Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(12): 13072–13091.
DOI:10.3934/math.2021756
Received: 08 June 2021
Accepted: 31 August 2021
Published: 14 September 2021

Research article

Relation-theoretic metrical coincidence theorems under weak C-contractions and K-contractions

Faruk Sk*, **Asik Hossain and Qamrul Haq Khan**

Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India

* **Correspondence:** Email: sk.faruk.amu@gmail.com; Tel: +917550992021.

Abstract: In this paper, we prove some coincidence point theorems for weak C-contractions and K-contractions involving a new auxiliary function in a metric space endowed with a locally f-transitive binary relation. In this context, we generalize some relevant fixed point results in the literature. Further, we give an example to substantiate the utility of our results.

Keywords: binary relation; locally f-transitive; $\mathcal{R}$-completeness; weak C-contraction; weak K-contraction
Mathematics Subject Classification: 47H10, 54H25

1. Introduction

The first significant result in metric fixed point theory about contractive mappings, the Banach contraction principle (BCP) was established by S. Banach [1] in 1922. Due to its simplicity, this theorem vest as a conventional research tool in many different fields of mathematics. Then, several researchers proposed various types of fixed point theorems concerning different kinds of contractive mappings, see [2–4] and references therein. In 1968, R. Kannan [5] introduced the concept of K-contractions, and then in 1972, S. K Chatterjea [6] initiated the idea of C-contractions.

Definition 1.1. [5, 6] Let (X, d) be a metric space and f a self-mapping on X. Then,

(i) A mapping f is said to be K-contraction if there exists $\beta \in [0, \frac{1}{2})$ such that

$$d(fx, fy) \leq \beta[d(x, fx) + d(y, fy)] \quad \forall\, x, y \in X;$$

(ii) A mapping f is said to be C-contraction if there exists $\beta \in [0, \frac{1}{2})$ such that

$$d(fx, fy) \leq \beta[d(x, fy) + d(y, fx)] \quad \forall\, x, y \in X.$$

Kannan [5] proved that every K-contraction mapping in a complete metric space has a unique fixed point. Chatterjea [6] proved a similar fixed theorem using C-contraction mappings.

Further, generalizations of C-contraction and K-contraction mappings were introduced by B. S. Choudhury [7] and Razani and Parvaneh [8] respectively as follows:

Definition 1.2. [7,8] Let (X, d) be a metric space, $f : X \to X$ a self-mapping and $\phi : [0, \infty)^2 \to [0, \infty)$ a continuous mapping with $\phi(x, y) = 0$ if and only if $x = y = 0$. Then,

(i) f is said to be a weak C-contraction if

$$d(fx, fy) \leq \frac{1}{2}[d(x, fy) + d(y, fx)] - \phi(d(x, fy), d(y, fx)) \; \forall \; x, y \in X;$$

(ii) f is said to be a weak K-contraction if

$$d(fx, fy) \leq \frac{1}{2}[d(x, fx) + d(y, fy)] - \phi(d(x, fx), d(y, fy)) \; \forall \; x, y \in X.$$

In 2009, Choudhury [7] established a fixed point theorem using weak C-contraction as follows:

Theorem 1.3. *[7] Every weak C-contraction in a complete metric space has a unique fixed point.*

On the other hand, in 2015, Alam and Imdad [9] established another generalization of the classical Banach contraction principle using an amorphous (arbitrary) binary relation. In this context, many relation-theoretic variants of existing fixed point results have been reported for both linear and nonlinear contractions, see [10–12] and references therein. For nonlinear contractions, the underlying binary relation should be transitive. To make the transitivity condition weaker, Alam and Imdad [13] introduced the concept of locally f-transitivity.

Fixed point theory is used as a requisite tool in investigating the existence and uniqueness of solutions of differential and integral equations, see [14–16] and references therein. Also, the fixed point theorems for contractive mappings are used in economics, game theory, and many branches of mathematics. For instance, consider the following integral equation

$$f(x) \quad = \quad g(x) + \int_0^x k(x, s)u(s, f(s))ds, x \in [0, 1]. \tag{1.1}$$

where $g : [0, 1] \to \mathbb{R}$, $k : [0, 1] \times [0, 1] \to [0, \infty)$ and $u : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are three continuous functions. The Eq (1.1) can be studied by means of fixed point theory in view of the fact that $f(x)$ is a solution of (1.1) if and only if $f(x)$ is a fixed point of T where T is defined by

$$Tf(x) = g(x) + \int_0^x k(x, s)u(s, f(s))ds, \; x \in [0, 1].$$

This paper aims to establish some coincidence point theorems for weak C-contraction and K-contraction mappings involving a new auxiliary function in a metric space endowed with a locally f-transitive binary relation. We also deduce related fixed point theorems. As a consequence, these results improve and sharpen some existing fixed point results. Further, we give an example that shows the effectiveness of our results.

2. Preliminaries

In this section, we recall some basic definitions which will be required in proving our main results. We denote $\mathbb{N} \cup \{0\}$ as $\mathbb{N}_0$ throughout the paper.

Definition 2.1. [17, 18] Let X be a nonempty set and (f, g) be a pair of self-mappings on X. Then

(i) an element $x \in X$ is called a coincidence point of f and g if

$$f(x) = g(x),$$

(ii) if $x \in X$ is a coincidence point of f and g and $\bar{x} \in X$ such that $\bar{x} = g(x) = f(x)$, then $\bar{x}$ is called a point of coincidence of f and g,

(iii) if $x \in X$ is a coincidence point of f and g such that $x = f(x) = g(x)$, then x is called a common fixed point of f and g,

(iv) the pair (f, g) is said to be commuting if

$$g(fx) = f(gx)\ \forall\ x \in X,$$

(v) the pair (f, g) is said to be weakly compatible (or partially commuting or coincidentally commuting) if f and g commute at their coincidence points, i.e.,

$$g(fx) = f(gx) \text{ whenever } g(x) = f(x).$$

Definition 2.2. [19] Let X be a nonempty set. A subset $\mathcal{R}$ of X^2 is called a binary relation on X. The subsets, X^2 and $\emptyset$ of X^2 are called the universal relation and empty relation respectively.

Definition 2.3. [9] Let $\mathcal{R}$ be a binary relation on a nonempty set X. For $x, y \in X$, we say that x and y are $\mathcal{R}$-comparative if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$. We denote it by $[x, y] \in \mathcal{R}$.

Definition 2.4. [19–24] A binary relation $\mathcal{R}$ defined on a nonempty set X is called

(i) amorphous if $\mathcal{R}$ has no specific property at all,

(ii) reflexive if $(x, x) \in \mathcal{R}\ \forall\ x \in X$,

(iii) symmetric if $(x, y) \in \mathcal{R} \implies (y, x) \in \mathcal{R}$,

(iv) anti-symmetric if $(x, y) \in \mathcal{R}$ and $(y, x) \in \mathcal{R} \implies x = y$,

(v) transitive if $(x, y) \in \mathcal{R}$ and $(y, w) \in \mathcal{R} \implies (x, w) \in \mathcal{R}$,

(vi) complete, connected or dichotomous if $[x, y] \in \mathcal{R}\ \ \forall\ x, y \in X$,

(vii) a partial order if $\mathcal{R}$ is reflexive, anti-symmetric and transitive.

Definition 2.5. [19] Let X be a nonempty set and $\mathcal{R}$ be a binary relation on X.

(i) The inverse, transpose or dual relation of $\mathcal{R}$, denoted by $\mathcal{R}^{-1}$ is defined by,

$$\mathcal{R}^{-1} = \{(x, y) \in X^2 : (y, x) \in \mathcal{R}\}.$$

(ii) Symmetric closure of $\mathcal{R}$, denoted by $\mathcal{R}^s$, is defined to be the set $\mathcal{R} \cup \mathcal{R}^{-1}$(i.e., $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$).

Proposition 2.6. *[9] For a binary relation $\mathcal{R}$ defined on a nonempty set X,*

$$(x, y) \in \mathcal{R}^s \implies [x, y] \in \mathcal{R}.$$

Definition 2.7. [25] Let X be a nonempty set, $E \subseteq X$ and $\mathcal{R}$ be a binary relation on X. Then, the restriction of $\mathcal{R}$ to E, denoted by $\mathcal{R}|_E$, is defined to be the set $\mathcal{R} \cap E^2$ (i.e. $\mathcal{R}|_E = \mathcal{R} \cap E^2$). Indeed, $\mathcal{R}|_E$ is a relation on E induced by $\mathcal{R}$.

Definition 2.8. [9] Let X be a nonempty set and $\mathcal{R}$ be a binary relation on X. A sequence $\{x_n\} \subset X$ is called $\mathcal{R}$-preserving if

$$(x_n, x_{n+1}) \in \mathcal{R} \quad \forall\, n \in \mathbb{N}_0.$$

Definition 2.9. [9] Let X be a nonempty set and f be a self-mapping on X. A binary relation $\mathcal{R}$ on X is called f-closed if $\forall\, x, y \in X$,

$$(x, y) \in \mathcal{R} \implies (fx, fy) \in \mathcal{R}.$$

Definition 2.10. [26] Let X be a nonempty set and f and g be two self-mappings in X. A binary relation $\mathcal{R}$ defined on X is called (f, g)-closed if $\forall\, x, y \in X$

$$(gx, gy) \in \mathcal{R} \implies (fx, fy) \in \mathcal{R}.$$

Note that under the restriction $g = I$, the identity mapping on X, Definition 2.10 reduces to the notion of f-closedness of $\mathcal{R}$ defined in Definition 2.9.

Definition 2.11. [26] Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation on X. We say that (X, d) is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in X converges.

Clearly, every complete metric space is $\mathcal{R}$-complete with respect to a binary relation $\mathcal{R}$ but not conversely. For instance, Suppose $X = (-1, 1]$ together with the usual metric d. Notice that (X, d) is not complete. Now endow X with the following relation:

$$\mathcal{R} = \{(x, y) \in X^2 : x, y \geq 0\}.$$

Then, (X, d) is a $\mathcal{R}$-complete metric space. Particularly, under the universal relation, the notion of $\mathcal{R}$-completeness coincides with usual completeness.

Definition 2.12. [26] Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation on X with $x \in X$. A mapping $f : X \to X$ is called $\mathcal{R}$-continuous at x if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, we have $f(x_n) \xrightarrow{d} f(x)$. Moreover, f is called $\mathcal{R}$-continuous if it is $\mathcal{R}$-continuous at each point of X.

Clearly, every continuous mapping is $\mathcal{R}$-continuous under any binary relation $\mathcal{R}$. Particularly, under the universal, relation the notion of $\mathcal{R}$-continuity coincides with usual continuity.

Definition 2.13. [27, 28] Let (X, d) be a metric space and (f, g) be a pair of self-mappings on X. The the pair (f, g) is said to be compatible if

$$\lim_{n\to\infty} d(gfx_n, fgx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_n).$$

Definition 2.14. [26] Let (X, d) be a metric space and $\mathcal{R}$ be a binary relation on X and let f and g be two self-mappings on X. Then the mappings f and g are $\mathcal{R}$-compatible if for any sequence $\{x_n\} \subset X$ such that $\{fx_n\}$ and $\{gx_n\}$ are $\mathcal{R}$-preserving and $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n$, we have

$$\lim_{n\to\infty} d(gfx_n, fgx_n) = 0.$$

Remark 2.15. [26] In a metric space (X, d) endowed with a binary elation $\mathcal{R}$,

$$\text{commutativity} \implies \text{compatibility} \implies \mathcal{R}\text{-compatibility} \implies \text{weak compatibility.}$$

In particular, under the universal relation, the notion of $\mathcal{R}$-compatibility coincides with usual compatibility.

Definition 2.16. [9] Let (X, d) be a metric space. A binary relation $\mathcal{R}$ defined on X is called d-self-closed if whenever $\{x_n\}$ is an $\mathcal{R}$-preserving sequence and

$$x_n \xrightarrow{d} x,$$

then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \mathcal{R}\ \forall\ k \in \mathbb{N}_0$.

Definition 2.17. [26] Let (X, d) be a metric space, and let g be a self-mapping on X. A binary relation $\mathcal{R}$ defined on X is called (g, d)-self-closed if for any $\mathcal{R}$-preserving sequence $\{x_n\}$ such that $\{x_n\} \to x$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[gx_{n_k}, gx] \in \mathcal{R}\ \forall\ k \in \mathbb{N}_0$.

Note that under the restriction $g = I$, the identity mapping on X, Definition 2.17 reduces to the notion of d-self-closedness of $\mathcal{R}$.

Definition 2.18. [29] Given a mapping $f : X \to X$, a binary relation $\mathcal{R}$ defined on X is called f-transitive if for any $x, y, z \in X$,

$$(fx, fy), (fy, fz) \in \mathcal{R} \implies (fx, fz) \in \mathcal{R}.$$

Later, Alam and Imdad [13] introduced the concept of locally f-transitivity.

Definition 2.19. [13] Let X be a nonempty set, and f be a self-mapping on X. A binary relation $\mathcal{R}$ on X is called locally f-transitive if for each (effectively) $\mathcal{R}$-preserving sequence $\{x_n\} \subset f(X)$ (with range $E = \{x_n \in \mathbb{N}_0\}$), the binary relation $\mathcal{R}|_E$ is transitive.

Clearly, for a given self-mapping f and a binary relation $\mathcal{R}$ on a nonempty set X,

$$\text{transitivity} \implies f\text{-transitivity} \implies \text{locally } f\text{-transitivity.}$$

Definition 2.20. [24] Let X be a nonempty set and $\mathcal{R}$ a binary relation on X. A subset E of X is called $\mathcal{R}$-directed if for each $x, y \in E$, there exists $z \in X$ such that $(x, z) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$.

Now we recollect the following lemma which will be required in the proof of our main results.

Lemma 2.21. *[30] Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and two subsequence $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that*

(i) $k < m_k < n_k \ \forall \ k \in \mathbb{N}$,

(ii) $d(x_{m_k}, x_{n_k}) \geq \epsilon$,

(iii) $d(x_{m_k}, x_{n_k-1}) < \epsilon$.

In addition to this, if $\{x_n\}$ also satisfies $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, then

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \lim_{k\to\infty} d(x_{m_k}, x_{n_k-1}) = \epsilon.$$

Now we are equipped to prove our main results.

3. Main results

In what follows, we define a family of functions as follows:

$$\Phi = \{\phi : [0,\infty)^2 \to [0,\infty) : \phi \text{ is a function such that} \liminf_{n\to\infty} \phi(a_n.b_n) > 0 \text{ whenever the pair } (a_n, b_n) \to (a, b) \neq (0, 0)\}.$$

Consider a function $\phi : [0,\infty)^2 \to [0,\infty)$ defined by

$$\phi(x, y) = \begin{cases} \frac{1}{6}(\frac{x}{6} + 3y + 1), & \text{when } x, y \in [0, 1] \\ 1, & \text{others.} \end{cases}$$

Notice that $\phi \in \Phi$ but ϕ is not continuous, which establishes the fact that the family of functions Φ is larger than the family of functions considered by Choudhury [7] and Razani and Parvaneh [8].

Now we present our first result on the existence of coincidence points under weak C-contractions.

Theorem 3.1. *Let (X, d) be a $\mathcal{R}$-complete metric space endowed with a binary relation $\mathcal{R}$ and f, g be two self-mappings on X. Suppose that the following conditions hold:*

(a) $f(X) \subseteq g(X)$,

(b) *$\mathcal{R}$ is (f, g)-closed and locally f-transitive,*

(c) *there exists $x_0 \in X$ such that $(gx_0, fx_0) \in \mathcal{R}$,*

(d) *f and g are $\mathcal{R}$-compatible,*

(e) *g is $\mathcal{R}$-continuous,*

(f) *either f is $\mathcal{R}$-continuous or $\mathcal{R}$ is (g, d)-self-closed,*

(g) *there exists $\phi \in \Phi$ such that*

$$d(fx, fy) \leq \tfrac{1}{2}[d(gx, fy) + d(gy, fx)] - \phi(d(gx, fy), d(gy, fx))$$

$\forall\ x, y \in X$ *with* $(gx, gy) \in \mathcal{R}$.

Then f and g have a coincidence point.

Proof. By assumption (c), there exists $x_0 \in X$ such that $(gx_0, fx_0) \in \mathcal{R}$. If $f(x_0) = g(x_0)$, then we are done. Otherwise, by (a) we can choose $x_1 \in X$ such that $g(x_1) = f(x_0)$. Again from $f(X) \subseteq g(X)$ there exists $x_2 \in X$ such that $g(x_2) = f(x_1)$. Continuing this process inductively, we can define a sequence $\{x_n\} \subset X$ of joint iterates such that

$$g(x_{n+1}) \quad = \quad f(x_n) \quad \forall\ n \in \mathbb{N}_0. \tag{3.1}$$

Now, we assert that $\{gx_n\}$ is a $\mathcal{R}$-preserving sequence, i.e.,

$$(gx_n, gx_{n+1}) \in \mathcal{R} \quad \forall\ n \in \mathbb{N}_0. \tag{3.2}$$

We prove the fact by mathematical induction. On using assumption (c) and Eq (3.1) with $n = 0$, we have

$$(gx_0, gx_1) \in \mathcal{R}.$$

Thus (3.2) holds for $n = 0$. Now suppose (3.2) holds for $n = r > 0$, i.e.,

$$(gx_r, gx_{r+1}) \in \mathcal{R}, \tag{3.3}$$

then we have to show that (3.2) holds for $n = r + 1$. since $\mathcal{R}$ is (f, g)-closed, we have from (3.3),

$$(gx_r, gx_{r+1}) \in \mathcal{R} \implies (fx_r, fx_{r+1}) \in \mathcal{R} \implies (gx_{r+1}, gx_{r+2}) \in \mathcal{R},$$

that is, (3.2) holds for $n = r + 1$ also. Thus, by induction (3.2) holds $\forall\ n \in \mathbb{N}_0$. In view of (3.1) and (3.2), the sequence $\{fx_n\}$ is also $\mathcal{R}$-preserving, i.e.,

$$(fx_n, fx_{n+1}) \in \mathcal{R} \quad \forall\ n \in \mathbb{N}_0.$$

If $g(x_{n_0}) = g(x_{n_0+1})$ for some $n_0 \in \mathbb{N}$, then using (3.1) we have $g(x_{n_0}) = f(x_{n_0})$, i.e., x_{n_0} is a coincidence point of f and g and hence we are done.
On the other hand, if $g(x_n) \neq g(x_{n+1})\ \forall\ n \in \mathbb{N}_0$ then $d(gx_n, gx_{n+1}) \neq 0$, then we can define a sequence $\{d_n\}_{n=0}^{\infty} \subset (0, \infty)$ by

$$d_n = d(gx_n, gx_{n+1}).$$

Applying (3.1), (3.2) and assumption (g),

$$\begin{aligned}
d_{n+1} &= d(gx_{n+1}, gx_{n+2}) \\
&= d(fx_n, fx_{n+1}) \\
&\leq \frac{1}{2}[d(gx_n, fx_{n+1}) + d(gx_{n+1}, fx_n)] - \phi(d(gx_n, fx_{n+1}), d(gx_{n+1}, fx_n)) \\
&\leq \frac{1}{2}[d(gx_n, gx_{n+2}) + d(gx_{n+1}, gx_{n+1})] - \phi(d(gx_n, gx_{n+2}), d(gx_{n+1}, gx_{n+1})) \\
&= \frac{1}{2}d(gx_n, gx_{n+2}) - \phi(d(gx_n, gx_{n+2}), 0)
\end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&\leq \frac{1}{2}d(gx_n, gx_{n+2}) &(3.5)\\
&\leq \frac{1}{2}d(gx_n, gx_{n+1}) + \frac{1}{2}d(gx_{n+1}, gx_{n+2}) &(3.6)
\end{aligned}
$$

which gives

$$
\begin{aligned}
d_{n+1} &\leq \frac{1}{2}d_n + \frac{1}{2}d_{n+1}\\
\implies d_{n+1} &\leq d_n.
\end{aligned}
$$

Therefore, the sequence $\{d_n\}$ is a decreasing sequence of non-negative real numbers and hence it is convergent. Suppose there exists $r \geq 0$ such that

$$d_n \to r, \ i.e., \ d(gx_n, gx_{n+1}) \to r. \tag{3.7}$$

We will show that $r = 0$. Suppose $r > 0$, then letting $n \to \infty$ in (3.5) and (3.6) we get

$$
\begin{aligned}
&r \leq \lim_{n\to\infty} \frac{1}{2}d(gx_n, gx_{n+2}) \leq \frac{1}{2}(r + r) = r\\
&\implies \lim_{n\to\infty} d(gx_n, gx_{n+2}) = 2r. \qquad (3.8)
\end{aligned}
$$

Again, taking upper limit in (3.4) and using (3.7) and (3.8), we get

$$\limsup_{n\to\infty} d(gx_{n+1}, gx_{n+2}) \leq \limsup_{n\to\infty} \frac{1}{2}d(gx_n, gx_{n+2}) + \limsup_{n\to\infty}(-\phi(d(gx_n, gx_{n+2}), 0)).$$

Using the fact that for any sequence $\{x_n\}$, $\limsup(-x_n) = -\liminf(x_n)$, we get

$$
\begin{aligned}
\limsup_{n\to\infty} d(gx_{n+1}, gx_{n+2}) &\leq \frac{1}{2}\limsup_{n\to\infty} d(gx_n, gx_{n+2}) - \liminf_{n\to\infty} \phi(d(gx_n, gx_{n+2}), 0)\\
\implies r &\leq \frac{1}{2}.2r - \liminf_{n\to\infty} \phi(d(gx_n, gx_{n+2}), 0)
\end{aligned}
$$

which gives $\liminf_{n\to\infty} \phi(d(gx_n, gx_{n+2}), 0) \leq 0$ which is a contradiction to the property of ϕ, since $(d(gx_n, gx_{n+2}), 0) \to (2r, 0) \neq (0, 0)$. Hence,

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0. \tag{3.9}$$

Now, we show that $\{gx_n\}$ is a Cauchy sequence. Suppose that $\{gx_n\}$ is not a Cauchy sequence. Therefore, by Lemma 2.21, there exist $\epsilon > 0$ and two subsequences $\{gx_{n_k}\}$ and $\{gx_{m_k}\}$ such that

$$d(gx_{m_k}, gx_{n_k}) \geq \epsilon \tag{3.10}$$

for $n_k > m_k > k$, further, corresponding to m_k we can choose n_k in such a way that it is the smallest integer $n_k > m_k$ and satisfy (3.10). Then,

$$d(gx_{m_k}, gx_{n_k-1}) < \epsilon. \tag{3.11}$$

Further, in view of (3.9), (3.10), (3.11) and Lemma 2.21, we have

$$\lim_{n\to\infty} d(gx_{m_k}, gx_{n_k}) = \lim_{n\to\infty} d(gx_{m_k}, gx_{n_k-1}) = \lim_{n\to\infty} d(gx_{m_k-1}, gx_{n_k}) = \epsilon. \tag{3.12}$$

As $\{gx_n\}$ is $\mathcal{R}$-preserving and $\{gx_n\} \subseteq f(X)$, by the local f-transitivity of $\mathcal{R}$, we have $(gx_{m_k}, gx_{n_k}) \in \mathcal{R}$. Hence, applying contractivity condition (g), we obtain

$$\begin{aligned} d(gx_{m_k}, gx_{n_k}) &= d(fx_{m_{k-1}}, fx_{n_{k-1}}) \\ &\leq \frac{1}{2}[d(gx_{m_k-1}, fx_{n_k-1}) + d(gx_{n_k-1}, fx_{m_k-1})] - \phi(d(gx_{m_k-1}, fx_{n_k-1}), d(gx_{n_k-1}, fx_{m_k-1})) \\ &= \frac{1}{2}[d(gx_{m_k-1}, gx_{n_k}) + d(gx_{n_k-1}, gx_{m_k})] - \phi(d(gx_{m_k-1}, gx_{n_k}), d(gx_{n_k-1}, gx_{m_k})). \end{aligned}$$

Taking the upper limit in the above equation and in view of (3.12), we get

$$\begin{aligned} \limsup_{n\to\infty} d(gx_{m_k}, gx_{n_k}) \leq \frac{1}{2}[\limsup_{n\to\infty} d(gx_{m_k-1}, fx_{n_k-1}) + \limsup_{n\to\infty} d(gx_{n_k-1}, fx_{m_k-1})] \\ + \limsup_{n\to\infty}(-\phi(d(gx_{m_k-1}, fx_{n_k-1}), d(gx_{n_k-1}, fx_{m_k-1}))) \end{aligned}$$

Since for any sequence $\{x_n\}$, $\limsup(-x_n) = -\liminf(x_n)$, we get using (3.12)

$$\begin{aligned} \epsilon &\leq \frac{1}{2}(\epsilon + \epsilon) - \liminf_{n\to\infty} \phi(d(gx_{m_k-1}, fx_{n_k-1}), d(gx_{n_k-1}, fx_{m_k-1})) \\ &\Longrightarrow \liminf_{n\to\infty} \phi(d(gx_{m_k-1}, fx_{n_k-1}), d(gx_{n_k-1}, fx_{m_k-1})) \leq 0 \end{aligned}$$

which is a contradiction to the property of ϕ, since

$$(d(gx_{m_k-1}, fx_{n_k-1}), d(gx_{n_k-1}, fx_{m_k-1})) \to (\epsilon, \epsilon) \neq (0, 0).$$

Hence, $\{gx_n\}$ is a Cauchy sequence. Now since $\{gx_n\}$ is a $\mathcal{R}$-preserving sequence and X is $\mathcal{R}$-complete, $\{gx_n\}$ converges to an element $z \in X$, i.e.,

$$\lim_{n\to\infty} g(x_n) = z. \tag{3.13}$$

Also, from (3.1),

$$\lim_{n\to\infty} f(x_n) = z. \tag{3.14}$$

By $\mathcal{R}$-continuity of g,

$$\lim_{n\to\infty} g(gx_n) = g(\lim_{n\to\infty} gx_n) = g(z). \tag{3.15}$$

Utilizing (3.14) and $\mathcal{R}$-continuity of g,

$$\lim_{n\to\infty} g(fx_n) = g(\lim_{n\to\infty} fx_n) = g(z). \tag{3.16}$$

Since $\{fx_n\}$ and $\{gx_n\}$ are $\mathcal{R}$-preserving and

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z,$$

by the $\mathcal{R}$-compatibility of f and g,

$$\lim_{n\to\infty} d(gfx_n, fgx_n) = 0. \tag{3.17}$$

Now, we prove that z is a coincidence point of f and g.
Suppose that f is $\mathcal{R}$-continuous. Using (3.2), (3.13) and $\mathcal{R}$-continuity of f,

$$\lim_{n\to\infty} f(gx_n) = f(\lim_{n\to\infty} gx_n) = f(z). \tag{3.18}$$

Applying (3.16), (3.17) and continuity of d,

$$\begin{aligned} d(gz, fz) &= d(\lim_{n\to\infty} gfx_n, \lim_{n\to\infty} fgx_n) \\ &= \lim_{n\to\infty} d(gfx_n, fgx_n) = 0 \end{aligned}$$

so that $g(z) = f(z)$. Hence, z is a coincidence point of f and g.
Alternately, suppose that $\mathcal{R}$ is (g, d)-self-closed. As $\{gx_n\}$ is $\mathcal{R}$-preserving and $gx_n \to z$, due to (g, d)-self-closedness of $\mathcal{R}$, there exists a subsequences $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$[ggx_{n_k}, gz] \in \mathcal{R} \quad k \in \mathbb{N}_0. \tag{3.19}$$

Since $gx_{n_k} \to z$, Eqs (3.13)–(3.17) also holds for $\{x_{n_k}\}$ instead of $\{x_n\}$. In view of (3.19) and using assumption (g), we get

$$\begin{aligned} d(fgx_{n_k}, fz) &\leq \frac{1}{2}[d(ggx_{n_k}, fz) + d(gz, fgx_{n_k})] - \phi(d(ggx_{n_k}, fz), d(gz, fgx_{n_k})) \\ &\leq \frac{1}{2}[d(ggx_{n_k}, fz) + d(gz, fgx_{n_k})] \\ &\leq \frac{1}{2}[d(ggx_{n_k}, fz) + d(gz, gfx_{n_k}) + d(gfx_{n_k}, fgx_{n_k})]. \end{aligned} \tag{3.20}$$

Now using triangle inequality and (3.20) we get

$$\begin{aligned} d(gz, fz) &\leq d(gz, gfx_{n_k}) + d(gfx_{n_k}, fgx_{n_k}) + d(fgx_{n_k}, fz) \\ &\leq d(gz, gfx_{n_k}) + d(gfx_{n_k}, fgx_{n_k}) + \frac{1}{2}[d(ggx_{n_k}, fz) + d(gz, gfx_{n_k}) + d(gfx_{n_k}, fgx_{n_k})]. \end{aligned}$$

Letting $k \to \infty$ in the above equation, we get

$$\begin{aligned} d(gz, fz) &\leq d(gz, gz) + \frac{1}{2}[d(gz, fz) + d(gz, gz)] \\ &\leq \frac{1}{2} d(gz, fz) \\ \implies \frac{1}{2} d(gz, fz) &\leq 0 \end{aligned}$$

which gives $d(gz, fz) = 0$, i.e, z is a coincidence point of f and g. □

Theorem 3.2. *In addition to Theorem 3.1, if we consider the following condition:*

(h) $f(X)$ *is* $\mathcal{R}^s_{g(X)}$*-directed,*

then f and g have a unique point of coincidence.

Proof. Suppose there exist $x, y \in X$ such that

$$f(x) = g(x) = \bar{x} \text{ and } f(y) = g(y) = \bar{y}. \tag{3.21}$$

We show that $\bar{x} = \bar{y}$. As $f(x), f(y) \in f(X) \subseteq g(X)$, by assumption ($h$), there exist $g(z)$ such that

$$[fx, gz] \in \mathcal{R} \implies [gx, gz] \in \mathcal{R} \tag{3.22}$$

and

$$[fy, gz] \in \mathcal{R} \implies [gy, gz] \in \mathcal{R}. \tag{3.23}$$

Now define the constant sequences: $z_n^0 = x$ and $z_n^2 = y$ $\forall n \in \mathbb{N}_0$, then using (3.21),

$$g(z_{n+1}^0) = f(z_n^0) = \bar{x}, \tag{3.24}$$

$$g(z_{n+1}^2) = f(z_n^2) = \bar{y}. \tag{3.25}$$

Suppose $z_0^1 = z$. Since $f(X) \subseteq g(X)$, we can define a sequence $\{z_n^1\}$ such that $g(z_{n+1}^1) = f(z_n^1)$ $\forall\, n \in \mathbb{N}_0$. Therefore,

$$g(z_{n+1}^i) = f(z_n^i) \;\forall\, n \in \mathbb{N}_0 \text{ and } i = 0, 1. \tag{3.26}$$

We claim that

$$[g(z_n^i), g(z_n^{i+1})] \in \mathcal{R} \;\forall\, n \in \mathbb{N}_0 \text{ and } i = 0, 1. \tag{3.27}$$

We prove this fact by mathematical induction. It follows from (3.22) and (3.23) that (3.27) holds for $n = 0$. Suppose that (3.27) holds for $n = r > 0$, i.e.,

$$[g(z_r^i), g(z_r^{i+1})] \in \mathcal{R} \text{ and } i = 0, 1.$$

As $\mathcal{R}$ is (f, g)-closed, we have

$$[f(z_r^i), f(z_r^{i+1})] \in \mathcal{R} \text{ and } i = 0, 1,$$

which gives in view of (3.26), we get

$$[g(z_{r+1}^i), g(z_{r+1}^{i+1})] \in \mathcal{R} \text{ and } i = 0, 1.$$

Therefore, (3.27) holds for $n = r + 1$ also. Hence, (3.27) holds $\forall\, n \in \mathbb{N}_0$. Now in view of (3.27) and using (3.26) and assumption (g), we get

$$\begin{aligned} d(gz_{n+1}^0, gz_{n+1}^1) &= d(fz_n^0, fz_n^1) \\ &\leq \frac{1}{2}[d(gz_n^0, fz_n^1) + d(gz_n^1, fz_n^0)] - \phi(d(gz_n^0, fz_n^1), d(gz_n^1, fz_n^0)) \\ &= \frac{1}{2}[d(gz_n^0, gz_{n+1}^1) + d(gz_n^1, gz_{n+1}^0)] - \phi(d(gz_n^0, gz_{n+1}^1), d(gz_n^1, gz_{n+1}^0)) \end{aligned}$$

which gives

$$d(gz_{n+1}^0, gz_{n+1}^1) \leq \frac{1}{2}[d(gz_{n+1}^0, gz_{n+1}^1) + d(gz_n^1, gz_n^0)] - \phi(d(gz_{n+1}^0, gz_{n+1}^1), d(gz_n^1, gz_n^0)) \tag{3.28}$$

which implies

$$d(gz^0_{n+1}, gz^1_{n+1}) \leq \frac{1}{2}[d(gz^0_{n+1}, gz^1_{n+1}) + d(gz^1_n, gz^0_n)]$$

which implies

$$d(gz^0_{n+1}, gz^1_{n+1}) \leq d(gz^0_n, gz^1_n).$$

Therefore, the sequence $\{d(gz^0_n, gz^1_n)\}$ is decreasing non-negative sequence which converges to a non-negative real number r, i.e,

$$\{d(gz^0_n, gz^1_n)\} \to r.$$

We will show that $r = 0$. Suppose that $r > 0$, then taking upper limit in (3.28) and utilizing the same method as used earlier in this proof, we get

$$r \leq \frac{1}{2}(r + r) - \liminf_{n\to\infty} \phi(d(gz^0_{n+1}, gz^1_{n+1}), d(gz^1_n, gz^0_n))$$

which gives $\liminf_{n\to\infty} \phi(d(gz^0_{n+1}, gz^1_{n+1}), d(gz^1_n, gz^0_n)) \leq 0$, which is a contradiction to the property of ϕ. Hence,

$$\lim_{n\to\infty} d(gz^0_n, gz^1_n) = 0. \tag{3.29}$$

Similarly, we can show that

$$\lim_{n\to\infty} d(gz^1_n, gz^2_n) = 0. \tag{3.30}$$

Now, using triangle inequality, (3.24) and (3.25), we get

$$\begin{aligned} d(\bar{x}, \bar{y}) &= d(gz^0_n, gz^2_n) \\ &\leq d(gz^0_n, gz^1_n) + d(gz^1_n, gz^2_n). \end{aligned}$$

Letting $n \to \infty$ and in view of (3.29) and (3.30), we get

$$d(\bar{x}, \bar{y}) = 0.$$

Hence, f and g have unique point of coincidence. □

Now we present coincidence point results for mappings satisfying weak K-contraction.

Theorem 3.3. *Let (X, d) be a $\mathcal{R}$-complete metric space endowed with a binary relation $\mathcal{R}$. Let f and g be two self-mappings on X. Suppose that the following conditions hold:*

(a) $f(X) \subseteq g(X)$,

(b) $\mathcal{R}$ is (f, g)-closed and locally f-transitive,

(c) there exists $x_0 \in X$ such that $(gx_0, fx_0) \in \mathcal{R}$,

(d) f and g are $\mathcal{R}$-compatible,

(e) g is $\mathcal{R}$-continuous,

(f) either f is $\mathcal{R}$-continuous or $\mathcal{R}$ is (g, d)-self-closed,

(g) there exists $\phi \in \Phi$ such that

$$d(fx, fy) \leq \tfrac{1}{2}[d(gx, fx) + d(gy, fy)] - \phi(d(gx, fx), d(gy, fy))$$

$\forall\, x, y \in X$ with $(gx, gy) \in \mathcal{R}$.

Then f and g have a coincidence point.

Proof. Following the proof of Theorem 3.1, we can construct the sequence $\{gx_n\}$ defined in (3.1) and claim that the sequences $\{gx_n\}$ and $\{fx_n\}$ are $\mathcal{R}$-preserving. If $g(x_{n_0}) = g(x_{n_0+1})$ for some $n_0 \in \mathbb{N}$, then using (3.1) we have $g(x_{n_0}) = f(x_{n_0})$ i.e. x_{n_0} is a coincidence point of f and g, hence we are done. On the other hand, if $g(x_n) \neq g(x_{n+1})\ \forall\, n \in \mathbb{N}_0$ then $d(gx_n, gx_{n+1}) \neq 0$. On using (3.1), (3.2) and assumption (g), we have

$$\begin{aligned} d(gx_{n+1}, gx_{n+2}) &= d(fx_n, fx_{n+1}) \\ &\leq \frac{1}{2}[d(gx_n, fx_n) + d(gx_{n+1}, fx_{n+1})] - \phi(d(gx_n, fx_n), d(gx_{n+1}, fx_{n+1})) \\ &= \frac{1}{2}[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2})] - \phi(d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2})) \end{aligned} \tag{3.31}$$

which gives

$$\begin{aligned} & d(gx_{n+1}, gx_{n+2}) \leq \frac{1}{2}[d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2})] \\ \Longrightarrow\ & d(gx_{n+1}, gx_{n+2}) \leq d(gx_n, gx_{n+1}). \end{aligned}$$

Hence, the sequence $\{d(gx_n, gx_{n+1})\}$ is a decreasing sequence which converges to $r \geq 0$. Using the same technique as in Theorem 3.1, we get

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0. \tag{3.32}$$

Now, we show that $\{gx_n\}$ is a Cauchy sequence. Following the lines of Theorem 3.1, there exist subsequences $\{gx_{n_k}\}$ and $\{gx_{m_k}\}$ of $\{gx_n\}$ such that $(gx_{m_k}, gx_{n_k}) \in \mathcal{R}$. Then applying condition (g), we obtain

$$\begin{aligned} d(gx_{m_k}, gx_{n_k}) &= d(fx_{m_{k-1}}, fx_{n_{k-1}}) \\ &\leq \frac{1}{2}[d(gx_{m_k-1}, fx_{m_k-1}) + d(gx_{n_k-1}, fx_{n_k-1})] - \phi(d(gx_{m_k-1}, fx_{m_k-1}), d(gx_{n_k-1}, fx_{n_k-1})) \\ &= \frac{1}{2}[d(gx_{m_k-1}, gx_{m_k}) + d(gx_{n_k-1}, gx_{n_k})] - \phi(d(gx_{m_k-1}, gx_{m_k}), d(gx_{n_k-1}, gx_{n_k})). \end{aligned}$$

Letting $k \to \infty$ in the above equation and using the property of ϕ, we get that the sequence $\{gx_n\}$ is Cauchy. The rest of the proof is similar to Theorem 3.1. □

Theorem 3.4. *In addition to Theorem 3.3, if we consider the following condition:*

(h) $f(X)$ is $\mathcal{R}^s_{g(X)}$-directed,

then f and g have a unique point of coincidence.

Proof. The proof is almost similar to Theorem 3.2. Here,

$$\begin{aligned} d(gz_{n+1}^0, gz_{n+1}^1) &= d(fz_n^0, fz_n^1) \\ &\leq \frac{1}{2}[d(gz_n^0, fz_n^0) + d(gz_n^1, fz_n^1)] - \phi(d(gz_n^0, fz_n^0), d(gz_n^1, fz_n^1)) \\ &= \frac{1}{2}[d(gz_n^0, gz_{n+1}^0) + d(gz_n^1, gz_{n+1}^1)] - \phi(d(gz_n^0, gz_{n+1}^0), d(gz_n^1, gz_{n+1}^1)) \\ &= \frac{1}{2}d(gz_n^1, gz_{n+1}^1) - \phi(0, d(gz_n^1, gz_{n+1}^1)) \end{aligned} \tag{3.33}$$

which gives

$$\begin{aligned} d(gz_{n+1}^0, gz_{n+1}^1) &\leq \frac{1}{2}d(gz_n^1, gz_{n+1}^1) && (3.34) \\ &\leq \frac{1}{2}[d(gz_n^1, gz_n^0) + d(gz_n^0, gz_{n+1}^1) \\ &\leq \frac{1}{2}[d(gz_n^1, gz_n^0) + d(gz_{n+1}^0, gz_{n+1}^1) && (3.35) \\ \Longrightarrow d(gz_{n+1}^0, gz_{n+1}^1) &\leq d(gz_n^1, gz_n^0). \end{aligned}$$

Therefore, the sequence $\{d(gz_n^1, gz_n^0)\}$ is decreasing non-negative sequence which converges to a non-negative real number, i.e.,

$$\lim_{n\to\infty} d(gz_n^1, gz_n^0) = r.$$

Also in view of (3.34) and (3.35), we get

$$\lim_{n\to\infty} d(gz_n^1, gz_{n+1}^1) = 2r.$$

Now taking the upper limit in (3.33) and using the property of ϕ, we get

$$\lim_{n\to\infty} d(gz_n^0, gz_n^1) = 0$$

Similarly,

$$\lim_{n\to\infty} d(gz_n^1, gz_n^2) = 0.$$

Now the conclusion is immediate following Theorem 3.2. □

Remark 3.5. Theorem 3.2 (Theorem 3.4) also guarantees the existence of a unique common fixed point of f and g.

Proof. Let x be a coincidence point of f and g. So we have, $g(x) = f(x) = \bar{x}$. Using Remark 2.15, every $\mathcal{R}$-compatible pair is weakly compatible. Therefore, $\bar{x}$ is also a coincidence point of f and g. In view of Theorem 3.2 (Theorem 3.4), we get $g(x) = g(\bar{x})$ which gives

$$\bar{x} = g(\bar{x}) = f(\bar{x}).$$

Hence, $\bar{x}$ is a common fixed point of f and g. To claim the uniqueness, we assume x' be another common fixed point of f and g. Using Theorem 3.2 (Theorem 3.4), we get

$$x' = g(x') = g(\bar{x}) = \bar{x}$$

which concludes the proof. □

Under universal relation (i.e. $\mathcal{R} = X^2$), Theorem 3.1 together with Theorems 3.2, 3.3, 3.4 reduce to the following coincidence point theorems

Corollary 3.6. *Let (X, d) be a complete metric space and f, g be two self-mappings on X. Suppose that the following conditions hold:*

(a) $f(X) \subseteq g(X)$,

(b) f *and* g *are compatible,*

(c) g *is continuous,*

(d) there exists $\phi \in \Phi$ *such that*

$$d(fx, fy) \leq \frac{1}{2}[d(gx, fy) + d(gy, fx)] - \phi\big(d(gx, fy), d(gy, fx)\big)$$

$\forall\, x, y \in X$.

Then f *and* g *have a unique point of coincidence.*

Corollary 3.7. *Let (X, d) be a complete metric space and f, g be two self-mappings on X. Suppose that the following conditions hold:*

(a) $f(X) \subseteq g(X)$,

(b) f *and* g *are compatible,*

(c) g *is continuous,*

(d) there exists $\phi \in \Phi$ *such that*

$$d(fx, fy) \leq \frac{1}{2}[d(gx, fx) + d(gy, fy)] - \phi\big(d(gx, fx), d(gy, fy)\big)$$

$\forall\, x, y \in X$.

Then f *and* g *have a unique point of coincidence.*

4. Related fixed point results

In this section, we derive several fixed point results of the existing literature as consequences of our newly proved results.

Taking $g = I$, the identity map on X in Theorems 3.1, 3.2, 3.3 and 3.4, we obtain

Corollary 4.1. *Let (X, d) be a $\mathcal{R}$-complete metric space endowed with a binary relation $\mathcal{R}$ and f a self-mappings on X. Suppose that the following conditions hold:*

(a) $\mathcal{R}$ is f-closed and locally f-transitive,

(b) there exists $x_0 \in X$ such that $(x_0, fx_0) \in \mathcal{R}$,

(c) either f is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed,

(d) there exists $\phi \in \Phi$ such that

$$d(fx, fy) \leq \frac{1}{2}[d(x, fy) + d(y, fx)] - \phi(d(x, fy), d(y, fx))$$

$\forall\ x, y \in X$ *with* $(x, y) \in \mathcal{R}$.

Then f has a fixed point. Moreover, if

(e) $f(X)$ is $\mathcal{R}^s$-directed,

then f has a unique fixed point.

Corollary 4.2. *Let (X, d) be a $\mathcal{R}$-complete metric space endowed with a binary relation $\mathcal{R}$ and f a self-mappings on X. Suppose that the following conditions hold:*

(a) $\mathcal{R}$ is f-closed and locally f-transitive,

(b) there exists $x_0 \in X$ such that $(x_0, fx_0) \in \mathcal{R}$,

(c) either f is $\mathcal{R}$-continuous or $\mathcal{R}$ is d-self-closed,

(d) there exists $\phi \in \Phi$ such that

$$d(fx, fy) \leq \frac{1}{2}[d(x, fx) + d(y, fy)] - \phi(d(x, fx), d(y, fy))$$

$\forall\ x, y \in X$ *with* $(x, y) \in \mathcal{R}$.

Then f has a fixed point. Moreover, if

(e) $f(X)$ is $\mathcal{R}^s$-directed,

then f has a unique fixed point.

Under universal relation (i.e. $\mathcal{R} = X^2$), taking $g = I$, the identity map on X in Theorems 3.1, 3.2, 3.3 and 3.4, we obtain:

Corollary 4.3. *Let (X, d) be a complete metric space and f be a self-mapping on X. If there exists $\phi \in \Phi$ such that*

$$d(fx, fy) \leq \frac{1}{2}[d(x, fy) + d(y, fx)] - \phi(d(x, fy), d(y, fx))$$

$\forall\ x, y \in X$*. Then, f has a unique fixed point.*

Corollary 4.4. *Let (X, d) be a complete metric space and f a self-mapping on X. If there exists $\phi \in \Phi$ such that*

$$d(fx, fy) \leq \tfrac{1}{2}[d(x, fx) + d(y, fy)] - \phi(d(x, fx), d(y, fy))$$

$\forall\, x, y \in X$. Then, f has a unique fixed point.

Taking $\mathcal{R} = \preceq$, the partial order and $g = I$, the identity map on X in Theorem 3.1 we obtain the following result:

Corollary 4.5. *Let $(X, d, \preceq)$ be a complete metric space endowed with usual partial order and f a self-mapping on X. Suppose the following conditions hold:*

(a) there exists $x_0 \in X$ such that $x_0 \preceq fx_0$,

(b) f is nondecreasing with respect to $\preceq$,

(c) f is continuous or $(X, d, \preceq)$ is regular,

(d) there exists $\phi \in \Phi$ such that

$$d(fx, fy) \leq \tfrac{1}{2}[d(x, fy) + d(y, fx)] - \phi(d(x, fy), d(y, fx)) \ \forall\, x \succeq y,$$

then, f has a fixed point.

Notice that Corollary 4.1 improves and sharpens Theorem 1.3 and Theorem 2.1 of Harjani et al. [30] since the auxiliary function ϕ considered here is relatively weaker than the existing one.

Remark 4.6. Taking $\phi(x, y) = k(x + y)$, where $0 < k < 1$ in Corollaries 4.3 and 4.4 we obtain the classical fixed point theorems given by Kannan [5] and Chatterjea [6] respectively.

5. An illustrative example

Consider the metric space (X, d) with $X = (-1, 1]$ with the usual metric d. Define two functions $f, g : X \to X$ by

$$f(x) = \begin{cases} \frac{1-x}{4}, & \text{if } x \in (-1, 0) \\ \frac{x}{3}, & \text{if } x \in [0, 1] \end{cases} \quad \text{and} \quad g(x) = \frac{3x}{4}.$$

Now endow X with the following binary relation

$$\mathcal{R} = \{(x, y) \in X^2 : x \geq y \geq 0\}.$$

Obviously, X is $\mathcal{R}$-complete metric space. Consider a function $\phi : [0, \infty)^2 \to [0, \infty)$ defined by

$$\phi(x, y) = \begin{cases} \frac{1}{4}(\frac{x}{5} + 2y), & \text{when } x, y \in [0, 1] \\ 1, & \text{others.} \end{cases}$$

For $x, y \in X$ with $(gx, gy) \in \mathcal{R}$,

$$\begin{aligned} d(fx, fy) &= \left| \frac{x}{3} - \frac{y}{3} \right| \\ &= \frac{1}{12}[4 \mid x - y \mid] \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{12}\Big[\frac{9}{20}\mid 9x-4y\mid\Big]\\
&\le \frac{1}{12}\Big[(\frac{1}{2}-\frac{1}{20})\mid 9x-4y\mid\Big]\\
&\le \frac{1}{12}\Big[(\frac{1}{2}-\frac{1}{20})\mid 9x-4y\mid +(\frac{1}{2}-\frac{1}{2})\mid 9y-4x\mid\Big]\\
&\le \frac{1}{12}\Big[\frac{1}{2}\{\mid 9x-4y\mid+\mid 9y-4x\mid\}-\frac{1}{4}\{\frac{1}{5}\mid 9x-4y\mid +2\mid 9y-4x\mid\}\Big]\\
&\le \frac{1}{2}\{\mid\frac{9x-4y}{12}\mid+\mid\frac{9y-4x}{12}\mid\}-\frac{1}{4}\{\frac{1}{5}\mid\frac{9x-4y}{12}\mid +2\mid\frac{9y-4x}{12}\mid\}\\
&= \frac{1}{2}[d(gx,fy)+d(gy,fx)]-\phi(d(gx,fy),d(gy,fx)).
\end{aligned}
$$

Therefore, the condition (g) of Theorem 3.1 is satisfied. Also, by routine calculation, it can be observed that all the other conditions of Theorems 3.1 and 3.2 are satisfied, and f and g have a unique point of coincidence, namely: $x = 0$, which is also a unique common fixed point of f and g in view of Remark 3.5.

Notice that condition (g) of Theorem 3.1 does not hold for the whole space (for example, take $x = 0$ and $y = 1$). Also, the used auxiliary function ϕ in this example is discontinuous. Therefore, this example cannot be solved by the existing results, which establishes the importance of our results.

6. Conclusions

In this paper, we observed that some of the conditions of the auxiliary function ϕ are not necessary for weak C-contractions and weak K-contractions. Moreover, we established the relation-theoretic variants of some coincidence point as well as fixed point theorems for these contractions. Also, the presented example shows the effectiveness of our newly proved results over corresponding several noted results. Now, for possible problems, one can attempt to prove related contractions like weakly f-Kannan, weakly f-Chatterjea etc., with the newly introduced family of auxiliary functions.

Acknowledgments

The authors would like to thank the anonymous referees for their comments that helped us improve this article. The first author would like to thank the Council of Scientific and Industrial Research (CSIR), Government of India for JRF (File No. 09/112(0604)/2018-EMR-I).

Conflict of interest

The authors declare no conflict of interest.

References

1. S. Banach, Sur les operations dans les ensembles abstraints er leur application aux equations intgrales, *Fund. Math.,* **3** (1922), 133–181.

2. A. Ali, H. Işık, H. Aydi, E. Ameer, J. R. Lee, M. Arshad, On multivalued SU-type -contractions and related applications, *Open Math.,* **18** (2020), 386–399.
3. A. Ali, F. Uddin, M. Arshad, M. Rashid, Hybrid fixed point results via generalized dynamic process for F-HRS type contractions with application, *Phys. A. Stat. Mech. Appl.,* **538** (2020), 122669.
4. A. Ali, M. Alansari, F. Uddin, M. Arshad, A. Asif, G. A. Basendwah, Set-valued SU-type fixed point theorems via Gauge function with applications, *J. Math.,* (2021), 6612448.
5. R. Kannan, Some results on fixed points, *Bull. Calcutta Math. Soc.,* **60** (1968), 71–76.
6. S. K. Chatterjea, Fixed-point theorems, *Comptes Rendus de l'Acadmie Bulgare des Sci.,* **25** (1972), 727–730.
7. B. S. Choudhury, Unique fixed point theorem for weak C-contractive mappings, *Kathmandu Univ. J. Sci., Eng. Technol.,* **5** (2009), 6–13.
8. A. Razani, V. Parvaneh, Some fixed point theorems for weakly T-Chatterjea and weakly T-Kannan-contractive mappings in complete metric spaces, *Russ. Math. (Izv. VUZ),* **57** (2013), 38–45.
9. A. Alam, M. Imdad, Relation-theoretic contraction principle, *J. Fixed Point Theory Appl.,* **17** (2015), 693–702.
10. M. U. Ali, Y. Guo, F. Uddin, H. Aydi, K. Javed, Z. Ma, On partial metric spaces and related fixed point results with applications, *J. Funct. Spaces,* (2020), 6671828.
11. P. Gopi, K. Deepak, Fixed point theorems in relational metric spaces with an application to boundary value problems, *J. Part. Differ. Equ.,* **34** (2021), 83–93.
12. F. Uddin, C. Park, K. Javed, M. Arshad, J. R. Lee, Orthogonal m-metric spaces and an application to solve integral equations, *Adv. Differ. Equ.,* (2021), 159.
13. A. Alam, M. Imdad, Nonlinear contractions in metric spaces under locally T-transitive binary relations, *Fixed Point Theory,* **19** (2018), 13–24.
14. H. A. Hammad, M. De la Sen, A solution of Fredholm integral equation by using the cyclic η_s^q-rational contractive mappings technique in b-metric-like spaces, *Symmetry,* **11** (2019), 1184.
15. H. A. Hammad, M. De la Sen, Solution of nonlinear integral equation via fixed point of cyclic α_L^ψ-rational contraction mappings in metric-like spaces, *Bull. Braz. Math. Soc. New Ser.,* **51** (2020), 81–105.
16. H. A. Hammad, M. De la Sen, A coupled fixed point technique for solving coupled systems of functional and nonlinear integral equations, *Mathematics,* **7** (2019), 634.
17. G. Jungck, Commuting maps and fixed points, *Amer. Math. Mon.,* **83** (1976), 261–263.
18. G. Jungck, Common fixed points for noncontinuous non self maps on non-metric space, *Far. East. J. Math. Sci.,* **4** (1996), 199–215.
19. S. Lipschutz, *Schaums outlines of theory and problems of set theory and related topics*, McGraw-Hill, New York, 1964.
20. R. D. Maddux, *Relation algebras, studies in logic and the foundations of mathematics*, Elsevier, Amsertdam, 2006.
21. V. Flaška, J. Ježek, T. Kepka, J. Kortelainen, Transitive closures of binary relations I, *Acta Univ. Carolin. Math. Phys.,* **48** (2007), 55–69.

22. H. L. Skala, Trellis theory, *Algebr. Univ.,* **1** (1971), 218–233.

23. A. Stouti, A. Maaden, Fixed point and common fixed point theorems in pseudo-ordered sets, *Proyecciones,* **32** (2013), 409–418.

24. B. Samet, M. Turinici, Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications, *Commun. Math. Anal.,* **13** (2012), 82–97.

25. B. Kolman, R. C. Busby, S. Ross, *Discrete mathematical structures*, 3 Eds, PHI Pvt. Ltd., New Delhi, 2000.

26. A. Alam, M. Imdad, Relation-theoretic metrical coincidence theorems, *Filomat,* **31** (2015), 693–702.

27. S. Sessa, On a weak commutativity conditon of mappings in fixed point consideration, *Publ. Inst. Math. Soc.,* **32** (1982), 149–153.

28. G. Jungck, Compatible mappings and common fixed points, *Int. J. Math. Sci.,* **9** (1986), 771–779.

29. A. F. Roldán-López-de-Hierro, E. Karapinar, M. de la sen, Coincidence point theorems in quasi-metric spaces without assuming the mixed monotone property and consequences in G-metric spaces, *Fixed Point Theory Appl.,* (2014), 184.

30. J. Harjani, B. López, K. Sadarangani, Fixed point theorems for weakly C-contractive mappings in ordered metric spaces, *Comput. Math. Appl.,* **61** (2011), 790–796.