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\},\ 0 < \nu < 1,\ p_1, p_2 > 0,\ u_0, v_0$ are constants, $f_1, f_2 : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $g_k : [s_{k-1}, w_k] \times \mathbb{R} \to \mathbb{R}$ is continuous for all $k = 1, 2, \ldots, m$.

In the second section of this paper, we create a uniform framework to originate appropriate formula of solutions for our proposed model. In section 3, we implement the concept of generalized Ulam-Hyers-Rassias stability of Eq (1.1). Finally, we give an example which supports our main result.

2. Preliminaries

Let $J = [0,T]$ and $C(J, \mathbb{R})$ be the space of all continuous functions from J to $\mathbb{R}$, and the piecewise continuous function space $PC(J, \mathbb{R}) = \{z : J \to \mathbb{R} : z \in C((w_k, w_{k-1}], \mathbb{R}), k = 0, \ldots, m$ and there exist $z(w_k^-)$ and $z(w_k^+),\ k = 1, 2, \ldots, m$ with $z(w_k^-) = z(w_k)\}$.

In the current section, we create a uniform framework to originate appropriate formula for the solution of impulsive fractional differential equation of the form:

$$\begin{cases} {}^{c}\mathcal{D}^{\nu}_{0,w}(\mathcal{D} + \lambda_1)u(w) = f_1(w), & w \in (w_k, s_k], \quad k = 0, 1, \ldots, m, \\ u(w) = g_k(w), & w \in (s_{k-1}, w_k], \quad k = 1, 2, \ldots, m, \\ u(0) = u_0, \quad u(T) = \int_0^{\eta_1} \frac{1}{\Gamma p_1}(\eta_1 - s)^{p_1-1}u(s)ds, & 0 < \eta_1 < T, \end{cases} \tag{2.1}$$

We recall some definitions of fractional calculus from [12] as follows:

Definition 2.1. The fractional integral of order ν from 0 to w for the function f is defined by

$$I_{0,w}^{\nu}f(w) = \frac{1}{\Gamma(\nu)}\int_0^w f(s)(w-s)^{\nu-1}ds, \quad w>0,\ \nu>0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. The Riemman-Liouville fractional derivative of fractional order ν from 0 to w for a function f can be written as

$${}^{L}\mathcal{D}_{0,w}^{\nu}f(w) = \frac{1}{\Gamma(n-\nu)}\frac{d^n}{dt^n}\int_0^w \frac{f(s)}{(w-s)^{\nu+1-n}}ds, \quad w>0,\ n-1<\nu<n,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.3. The Caputo derivative of fractional order ν from 0 to w for a function f can be defined as

$${}^{c}\mathcal{D}_{0,w}^{\nu}f(w) = \frac{1}{\Gamma(n-\nu)}\int_0^w (w-s)^{n-\nu-1}f^{(n)}(s)ds, \quad \text{where } n=[\nu]+1.$$

Definition 2.4. The general form of classical Caputo derivative of order ν of a function f can be given as

$${}^{c}\mathcal{D}_{0,w}^{\nu} = {}^{L}\mathcal{D}_{0,w}^{\nu}\Big(f(w) - \sum_{k=0}^{n-1}\frac{w^k}{k!}f^{(k)}(0)\Big), \quad w>0,\ n-1<\nu<n.$$

Remark 2.5. **(i)** If $f(\cdot)\in C^m([0,\infty),\mathbb{R})$, then

$${}^{L}\mathcal{D}_{0,w}^{\nu}f(w) = \frac{1}{\Gamma(m-\nu)}\int_0^w \frac{f^{(m)}(s)}{(w-s)^{\nu+1-m}}ds = I_{0,w}^{m-\nu}f^{(m)}(w), \quad w>0,\ m-1<\nu<m.$$

(ii) In Definition 2.4, the integrable function f can be discontinuous function. This fact can support us to consider impulsive fractional problems in the sequel.

Definition 2.6. The Hilfer fractional derivative of order $0<\alpha<1$ and $0\le\gamma\le1$ of function $f(x)$ is

$$D^{\alpha,\gamma}f(x) = (I^{\gamma(1-\alpha)}D(I^{(1-\gamma)(1-\alpha)}(f))(x).$$

The Hilfer fractional derivative is used as an interpolator between the Riemman-Liouville and Caputo derivative.

Remark 2.7. **(a)** Operator $D^{\alpha,\gamma}$ also can be written as

$$D^{\alpha,\gamma}f(x) = (I^{\gamma(1-\alpha)}D(I^{(1-\gamma)(1-\alpha)}f)(x)) = I^{\gamma(1-\alpha)}D^{\eta}f(x), \eta=\alpha+\gamma-\alpha\gamma.$$

(b) If $\gamma=0$, then $D^{\alpha,\gamma}=D^{\alpha,0}$ is called the Riemman-Liouville fractional derivative.

(c) If $\gamma=1$, then $D^{\alpha,\gamma}=I^{1-\alpha}D$ is called the Caputo fractional derivative.

Lemma 2.8. *[12] The fractional differential equation ${}^{c}D^{\alpha}f(x)=0$ with $\alpha>0$, involving Caputo differential operator ${}^{c}D^{\alpha}$ have a solution in the following form:*

$$f(x) = c_0 + c_1x + c_2x^2 + \cdots + c_{m-1}x^{m-1},$$

where $\quad c_k\in\mathbb{R}, \quad k=0,1,\ldots,m-1 \quad$ *and* $\quad m=[\alpha]+1.$

Lemma 2.9. *[17] Let $\alpha > 0$ and $\gamma > 0$, $f \in L^1([a,b])$.*

Then $I^{\alpha} I^{\gamma} f(x) = I^{\alpha+\gamma} f(x)$, ${}^{c}D^{\alpha}_{0,x}({}^{c}D^{\gamma}_{0,x} f(x)) = {}^{c}D^{\alpha+\gamma}_{0,x} f(x)$ *and* $I^{\alpha} D^{\alpha}_{0,x} f(x) = f(x), x \in [a,b]$.

Lemma 2.10. *[24] The function $u \in PC(J,\mathbb{R})$ is a solution of* (2.1) *if and only if*

$$u(w) = \begin{cases} \begin{cases} \int_0^w e^{-\lambda_1(w-s)} I^{\nu} f_1(s)ds + \frac{A_{11}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^{\nu} f_1(s)ds - A_{11} \int_0^T e^{-\lambda_1(T-s)} I^{\nu} f_1(s)ds \\ + \Big(A_{11}(\eta_1^{p_1} E_{(1,p_1+1)}(aw) - e^{\lambda_1 T}) + e^{\lambda_1 T} \Big) u_0, \quad w \in (0, s_0], \end{cases} \\ \begin{cases} g_k(w), \qquad w \in (s_{k-1}, w_k], \quad k = 1,2,\dots,m, \end{cases} \\ \begin{cases} \int_0^w e^{-\lambda_1(w-s)} I^{\nu} f_1(s)ds + \frac{M_{k_1}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^{\nu} f_1(s)ds - M_{k_1} \int_0^T e^{-\lambda_1(T-s)} I^{\nu} f_1(s)ds \\ + N_{k_1} \int_0^{w_k} e^{-\lambda_1(w_k - s)} I^{\nu} f_1(s)ds + N_{k_1} g_k(w_k), \quad w \in (w_k, s_k], \quad k = 0,1,\dots,m, \end{cases} \end{cases}$$

where

$$\begin{aligned}
A_{11} &= \frac{\lambda\Gamma(p_1+1)}{(1-e^{-\lambda T})\Gamma(p_1+1) - {\eta_1}^{p_1} + \Gamma(p_1+1){\eta_1}^{p_1} E_{(1,p_1+1)}(aw)}, \\
B_{k_1} &= \frac{\lambda\Gamma(p_1+1)\Big({\eta_1}^{p_1} E_{(1,p_1+1)}(aw) - e^{-\lambda T}\Big)}{\delta_{k_1}}, \\
A_{k_1} &= \frac{\delta_{k_1}\lambda\Gamma(p_1+1) - \Gamma(p_1+1)(1-e^{\lambda w_k})\Big(\lambda\Gamma(p_1+1)({\eta_1}^{p_1} E_{(1,p_1+1)}(aw) - e^{-\lambda T})\Big)}{\delta_{k_1}\Big((1-e^{-\lambda T})\Gamma(p_1+1) - {\eta_1}^{p_1} + \Gamma(p_1+1){\eta_1}^{p_1} E_{(1,p_1+1)}(aw)\Big)}, \\
M_{k_1} &= \frac{A_{k_1}(1-e^{-\lambda w})}{\lambda} - \frac{\Gamma(p_1+1)e^{-\lambda w}(1-e^{-\lambda w_k})}{\delta_{k_1}}, \\
N_{k_1} &= \frac{B_{k_1}(1-e^{-\lambda w})}{\lambda} - \frac{(1-e^{-\lambda T})\Gamma(p_1+1) - {\eta_1}^{p_1} + \Gamma(p_1+1){\eta_1}^{p_1} E_{(1,p_1+1)}(aw)}{\delta_{k_1} e^{\lambda w}}, \\
\delta_{k_1} &= 2\Gamma(p_1+1)\Big(e^{-\lambda w_k} - e^{-\lambda(w_k+T)} + {\eta_1}^{p_1} E_{(1,p_1+1)}(aw)e^{-\lambda w_k}\Big) - {\eta_1}^{p_1} e^{-\lambda w_k} - \Gamma(p_1+1)E_{(1,p_1+1)}(aw).
\end{aligned}$$

In view of Lemma 2.10 the solution form of proposed system (1.1) is given by

$$\begin{cases} u(w) = \begin{cases} \int_0^w e^{-\lambda_1(w-s)} I^\nu f_1(s, v(s), u(s))ds + \frac{A_{11}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^\nu f_1(s, v(s), u(s))ds \\ - A_{11} \int_0^T e^{-\lambda_1(T-s)} I^\nu f_1(s, v(s), u(s))ds + \Big(A_{11}(\eta_1^{p_1} E_{(1,p_1+1)}(aw) - e^{\lambda_1 T}) + e^{\lambda_1 T}\Big)u_0, \ w \in (0, s_0], \\ g_k(w, u(w)), \qquad w \in (s_{k-1}, w_k], \quad k = 1, 2, \dots, m, \\ \int_0^w e^{-\lambda_1(w-s)} I^\nu f_1(s, v(s), u(s))ds + \frac{M_{k_1}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^\nu f_1(s, v(s), u(s))ds \\ - M_{k_1} \int_0^T e^{-\lambda_1(T-s)} I^\nu f_1(s, v(s), u(s))ds + N_{k_1} \int_0^{w_k} e^{-\lambda_1(w_k - s)} I^\nu f_1(s, v(s), u(s))ds \\ + N_{k_1} g_k(w_k, u(w_k)), \quad w \in (w_k, s_k], \quad k = 0, 1, \dots, m, \end{cases} \\ v(w) = \begin{cases} \int_0^w e^{-\lambda_2(w-s)} I^\mu f_2(s, u(s), v(s))ds + \frac{A_{22}}{\Gamma(p_2+1)} \int_0^{\eta_2} (\eta_2 - s)^{p_2} e^{-\lambda_2(\eta_2 - s)} I^\mu f_2(s, u(s), v(s))ds \\ - A_{22} \int_0^T e^{-\lambda_2(T-s)} I^\mu f_2(s, u(s), v(s))ds + \Big(A_{22}(\eta_2^{p_2} E_{(1,p_2+1)}(aw) - e^{\lambda_2 T}) + e^{\lambda_2 T}\Big)v_0, \ w \in (0, s_0], \\ g_k(w, v(w)), \qquad w \in (s_{k-1}, w_k], \quad k = 1, 2, \dots, m, \\ \int_0^w e^{-\lambda_2(w-s)} I^\mu f_2(s, u(s), v(s))ds + \frac{M_{k_2}}{\Gamma(p_2+1)} \int_0^{\eta_2} (\eta_2 - s)^{p_2} e^{-\lambda_2(\eta_2 - s)} I^\mu f_2(s, u(s), v(s))ds \\ - M_{k_2} \int_0^T e^{-\lambda_2(T-s)} I^\mu f_2(s, u(s), v(s))ds + N_{k_2} \int_0^{w_k} e^{-\lambda_2(w_k - s)} I^\mu f_2(s, u(s), v(s))ds \\ + N_{k_2} g_k(w_k, v(w_k)), \quad w \in (w_k, s_k], \quad k = 0, 1, \dots, m, \end{cases} \end{cases} \tag{2.2}$$

3. Generalized Ulam-Hyers-Rassias stability concept

By the ideas of stability in [25, 29], we can generate a generalized Ulam-Hyers-Rassias stability concept for Eq (1.1).

$$\begin{cases} \Big|{}^c\mathcal{D}^\nu_{0,w}(\mathcal{D} + \lambda_1)u(w) - f_1(w, v(w), u(w))\Big| \le \varphi_u(w), \quad w \in (w_k, s_k], \quad k = 0, 1, \dots, m, \ 0 < \nu < 1, \\ \Big|u(w) - N_{k_1} g_k(w, u(w))\Big| \le \psi, \quad w \in (s_{k-1}, w_k], \ k = 1, 2, \dots, m. \\ \Big|{}^c\mathcal{D}^\mu_{0,w}(\mathcal{D} + \lambda_2)v(w) - f_2(w, u(w), v(w))\Big| \le \varphi_v(w), \quad w \in (w_k, s_k], \quad k = 0, 1, \dots, m, \ 0 < \mu < 1, \\ \Big|v(w) - N_{k_2} g_k(w, v(w))\Big| \le \psi, \quad w \in (s_{k-1}, w_k], \ k = 1, 2, \dots, m. \end{cases} \tag{3.1}$$

Definition 3.1. Equation (1.1) is generalized Ulam-Hyers-Rassias stable with respect to $(\varphi_u, \varphi_v, \psi)$ if there exists $C_u, C_v > 0$ such that for each solution $(u, v) \in PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$ of inequality (3.1) there is a solution $(u_0, v_0) \in PC(J, \mathbb{R}) \times PC(J, \mathbb{R})$ of Eq (1.1) with

$$\Big|(u, v)(w) - (u_0, v_0)(w)\Big| \le C_\varphi(\varphi_u(w) + \varphi_v(w)) + L_{gk}(C_u + C_v)\psi, \quad w \in J.$$

Remark 3.2. A function $u, v \in PC(J, \mathbb{R})$ is a solution of the inequality (3.1) if and only if there is $G \in PC(J, \mathbb{R})$ and a sequence $G_k,\ k = 1, 2, \dots, m$ (which depends on z) such that

(i) $|G(w)| \leq \varphi(w),\ w \in J$ and $|G_k| \leq \psi, \quad k = 1, 2, \ldots, m,$

(ii) ${}^{c}\mathcal{D}^{\nu}_{0,w}(\mathcal{D} + \lambda_1)u(w) = f_1(w, v(w), u(w)) + G(w),\ w \in (w_k, s_k], \quad k = 0, 1, \ldots, m,$

(iii) ${}^{c}\mathcal{D}^{\mu}_{0,w}(\mathcal{D} + \lambda_2)v(w) = f_2(w, u(w), v(w)) + G(w),\ w \in (w_k, s_k], \quad k = 0, 1, \ldots, m,$

(iv) $u(w) = N_{k_1} g_k(w, u(w)) + G_k,\ w \in (s_{k-1}, w_k], \quad k = 1, 2,\ \ldots, m.$

Remark 3.3. **(1)** If $u \in PC(J, \mathbb{R})$ is a solution of the inequality (3.1) then u is a solution of the following integral inequality,

$$\begin{cases} \begin{cases} \Big|u(w) - \int_0^w e^{-\lambda_1(w-s)} I^{\nu} f_1(w, v(w), u(w))ds - \frac{A_{11}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^{\nu} f_1(s, v(s), u(s))ds \\ \quad + A_{11} \int_0^T e^{-\lambda_1(T-s)} I^{\nu} f_1(s, v(s), u(s))ds - \Big(A_{11}({\eta_1}^{p} E_{(1,p_1+1)}(aw) - e^{\lambda_1 T}) + e^{\lambda_1 T}\Big)u_0\Big|, \\ \leq \int_0^w e^{-\lambda_1(w-s)} I^{\nu} \varphi_u(s)ds - \frac{A_{11}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^{\nu} \varphi_u(s)ds \\ \quad + A_{11} \int_0^T e^{-\lambda_1(T-s)} I^{\nu} \varphi_u(s)ds, \qquad w \in (0, s_0]; \end{cases} \\ \Big\{\Big|u(w) - N_{k_1} g_k(w, u(w))\Big| \leq \psi, \quad w \in (s_{k-1}, w_k], \quad k = 1, 2, \ldots, m; \\ \begin{cases} \Big|u(w) - \int_0^w e^{-\lambda_1(w-s)} I^{\nu} f_1(s, v(s), u(s))ds - \frac{M_{k_1}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^{\nu} f_1(s, v(s), u(s))ds \\ \quad + M_{k_1} \int_0^T e^{-\lambda_1(T-s)} I^{\nu} f_1(s, v(s), u(s))ds - N_{k_1} \int_0^{w_k} e^{-\lambda_1(w_k - s)} I^{\nu} f_1(s, v(s), u(s))ds - N_{k_1} g_k(w_k, u(w_k))\Big| \\ \leq \int_0^w e^{-\lambda_1(w-s)} I^{\nu} \varphi_u(s)ds + \frac{M_{k_1}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^{\nu} \varphi_u(s)ds \\ \quad + M_{k_1} \int_0^T e^{-\lambda_1(T-s)} I^{\nu} \varphi_u(s)ds + N_{k_1} \int_0^{w_k} e^{-\lambda_1(w_k - s)} I^{\nu} \varphi_u(s)ds + \psi,\ w \in (w_k, s_k],\ k = 0, 1, \ldots, m. \end{cases} \end{cases} \tag{3.2}$$

In fact by Remark 3.2, we get

$$\begin{cases} {}^{c}\mathcal{D}^{\nu}_{0,w}(\mathcal{D} + \lambda_1)u(w) = f_1(w, v(w), u(w)) + G(w), \quad w \in (w_k, s_k],\ k = 0, 1, \ldots, m, \quad 0 < \nu < 1, \\ u(w) = N_{k_1} g_k(w, u(w)) + G_k, \quad w \in (s_{k-1}, w_k], \quad k = 1, 2, \ldots, m. \end{cases} \tag{3.3}$$

Clearly, the solution of Eq (3.5) is given by

$$u(w)=\begin{cases}\begin{cases}\int_0^w e^{-\lambda_1(w-s)}I^{\nu}(f_1(s,v(s),u(s))+G(s))ds\\ +\dfrac{A_{11}}{\Gamma(p_1+1)}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^{\nu}(f_1(s,v(s),u(s))+G(s))ds\\ -A_{11}\int_0^T e^{-\lambda_1(T-s)}I^{\nu}(f_1(s,v(s),u(s))+G(s))ds\\ +\Big(A_{11}(\eta_1{}^{p_1}E_{(1,p_1+1)}(aw)-e^{\lambda_1 T})+e^{\lambda_1 T}\Big)u_0, \quad w\in(0,s_0];\end{cases}\\ \Big\{N_{k_1}g_k(w,u(w)), \quad w\in(s_{k-1},w_k], \quad k=1,2,\dots,m;\\ \begin{cases}\int_0^w e^{-\lambda_1(w-s)}I^{\nu}(f_1(s,v(s),u(s))+G(s))ds\\ +\dfrac{M_{k_1}}{\Gamma(p_1+1)}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^{\nu}(f_1(s,v(s),u(s))+G(s))ds\\ -M_{k_1}\int_0^T e^{-\lambda_1(T-s)}I^{\nu}(f_1(s,v(s),u(s))+G(s))ds\\ +N_{k_1}\int_0^{w_k} e^{-\lambda_1(w_k-s)}I^{\nu}(f_1(s,v(s),u(s))+G(s))ds\\ +N_{k_1}g_k(w_k,u(w_k))+G_k, \quad w\in(w_k,s_k], \quad k=0,1,\dots,m.\end{cases}\end{cases}$$

For $w\in(w_k,s_k], \quad k=0,1,\dots,m$, we get

$$\begin{aligned}&\Big|u(w)-\int_0^w e^{-\lambda_1(w-s)}I^{\nu}f_1(s,v(s),u(s))ds-\frac{M_{k_1}}{\Gamma(p_1+1)}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^{\nu}f_1(s,v(s),u(s))ds\\ &\quad +M_{k_1}\int_0^T e^{-\lambda_1(T-s)}I^{\nu}f_1(s,v(s),u(s))ds-N_{k_1}\int_0^{w_k}e^{-\lambda_1(w_k-s)}I^{\nu}f_1(s,v(s),u(s))ds-N_{k_1}g_k(w_k,u(w_k))\Big|\\ &\le\Big|\int_0^w e^{-\lambda_1(w-s)}I^{\nu}G(s)ds\Big|+\Big|\frac{M_{k_1}}{\Gamma(p_1+1)}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^{\nu}G(s)ds\Big|\\ &\quad +\Big|M_{k_1}\int_0^T e^{-\lambda_1(T-s)}I^{\nu}G(s)ds\Big|+\Big|N_{k_1}\int_0^{w_k}e^{-\lambda_1(w_k-s)}I^{\nu}G(s)ds\Big|+\Big|G_k\Big|\\ &\le\int_0^w e^{-\lambda_1(w-s)}I^{\nu}\varphi_u(s)ds+\frac{M_{k_1}}{\Gamma(p_1+1)}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^{\nu}\varphi_u(s)ds\\ &\quad +M_{k_1}\int_0^T e^{-\lambda_1(T-s)}I^{\nu}\varphi(s)ds+N_{k_1}\int_0^{w_k}e^{-\lambda_1(w_k-s)}I^{\nu}\varphi_u(s)ds+\psi.\end{aligned}$$

Proceeding the above, we derive that

$$\Big|u(w)-N_{k_1}g_k(w,u(w))\Big|\le|G_k|\le\psi, \quad w\in(s_{k-1},w_k], \quad k=1,2,\dots,m,$$

and

$$\Big|u(w) - \int_0^w e^{-\lambda_1(w-s)} I^\nu f_1(s, v(s), u(s))ds - \frac{A_{11}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^\nu f_1(s, v(s), u(s))ds$$
$$+A_{11} \int_0^T e^{-\lambda_1(T-s)} I^\nu f_1(s, v(s), u(s))ds - \Big(A_{11}(\eta_1{}^{p_1} E_{(1,p_1+1)}(aw) - e^{\lambda_1 T}) + e^{\lambda_1 T}\Big) u_0\Big|$$
$$\leq \Big| \int_0^w e^{-\lambda_1(w-s)} I^\nu G(s)ds\Big| + \Big|\frac{A_{11}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^\nu G(s)ds\Big|$$
$$+\Big|A_{11} \int_0^T e^{-\lambda_1(T-s)} I^\nu G(s)ds\Big|$$
$$\leq \int_0^w e^{-\lambda_1(w-s)} I^\nu \varphi_u(s)ds + \frac{A_{11}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^\nu \varphi_u(s)ds$$
$$+A_{11} \int_0^T e^{-\lambda_1(T-s)} I^\nu \varphi_u(s)ds, \; w \in (0, s_0].$$

Similarly

(2) If $v \in PC(J, \mathbb{R})$ is a solution of the inequality (3.1) then v is a solution of the following integral inequality,

$$\begin{cases} \begin{cases} \Big|v(w) - \int_0^w e^{-\lambda_2(w-s)} I^\mu f_2(s, u(s), v(s)ds - \frac{A_{22}}{\Gamma(p_2+1)} \int_0^{\eta_2} (\eta_2 - s)^{p_2} e^{-\lambda_2(\eta_2 - s)} I^\mu f_2(s, u(s), v(s)ds \\ \quad + A_{22} \int_0^T e^{-\lambda_2(T-s)} I^\mu f_2(s, u(s), v(s)ds - \Big(A_{22}(\eta_2{}^{p_2} E_{(1,p_2+1)}(aw) - e^{\lambda_2 T}) + e^{\lambda_2 T}\Big) v_0\Big|, \\ \leq \int_0^w e^{-\lambda_2(w-s)} I^\mu \varphi_v(s)ds - \frac{A_{22}}{\Gamma(p_2+1)} \int_0^{\eta_2} (\eta_2 - s)^{p_2} e^{-\lambda_2(\eta_2 - s)} I^\mu \varphi_v(s)ds \\ \quad + A_{22} \int_0^T e^{-\lambda_2(T-s)} I^\mu \varphi_v(s)ds, \qquad w \in (0, s_0]; \end{cases} \\ \Big\{ \Big|v(w) - N_{k_2} g_k(w, v(w))\Big| \leq \psi, \quad w \in (s_{k-1}, w_k], \quad k = 1, 2, \ldots, m; \\ \begin{cases} \Big|v(w) - \int_0^w e^{-\lambda_2(w-s)} I^\mu f_2(s, u(s), v(s)ds - \frac{M_{k_2}}{\Gamma(p_2+1)} \int_0^{\eta_2} (\eta_2 - s)^{p_2} e^{-\lambda_2(\eta_2 - s)} I^\mu f_2(s, u(s), v(s)ds \\ \quad + M_{k_2} \int_0^T e^{-\lambda_2(T-s)} I^\mu f_2(s, u(s), v(s)ds - N_{k_2} \int_0^{w_k} e^{-\lambda_2(w_k - s)} I^\mu f_2(s, u(s), v(s)ds - N_{k_2} g_k(w_k, v(w_k))\Big| \\ \leq \int_0^w e^{-\lambda_2(w-s)} I^\mu \varphi_v(s)ds + \frac{M_{k_2}}{\Gamma(p_2+1)} \int_0^{\eta_2} (\eta_2 - s)^{p_2} e^{-\lambda_2(\eta_2 - s)} I^\mu \varphi_v(s)ds \\ \quad + M_{k_2} \int_0^T e^{-\lambda_2(T-s)} I^\mu \varphi_v(s)ds + N_{k_2} \int_0^{w_k} e^{-\lambda_2(w_k - s)} I^\mu \varphi_v(s)ds + \psi, \; w \in (w_k, s_k], \; k = 0, 1, \ldots, m. \end{cases} \end{cases} \tag{3.4}$$

In fact by Remark 3.2, we get

$$\begin{cases} {}^c\mathcal{D}^\mu_{0,w}(\mathcal{D} + \lambda_2)v(w) = f_2(w, u(w), v(w)) + G(w), \quad w \in (w_k, s_k], \; k = 0, 1, \ldots, m, \quad 0 < \mu < 1, \\ v(w) = N_{k_2} g_k(w, v(w)) + G_k, \quad w \in (s_{k-1}, w_k], \quad k = 1, 2, \ldots, m. \end{cases} \tag{3.5}$$

Clearly, the solution of Eq (3.5) is given by

$$v(w)=\begin{cases}\begin{cases}\int_0^w e^{-\lambda_2(w-s)}I^{\mu}(f_2(s,u(s),v(s)+G(s))ds\\+\dfrac{A_{22}}{\Gamma(p_2+1)}\int_0^{\eta_2}(\eta_2-s)^{p_2}e^{-\lambda_2(\eta_2-s)}I^{\mu}(f_2(s,u(s),v(s)+G(s))ds\\-A_{22}\int_0^T e^{-\lambda_2(T-s)}I^{\mu}(f_2(s,u(s),v(s)+G(s))ds\\+\Big(A_{22}({\eta_2}^{p_2}E_{(1,p_2+1)}(aw)-e^{\lambda_2 T})+e^{\lambda_2 T}\Big)u_0,\quad w\in(0,s_0];\end{cases}\\ \big\{N_{k_2}g_k(w,v(w)),\quad w\in(s_{k-1},w_k],\quad k=1,2,\dots,m;\\ \begin{cases}\int_0^w e^{-\lambda_2(w-s)}I^{\mu}(f_2(s,u(s),v(s)+G(s))ds\\+\dfrac{M_{k_2}}{\Gamma(p_2+1)}\int_0^{\eta_2}(\eta_2-s)^{p_2}e^{-\lambda_2(\eta_2-s)}I^{\mu}(f_2(s,u(s),v(s)+G(s))ds\\-M_{k_2}\int_0^T e^{-\lambda_2(T-s)}I^{\mu}(f_2(s,u(s),v(s)+G(s))ds\\+N_{k_2}\int_0^{w_k} e^{-\lambda_2(w_k-s)}I^{\mu}(f_2(s,u(s),v(s)+G(s))ds\\+N_{k_2}g_k(w_k,v(w_k))+G_k,\quad w\in(w_k,s_k],\quad k=0,1,\dots,m.\end{cases}\end{cases}$$

For $w\in(w_k,s_k],\quad k=0,1,\dots,m$, we get

$$\begin{aligned}&\Big|v(w)-\int_0^w e^{-\lambda_2(w-s)}I^{\mu}f_2(s,u(s),v(s)ds-\frac{M_{k_2}}{\Gamma(p_2+1)}\int_0^{\eta_2}(\eta_2-s)^{p_2}e^{-\lambda_2(\eta_2-s)}I^{\mu}f_2(s,u(s),v(s)ds\\&\quad+M_{k_2}\int_0^T e^{-\lambda_2(T-s)}I^{\mu}f_2(s,u(s),v(s)ds-N_{k_2}\int_0^{w_k} e^{-\lambda_2(w_k-s)}I^{\mu}f_2(s,u(s),v(s)ds-N_{k_2}g_k(w_k,v(w_k))\Big|\\&\le\Big|\int_0^w e^{-\lambda_2(w-s)}I^{\mu}G(s)ds\Big|+\Big|\frac{M_{k_2}}{\Gamma(p_2+1)}\int_0^{\eta_2}(\eta_2-s)^{p_2}e^{-\lambda_2(\eta_2-s)}I^{\mu}G(s)ds\Big|\\&\quad+\Big|M_{k_2}\int_0^T e^{-\lambda_2(T-s)}I^{\mu}G(s)ds\Big|+\Big|N_{k_2}\int_0^{w_k} e^{-\lambda_2(w_k-s)}I^{\mu}G(s)ds\Big|+\Big|G_k\Big|\\&\le\int_0^w e^{-\lambda_2(w-s)}I^{\mu}\varphi_v(s)ds+\frac{M_{k_2}}{\Gamma(p_2+1)}\int_0^{\eta_2}(\eta_2-s)^{p_2}e^{-\lambda_2(\eta_2-s)}I^{\mu}\varphi_v(s)ds\\&\quad+M_{k_2}\int_0^T e^{-\lambda_2(T-s)}I^{\mu}\varphi_v(s)ds+N_{k_2}\int_0^{w_k} e^{-\lambda_2(w_k-s)}I^{\mu}\varphi_v(s)ds+\psi.\end{aligned}$$

Proceeding the above, we derive that

$$\Big|v(w)-N_{k_2}g_k(w,v(w))\Big|\le|G_k|\le\psi,\quad w\in(s_{k-1},w_k],\quad k=1,2,\dots,m,$$

and

$$
\begin{aligned}
&\Big|v(w)-\int_0^w e^{-\lambda_2(w-s)}I^\mu f_2(s,u(s),v(s))ds-\frac{A_{22}}{\Gamma(p_2+1)}\int_0^{\eta_2}(\eta_2-s)^{p_2}e^{-\lambda_2(\eta_2-s)}I^\mu f_2(s,u(s),v(s))ds\\
&+A_{22}\int_0^T e^{-\lambda_2(T-s)}I^\mu f_2(s,u(s),v(s))ds-\Big(A_{22}({\eta_2}^{p_2}E_{(1,p_2+1)}(aw)-e^{\lambda_2 T})+e^{\lambda_2 T}\Big)v_0\Big|\\
&\le\Big|\int_0^w e^{-\lambda_2(w-s)}I^\mu G(s)ds\Big|+\Big|\frac{A_{22}}{\Gamma(p_2+1)}\int_0^{\eta_2}(\eta_2-s)^{p_2}e^{-\lambda_2(\eta_2-s)}I^\mu G(s)ds\Big|\\
&+\Big|A_{22}\int_0^T e^{-\lambda_2(T-s)}I^\mu G(s)ds\Big|\\
&\le\int_0^w e^{-\lambda_2(w-s)}I^\mu\varphi_v(s)ds+\frac{A_{22}}{\Gamma(p_2+1)}\int_0^{\eta_2}(\eta_2-s)^{p_2}e^{-\lambda_2(\eta_2-s)}I^\mu\varphi_v(s)ds\\
&+A_{22}\int_0^T e^{-\lambda_2(T-s)}I^\mu\varphi_v(s)ds,\ w\in(0,s_0].
\end{aligned}
$$

4. Main results via fixed point methods

In order to apply a fixed point theorem of the alternative, for contractions on a generalized complete metric space to achieve our main result, we want to collect the following realities.

Definition 4.1. For a non empty set V, a function $d: V\times V\to[0,\infty]$ is called a generalized metric on V if and only if d satisfies:

- ⋄ $d(\upsilon_1,\upsilon_2)=0$ if and only if $\upsilon_1=\upsilon_2$;
- ⋄ $d(\upsilon_1,\upsilon_2)=d(\upsilon_2,\upsilon_1)$ for all $\upsilon_1,\upsilon_2\in V$;
- ⋄ $d(\upsilon_1,\upsilon_3)\le d(\upsilon_1,\upsilon_2)+d(\upsilon_2,\upsilon_3)$ for all $\upsilon_1,\upsilon_2,\upsilon_3\in V$.

Lemma 4.2. *(see [9] (Generalized Diaz-Margolis's fixed point theorem)). Let (V,d) be a generalized complete metric space. Assume that $T: V\to V$ is a strictly contractive operator with the Lipschitz constant $L<1$. If there exists a $k\ge 0$ such that $d(T^{k+1}v,T^kv)<\infty$ for some v in V, then the followings statements are true:*

(B_1) *The sequence $\{T^nv\}$ converges to a fixed point v^* of T;*

(B_2) *The unique fixed point of T is $v^*\in V^*=\Big\{u\in V \text{ such that } d(T^kv,u)<\infty\Big\}$;*

(B_3) *$u\in V^*$, then $d(u,v^*)\le\dfrac{1}{1-L}d(Tu,u)$.*

We introduced some assumptions as follows:

(H_1) $f\in C(J\times\mathbb{R},\mathbb{R})$.

(H_2)
- *There exists positive constants $0<L_{f_{u_1}}<1$ and $0<L_{f_{u_2}}<1$, such that* $\big|f_1(w,u,m)-f_1(w,v,n)\big|\le L_{f_{u_1}}|u-v|+L_{f_{u_2}}|m-n|,$ *for each $w\in J$ and all $u,v,m,n\in\mathbb{R}$.*
- *There exists positive constants $0<L_{f_{v_1}}<1$ and $0<L_{f_{v_2}}<1$, such that* $\big|f_2(w,u,m)-f_2(w,v,n)\big|\le L_{f_{v_1}}|u-v|+L_{f_{v_2}}|m-n|,$ *for each $w\in J$ and all $u,v,m,n\in\mathbb{R}$.*

(H_3) • $g_k \in C((s_{k-1}, w_k] \times \mathbb{R}, \mathbb{R})$ *and there are positive constant* L_{gk_1}, $k = 1, 2, \ldots, m$ *such that* $\left|g_k(w, v) - g_k(w, v)\right| \le L_{gk_1}|u - v|,$ *for each* $w \in (s_{k-1}, w_k]$, *and all* $u, v \in \mathbb{R}$.

• $g_k \in C((s_{k-1}, w_k] \times \mathbb{R}, \mathbb{R})$ *and there are positive constant* L_{gk_2}, $k = 1, 2, \ldots, m$ *such that* $\left|g_k(w, v) - g_k(w, v)\right| \le L_{gk_2}|u - v|,$ *for each* $w \in (s_{k-1}, w_k]$, *and all* $u, v \in \mathbb{R}$.

(H_4) • *Let* $\varphi_u \in C(J, \mathbb{R}_+)$ *be a nondecreasing function, there exists* $c_\varphi > 0$ *such that*

$$\int_0^w I^\nu(\varphi(s))ds \le C_\varphi \varphi_u(w) \qquad \text{for each} \quad w \in J.$$

• *Let* $\varphi_v \in C(J, \mathbb{R}_+)$ *be a nondecreasing function, there exists* $c_\varphi > 0$ *such that*

$$\int_0^w I^\nu(\varphi(s))ds \le C_\varphi \varphi_v(w) \qquad \text{for each} \quad w \in J.$$

Theorem 4.3. *Suppose that* (H_1)–(H_4) *are satisfied and also a function* $u, v \in PC(J, \mathbb{R})$ *satisfies* (3.1). *Then there exists unique solutions* u_0, v_0 *of Eq* (1.1) *such that*

$$u_0(w) = \begin{cases} \begin{cases} \int_0^w e^{-\lambda_1(w-s)} I^\nu f_1(s, v_0(s), u_0(s))ds + \frac{A_{11}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)_1^p e^{-\lambda_1(\eta_1 - s)} I^\nu f_1(s, v_0(s), u_0(s))ds \\ - A_{11} \int_0^T e^{-\lambda_1(T-s)} I^\nu f_1(s, v_0(s), u_0(s))ds \\ + \Big(A_{11}({\eta_1}^{p_1} E_{(1,p_1+1)}(aw) - e^{\lambda_1 T}) + e^{\lambda_1 T}\Big) u_0, \quad w \in (0, s_0]; \end{cases} \\ \big\{ g_k(w, u_0(w)), \quad w \in (s_{k-1}, w_k], \quad k = 1, 2, \ldots, m; \\ \begin{cases} \int_0^w e^{-\lambda_1(w-s)} I^\nu f_1(s, v_0(s), u_0(s))ds + \frac{M_{k_1}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)_1^p e^{-\lambda_1(\eta_1 - s)} I^\nu f_1(s, v_0(s), u_0(s))ds \\ - M_{k_1} \int_0^T e^{-\lambda_1(T-s)} I^\nu f_1(s, v_0(s), u_0(s))ds + N_{k_1} \int_0^{w_k} e^{-\lambda_1(w_k - s)} I^\nu f_1(s, v_0(s), u_0(s))ds \\ + N_{k_1} g_k(w_k, u_0(w_k)), \quad w \in (w_k, s_k], \quad k = 0, 1, \ldots, m, \end{cases} \end{cases} \tag{4.1}$$

similarly

$$v_0(w) = \begin{cases} \begin{cases} \int_0^w e^{-\lambda_2(w-s)} I^\nu f_2(s, u_0(s), v_0(s))ds + \frac{A_{22}}{\Gamma(p_2+1)} \int_0^{\eta_2} (\eta_2 - s)_2^p e^{-\lambda_2(\eta_2 - s)} I^\nu f_2(s, u_0(s), v_0(s))ds \\ - A_{22} \int_0^T e^{-\lambda_2(T-s)} I^\nu f_2(s, u_0(s), v_0(s))ds \\ + \Big(A_{22}({\eta_2}^{p_2} E_{(1,p_2+1)}(aw) - e^{\lambda_2 T}) + e^{\lambda_2 T}\Big) v_0, \quad w \in (0, s_0]; \end{cases} \\ \big\{ g_k(w, v_0(w)), \quad w \in (s_{k-1}, w_k], \quad k = 1, 2, \ldots, m; \\ \begin{cases} \int_0^w e^{-\lambda_2(w-s)} I^\nu f_2(s, u_0(s), v_0(s))ds + \frac{M_{k_2}}{\Gamma(p_2+1)} \int_0^{\eta_2} (\eta_2 - s)_2^p e^{-\lambda_2(\eta_2 - s)} I^\nu f_2(s, u_0(s), v_0(s))ds \\ - M_{k_2} \int_0^T e^{-\lambda_2(T-s)} I^\nu f_2(s, u_0(s), v_0(s))ds + N_{k_2} \int_0^{w_k} e^{-\lambda_2(w_k - s)} I^\nu f_2(s, u_0(s), v_0(s))ds \\ + N_{k_2} g_k(w_k, v_0(w_k)), \quad w \in (w_k, s_k], \quad k = 0, 1, \ldots, m, \end{cases} \end{cases} \tag{4.2}$$

and

$$
\begin{aligned}
&\Big|(u,v)(w)-(u_0,v_0)(w)\Big| \\
\leq\ & \Bigg\{\Bigg(\Big(\frac{1-e^{-\lambda w}}{\lambda}\Big)+\frac{M_k}{\Gamma(p+1)}\frac{\eta^{p+1}}{p+1}\Big(\frac{1-e^{-\lambda\eta}}{\lambda}\Big)+M_k\Big(\frac{1-e^{-\lambda T}}{\lambda}\Big) \\
&+N_k\Big(\frac{1-e^{-\lambda w_k}}{\lambda}\Big)\Bigg)\Big(\frac{C_uL_{f_{u_1}}}{1-L_{f_{u_2}}}+\frac{C_vL_{f_{v_1}}}{1-L_{f_{v_2}}}\Big)\Bigg\}\Big(\frac{C_\varphi(\varphi_u(w)+\varphi_v(w))+L_{gk}(C_u+C_v)\psi}{1-Ł}\Big).
\end{aligned}
\tag{4.3}
$$

for all $w\in J$, *if* $0<\nu<1$ *and*

$$
Ł=\max\{Ł_1,Ł_2\}<1, \tag{4.4}
$$

where

$$
\begin{aligned}
Ł_1 &= \max\Bigg\{\Bigg(\Big(\frac{1-e^{-\lambda w}}{\lambda}\Big)+\frac{M_k}{\Gamma(p+1)}\frac{\eta^{p+1}}{p+1}\Big(\frac{1-e^{-\lambda\eta}}{\lambda}\Big)+M_k\Big(\frac{1-e^{-\lambda T}}{\lambda}\Big) \\
&\quad +N_k\Big(\frac{1-e^{-\lambda w_k}}{\lambda}\Big)\Bigg)\Big(\frac{C_uL_{f_{u_1}}}{1-L_{f_{u_2}}}+\frac{C_vL_{f_{v_1}}}{1-L_{f_{v_2}}}\Big)\quad \textit{such that}\quad k=1,2,\ldots,m\Bigg\}, \\
Ł_2 &= \max\Bigg\{\Bigg(\Big(\frac{1-e^{-\lambda w}}{\lambda}\Big)\Big(\frac{w^\nu}{\Gamma(\nu+1)}\Big)+\frac{M_k}{\Gamma(p+1)}\frac{\eta^{p+1}}{p+1}\Big(\frac{1-e^{-\lambda\eta}}{\lambda}\Big)\Big(\frac{\eta^\nu}{\Gamma(\nu+1)}\Big) \\
&\quad +M_k\Big(\frac{1-e^{-\lambda T}}{\lambda}\Big)\Big(\frac{T^\nu}{\Gamma(\nu+1)}\Big)+N_k\Big(\frac{1-e^{-\lambda w_k}}{\lambda}\Big)\Big(\frac{w_k^\nu}{\Gamma(\nu+1)}\Big)\Bigg)\Big(\frac{C_uL_{f_{u_1}}}{1-L_{f_{u_2}}}+\frac{C_vL_{f_{v_1}}}{1-L_{f_{v_2}}}\Big), \\
&\qquad \textit{such that}\quad k=0,1,\ldots,m\Bigg\}.
\end{aligned}
$$

Proof. Consider the space of piecewise continuous functions

$$
V=\Big\{u,v:J\to\mathbb{R}\text{ such that }u,v\in PC(J,\mathbb{R})\Big\},
$$

endowed with the generalized metric on V, defined by

$$
\begin{aligned}
d((u,v)-(\bar{u},\bar{v})) &= \inf\Big\{C_u+C_v\in[0,+\infty]\quad\text{such that}\quad |(u,v)(w)-(\bar{u},\bar{v})(w)| \\
&\qquad \leq C_\varphi(\varphi_u(w)+\varphi_v(w))+L_{gk}(C_u+C_v)\psi\quad\text{for all}\quad w\in J\Big\},
\end{aligned}
\tag{4.5}
$$

where

$$
C_u\in\{C\in[0,\infty]\ :\ |(u,v)(w)-(\bar{u},\bar{v})(w)|\leq C_\varphi(\varphi_u(w)+\varphi_v(w))\ \forall\ w\in(w_k,s_k],\ k=0,1,\ldots,m\}
$$

and

$$
C_v\in\{C\in[0,\infty]\ :\ |(u,v)(w)-(\bar{u},\bar{v})(w)|\leq(C_u+C_v)\psi\quad\forall\ w\in(s_{k-1},w_k],\ k=1,2,\ldots,m\}.
$$

It is easy to verify that (V,d) is a complete generalized metric space [14].

Define an operator $\Lambda : V \to V$ by

$$(\Lambda u)(w) = \begin{cases} \begin{cases} \int_0^w e^{-\lambda_1(w-s)} I^\nu f_1(s, v(s), u(s))ds + \frac{A_{11}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^\nu f_1(s, v(s), u(s))ds \\ - A_{11} \int_0^T e^{-\lambda_1(T-s)} I^\nu f_1(s, v(s), u(s))ds \\ + \Big(A_{11}({\eta_1}^{p_1} E_{(1,p_1+1)}(aw) - e^{\lambda_1 T}) + e^{\lambda_1 T}\Big) u_0, \quad w \in (0, s_0]; \end{cases} \\ \Big\{ g_k(w, u(w)), \quad w \in (s_{k-1}, w_k], \quad k = 1, 2, \dots, m; \\ \begin{cases} \int_0^w e^{-\lambda_1(w-s)} I^\nu f_1(s, v(s), u(s))ds + \frac{M_{k_1}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^\nu f_1(s, v(s), u(s))ds \\ - M_{k_1} \int_0^T e^{-\lambda_1(T-s)} I^\nu f_1(s, v(s), u(s))ds + N_{k_1} \int_0^{w_k} e^{-\lambda_1(t_k - s)} I^\nu f_1(s, v(s), u(s))ds \\ + N_{k_1} g_k(w_k, u(w_k)), \quad w \in (w_k, s_k], \quad k = 0, 1, \dots, m, \end{cases} \end{cases} \tag{4.6}$$

for all u belongs to V and $w \in J$. Obviously, according to (H_1), Λ is well defined operator.

Next we shall verify that Λ is strictly contractive on V. Note that according to definition of (V, d), for any $\mu, \upsilon \in V$, it is possible to find $C_1, C_2 \in [0, \infty]$ such that

$$\begin{cases} \left|u(w) - \bar{u}(w)\right| \le \begin{cases} C_u \varphi_u(w), & w \in (w_k, s_k], \quad k = 0, \dots, m, \\ C_u \psi, & w \in (s_{k-1}, w_k], \quad k = 1, \dots, m. \end{cases} \\ \left|v(w) - \bar{v}(w)\right| \le \begin{cases} C_v \varphi_v(w), & w \in (w_k, s_k], \quad k = 0, \dots, m, \\ C_v \psi, & w \in (s_{k-1}, w_k], \quad k = 1, \dots, m. \end{cases} \end{cases} \tag{4.7}$$

From the definition of Λ in Eq (4.6), (H_2), (H_3) and (4.7), we obtain that

Case 1: For $w \in [0, s_0]$,

$$\begin{aligned} |(\Lambda u)(w) - (\Lambda \bar{u})(w)| = & \Big| \int_0^w e^{-\lambda_1(w-s)} I^\nu u(s)ds + \frac{A_{11}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^\nu u(s)ds \\ & - A_{11} \int_0^T e^{-\lambda_1(T-s)} I^\nu u(s)ds + \Big(A_{11}({\eta_1}^{p_1} E_{(1,p_1+1)}(aw) - e^{\lambda_1 T}) + e^{\lambda_1 T}\Big) u_0 \\ & - \int_0^w e^{-\lambda_1(w-s)} I^\nu \bar{u}(s)ds - \frac{A_{11}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^\nu \bar{u}(s)ds \\ & + A_{11} \int_0^T e^{-\lambda_1(T-s)} I^\nu \bar{u}(s)ds - \Big(A_{11}({\eta_1}^{p_1} E_{(1,p_1+1)}(aw) - e^{\lambda_1 T}) + e^{\lambda_1 T}\Big) u_0 \Big| \end{aligned}$$

$$\begin{aligned} \le & \Big| \int_0^w e^{-\lambda_1(w-s)} I^\nu u(s)ds - \int_0^w e^{-\lambda_1(w-s)} I^\nu \bar{u}(s)ds \Big| \\ & + \Big| \frac{A_{11}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^\nu u(s)ds - \frac{A_{11}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^\nu \bar{u}(s)ds \Big| \\ & + \Big| A_{11} \int_0^T e^{-\lambda_1(T-s)} I^\nu \bar{u}(s)ds - A_{11} \int_0^T e^{-\lambda_1(T-s)} I^\nu u(s)ds \Big| \end{aligned}$$

$$
\begin{aligned}
|(\Lambda u)(w)-(\Lambda\bar{u})(w)| \;\leq\; & \int_0^w e^{-\lambda_1(w-s)}I^\nu\Big|u(s)-\bar{u}(s)\Big|ds + A_{11}\int_0^T e^{-\lambda_1(T-s)}I^\nu\Big|\bar{u}(s)-u(s)\Big|ds \\
& +\frac{A_{11}}{\Gamma(p_1+1)}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^\nu\Big|u(s)-\bar{u}(s)\Big|ds
\end{aligned}
\tag{4.8}
$$

where, we use the following notations for convenience

$$
u(w) := f_1(w, v(w), u(w))
$$

$$
\bar{u}(w) := f_1(w, v(w), \bar{u}(w))
$$

$$
\begin{aligned}
|u(w)-\bar{u}(w)| &= |f_1(w, v(w), u(w)) - f_1(w, \bar{v}(w), \bar{u}(w))| \\
&\leq L_{f_{u_1}}|v(w)-\bar{v}(w)| + L_{f_{u_2}}|u(w)-\bar{u}(w)|,
\end{aligned}
$$

which further gives

$$
|u(w)-\bar{u}(w)| \leq \frac{L_{f_{u_1}}}{1-L_{f_{u_2}}}|v(w)-\bar{v}(w)|,
\tag{4.9}
$$

similarly

$$
|v(w)-\bar{v}(w)| \leq \frac{L_{f_{v_1}}}{1-L_{f_{v_2}}}|u(w)-\bar{u}(w)|.
$$

Put (4.9) in (4.8), we obtain

$$
\begin{aligned}
|(\Lambda u)(w)-(\Lambda\bar{u})(w)| \;\leq\; & \frac{L_{f_{u_1}}}{1-L_{f_{u_2}}}\int_0^w e^{-\lambda_1(w-s)}I^\nu|v(s)-\bar{v}(s)|ds + \frac{A_{11}L_{f_{u_1}}}{1-L_{f_{u_2}}}\int_0^T e^{-\lambda_1(T-s)}I^\nu|v(s)-\bar{v}(s)|ds \\
& +\frac{A_{11}L_{f_{u_1}}}{\Gamma(p_1+1)1-L_{f_{u_2}}}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^\nu|v(s)-\bar{v}(s)|ds \\
\leq\; & \frac{C_vL_{f_{u_1}}}{1-L_{f_{u_2}}}\int_0^w e^{-\lambda_1(w-s)}I^\nu|\varphi_v(s)|ds + \frac{C_vA_{11}L_{f_{u_1}}}{1-L_{f_{u_2}}}\int_0^T e^{-\lambda_1(T-s)}I^\nu|\varphi_v(s)|ds \\
& +\frac{C_vA_{11}L_{f_{u_1}}}{\Gamma(p_1+1)1-L_{f_{u_2}}}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^\nu|\varphi_v(s)|ds \\
\leq\; & \frac{C_vL_{f_{u_1}}}{1-L_{f_{u_2}}}\Big(\int_0^w e^{-\lambda_1(w-s)}ds\Big)\Big(\int_0^w I^\nu|\varphi_v(s)|ds\Big) \\
& +\frac{C_vA_{11}L_{f_{u_1}}}{1-L_{f_{u_2}}}\Big(\int_0^T e^{-\lambda_1(T-s)}ds\Big)\Big(\int_0^T I^\nu|\varphi_v(s)|ds\Big) \\
& +\frac{C_vA_{11}L_{f_{u_1}}}{\Gamma(p_1+1)1-L_{f_{u_2}}}\Big(\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}ds\Big)\Big(\int_0^{\eta_1} I^\nu|\varphi_v(s)|ds\Big)
\end{aligned}
$$

$$
\leq \frac{C_vL_{f_{u_1}}}{1-L_{f_{u_2}}}\Big(\frac{1-e^{-\lambda_1 w}}{\lambda_1}\Big)C_\varphi\varphi_v(w) + \frac{C_vA_{11}L_{f_{u_1}}}{1-L_{f_{u_2}}}\Big(\frac{1-e^{-\lambda_1 T}}{\lambda_1}\Big)C_\varphi\varphi_v(w)
$$

$$+\frac{C_v A_{11} L_{f_{u_1}}(\eta_1{}^{p_1+1})}{\Gamma(p_1+1)(1-L_{f_{u_2}})(p_1+1)}\Big(\frac{1-e^{-\lambda_1\eta_1}}{\lambda_1}\Big)C_\varphi\varphi_v(w)$$

$$\leq \left(\Big(\frac{1-e^{-\lambda_1 w}}{\lambda_1}\Big)+A_{11}\Big(\frac{1-e^{-\lambda_1 T}}{\lambda_1}\Big)+\frac{A_{11}(\eta_1{}^{p_1+1})}{\Gamma(p_1+1)(p_1+1)}\Big(\frac{1-e^{-\lambda_1\eta_1}}{\lambda_1}\Big)\right) \times\frac{C_v L_{f_{u_1}} C_\varphi\varphi_v(w)}{1-L_{f_{u_2}}}$$

$$|(\Lambda u)(w)-(\Lambda\bar{u})(w)| \leq \left(\Big(\frac{1-e^{-\lambda_1 w}}{\lambda_1}\Big)+A_{11}\Big(\frac{1-e^{-\lambda_1 T}}{\lambda_1}\Big)+\frac{A_{11}(\eta_1{}^{p_1+1})}{\Gamma(p_1+1)(p_1+1)}\Big(\frac{1-e^{-\lambda_1\eta_1}}{\lambda_1}\Big)\right) \times\frac{C_v L_{f_{u_1}} C_\varphi\varphi_v(w)}{1-L_{f_{u_2}}}. \tag{4.10}$$

On the similar way, we can obtain

$$|(\Lambda v)(w)-(\Lambda\bar{v})(w)| \leq \left(\Big(\frac{1-e^{-\lambda_2 w}}{\lambda_2}\Big)+A_{22}\Big(\frac{1-e^{-\lambda_2 T}}{\lambda_2}\Big)+\frac{A_{22}(\eta_2{}^{p_2+1})}{\Gamma(p_2+1)(p_2+1)}\Big(\frac{1-e^{-\lambda_2\eta_2}}{\lambda_2}\Big)\right) \times\frac{C_u L_{f_{v_2}} C_\varphi\varphi_u(w)}{1-L_{f_{v_2}}}. \tag{4.11}$$

Therefore from (4.10) and (4.11), we get the following result

$$\begin{aligned}&|\Lambda(u,v)-\Lambda(\bar{u},\bar{v})|\\ \leq\ & \left(\left\{\Big(\frac{1-e^{-\lambda_1 w}}{\lambda_1}\Big)+A_{11}\Big(\frac{1-e^{-\lambda_1 T}}{\lambda_1}\Big)+\frac{A_{11}}{\Gamma(p_1+1)}\frac{\eta_1{}^{p_1+1}}{p_1+1}\Big(\frac{1-e^{-\lambda_1\eta_1}}{\lambda_1}\Big)\right\}\frac{L_{f_{u_1}}}{1-L_{f_{u_2}}}+\right.\\ &\left\{\Big(\frac{1-e^{-\lambda_2 w}}{\lambda_2}\Big)+A_{22}\Big(\frac{1-e^{-\lambda_2 T}}{\lambda_1}\Big)+\frac{A_{22}}{\Gamma(p_2+1)}\frac{\eta_2{}^{p_2+1}}{p_2+1}\Big(\frac{1-e^{-\lambda_2\eta_2}}{\lambda_2}\Big)\right\}\\ &\left.\times\frac{L_{f_{v_2}}}{1-L_{f_{v_2}}}\right)\Big(C_u C_\varphi\varphi_u(w)+C_v C_\varphi\varphi_v(w)\Big).\end{aligned}$$

Suppose that $\max\{\lambda_1,\lambda_2\}=\lambda$, $\max\{p_1,p_2\}=p$, $\max\{A_{11},A_{22}\}=A$ and $\max\{\eta_1,\eta_2\}=\eta$

$$\begin{aligned}|\Lambda(u,v)-\Lambda(\bar{u},\bar{v})| \leq\ & \left(\Big(\frac{1-e^{-\lambda w}}{\lambda}\Big)+A\Big(\frac{1-e^{-\lambda T}}{\lambda}\Big)+\frac{A}{\Gamma(p+1)}\frac{\eta^{p+1}}{p+1}\Big(\frac{1-e^{-\lambda\eta}}{\lambda}\Big)\right)\\ &\left(\frac{L_{f_{u_1}}}{1-L_{f_{u_2}}}+\frac{L_{f_{v_1}}}{1-L_{f_{v_2}}}\right)\Big(C_u C_\varphi\varphi_u(w)+C_v C_\varphi\varphi_v(w)\Big).\end{aligned}$$

Case 2: For $w\in(s_{k-1},w_k]$, we have

$$|(\Lambda u)(w)-(\Lambda\bar{u})(w)|=|g_k(w,u(w))-g_k(w,\bar{u})|\leq L_{gk_1}|u(w)-\bar{u}(w)|\leq L_{gk_1}C_u\psi. \tag{4.12}$$

On the similar way, we can obtain

$$|(\Lambda v)(w)-(\Lambda\bar{v})(w)|=|g_k(w,v(w))-g_k(w,\bar{v})|\leq L_{gk_2}|v(w)-\bar{v}(w)|\leq L_{gk_2}C_v\psi. \tag{4.13}$$

Therefore from (4.12) and (4.13), we get the result given as

$$|\Lambda(u,v)-\Lambda(\bar{u},\bar{v})| \leq L_{gk_1}C_u\psi + L_{gk_2}C_v\psi.$$

Case 3: For $w \in (w_k, s_k]$, and $s \in (w_k, s_k]$,

$$\begin{aligned}
&|(\Lambda u)(w)-(\Lambda\bar{u})(w)|\\
&= \Big|\int_0^w e^{-\lambda_1(w-s)} I^\nu u(s)ds + \frac{M_{k_1}}{\Gamma(p_1+1)}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^\nu u(s)ds\\
&\quad -M_{k_1}\int_0^T e^{-\lambda_1(T-s)}I^\nu u(s)ds + N_{k_1}\int_0^{w_k} e^{-\lambda_1(w_k-s)}I^\nu u(s)ds\\
&\quad +N_{k_1}g_k(w_k,u(w_k)) - N_{k_1}g_k(w_k,\bar{u}(w_k)) - \int_0^w e^{-\lambda_1(w-s)}I^\nu f(s,\upsilon(s))ds\\
&\quad -\frac{M_{k_1}}{\Gamma(p_1+1)}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^\nu\bar{u}(s)ds\\
&\quad +M_{k_1}\int_0^T e^{-\lambda_1(T-s)}I^\nu\bar{u}(s)ds - N_{k_1}\int_0^{w_k} e^{-\lambda_1(w_k-s)}I^\nu\bar{u}(s)ds\Big|
\end{aligned}$$

$$\begin{aligned}
\leq\;& \Big|\int_0^w e^{-\lambda_1(w-s)}I^\nu u(s)ds - \int_0^w e^{-\lambda_1(w-s)}I^\nu\bar{u}(s)ds\Big|\\
&+\Big|\frac{M_{k_1}}{\Gamma(p_1+1)}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^\nu u(s)ds - \frac{M_{k_1}}{\Gamma(p_1+1)}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^\nu\bar{u}(s)ds\Big|\\
&+\Big|M_{k_1}\int_0^T e^{-\lambda_1(T-s)}I^\nu\bar{u}(s)ds - M_{k_1}\int_0^T e^{-\lambda_1(T-s)}I^\nu u(s)ds\Big|\\
&+\Big|N_{k_1}\int_0^{w_k} e^{-\lambda_1(w_k-s)}I^\nu u(s)ds - N_{k_1}\int_0^{w_k} e^{-\lambda_1(w_k-s)}I^\nu\bar{u}(s)ds\Big|\\
&+\Big|N_{k_1}g_k(w_k,u(w_k)) - N_{k_1}g_k(w_k,\bar{u}(w_k))\Big|
\end{aligned}$$

$$\begin{aligned}
\leq\;& \int_0^w e^{-\lambda_1(w-s)}I^\nu\Big|u(s)-\bar{u}(s)\Big|ds\\
&+\frac{M_{k_1}}{\Gamma(p_1+1)}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^\nu\Big|u(s)-\bar{u}(s)\Big|ds + L_{gk_1}C_u\psi\\
&+M_{k_1}\int_0^T e^{-\lambda_1(T-s)}I^\nu\Big|\bar{u}(s)-u(s)\Big|ds + N_{k_1}\int_0^{w_k} e^{-\lambda_1(w_k-s)}I^\nu\Big|u(s)-\bar{u}(s)\Big|ds
\end{aligned}$$

$$\begin{aligned}
\leq\;& \frac{L_{f_{u_1}}}{1-L_{f_{u_2}}}\int_0^w e^{-\lambda_1(w-s)}I^\nu|v(w)-\bar{v}(w)|ds\\
&+\frac{M_{k_1}L_{f_{u_1}}}{\Gamma(p_1+1)(1-L_{f_{u_2}})}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^\nu|v(w)-\bar{v}(w)|ds + L_{gk_1}C_u\psi\\
&+\frac{M_{k_1}L_{f_{u_1}}}{1-L_{f_{u_2}}}\int_0^T e^{-\lambda_1(T-s)}I^\nu|v(w)-\bar{v}(w)|ds + \frac{N_{k_1}L_{f_{u_1}}}{1-L_{f_{u_2}}}\int_0^{w_k} e^{-\lambda_1(w_k-s)}I^\nu|v(w)-\bar{v}(w)|ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C_v L_{f_{u_1}}}{1-L_{f_{u_2}}}\int_0^w e^{-\lambda_1(w-s)} I^\nu |\varphi_v(s)| ds \\
&+\frac{C_v M_{k_1} L_{f_{u_1}}}{\Gamma(p_1+1)(1-L_{f_{u_2}})}\int_0^{\eta_1}(\eta_1-s)^{p_1} e^{-\lambda_1(\eta_1-s)} I^\nu |\varphi_v(s)| ds + L_{gk_1} C_u \psi \\
&+\frac{C_v M_{k_1} L_{f_{u_1}}}{1-L_{f_{u_2}}}\int_0^T e^{-\lambda_1(T-s)} I^\nu |\varphi_v(s)| ds + \frac{C_v N_{k_1} L_{f_{u_1}}}{1-L_{f_{u_2}}}\int_0^{w_k} e^{-\lambda_1(w_k-s)} I^\nu |\varphi_v(s)| ds \\
&\leq \frac{C_v L_{f_{u_1}}}{1-L_{f_{u_2}}}\Big(\int_0^w e^{-\lambda_1(w-s)} ds\Big)\Big(\int_0^w I^\nu |\varphi_v(s)| ds\Big) \\
&+\frac{C_v M_{k_1} L_{f_{u_1}}}{\Gamma(p_1+1)(1-L_{f_{u_2}})}\Big(\int_0^{\eta_1}(\eta_1-s)^{p_1} e^{-\lambda_1(\eta_1-s)} ds\Big)\Big(\int_0^{\eta_1} I^\nu |\varphi_v(s)| ds\Big) + L_{gk_1} C_u \psi \\
&+\frac{C_v M_{k_1} L_{f_{u_1}}}{1-L_{f_{u_2}}}\Big(\int_0^T e^{-\lambda_1(T-s)} ds\Big)\Big(\int_0^T I^\nu |\varphi_v(s)| ds\Big) + \frac{C_v N_{k_1} L_{f_{u_1}}}{1-L_{f_{u_2}}}\Big(\int_0^{w_k} e^{-\lambda_1(w_k-s)} ds\Big)\Big(\int_0^{w_k} I^\nu |\varphi_v(s)| ds\Big),
\end{aligned}
$$

that is,

$$
\begin{aligned}
&|(\Lambda u)(w)-(\Lambda \bar{u})(w)| \\
\leq\ & \frac{C_v L_{f_{u_1}}}{1-L_{f_{u_2}}}\Big(\frac{1-e^{-\lambda_1 w}}{\lambda_1}\Big) C_\varphi \varphi_v(w) + \frac{C_v M_{k_1} L_{f_{u_1}}}{\Gamma(p_1+1)(1-L_{f_{u_2}})}\frac{\eta_1{}^{p_1+1}}{p_1+1}\Big(\frac{1-e^{-\lambda_1 \eta_1}}{\lambda_1}\Big) C_\varphi \varphi_v(w) \\
&+\frac{C_v M_{k_1} L_{f_{u_1}}}{1-L_{f_{u_2}}}\Big(\frac{1-e^{-\lambda_1 T}}{\lambda_1}\Big) C_\varphi \varphi_v(w) + \frac{C_v N_{k_1} L_{f_{u_1}}}{1-L_{f_{u_2}}}\Big(\frac{1-e^{-\lambda_1 w_k}}{\lambda_1}\Big) C_\varphi \varphi_v(w) + L_{gk_1} C_u \psi
\end{aligned}
$$

$$
\begin{aligned}
|(\Lambda u)(w)-(\Lambda \bar{u})(w)| \leq\ & \Bigg(\Big(\frac{1-e^{-\lambda_1 w}}{\lambda_1}\Big) + \frac{M_{k_1}}{\Gamma(p_1+1)}\frac{\eta_1{}^{p_1+1}}{p_1+1}\Big(\frac{1-e^{-\lambda_1 \eta_1}}{\lambda_1}\Big) \\
&+M_{k_1}\Big(\frac{1-e^{-\lambda_1 T}}{\lambda_1}\Big) + N_{k_1}\Big(\frac{1-e^{-\lambda_1 w_k}}{\lambda_1}\Big)\Bigg)\frac{C_v L_{f_{u_1}}}{1-L_{f_{u_2}}} C_\varphi \varphi_v(w) + L_{gk_1} C_u \psi
\end{aligned}
$$

$$
\begin{aligned}
|(\Lambda u)(w)-(\Lambda \bar{u})(w)| \leq\ & \Bigg(\Big(\frac{1-e^{-\lambda_1 w}}{\lambda_1}\Big) + \frac{M_{k_1}}{\Gamma(p_1+1)}\frac{\eta_1{}^{p_1+1}}{p_1+1}\Big(\frac{1-e^{-\lambda_1 \eta_1}}{\lambda_1}\Big) + M_{k_1}\Big(\frac{1-e^{-\lambda_1 T}}{\lambda_1}\Big) \\
&+N_{k_1}\Big(\frac{1-e^{-\lambda_1 w_k}}{\lambda_1}\Big)\Bigg)\frac{C_v L_{f_{u_1}}}{1-L_{f_{u_2}}}\Big(C_\varphi \varphi_v(w) + L_{gk_1} C_u \psi\Big). \qquad (4.14)
\end{aligned}
$$

On the similar way, we can obtain

$$
\begin{aligned}
|(\Lambda v)(w)-(\Lambda \bar{v})(w)| \leq\ & \Bigg(\Big(\frac{1-e^{-\lambda_2 w}}{\lambda_2}\Big) + \frac{M_{k_2}}{\Gamma(p_2+1)}\frac{\eta_2{}^{p_2+1}}{p_2+1}\Big(\frac{1-e^{-\lambda_2 \eta_2}}{\lambda_2}\Big) + M_{k_2}\Big(\frac{1-e^{-\lambda_2 T}}{\lambda_2}\Big) \\
&+N_{k_2}\Big(\frac{1-e^{-\lambda_2 w_k}}{\lambda_2}\Big)\Bigg)\frac{C_v L_{f_{v_1}}}{1-L_{f_{v_2}}}\Big(C_\varphi \varphi_u(w) + L_{gk_2} C_v \psi\Big). \qquad (4.15)
\end{aligned}
$$

Therefore from (4.14) and (4.15), we get the result given as

$$
\begin{aligned}
|\Lambda(u,v)-\Lambda(\bar{u},\bar{v})| \leq\ & \Bigg(\Big(\frac{1-e^{-\lambda_1 w}}{\lambda_1}\Big)+\frac{M_{k_1}}{\Gamma(p_1+1)}\frac{{\eta_1}^{p_1+1}}{p_1+1}\Big(\frac{1-e^{-\lambda_1\eta_1}}{\lambda_1}\Big)+M_{k_1}\Big(\frac{1-e^{-\lambda_1 T}}{\lambda_1}\Big) \\
&+N_{k_1}\Big(\frac{1-e^{-\lambda_1 w_k}}{\lambda_1}\Big)\Bigg)\frac{C_v L_{f_{u_1}}}{1-L_{f_{u_2}}}\Big(C_\varphi \varphi_v(w)+L_{gk_1}C_u\psi\Big) \\
&+\Bigg(\Big(\frac{1-e^{-\lambda_2 w}}{\lambda_2}\Big)+\frac{M_{k_2}}{\Gamma(p_2+1)}\frac{{\eta_2}^{p_2+1}}{p_2+1}\Big(\frac{1-e^{-\lambda_2\eta_2}}{\lambda_2}\Big)+M_{k_2}\Big(\frac{1-e^{-\lambda_2 T}}{\lambda_2}\Big) \\
&+N_{k_2}\Big(\frac{1-e^{-\lambda_2 w_k}}{\lambda_2}\Big)\Bigg)\frac{C_v L_{f_{v_1}}}{1-L_{f_{v_2}}}\Big(C_\varphi \varphi_u(w)+L_{gk_2}C_v\psi\Big).
\end{aligned}
$$

Suppose that $\max\{\lambda_1,\lambda_2\}=\lambda$, $\max\{p_1,p_2\}=p$, $\max\{M_{K_1},M_{K_2}\}=M_K$, $\max\{N_{K_1},N_{K_2}\}=N_K$ $\max\{L_{gk_1},L_{gk_2}\}=L_{gk}$ and $\max\{\eta_1,\eta_2\}=\eta$

$$
\begin{aligned}
|\Lambda(u,v)-\Lambda(\bar{u},\bar{v})| \leq\ & \Bigg(\Big(\frac{1-e^{-\lambda w}}{\lambda}\Big)+\frac{M_k}{\Gamma(p+1)}\frac{\eta^{p+1}}{p+1}\Big(\frac{1-e^{-\lambda\eta}}{\lambda}\Big)+M_k\Big(\frac{1-e^{-\lambda T}}{\lambda}\Big) \\
&+N_k\Big(\frac{1-e^{-\lambda w_k}}{\lambda}\Big)\Bigg)\Big(\frac{C_u L_{f_{u_1}}}{1-L_{f_{u_2}}}+\frac{C_v L_{f_{v_1}}}{1-L_{f_{v_2}}}\Big) \\
&\times\Big(C_\varphi(\varphi_u(w)+\varphi_v(w))+L_{gk}(C_u+C_v)\psi\Big).
\end{aligned}
$$

Also, for $w\in(w_k,s_k]$, and $s\in(s_{k-1},w_k]$,

$$
\begin{aligned}
|(\Lambda u)(w)-(\Lambda\bar{u})(w)| =\ & \Big|\int_0^w e^{-\lambda_1(w-s)}I^\nu u(s)ds+\frac{M_{k_1}}{\Gamma(p_1+1)}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^\nu u(s)ds \\
&-M_{k_1}\int_0^T e^{-\lambda_1(T-s)}I^\nu u(s)ds+N_{k_1}\int_0^{w_k}e^{-\lambda_1(w_k-s)}I^\nu u(s)ds \\
&+N_{k_1}g_k(w_k,u(w_k))-N_{k_1}g_k(w_k,\bar{u}(w_k))-\int_0^w e^{-\lambda_1(w-s)}I^\nu f(s,\upsilon(s))ds \\
&-\frac{M_{k_1}}{\Gamma(p_1+1)}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^\nu\bar{u}(s)ds \\
&+M_{k_1}\int_0^T e^{-\lambda_1(T-s)}I^\nu\bar{u}(s)ds-N_{k_1}\int_0^{w_k}e^{-\lambda_1(w_k-s)}I^\nu\bar{u}(s)ds\Big|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \Big|\int_0^w e^{-\lambda_1(w-s)}I^\nu u(s)ds-\int_0^w e^{-\lambda_1(w-s)}I^\nu\bar{u}(s)ds\Big| \\
&+\Big|\frac{M_{k_1}}{\Gamma(p_1+1)}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^\nu u(s)ds-\frac{M_{k_1}}{\Gamma(p_1+1)}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^\nu\bar{u}(s)ds\Big| \\
&+\Big|M_{k_1}\int_0^T e^{-\lambda_1(T-s)}I^\nu\bar{u}(s)ds-M_{k_1}\int_0^T e^{-\lambda_1(T-s)}I^\nu u(s)ds\Big| \\
&+\Big|N_{k_1}\int_0^{w_k}e^{-\lambda_1(w_k-s)}I^\nu u(s)ds-N_{k_1}\int_0^{w_k}e^{-\lambda_1(w_k-s)}I^\nu\bar{u}(s)ds\Big|
\end{aligned}
$$

$$
\begin{aligned}
&+\Big|N_{k_1}g_k(w_k,u(w_k))-N_{k_1}g_k(w_k,\bar{u}(w_k))\Big|\\
\leq\ & \int_0^w e^{-\lambda_1(w-s)}I^\nu\Big|u(s)-\bar{u}(s)\Big|ds+\frac{M_{k_1}}{\Gamma(p_1+1)}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^\nu\Big|u(s)-\bar{u}(s)\Big|ds+L_{gk_1}C_u\psi\\
&+M_{k_1}\int_0^T e^{-\lambda_1(T-s)}I^\nu\Big|\bar{u}(s)-u(s)\Big|ds+N_{k_1}\int_0^{w_k}e^{-\lambda_1(w_k-s)}I^\nu\Big|u(s)-\bar{u}(s)\Big|ds\\
\leq\ & \frac{L_{f_{u_1}}}{1-L_{f_{u_2}}}\int_0^w e^{-\lambda_1(w-s)}I^\nu|v(w)-\bar{v}(w)|ds+\frac{M_{k_1}L_{f_{u_1}}}{\Gamma(p_1+1)(1-L_{f_{u_2}})}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^\nu|v(w)-\bar{v}(w)|ds\\
&+\frac{M_{k_1}L_{f_{u_1}}}{1-L_{f_{u_2}}}\int_0^T e^{-\lambda_1(T-s)}I^\nu|v(w)-\bar{v}(w)|ds+\frac{N_{k_1}L_{f_{u_1}}}{1-L_{f_{u_2}}}\int_0^{w_k}e^{-\lambda_1(w_k-s)}I^\nu|v(w)-\bar{v}(w)|ds+L_{gk_1}C_u\psi\\
\leq\ & \frac{C_v\psi L_{f_{u_1}}}{1-L_{f_{u_2}}}\int_0^w e^{-\lambda_1(w-s)}I^\nu(1)ds+\frac{C_v\psi M_{k_1}L_{f_{u_1}}}{\Gamma(p_1+1)(1-L_{f_{u_2}})}\int_0^{\eta_1}(\eta_1-s)^{p_1}e^{-\lambda_1(\eta_1-s)}I^\nu(1)ds\\
&+\frac{C_v\psi M_{k_1}L_{f_{u_1}}}{1-L_{f_{u_2}}}\int_0^T e^{-\lambda_1(T-s)}I^\nu(1)ds+\frac{C_v\psi N_{k_1}L_{f_{u_1}}}{1-L_{f_{u_2}}}\int_0^{w_k}e^{-\lambda_1(w_k-s)}I^\nu(1)ds+L_{gk_1}C_u\psi,
\end{aligned}
$$

that is

$$
\begin{aligned}
|(\Lambda u)(w)-(\Lambda\bar{u})(w)|\ \leq\ & \Big\{\Big(\frac{1-e^{-\lambda_1 w}}{\lambda_1}\Big)\Big(\frac{w^\nu}{\Gamma(\nu+1)}\Big)+\frac{M_{k_1}}{\Gamma(p_1+1)}\frac{{\eta_1}^{p_1+1}}{p_1+1}\Big(\frac{1-e^{-\lambda_1\eta_1}}{\lambda_1}\Big)\Big(\frac{\eta_1^\nu}{\Gamma(\nu+1)}\Big)\\
&+M_{k_1}\Big(\frac{1-e^{-\lambda_1 T}}{\lambda_1}\Big)\Big(\frac{T^\nu}{\Gamma(\nu+1)}\Big)+N_{k_1}\Big(\frac{1-e^{-\lambda_1 w_k}}{\lambda_1}\Big)\Big(\frac{w_k^\nu}{\Gamma(\nu+1)}\Big)\Big\}\\
&\times\frac{C_vL_{f_{u_1}}}{1-L_{f_{u_2}}}\Big(C_\varphi\varphi_v(w)+L_{gk_1}C_u\psi\Big). \qquad (4.16)
\end{aligned}
$$

On the similar way, we can obtain

$$
\begin{aligned}
|(\Lambda v)(w)-(\Lambda\bar{v})(w)|\ \leq\ & \Big\{\Big(\frac{1-e^{-\lambda_2 w}}{\lambda_2}\Big)\Big(\frac{w^\nu}{\Gamma(\nu+1)}\Big)+\frac{M_{k_2}}{\Gamma(p_2+1)}\frac{{\eta_2}^{p_2+1}}{p_2+1}\Big(\frac{1-e^{-\lambda_2\eta_2}}{\lambda_2}\Big)\Big(\frac{\eta_2^\nu}{\Gamma(\nu+1)}\Big)\\
&+M_{k_2}\Big(\frac{1-e^{-\lambda_2 T}}{\lambda_2}\Big)\Big(\frac{T^\nu}{\Gamma(\nu+1)}\Big)+N_{k_2}\Big(\frac{1-e^{-\lambda_2 w_k}}{\lambda_2}\Big)\Big(\frac{w_k^\nu}{\Gamma(\nu+1)}\Big)\Big\}\\
&\times\frac{C_uL_{f_{v_1}}}{1-L_{f_{v_2}}}\Big(C_\varphi\varphi_u(w)+L_{gk_2}C_v\psi\Big). \qquad (4.17)
\end{aligned}
$$

Therefore from (4.16) and (4.17), we get the following result

$$
\begin{aligned}
|\Lambda(u,v)-\Lambda(\bar{u},\bar{v})|\ \leq\ & \Big\{\Big(\frac{1-e^{-\lambda_1 w}}{\lambda_1}\Big)\Big(\frac{w^\nu}{\Gamma(\nu+1)}\Big)+\frac{M_{k_1}}{\Gamma(p_1+1)}\frac{{\eta_1}^{p_1+1}}{p_1+1}\Big(\frac{1-e^{-\lambda_1\eta_1}}{\lambda_1}\Big)\Big(\frac{\eta_1^\nu}{\Gamma(\nu+1)}\Big)\\
&+M_{k_1}\Big(\frac{1-e^{-\lambda_1 T}}{\lambda_1}\Big)\Big(\frac{T^\nu}{\Gamma(\nu+1)}\Big)+N_{k_1}\Big(\frac{1-e^{-\lambda_1 w_k}}{\lambda_1}\Big)\Big(\frac{w_k^\nu}{\Gamma(\nu+1)}\Big)\Big\}\\
&\times\frac{C_vL_{f_{u_1}}}{1-L_{f_{u_2}}}\Big(C_\varphi\varphi_v(w)+L_{gk_1}C_u\psi\Big)
\end{aligned}
$$

$$+\bigg\{\Big(\frac{1-e^{-\lambda_2 w}}{\lambda_2}\Big)\Big(\frac{w^\nu}{\Gamma(\nu+1)}\Big)+\frac{M_{k_2}}{\Gamma(p_2+1)}\frac{{\eta_2}^{p_2+1}}{p_2+1}\Big(\frac{1-e^{-\lambda_2\eta_2}}{\lambda_2}\Big)\Big(\frac{\eta_2^\nu}{\Gamma(\nu+1)}\Big)$$
$$+M_{k_2}\Big(\frac{1-e^{-\lambda_2 T}}{\lambda_2}\Big)\Big(\frac{T^\nu}{\Gamma(\nu+1)}\Big)+N_{k_2}\Big(\frac{1-e^{-\lambda_2 w_k}}{\lambda_2}\Big)\Big(\frac{w_k^\nu}{\Gamma(\nu+1)}\Big)\bigg\}$$
$$\times\frac{C_u L_{f_{v_1}}}{1-L_{f_{v_2}}}\Big(C_\varphi\varphi_u(w)+L_{gk_2}C_v\psi\Big).$$

Suppose that $\max\{\lambda_1,\lambda_2\}=\lambda$, $\max\{p_1,p_2\}=p$, $\max\{M_{K_1},M_{K_2}\}=M_K$, $\max\{N_{K_1},N_{K_2}\}=N_K$ $\max\{L_{gk_1},L_{gk_2}\}=L_{gk}$ and $\max\{\eta_1,\eta_2\}=\eta$

$$\begin{aligned}|\Lambda(u,v)-\Lambda(\bar{u},\bar{v})| \;\leq\; & \bigg(\Big(\frac{1-e^{-\lambda w}}{\lambda}\Big)\Big(\frac{w^\nu}{\Gamma(\nu+1)}\Big)+\frac{M_k}{\Gamma(p+1)}\frac{\eta^{p+1}}{p+1}\Big(\frac{1-e^{-\lambda\eta}}{\lambda}\Big)\Big(\frac{\eta^\nu}{\Gamma(\nu+1)}\Big)\\ &+M_k\Big(\frac{1-e^{-\lambda T}}{\lambda}\Big)\Big(\frac{T^\nu}{\Gamma(\nu+1)}\Big)+N_k\Big(\frac{1-e^{-\lambda w_k}}{\lambda}\Big)\Big(\frac{w_k^\nu}{\Gamma(\nu+1)}\Big)\bigg)\\ &\times\Big(\frac{C_u L_{f_{u_1}}}{1-L_{f_{u_2}}}+\frac{C_v L_{f_{v_1}}}{1-L_{f_{v_2}}}\Big)\Big(C_\varphi(\varphi_u(w)+\varphi_v(w))+L_{gk}(C_u+C_v)\psi\Big).\end{aligned}$$

From above, we have

$$|(\Lambda(u,v))(w)-(\Lambda(\bar{u},\bar{v}))(w)|\leq Ł\Big(C_\varphi(\varphi_u(w)+\varphi_v(w))+L_{gk}(C_u+C_v)\psi\Big),\qquad w\in[0,\tau],$$

that is,

$$d\big(\Lambda(u,v),\Lambda(\bar{u},\bar{v})\big)\leq Ł(C_\varphi(\varphi_u(w)+\varphi_v(w))+L_{gk}(C_u+C_v)\psi).$$

Hence, we conclude that

$$d(\Lambda(u,v),\Lambda(\bar{u},\bar{v}))\leq Łd((u,v)-(\bar{u},\bar{v})),$$

for any $(u,v),(\bar{u},\bar{v})\in V$, since the condition (4.4) is strictly contraction property is shown.

Now we take $(u_0,v_0)\in V$. From the piecewise continuous property of (u_0,v_0) and $\Lambda(u_0,v_0)$, it follows that there exists a constant $0<G_1<\infty$, such that

$$\begin{aligned}&|(\Lambda(u_0,v_0))(w)-(u_0,v_0)(w)|\\ \leq\;& \Big|\int_0^w e^{-\lambda(w-s)}I^\nu f(s,(u_0,v_0)(s))ds+\frac{A_{11}}{\Gamma(p+1)}\int_0^\eta(\eta-s)^p e^{-\lambda(\eta-s)}I^\nu f(s,(u_0,v_0)(s))ds\\ &-A_{11}\int_0^T e^{-\lambda(T-s)}I^\nu f(s,(u_0,v_0)(s))ds+\Big(A_{11}(\eta^p E_{(1,p+1)}(aw)-e^{\lambda T})+e^{\lambda T}\Big)z_0-(u_0,v_0)(w)\Big|,\\ \leq\;& G_1\varphi(w)\leq G_1(C_\varphi(\varphi_u(w)+\varphi_v(w))+L_{gk}(C_u+C_v)\psi),\qquad w\in(0,s_0].\end{aligned}$$

There exists a constant $0<G_2<\infty$, such that

$$\begin{aligned}|(\Lambda(u_0,v_0))(w)-(u_0,v_0)(w)| \;=\;& |g_k(w,(u_0,v_0)(w))-(u_0,v_0)(w)|\\ \leq\;& G_2\psi\leq G_2\big(C_\varphi(\varphi_u(w)+\varphi_v(w))+L_{gk}(C_u+C_v)\psi\big),\end{aligned}$$

where $w \in (s_{k-1}, w_k],\ k = 1, 2, \ldots, m$.

Also we can find a constant $0 < G_3 < \infty$, such that

$$\begin{aligned}
&|(\Lambda(u_0, v_0))(w) - (u_0, v_0)(w)| \\
\leq\ & \Big| \int_0^w e^{-\lambda(w-s)} I^{\nu} f(s, (u_0, v_0)(s))ds + \frac{M_k}{\Gamma(p+1)} \int_0^{\eta} (\eta - s)^p e^{-\lambda(\eta-s)} I^{\nu} f(s, (u_0, v_0)(s))ds \\
&-M_k \int_0^T e^{-\lambda(T-s)} I^{\nu} f(s, (u_0, v_0)(s))ds + N_k \int_0^{w_k} e^{-\lambda(w_k-s)} I^{\nu} f(s, (u_0, v_0)(s))ds \\
&+N_k g_k(w_k, (u_0, v_0)(w_k)) - (u_0, v_0)(w)\Big|, \\
\leq\ & G_3 \varphi(w) \leq G_3(C_{\varphi}(\varphi_u(w) + \varphi_v(w)) + L_{gk}(C_u + C_v)\psi), \quad w \in (w_k, s_k], \quad k = 1, 2, \ldots, m.
\end{aligned}$$

Since $f,\ (u_k, v_k)$ and (u_0, v_0) are bounded on J and $(C_{\varphi}(\varphi_u(w) + \varphi_v(w)) + L_{gk}(C_u + C_v)\psi) > 0$. Thus (4.5) implies that $d(\Lambda(u_0, v_0), (u_0, v_0)) < \infty$.

By using Banach fixed point theorem, there exists a continuous function $u_0, v_0 : J \to \mathbb{R}$ such that $\Lambda^n(u_0, v_0) \to (u_0, v_0)$ in (V, d) as $n \to \infty$ and $\Lambda(u_0, v_0) = (u_0, v_0)$, that is, $u_0,\ v_0$ satisfies Eqs (4.1) and (4.2) for every $w \in J$.

Now we show that $\{u, v \in V$ such that $d((u_0, v_0), (u_0, v_0)) < \infty\} = V$. For any $u, v \in V$, since u, v and u_0, v_0 are bounded on J and $\min_{w \in J} (C_{\varphi}(\varphi_u(w) + \varphi_v(w)) + L_{gk}(C_u + C_v)\psi) > 0$, there exists a constant $0 < C_{(u,v)} < \infty$ such that $|(u_0, v_0)(w) - (u, v)(w)| \leq C_{(u,v)}(C_{\varphi}(\varphi_u(w) + \varphi_v(w)) + L_{gk}(C_u + C_v)\psi)$, for any $w \in J$. Hence, we have $d((u_0, v_0), (u, v)) < \infty$ for all $u, v \in V$, that is $\{(u, v) \in V$ such that $d((u_0, v_0), (u, v)) < \infty\} = V$. Thus, we determine that u, v are the unique continuous functions with the Eqs (4.1), and (4.2) respectively. From (3.2), (3.4) and (H_4), we can write

$$\begin{aligned}
d((u, v), \Lambda(u_0, v_0)) \leq\ & \bigg(\Big(\frac{1 - e^{-\lambda w}}{\lambda}\Big) + \frac{M_k}{\Gamma(p+1)} \frac{\eta^{p+1}}{p+1} \Big(\frac{1 - e^{-\lambda \eta}}{\lambda}\Big) + M_k \Big(\frac{1 - e^{-\lambda T}}{\lambda}\Big) \\
&+N_k \Big(\frac{1 - e^{-\lambda w_k}}{\lambda}\Big)\bigg)\Big(\frac{C_u L_{f_{u_1}}}{1 - L_{f_{u_2}}} + \frac{C_v L_{f_{v_1}}}{1 - L_{f_{v_2}}}\Big).
\end{aligned}$$

Summarizing we have

$$\begin{aligned}
d((u_0, v_0), (u, v)) \leq\ & \frac{d(\Lambda(u, v), (u, v))}{1 - Ł} \\
\leq\ & \bigg\{\bigg(\Big(\frac{1 - e^{-\lambda w}}{\lambda}\Big) + \frac{M_k}{\Gamma(p+1)} \frac{\eta^{p+1}}{p+1} \Big(\frac{1 - e^{-\lambda \eta}}{\lambda}\Big) + M_k \Big(\frac{1 - e^{-\lambda T}}{\lambda}\Big) \\
&+N_k \Big(\frac{1 - e^{-\lambda w_k}}{\lambda}\Big)\bigg)\Big(\frac{C_u L_{f_{u_1}}}{1 - L_{f_{u_2}}} + \frac{C_v L_{f_{v_1}}}{1 - L_{f_{v_2}}}\Big)\bigg\}\Big(\frac{1}{1 - Ł}\Big).
\end{aligned}$$

This shows that (4.19) is true for $w \in J$. □

Finally we give an example to illustrate our main result.

Example 4.4.

$$
\begin{cases}
\begin{cases}
{}^{c}\mathcal{D}^{\frac{1}{2}}_{0,w}(\mathcal{D}+2)u(w) = \dfrac{|u(w)|}{8+e^{w}+w^{2}}, & w \in (0,1] \cup (2,3], \\
u(w) = \dfrac{u(w)}{(3+w^{2})(1+|u(w)|)}, & w \in (1,2], \\
u(0) = \dfrac{\sqrt{2}}{3}, \qquad u(1) = \dfrac{5}{6}\displaystyle\int_{0}^{\frac{1}{4}} \dfrac{(\frac{1}{4}-s)}{\Gamma\frac{4}{3}}ds & 0<\eta<1
\end{cases} \\
\begin{cases}
{}^{c}\mathcal{D}^{\frac{1}{2}}_{0,w}(\mathcal{D}+2)v(w) = \dfrac{|v(w)|}{8+e^{w}+w^{2}}, & w \in (0,1] \cup (2,3], \\
v(w) = \dfrac{v(w)}{(3+w^{2})(1+|v(w)|)}, & w \in (1,2], \\
v(0) = \dfrac{\sqrt{2}}{3}, \qquad v(1) = \dfrac{5}{6}\displaystyle\int_{0}^{\frac{1}{4}} \dfrac{(\frac{1}{4}-s)}{\Gamma\frac{4}{3}}ds & 0<\eta<1
\end{cases}
\end{cases}
\tag{4.18}
$$

and

$$
\begin{cases}
\begin{cases}
\left|{}^{c}\mathcal{D}^{\frac{1}{2}}_{0,w}(\mathcal{D}+2)u(w) - \dfrac{|u(w)|}{8+e^{w}+w^{2}}\right| \le e^{w}, & w \in (0,1] \cup (2,3], \\
\left|u(w) - \dfrac{u(w)}{(3+w^{2})(1+|u(w)|)}\right| \le 1, & w \in (1,2].
\end{cases} \\
\begin{cases}
\left|{}^{c}\mathcal{D}^{\frac{1}{2}}_{0,w}(\mathcal{D}+2)v(w) - \dfrac{|v(w)|}{8+e^{w}+w^{2}}\right| \le e^{w}, & w \in (0,1] \cup (2,3], \\
\left|v(w) - \dfrac{v(w)}{(3+w^{2})(1+|v(w)|)}\right| \le 1, & w \in (1,2].
\end{cases}
\end{cases}
$$

Let $J = [0,3]$, $\mu = \nu = \frac{1}{2}$, $p_1 = p_2 = p = \frac{4}{3}$, $\eta_1 = \eta_2 = \eta = \frac{1}{4}$ and $0 = w_0 < s_0 = 1 < w_1 = 2 < s_1 = \tau = T = 3$. Denote $f_1(w,u(w)) = f_2(w,v(w)) = \frac{|z(w)|}{8+e^{w}+w^{2}}$ with $L_{f_{u_1}} = L_{f_{u_2}} == L_{f_{v_1}} = L_{f_{v_2}} = \frac{1}{4}$ for $w \in (0,1] \cup (2,3]$ and $g_k(w,u(w)) = \frac{u(w)}{(3+w^{2})(1+|u(w)|)}$, $g_k(w,v(w)) = \frac{v(w)}{(3+w^{2})(1+|v(w)|)}$ with $L_{g_k} = 1$ for $w \in (1,2]$. Putting $Ł_f = \frac{1}{4}$, $\varphi_u(w) = \varphi_v(w) = e^{w}$ and $C_\varphi = C_u = C_v = 1$, we have $\int_0^w I^{\frac{1}{2}} e^{s} ds \le e^{w}$ and $L_1 \approx 0.1012$, $L_2 \approx 0.9501$, so $L \approx 0.9501 < 1$.

By Theorem 4.3, there exists a unique solution $(u, v) : [0, 3] \to \mathbb{R}$ such that

$$\begin{cases} u_0(w) = \begin{cases} \int_0^w e^{-\lambda(w-s)} I^\nu \frac{|u_0(w)|}{8 + e^w + w^2} ds + \frac{A_{11}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^\nu \frac{|u_0(w)|}{8 + e^w + w^2} ds \\ - A_{11} \int_0^T e^{-\lambda_1(T-s)} I^\nu \frac{|u_0(w)|}{8 + e^w + w^2} ds + \Big(A_{11}(\eta_1^{p_1} E_{(1,p_1+1)}(aw) - e^{\lambda_1 T}) + e^{\lambda_1 T}\Big) u_0, \ w \in [0,1] \\ \frac{u_0(w)}{(3+w^2)(1+|u_0(w)|)}, \quad w \in (1,2], \quad k = 0, 1, \ldots, m, \\ \int_0^w e^{-\lambda_1(w-s)} I^\nu \frac{|u_0(w)|}{8 + e^w + w^2} ds + \frac{M_{k_1}}{\Gamma(p_1+1)} \int_0^{\eta_1} (\eta_1 - s)^{p_1} e^{-\lambda_1(\eta_1 - s)} I^\nu \frac{|u_0(w)|}{8 + e^w + w^2} ds \\ - M_{k_1} \int_0^T e^{-\lambda_1(T-s)} I^\nu \frac{|u_0(w)|}{8 + e^w + w^2} ds + N_{k_1} \int_0^{w_k} e^{-\lambda_1(w_k - s)} I^\nu \frac{|u_0(w)|}{8 + e^w + w^2} ds \\ + N_{k_1} \frac{u_0(w)}{(3+w^2)(1+|u_0(w)|)}, \quad w \in (2,3], \end{cases} \\ v_0(w) = \begin{cases} \int_0^w e^{-\lambda_2(w-s)} I^\mu \frac{|v_0(w)|}{8 + e^w + w^2} ds + \frac{A_{22}}{\Gamma(p_2+1)} \int_0^{\eta_2} (\eta_2 - s)^{p_2} e^{-\lambda_2(\eta_2 - s)} I^\mu \frac{|v_0(w)|}{8 + e^w + w^2} ds \\ - A_{22} \int_0^T e^{-\lambda_2(T-s)} I^\mu \frac{|v_0(w)|}{8 + e^w + w^2} ds + \Big(A_{22}(\eta_2^{p_2} E_{(1,p_2+1)}(aw) - e^{\lambda_2 T}) + e^{\lambda_2 T}\Big) v_0, \ w \in [0,1] \\ \frac{v_0(w)}{(3+w^2)(1+|v_0(w)|)}, \quad w \in (1,2], \quad k = 0, 1, \ldots, m, \\ \int_0^w e^{-\lambda_2(w-s)} I^\mu \frac{|v_0(w)|}{8 + e^w + w^2} ds + \frac{M_{k_2}}{\Gamma(p_2+1)} \int_0^{\eta_2} (\eta_2 - s)^{p_2} e^{-\lambda_2(\eta_2 - s)} I^\mu \frac{|v_0(w)|}{8 + e^w + w^2} ds \\ - M_{k_2} \int_0^T e^{-\lambda_2(T-s)} I^\mu \frac{|v_0(w)|}{8 + e^w + w^2} ds + N_{k_2} \int_0^{w_k} e^{-\lambda_2(w_k - s)} I^\mu \frac{|v_0(w)|}{8 + e^w + w^2} ds \\ + N_{k_2} \frac{v_0(w)}{(3+w^2)(1+|v_0(w)|)}, \quad w \in (2,3], \end{cases} \end{cases}$$

$$\begin{aligned} & \Big|(u,v)(w) - (u_0, v_0)(w)\Big| \\ \leq \ & \Bigg\{\Bigg(\Big(\frac{1 - e^{-\lambda w}}{\lambda}\Big) + \frac{M_k}{\Gamma(p+1)} \frac{\eta^{p+1}}{p+1} \Big(\frac{1 - e^{-\lambda \eta}}{\lambda}\Big) + M_k \Big(\frac{1 - e^{-\lambda T}}{\lambda}\Big) \\ & + N_k \Big(\frac{1 - e^{-\lambda w_k}}{\lambda}\Big)\Bigg)\Big(\frac{C_u L_{f_{u_1}}}{1 - L_{f_{u_2}}} + \frac{C_v L_{f_{v_1}}}{1 - L_{f_{v_2}}}\Big)\Bigg\}\Big(\frac{C_\varphi(\varphi_u(w) + \varphi_v(w)) + L_{gk}(C_u + C_v)\psi}{1 - Ł}\Big). \end{aligned}$$

putting maximum of $w = w_k = T = \eta = \tau$

$$\begin{aligned} & \Big|(u,v)(w) - (u_0, v_0)(w)\Big| \\ \leq \ & \Bigg\{\Bigg(\Big(\frac{1 - e^{-\lambda \tau}}{\lambda}\Big) + \frac{M_k}{\Gamma(p+1)} \frac{\eta^{p+1}}{p+1} \Big(\frac{1 - e^{-\lambda \tau}}{\lambda}\Big) + M_k \Big(\frac{1 - e^{-\lambda \tau}}{\lambda}\Big) \\ & + N_k \Big(\frac{1 - e^{-\lambda \tau}}{\lambda}\Big)\Bigg)\Big(\frac{C_u L_{f_{u_1}}}{1 - L_{f_{u_2}}} + \frac{C_v L_{f_{v_1}}}{1 - L_{f_{v_2}}}\Big)\Bigg\}\Big(\frac{C_\varphi(\varphi_u(w) + \varphi_v(w)) + L_{gk}(C_u + C_v)\psi}{1 - Ł}\Big). \end{aligned}$$

Now putting the values, we get

$$\left|(u,v)(w)-(u_0,v_0)(w)\right| \leq 0.8840\Big(\frac{2e^{w}+2}{1-0.9501}\Big),$$

$$\left|(u,v)(w)-(u_0,v_0)(w)\right| \leq 0.8840\Big(\frac{2(e^{w}+1)}{1-0.9501}\Big),$$

$$\left|(u,v)(w)-(u_0,v_0)(w)\right| \leq 35.4308(e^{w}+1), \qquad \text{for all} \quad w\in[0,3].$$

Thus the problem (4.18) is Ulam-Hyers-Rassias stability.

5. Conclusions

In this article, we considered switched coupled system of nonlinear impulsive Langevin equations with mixed derivatives and Some sufficient conditions are constructed to observe the existence, uniqueness and generalized Ulam-Hyers-Rassias stability. After introduction we built a uniform structure to originate a formula of solutions for our proposed model. We implemented the new concept of generalized Ulam-Hyers-Rassias stability to our proposed model, finally we solved a particular example for our proposed model.

Conflict of interest

The authors declare that they have no competing interest regarding this research work.

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