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Research article

Ground state solutions for periodic Discrete nonlinear Schrödinger equations

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Abstract: In this paper, we consider the following periodic discrete nonlinear Schrödinger equation

$$Lu_n - \omega u_n = g_n(u_n), \qquad n = (n_1, n_2, ..., n_m) \in \mathbb{Z}^m,$$

where $\omega \notin \sigma(L)$(the spectrum of L) and $g_n(s)$ is super or asymptotically linear as $|s| \to \infty$. Under weaker conditions on g_n, the existence of ground state solitons is proved via the generalized linking theorem developed by Li and Szulkin and concentration-compactness principle. Our result sharply extends and improves some existing ones in the literature.

Keywords: discrete nonlinear Schrödinger equation; ground state; superlinear; asymptotically linear; periodic potential
Mathematics Subject Classification: Primary: 35Q55; Secondary: 35Q51, 39A12, 39A70

1. Introduction and main results

In this paper, we are interested in the following periodic discrete nonlinear Schrödinger (DNLS) equation in m dimensional lattices

$$i\dot{\psi}_n = -\Delta\psi_n + \varepsilon_n\psi_n - g_n(\psi_n), \quad n \in \mathbb{Z}^m, \tag{1.1}$$

where

$$\begin{aligned}\Delta\psi_n =& \psi_{(n_1+1,n_2,...,n_m)} + \psi_{(n_1,n_2+1,...,n_m)} + \cdots + \psi_{(n_1,n_2,...,n_m+1)} \\ &- 2m\psi_{(n_1,n_2,...,n_m)} + \psi_{(n_1-1,n_2,...,n_m)} + \psi_{(n_1,n_2-1,...,n_m)} + \cdots + \psi_{(n_1,n_2,...,n_m-1)}\end{aligned}$$

is the discrete Laplacian in m spatial dimension and $\{\varepsilon_n\}$ is a real-valued and T-periodic in n, i.e., for $n = (n_1, n_2, ..., n_m) \in \mathbb{Z}^m$,

$$\varepsilon_{(n_1+T_1,n_2,...,n_m)} = \varepsilon_{(n_1,n_2+T_2,...,n_m)} = \cdots = \varepsilon_{(n_1,n_2,...,n_m+T_m)} = \varepsilon_{(n_1,n_2,...,n_m)},$$

here $T = (T_1, T_2, ..., T_m)$, T_i is a positive integer, $i = 1, 2, ..., m$. Moreover, we assume the nonlinearity $g_n(u)$ is T-periodic in n and gauge invariant in u, i.e.,

$$g_n(e^{i\theta}u) = e^{i\theta}g_n(u), \quad \theta \in \mathbb{R}.$$

Since breathers are spatially localized time-periodic solutions and decay to zero at infinity, we know ψ_n has the form

$$\psi_n = u_n e^{-i\omega t} \quad \text{and} \quad \lim_{|n|\to\infty} \psi_n = 0.$$

Then (1.1) becomes to

$$-\Delta u_n + \varepsilon_n u_n - \omega u_n = g_n(u_n), \qquad n \in \mathbb{Z}^m, \tag{1.2}$$

and

$$\lim_{|n|\to\infty} u_n = 0, \tag{1.3}$$

where $|n| = |n_1| + |n_2| + \cdots + |n_m|$ is the length of multi-index n, u_n is a real-valued sequence, m is a positive integer and $\omega \in \mathbb{R}$ is the temporal frequency. If $u_n \not\equiv 0$, then u is called a nontrivial solitons. Usually, if (1.3) holds, we say that a solution $u = \{u_n\}$ of (1.2) is homoclinic to 0.

The DNLS equation has been widely investigated because it is one of the most important inherently discrete models, ranging from solid state and condensed matter physics to biology. For examples, it has a physical meaning as a quantum Hamiltonian for a conduction electron in a magnetic field in a particular case of so-called tight-binding model (Aubry-Andre-Azbel-Harper model) [2, 4, 15], and also has been studied widely in nonlinear optics [8], biomolecular chains [20], Bose-Einstein condensates [23, 35] and the denaturation of the DNA double strand [19]. Here, we would like to mention the survey paper of Hennig and Tsironis [16] on wave transmission properties in one dimensional nonlinear lattices for an overview and more references cited in, some latest advances in both theory and applications of chaotic breather formation [17, 27], and some important works on discrete breathers [10–12]. We refer to [1, 3, 9, 13, 14] and the references cited therein for more physical background.

In recent years, there have enormous results devoting to the existence and multiplicity of discrete solitons of the DNLS equations. Results are obtained for such equations with superlinear nonlinearity [5, 24–26, 28, 29, 36] and saturable nonlinearity [6, 7, 30, 37, 38].

First, we deal with the suplinear case. To our knowledge, it seems that the earlier works investigating the existence of ground state solutions for problem (1.2) with superquadratic potentials is [25, 26]. In [25], Mai and Zhou considered equation (1.2) with $m = 1$ and superquadratic nonlinearity, they required the condition

(V_1) $\omega \notin \sigma(L)$, the spectrum of $L := -\Delta + \varepsilon_n$ and ω belonging to a finite gap (α, β);
(g_1) $g_n \in C(\mathbb{R}, \mathbb{R})$ and $g_n(\cdot) = g_{n+T}(\cdot)$ for all $n \in \mathbb{Z}^m$;
(g_2) $g_n(s) = o(s)$ as $|s| \to 0$ uniformly for all $n \in \mathbb{Z}^m$;
(g_3) $\frac{G_n(s)}{|s|^2} \to \infty$ as $|s| \to \infty$ uniformly for all $n \in \mathbb{Z}^m$, where $G_n(s) = \int_0^s g_n(t)\mathrm{d}t$;
(S_1) There exists $p > 2$ and $c > 0$ such that $|g_n(s)| \le c(1 + |s|^{p-1})$ for all $n \in \mathbb{Z}^m$,
(S_2) $s \mapsto \frac{g_n(s)}{|s|}$ is strictly increasing$(-\infty, 0) \cup (0, +\infty)$ for all $n \in \mathbb{Z}^m$.

and obtained ground state solutions by the generalized Nehari manifold approach developed by Szulkin and Weth [32]. In [26], Mai and Zhou considered equation (1.2) in m dimensional lattices replacing (S_2) by the weaker condition

(S_3) $s \mapsto \frac{g_n(s)}{|s|}$ is increasing$(-\infty, 0) \cup (0, +\infty)$ for all $n \in \mathbb{Z}^m$,

and followed the same way as in [22] to obtain the ground state solutions.

Motivated by papers [26,33], we will study the existence of ground state solutions of problem (1.2) under more general assumptions on g_n. More precisely, we make the following hypothesis:

(g_4) for each $n \in \mathbb{Z}^m$, $sg_n(s) \geq 0$, and there exists a constant $\eta_0 \in (0, 1)$ such that

$$\frac{1-\eta^2}{2} sg_n(s) \geq \int_{\eta s}^{s} g_n(\theta) \mathrm{d}\theta, \qquad \forall \eta \in [0, \eta_0].$$

Under our assumptions (g_1)-(g_2), the functional associated to (1.1)

$$\Phi(u) = \frac{1}{2}(Au, u)_E - \sum_{n \in \mathbb{Z}^m} G_n(u_n),$$

is of class C^1 on $E := l^2(\mathbb{Z}^m)$ (see section 2), and the critical points of Φ are weak solutions of (1.2).

Our main result is the following theorem.

Theorem 1.1. *If* (V_1), $(g_1) - (g_4)$ *hold, then* (1.2) *has a ground state soliton, i.e. a nontrivial soliton* v *such that* $\Phi(v) = \inf\{\Phi(u) \mid u \neq 0,\ \Phi'(u) = 0\}$. *Moreover, the solution* u *decays exponentially at infinity, that is, there exist constants* $C > 0$ *and* $\gamma > 0$ *such that*

$$|u_n| \leq Ce^{-\gamma|n|}, \quad n \in \mathbb{Z}^m.$$

Now we handle the asymptotically linear case. More precisely, we assume

(g_5) $g_n(s) - V_n s = y_n(s)$ with $\inf V_n > \beta - \alpha$, $y_n(s) = o(|s|)$ as $|s| \to \infty$ for all $n \in \mathbb{Z}^m$.

Theorem 1.2. *If* (V_1), (g_1), (g_2), (g_4), (g_5) *hold, then* (1.2) *has a ground state soliton, i.e. a nontrivial soliton* v *such that* $\Phi(v) = \inf\{\Phi(u) \mid u \neq 0,\ \Phi'(u) = 0\}$. *Moreover, the solution* u *decays exponentially at infinity, that is, there exist constants* $C > 0$ *and* $\gamma > 0$ *such that*

$$|u_n| \leq Ce^{-\gamma|n|}, \quad n \in \mathbb{Z}^m.$$

Remark 1.3. It is worth mentioning that we have removed condition (S_1) which has been commonly assumed to be satisfied in all existing works mentioned above. Next, it is easy to check that (S_3) implies (g_4). Moreover, let $g_n(s) = \varphi(s)$ for all n, where $\varphi(-s) = -\varphi(s)$ for $s \geq 0$, and

$$\varphi(s) = \begin{cases} 0 & 0 \leq s < \frac{1}{2}, \\ 2(s - \frac{1}{2}) & \frac{1}{2} \leq s < 1, \\ 3s^2 - \frac{15}{2}s^{\frac{3}{2}} + \frac{11}{2}s & s \geq 1, \end{cases}$$

it is not difficult to verify that $g_n(u)$ satisfies condition (g_4) with $\theta_0 = \frac{1}{10}$, but does not satisfy condition(S_3). This shows that condition (g_4) is weaker than condition (S_3). We remark that assumptions (S_1) and (S_3) play essential roles in verifying the isolation of trivial critical point and boundedness of Cerami seuqences for the energy functional Φ in [26]. Furthermore, it is not difficult to verify that (g_4) implies that $\mathcal{H}_n(s) := \frac{1}{2}g_n(s)u_n - G_n(s) \geq 0$, which is weaker than

(H) $\mathcal{H}_n(s) > 0$ if $s \neq 0$ and $\liminf_{|s|\to\infty} \mathcal{H}_n(s) > 0$,

We mention that (H) has made significant contributions to dealing with the asymptotically linear case, for example, see [6,31,37] along this direction. Therefore, our results would be applied to more general situations. To the best of our knowledge, there is no work devoted to handling with problem (1.2) under our weaker conditions. Hence our result is new and sharply extends and improves many existing ones in the literature, such as those in [6, 25, 26, 29, 31, 37].

Before we close this section, let us outline the proof of our main results and explain the difficulties we will encounter. To prove the existence of ground state solutions, we will adopt a technique developed in [22] (see also [18]. We set $m = \inf\{I(u) : u \in \mathcal{K} \setminus \{0\}\}$ where $\mathcal{K}$ denotes the critical points set of Φ. If $\mathcal{K} \setminus \{0\} \neq \emptyset$, then one can find a minimizing Cerami sequence for Φ on $\mathcal{K} \setminus \{0\}$. Next, the main task is to verify boundedness of this Cerami sequence. Finally, we must show that m is achieved at a nontrivial critical point. Fortunately, we can prove that all Cerami sequences of Φ are bounded (see Lemma 3.3 bellow). Then $\mathcal{K} \setminus \{0\} \neq \emptyset$ follows directly from the generalized linking theorem due to Li and Szulkin [21]. Furthermore, we can show that the zero function $\mathbf{0}$ is an isolated critical point of Φ (see Lemma 2.5 in section 2) which plays a crucial role in verifying that m is achieved at a nontrivial critical point.

The remaining of this paper is organized as follows. In Section 2, we establish the variational framework associated with (1.2), and give some preliminary lemmas. In Section 3, we present the detailed proofs of our main results.

2. Variational setting and preliminary lemmas

First, we will establish the variational framework associated with (1.2). Let

$$l^p \equiv l^p(\mathbb{Z}^m) = \Big\{u = \{u_n\}_{n\in\mathbb{Z}^m} : \forall n \in \mathbb{Z}^m, u_n \in \mathbb{R}, \|u\|_{l^p} = (\sum_{n\in\mathbb{Z}^m} |u_n|^p)^{\frac{1}{p}} < \infty\Big\}$$

Then the following embedding between l^p spaces holds,

$$l^q \subset l^p, \quad \|u\|_p \leq \|u\|_q, \quad 1 \leq q \leq p \leq \infty.$$

Let $L := -\Delta + \varepsilon_n$, $A := L - \omega$, $E := l^2(\mathbb{Z}^m)$, $(\cdot,\cdot)_E$ is the inner product in E, and the corresponding norm in E is denoted by $\|\cdot\|_E$. Then the derivative of Φ has the following formula,

$$\langle \Phi'(u), v\rangle = (Au, v)_E - \sum_{n\in\mathbb{Z}^m} g_n(u_n)v_n, \quad \forall v \in E. \tag{2.1}$$

Note that A is bounded and self-adjoint in E, we have $\sigma(A) \subset \mathbb{R}\backslash(\alpha-\omega, \beta-\omega)$ by (V_1). Therefore, E possesses the orthogonal decomposition

$$E = E^- \oplus E^+$$

corresponding to the spectrum of A such that

$$\begin{aligned} (Au, u) &\geq (\beta - \omega)\|u\|_E^2, \quad u \in E^+, \\ (Au, u) &\leq (\alpha - \omega)\|u\|_E^2, \quad u \in E^-. \end{aligned} \tag{2.2}$$

Obviously, $(Au, u)_E$ is positive on E^+ and negative on E^-, respectively. Moreover, we may define an equivalent norm $\|\cdot\|$ on $E^\pm$ by $\|u^\pm\|^2 = \pm(Au^\pm, u^\pm)_E$, $\forall u^\pm \in E^\pm$. Therefore, Φ can be rewritten as

$$\Phi(u) = \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \Psi(u), \tag{2.3}$$

where $u = u^- + u^+ \in E = E^- \oplus E^-$, $\Psi(u) = \sum_{n\in\mathbb{Z}^m} G_n(u_n)$. Then $\Phi, \Psi \in C^1(E, \mathbb{R})$ and the derivative is given by

$$\langle \Phi'(u), v\rangle = (u^+, v^+) - (u^-, v^-) - \langle \Psi'(u), v\rangle,$$

$$\langle \Psi'(u), v\rangle = \sum_{n\in\mathbb{Z}^m} g_n(u_n)v_n.$$

To state the generalized linking theorem of Li and Szulkin [21], we introduce some notation. let $R > r > 0$ and let $z_0 \in E^+$,$\|z_0\| = 1$. Set

$$N = \{u \in E^+ \mid \|u\| = r\}, \quad M = \{u \in E^- \oplus \mathbb{R}^+ z_0 \mid \|u\| \le R\}.$$

Then M is a submanifold of $E^- \oplus \mathbb{R}^+ z_0$ with boundary ∂M. The generalized linking theorem is stated as follows:

Proposition 2.1. *Assume that $\Psi \in C^1(E, \mathbb{R})$ is bounded from below, weakly sequentially lower semi-continuous and Ψ' is weakly sequentially continuous. Let Φ be a functional on E of the form* (2.3)*. If*

$$\kappa := \inf_N \Phi > \sup_{\partial M} \Phi,$$

then for some $\kappa \le c \le \sup_M \Phi$, there is a sequence $\{u^{(j)}\} \subset E$ such that

$$\Phi(u^{(j)}) \to c, \quad (1 + \|u^{(j)}\|)\|\Phi'(u^{(j)})\| \to 0. \tag{2.4}$$

Such a sequence is called a Cerami sequence on the level c, or a $(C)_c$ sequence.

We assume that (V_1) and (g_1), (g_2), (g_4) and (g_3) (or (g_5)) are satisfied from now on.

Lemma 2.2. Ψ *is non-negative and weakly sequentially lower semi-continuous, and* Ψ' *is weakly sequentially continuous.*

Proof. From (g_2) and (g_4), it is easy to see that $G_n(u_n) \ge 0$, hence $\Psi(u) \ge 0$ for all $u \in E$. Let $u^{(j)} \rightharpoonup u$ in E, one has $u_n^{(j)} \to u_n$ as $j \to \infty$ for all $n \in \mathbb{Z}^m$. Then $G_n(u_n^{(j)}) \to G_n(u_n)$ for all $n \in \mathbb{Z}^m$, since $G_n \in C^1(\mathbb{R}, \mathbb{R})$. Thus, by Fatou lemma, we deduce that

$$\Psi(u) = \sum_{n\in\mathbb{Z}^m} \lim_{j\to\infty} G_n(u_n^{(j)}) \le \liminf_{j\to\infty} \sum_{n\in\mathbb{Z}^m} G_n(u_n^{(j)}) = \liminf_{j\to\infty} \Psi(u^{(j)}),$$

proving that Ψ is weakly sequentially lower semi-continuous.

To show that Ψ' is weakly sequentially continuous, let $u^{(j)} \rightharpoonup u$ in E. We have that $u_n^{(j)} \to u_n$ as $j \to \infty$ for all $n \in \mathbb{Z}^m$, and there exists $C > 0$ such that $\|u^{(j)}\|_E \le C$ and $\|u\|_E \le C$. By (g_1) and (g_2), there exists $M > 0$, such that $|g_n(s)| \le M|s|$ for $|s| \le C$ and all $n \in \mathbb{N}$.

Given $v \in E$, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sum_{|n|>N} |v_n|^2 < \frac{\varepsilon^2}{16M^2C^2}. \tag{2.5}$$

For such N, by (g_1), for large j, we have

$$\Big| \sum_{|n|\le N} \Big(g_n(u_n^{(j)}) - g_n(u_n)\Big) v_n \Big| < \frac{\varepsilon}{2}. \tag{2.6}$$

Then using (2.5), (2.6) and Hölder's inequality in l^2, for j large enough, one has

$$\begin{aligned} |\Psi'(u^{(j)})v - \Psi'(u)v| &\le \Big| \sum_{|n|\le N} \Big(g_n(u_n^{(j)}) - g_n(u_n)\Big) v_n \Big| + \Big| \sum_{|n|>N} \Big(g_n(u_n^{(j)}) - g_n(u_n)\Big) v_n \Big| \\ &\le \frac{\varepsilon}{2} + M(\|u^{(j)}\|_E + \|u\|_E) \Big(\sum_{|n|>N} |v_n|^2 \Big)^{\frac{1}{2}} \\ &< \varepsilon, \end{aligned}$$

which implies the weakly sequentially continuity of Ψ'. □

Next, we discuss the linking structure of the function Φ. In the following, for the asymptotically quadratic case we set $\theta = \inf V_n$ and for the superquadratic case we choose $\theta = 2(\beta - \omega)$. Take a number μ satisfying

$$\beta - \omega < \mu < \theta. \tag{2.7}$$

Since $\sigma(H)$ is absolutely continuous, the subspace $Y_0 := \Big(P_\mu - P_{\beta-\omega}\Big) l^2$ is infinite dimensional, where $(P_\lambda)_{\lambda\in\mathbb{R}}$ denotes the spectrum family of H. By definition and (2.2),

$$Y_0 \subset E^+ \quad \text{and} \quad (\beta - \omega)\|u\|_2^2 \le \|u\|^2 \le \mu\|u\|_2^2, \quad \forall u \in Y_0.$$

Lemma 2.3. *The following statements hold true:*

(a) *There exists $r > 0$ such that $\kappa := \inf_N \Phi > 0$;*
(b) *For fixed $z_0 \in Y_0$ with $\|z_0\| = 1$, there is $R > r > 0$ such that $\sup_{\partial M} \Phi \le 0$.*

Proof. (a) For any $u \in E^+$, $\Phi(u) = \frac{1}{2}\|u\|^2 - \sum_{n\in\mathbb{Z}^m} G_n(u_n)$. Under assumption (g_1) and (g_2), there exists $\eta(\varepsilon) = o(\varepsilon)$ such that $|G_n(x)| \le \eta(\varepsilon)|x|^2$ holds for all $|x| \le \varepsilon$ and $n \in \mathbb{Z}^m$. Then it is easy obtain that $\sum_{n\in\mathbb{Z}^m} G_n(u_n) = o(\|u\|^2)$. So the inequality follows if we assume that $r > 0$ is sufficiently small.

(b) Arguing indirectly, suppose that, there exists $u^{(j)} = s^{(j)} z_0 + u^{(j)^-}$ with $\|u^{(j)}\| \to \infty$ as $j \to \infty$ such that $\Phi(u^{(j)}) \ge 0$ for all $j \in \mathbb{N}$. Setting $v^{(j)} = u^{(j)}/\|u^{(j)}\|$, we have $\|v^{(j)}\| = 1$. Then

$$0 \le \frac{\Phi(u^{(j)})}{\|u^{(j)}\|^2} = \frac{1}{2}(s^{(j)})^2 - \frac{1}{2}\|v^{(j)^-}\|^2 - \sum_{n\in\mathbb{Z}^m} \frac{G_n(u_n^{(j)})}{(u_n^{(j)})^2}(v_n^{(j)})^2). \tag{2.8}$$

Since $G_n(u_n) \ge 0$, we have

$$\|v^{(j)^-}\|^2 \le (s^{(j)})^2 = 1 - \|v^{(j)^-}\|^2,$$

which implies that $\|v^{(j)^-}\| \leq \frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}} \leq s^{(j)} \leq 1$. Passing to a subsequence, we may assume that $s^{(j)} \to s \in [\frac{1}{\sqrt{2}}, 1]$, $v^{(j)} \rightharpoonup v$ in E, $v^{(j)^-} \rightharpoonup v^-$, and $v_n^{(j)} \to v_n$ for every n. It follows that $v = sz_0 + v^- \neq 0$.

First we consider the super linear case. Recalling that $v \neq 0$, there exists $n_0 \in \mathbb{Z}^m$ such that $v_{n_0} \neq 0$ and $|u_{n_0}^{(j)}| = \|u^{(j)}\| \cdot |v_{n_0}^{(j)}| \to \infty$ as $k \to \infty$. Hence, it follows from (g_3) and Fatou's lemma that

$$\sum_{n\in\mathbb{Z}^m} \frac{G_n(u_n^{(j)})}{(u_n^{(j)})^2}(v_n^{(j)})^2 \to \infty,$$

contradicting with (2.8).

Next we consider the asymptotically linear case. Noting that $v^+ = sz_0 \neq 0$, by (g_5) and (2.7), there holds

$$\begin{aligned}\left\|v^+\right\|^2 - \left\|v^-\right\|^2 - \sum_{n\in\mathbb{Z}^m} V_n v_n^2 &\leq \left\|v^+\right\|^2 - \left\|v^-\right\|^2 - \theta|v|_2^2 \\ &\leq -\left(\left(\frac{\theta}{\mu} - 1\right)\left\|v^+\right\|^2 + \left\|v^-\right\|^2\right) \\ &< 0.\end{aligned}$$

Hence there exixts a finite set $A \subset \mathbb{Z}^m$ such that

$$\left\|v^+\right\|^2 - \left\|v^-\right\|^2 - \sum_{n\in A} V_n v_n^2 < 0. \tag{2.9}$$

Set

$$F_n(t) := G_n(t) - \frac{1}{2}V_n t^2.$$

By (g_1), (g_2) and (g_5), one gets that $|F_n(t)| \leq C|t|^2$ and $F_n(t)/t^2 \to 0$ as $|t| \to \infty$ uniformly in n. Noting that

$$\left|\sum_{n\in A} \frac{F_n\left(u_n^{(j)}\right)}{\left\|u^{(j)}\right\|^2}\right| = \left|\sum_{n\in A} \frac{F_n\left(u_n^{(j)}\right)\left|v_n^{(j)}\right|^2}{\left|u_n^{(j)}\right|^2}\right| \leq \sum_{n\in A} \frac{\left|F_n\left(u_n^{(j)}\right)\right|\left|v_n^{(j)}\right|^2}{\left|u_n^{(j)}\right|^2},$$

it follows from Lebesgue's dominated convergence theorem and the fact $\sum_{n\in A} |v_n^{(j)} - v_n|^2 \to 0$ that

$$\sum_{n\in A} \frac{F_n\left(u_n^{(j)}\right)\left|v_n^{(j)}\right|^2}{\left|u_n^{(j)}\right|^2} \to 0.$$

Therefore, by (2.8) and (2.9), we conclude

$$\begin{aligned}0 &\leq \lim_{j\to\infty}\left(\frac{1}{2}\left\|v^{(j)^+}\right\|^2 - \frac{1}{2}\left\|v^{(j)^-}\right\|^2 - \frac{\sum_{n\in A} G_n\left(u_n^{(j)}\right)}{\left\|u^{(j)}\right\|^2}\right) \\ &\leq \frac{1}{2}\left(\left\|v^+\right\|^2 - \left\|v^-\right\|^2 - \sum_{n\in A} V_n v_n^2\right) \\ &< 0.\end{aligned}$$

a contradiction. □

According to Lemma 2.3 and proposition 2.1, we obtain the following result.

Lemma 2.4. *Suppose that* (V_1) *and* (g_1), (g_2), (g_4) *and* (g_3) *(or* (g_5)*) are satisfied, then for the functional* Φ, *there exists a* $(C)_c$ *sequence* $\{u^{(m)}\}$ *with* $c > 0$.

Let $\mathcal{K} := \{u \in E : \Phi'(u) = 0\}$ denote the set of all the critical points of Φ. We have the following result which plays an important role in showing the ground state solution is nontrivial.

Lemma 2.5. *The zero function* ***0*** *is an isolated critical point of* Φ. *i.e.,* $\nu := \inf\{\|u\| : u \in \mathcal{K} \setminus \{0\}\} > 0$.

Proof. Arguing indirectly, assume that there is a sequence $\{u^{(i)}\} \subset \mathcal{K} \setminus 0$ such that $\|u^{(i)}\| \to 0$. By the embedding of E into l^∞,

$$\|u^{(i)}\|_\infty \to 0. \tag{2.10}$$

Since $\|u^{(i)^\pm}\|_2 \le C\|u^{(i)}\|$, jointly with $\Phi'(u^{(i)})u^{(i)^\pm} = 0$, by using (2.10), (g_1), (g_2) and Hölder's inequality, for i large enough and ε sufficiently small, we conclude that

$$\begin{aligned}
\|u^{(i)}\|^2 &= \sum_{n\in\mathbb{Z}^m} g_n(u_n^{(i)})(u_n^{(i)^+} - u_n^{(i)^-}) \\
&\le \varepsilon \sum_{n\in\mathbb{Z}^m} |u_n^{(i)}||u_n^{(i)^+} - u_n^{(i)^-}| \\
&\le \varepsilon\Big(\sum_{n\in\mathbb{Z}^m} |u_n^{(i)}|^2\Big)^{\frac{1}{2}}\Big(\sum_{n\in\mathbb{Z}^m} |u_n^{(i)^+}|^2\Big)^{\frac{1}{2}} + \varepsilon\Big(\sum_{n\in\mathbb{Z}^m} |u_n^{(i)}|^2\Big)^{\frac{1}{2}}\Big(\sum_{n\in\mathbb{Z}^m} |u_n^{(i)^-}|^2\Big)^{\frac{1}{2}} \\
&\le 2C\varepsilon\|u^{(i)}\|^2,
\end{aligned}$$

a contradiction. □

3. Proof of main Theorems

In order to prove the boundedness of $(C)_c$-sequence of Φ, we need the following important technical lemma.

Lemma 3.1. *Suppose that* (g_2) *and* (g_4) *are satisfied. Let* $u, v, r \in \mathbb{R}$ *be numbers with* $r \ge 0$ *and* $|rv| \le \eta_0|u|$. *Then for every n,*

$$\frac{1+r^2}{2}g_n(u)u - r^2 g_n(u)v + G_n(rv) - G_n(u) \ge 0.$$

Proof. Fix $u, v \in \mathbb{R}$. Set

$$\kappa_n(r) = \frac{1+r^2}{2}g_n(u)u - r^2 g_n(u)v + G_n(rv) - G_n(u).$$

If $uv \le 0$, then for $r \ge 0$, it follows from (g_2) and (g_4) that

$$\begin{aligned}
\kappa_n(r) &= \frac{1+r^2}{2}g_n(u)u - r^2 g_n(u)v + G_n(rv) - G_n(u) \\
&\ge \frac{1+r^2}{2}g_n(u)u - G_n(u) \\
&\ge 0.
\end{aligned} \tag{3.1}$$

If $uv \geq 0$, set $\eta = \frac{rv}{u}$. Assuming $\eta \leq \eta_0$, for $r \geq 0$, it follows from (g_2) and (g_4) that

$$\begin{aligned}\kappa_n(r) &= \frac{1+r^2}{2}g_n(u)u - r^2 g_n(u)v + G_n(rv) - G_n(u)\\ &= \frac{1+r^2-2\eta r}{2}g_n(u)u - \int_{\eta u}^{u} g_n(\theta)\mathrm{d}\theta\\ &= \frac{(\eta-r)^2}{2}g_n(u)u + \frac{1-\eta^2}{2}g_n(u)u - \int_{\eta u}^{u} g_n(\theta)\mathrm{d}\theta\\ &\geq \frac{1-\eta^2}{2}g_n(u)u - \int_{\eta u}^{u} g_n(\theta)\mathrm{d}\theta\\ &\geq 0.\end{aligned} \tag{3.2}$$

Then, (3.1) and (3.2) imply this lemma. □

Lemma 3.2. *Suppose that* (g_2) *and* (g_4) *are satisfied. Then for* $u \in E$*, there holds*

$$\Phi(u) - \Phi(su^+) \geq \frac{s^2\|u^-\|^2}{2} + \frac{(1-s^2)}{2}\langle \Phi'(u), u\rangle + s^2\langle \Phi'(u), u^-\rangle - \sum_{n\in B} s^2 g_n(u_n)u_n^+,$$

where $s \geq 0$*,* $B := \{n \mid |su_n^+| > \eta_0|u_n|,\ n \in \mathbb{Z}^m\}$ *and* η_0 *is given in* (g_4)*.*

Proof. Using Lemma (3.1), for $s \geq 0$, we have

$$\begin{aligned}\Phi(u) - \Phi(su^+) &= \frac{1}{2}((Au,u) - (Asu^+, su^+)) + \sum_{n\in\mathbb{Z}^m} G_n(su_n^+) - \sum_{n\in\mathbb{Z}^m} G_n(u_n)\\ &= \frac{1}{2}((1-s^2)(Au,u) + s^2(Au,u^-)) + \sum_{n\in\mathbb{Z}^m} G_n(su_n^+) - \sum_{n\in\mathbb{Z}^m} G_n(u_n)\\ &= \frac{s^2\|u^-\|^2}{2} + \frac{(1-s^2)}{2}\langle \Phi'(u), u\rangle + s^2\langle \Phi'(u), u^-\rangle\\ &\quad + \sum_{n\in\mathbb{Z}^m}\Big(\frac{(1-s^2)}{2}g_n(u_n)u_n + s^2 g_n(u_n)u_n^- + G_n(su_n^+) - G_n(u_n)\Big)\\ &= \frac{s^2\|u^-\|^2}{2} + \frac{(1-s^2)}{2}\langle \Phi'(u), u\rangle + s^2\langle \Phi'(u), u^-\rangle\\ &\quad + \sum_{n\in\mathbb{Z}^m}\Big(\frac{(1+s^2)}{2}g_n(u_n)u_n - s^2 g_n(u_n)u_n^+ + G_n(su_n^+) - G_n(u_n)\Big)\\ &= \frac{s^2\|u^-\|^2}{2} + \frac{(1-s^2)}{2}\langle \Phi'(u), u\rangle + s^2\langle \Phi'(u), u^-\rangle\\ &\quad + \sum_{n\in\mathbb{Z}^m\setminus B}\Big(\frac{(1+s^2)}{2}g_n(u_n)u_n - s^2 g_n(u_n)u_n^+ + G_n(su_n^+) - G_n(u_n)\Big)\\ &\quad + \sum_{n\in B}\Big(\frac{(1+s^2)}{2}g_n(u_n)u_n - s^2 g_n(u_n)u_n^+ + G_n(su_n^+) - G_n(u_n)\Big)\\ &\geq \frac{s^2\|u^-\|^2}{2} + \frac{(1-s^2)}{2}\langle \Phi'(u), u\rangle + s^2\langle \Phi'(u), u^-\rangle - \sum_{n\in B} s^2 g_n(u_n)u_n^+.\end{aligned}$$

□

Lemma 3.3. *Let $c \in \mathbb{R}$. Then any $(C)_c$ sequence of Φ is bounded.*

Proof. Let $\{u^{(j)}\}$ be a $(C)_c$ sequence of Φ, that is, $\{u^{(j)}\}$ satisfying (2.3). Suppose $\{u^{(j)}\}$ is unbounded. Then, passing to a subsequence, we assume that $\|u^{(j)}\| \to \infty$. Setting $h^{(j)} = u^{(j)}/\|u^{(j)}\|$, we have $\|h^{(j)}\| = 1$. Then, up to a subsequence, there exists $h \in E$ such that $h^{(j)} \rightharpoonup h$ in E. Let $h^{(j)^{\pm}}$ be the orthogonal projection of $h^{(j)}$ on $E^{\pm}$, respectively. Since $G_n(s) \geq 0$, for large j we have

$$c - 1 \leq \Phi(u^{(j)}) \leq \frac{1}{2}(\|u^{(j)^+}\|^2 - \|u^{(j)^-}\|^2).$$

This and $\|h^{(j)^+}\|^2 + \|h^{(j)^-}\|^2 = 1$ imply that $\|h^{(j)^+}\|^2 \geq \frac{1}{4}$. Given any $R > \sqrt{8(|c|+4)}$, noting that $\|h^{(j)^+}\|$ is bounded in E, there exists $\delta > 0$ and $n_j \in \mathbb{Z}^m$ such that

$$|h_{n_j}^{(j)^+}| \geq \delta. \tag{3.3}$$

Indeed, if not, then $h^{(j)^+} \to 0$ in l^∞ as $m \to \infty$. Under assumptions (g_1) and (g_2), there exists $\mu(\varepsilon) = o(\varepsilon)$ such that $|G_n(x)| \leq \mu(\varepsilon)|x|^2$ and $|g_n(x)| \leq \mu(\varepsilon)|x|$ hold for all $|x| \leq \varepsilon$ and $n \in \mathbb{Z}^m$. This implies that, as $j \to \infty$,

$$\sum_{n\in\mathbb{Z}^m} G_n(Rh_n^{(j)^+}) \leq \mu(\varepsilon)R^2\|h^{(j)^+}\|_2^2 \to 0, \tag{3.4}$$

and

$$\left|\sum_{n\in B} g_n(h_n^{(j)})h_n^{(j)^+}\right| \leq \mu(\varepsilon) \sum_{n\in\mathbb{Z}^m} |h_n^{(j)}||h_n^{(j)^+}| \leq \mu(\varepsilon)\|h^{(j)}\|_2\|h^{(j)^+}\|_2 \to 0. \tag{3.5}$$

Set $s^{(j)} = \frac{R}{\|u^{(j)}\|}$, by using (3.4), (3.5) and Lemma 3.2, for large j we have

$$\begin{aligned}
|c| + 1 \geq \Phi(u^{(j)}) &\geq \Phi(s^{(j)}u^{(j)^+}) + \frac{(s^{(j)})^2\|u^{(j)^-}\|^2}{2} + \frac{(1-(s^{(j)})^2)}{2}\langle\Phi'(u), u\rangle \\
&\quad + (s^{(j)})^2\langle\Phi'(u), u^-\rangle - \sum_{n\in B}(s^{(j)})^2 g_n(u_n^{(j)})u_n^{(j)^+} \\
&= \Phi(Rh^{(j)^+}) + \frac{R^2\|h^{(j)^-}\|^2}{2} - \frac{R^2}{\|u^{(j)}\|}\sum_{n\in B} g_n(h_n^{(j)})h_n^{(j)^+} + o(1) \\
&= \frac{R^2}{2} - \frac{R^2}{\|u^{(j)}\|}\sum_{n\in B} g_n(h_n^{(j)})h_n^{(j)^+} - \sum_{n\in\mathbb{Z}^m} G_n(Rh_n^{(j)^+}) + o(1) \\
&\geq \frac{R^2}{2} - 1,
\end{aligned}$$

this yields a contradiction since $R > \sqrt{8(|c|+4)}$.

For the super linear case, from the periodicity of the coefficients, we know Φ and Φ' are both invariant under translation, i.e., $v^{(j)} = \{v_n^{(j)}\} = \{h_{n+KT}^{(j)}\}$. Making such shifts, we can assume that $1 \leq n_j \leq T - 1$ in (3.3), moreover passing to a subsequence, we can assume that $n_j = n_0$ is independent of j. Next we may extract a subsequence, still denoted by $h^{(j)}$, such that $h^{(j)^+} \to h^+$ and $h_n^{(j)^+} \to h_n^+$ for all n. Specially, for $n = n_0$, inequality (3.3) shows that $|h_{n_0}^+| \geq \delta$. Moreover, we have

$$\sum_{n\in\mathbb{Z}^m} \frac{G_n(u_n^{(j)})}{(u_n^{(j)})^2}(h_n^{(j)^+})^2 \to \infty \quad \text{as } j \to \infty.$$

Consequently, we obtain

$$0 \leq \frac{\Phi(u^{(j)})}{\|u^{(j)}\|^2} = \frac{1}{2}\|h^{(j)^+}\|^2 - \frac{1}{2}\|h^{(j)^-}\|^2 - \sum_{n\in\mathbb{Z}^m} \frac{G_n(u_n^{(j)})}{(u_n^{(j)})^2}(h_n^{(j)^+})^2 \to -\infty,$$

as $j \to \infty$, which is a contradiction.

For the asymptotically linear case, let $n \in \mathbb{Z}^m$ be such that $h_n \neq 0$, then $\left|u_n^{(j)}\right| = \left|h_n^{(j)}\right| \cdot \left\|u^{(j)}\right\| \to \infty$ as $j \to \infty$. We denote l_0 the vector space of all finite sequences, i.e., sequences $u = \{u(n)\}$ such that $suppu = \{n \in \mathbb{Z}^m : u(n) \neq 0\}$ is a finite set. It is well known that l_0 is a dense subspace of l^p with $1 \leq p < \infty$. For any $\phi \in l_0$, we have

$$\begin{aligned}\left\langle \Phi'\left(u^{(j)}\right), \phi\right\rangle &= \left(u^{(j)^+} - u^{(j)^-}, \phi\right) - \sum_{n\in\mathbb{Z}^m} V_n u_n^{(j)}\phi_n - \sum_{n\in\mathbb{Z}^m} y_n\left(u_n^{(j)}\right)\phi_n \\ &= \left\|u^j\right\|\left[\left(\left(h^{(j)}\right)^+ - \left(h^{(j)}\right)^-, \phi\right) - \sum_{n\in\mathbb{Z}^m} V_n h_n^{(j)}\phi_n - \sum_{n\in\mathbb{Z}^m} \frac{y_n\left(u_n^{(j)}\right)}{u_n^{(j)}} h_n^{(j)}\phi_n\right].\end{aligned}$$

From (2.4), we derive

$$\left(h^{(j)^+} - h^{(j)^-}, \phi\right) - \sum_{n\in\mathbb{Z}^m} V_n h_n^{(j)}\phi_n - \sum_{n\in\mathbb{Z}^m} \frac{y_n\left(u_n^{(j)}\right)}{u_n^{(j)}} h_n^{(j)}\phi_n = o(1).$$

Note that

$$\begin{aligned}\left|\sum_{n\in\mathbb{Z}^m} \frac{y_n\left(u_n^{(j)}\right)}{u_n^{(j)}} h_n^{(j)}\phi_n\right| &\leq \sum_{n\in\mathbb{Z}^m} \left|\frac{y_n\left(u_n^{(j)}\right)}{u_n^{(j)}}\right| \left|h_n^{(j)} - h_n\right| |\phi_n| + \sum_{n\in\mathbb{Z}^m} \left|\frac{y_n\left(u_n^{(j)}\right)}{u_n^{(j)}}\right| |h_n|\, |\phi_n| \\ &\leq C \sum_{n\in \mathrm{supp}\phi} \left|h_n^{(j)} - h_n\right| |\phi_n| + \sum_{\{n\in\mathbb{Z}^m : h_n\neq 0\}} \left|\frac{y_n\left(u_n^{(j)}\right)}{u_n^{(j)}}\right| |h_n|\, |\phi_n| \\ &= o(1).\end{aligned}$$

Therefore,

$$(h^+ - h^-, \phi) - \sum_{n\in\mathbb{Z}^m} V_n h_n \phi_n = 0,$$

i.e.,

$$((L-\omega)h, \phi)_E = \sum_{n\in\mathbb{Z}^m} V_n h_n \phi_n.$$

This gives a contradiction since it is well known that the operator $L - \omega - V$ has no eigenvalue in E, where the operator V is defined as follows:

$$V : E \to E, \quad (Vu)_n = V_n u_n.$$

Thus $\left\{u^{(j)}\right\}$ is bounded and so the lemma is proved. □

Now we are in a position to give the proof of our Theorems.

Proof of Theorem 1.1. By Lemmas 2.5 and 3.3, Φ has a bounded $(C)_c$ sequence $\{u^{(j)}\}$, where $c > 0$. Noting that $\|u^{(j)^+}\|$ is bounded in E, there exists $\delta > 0$ and $n_j \in \mathbb{Z}^m$ such that

$$|u_{n_j}^{(j)^+}| \geq \delta. \tag{3.6}$$

Indeed, if not, then $u^{(j)^+} \to 0$ in l^∞ as $j \to \infty$. Under assumption (g_1) and (g_2), there exists $\mu(\varepsilon) = o(\varepsilon)$ such that $|g_n(x)| \leq \mu(\varepsilon)|x|$ holds for all $|x| \leq \varepsilon$ and $n \in \mathbb{N}$. This implies that, as $j \to \infty$,

$$\sum_{n\in\mathbb{Z}^m} |g_n(u_n^{(j)})u_n^{(j)^+}| \leq \mu(\varepsilon) \sum_{n\in\mathbb{Z}^m} |u_n^{(j)}||u_n^{(j)^+}| \leq \mu(\varepsilon)\|u^{(j)}\|_2\|u^{(j)^+}\|_2 \to 0.$$

Therefore, we have

$$\Phi(u^{(j)}) \leq \frac{1}{2}\|u^{(j)^+}\|^2 = \frac{1}{2}\langle\Phi'(u^{(j)}), u^{(j)^+}\rangle + \frac{1}{2}\sum_{n\in\mathbb{Z}^m} g_n(u_n^{(j)})u_n^{(j)^+} \to 0, \quad \text{as } j \to \infty.$$

This contradicts with the fact that $\Phi(u^{(j)}) \geq \kappa$. From the periodicity of the coefficients, we know Φ and Φ' are both invariant under translation, up to a translation of indices, we can assume that $1 \leq n_j \leq T-1$ in (3.6). Passing to a subsequence, we can assume that $n_j = n_0$ is independent of j. Next we may assume that $u^{(j)} \rightharpoonup \widetilde{u}$ in E and $u_n^{(j)^+} \to \widetilde{u}_n^+$ for all n. From (3.6), one gets $|\widetilde{u}_{n_0}^+| \geq \delta$. Moreover, we have

$$\langle\Phi'(\widetilde{u}), \varphi\rangle = \lim_{j\to\infty}\langle\Phi'(u^{(j)}), \varphi\rangle = 0, \quad \forall\varphi \in E,$$

that is, $\widetilde{u}$ is a nontrivial critical point of Φ.

To get ground state solution, set $c = \inf\{\Phi(u) \mid u \in \mathcal{K} \setminus \{\mathbf{0}\}\}$. By (g_4), a straightforward computation deduces that

$$\mathcal{H}_n(u_n) = \frac{1}{2}g_n(u_n)u_n - G_n(u_n) \geq 0. \tag{3.7}$$

Therefore for any $u \in \mathcal{K}$, we have

$$\Phi(u) = \Phi(u) - \frac{1}{2}\Phi'(u)u = \sum_{n\in\mathbb{Z}^m} (\frac{1}{2}g_n(u_n)u_n - G_n(u_n)) \geq 0.$$

That is, $0 \leq c \leq \Phi(\widetilde{u})$, where $\widetilde{u}$ is the nontrivial critical point found before.

Suppose $\{u^{(j)}\} \subset \mathcal{K}$ such that $\Phi(u^{(j)}) \to c$. Then $\{u^{(j)}\}$ is a $(C)_c$ sequence. By Lemma 3.3, $\{u^{(j)}\}$ is bounded. For this sequence,we claim that there exists $\delta > 0$ and $n_j \in \mathbb{Z}^m$ such that $|u_{n_j}^{(j)}| \geq \delta$. Indeed, if not, then $u^{(j)^\pm} \to 0$ in l^∞ as $j \to \infty$. With the same argument in (3.5), we know

$$\begin{aligned}\sum_{n\in\mathbb{Z}^m} |g_n(u_n^{(j)})u_n^{(j)^\pm}| &\leq \mu(\varepsilon) \sum_{n\in\mathbb{Z}^m} |u_n^{(j)}||u_n^{(j)^\pm}| \\ &\leq \mu(\varepsilon)\|u^{(j)}\|_2\|u^{(j)^\pm}\|_2 \to 0,\end{aligned}$$

which implies

$$\|u^{(j)}\|^2 = \Phi'(u^{(j)})(u^{(j)^+} - u^{(j)^-}) + \sum_{n\in\mathbb{Z}^m} g_n(u_n^{(j)})(u_n^{(j)^+} - u_n^{(j)^-}) \to 0.$$

This contradicts with Lemma 2.5. Therefore, $|u_{n_j}^{(j)}| \geq \delta$. With the same argument above, after a suitable translation, a subsequence of $\{u^{(j)}\}$ converges weakly to some $v \neq 0$, a nontrivial critical of Φ. Then by (3.7) and Fatou's lemma, we have

$$\begin{aligned}\Phi(v) = \Phi(v) - \frac{1}{2}\Phi'(v)v = \sum_{n\in\mathbb{Z}^m} \mathcal{H}_n(v_n) &\leq \lim_{j\to\infty} \sum_{n\in\mathbb{Z}^m} \mathcal{H}_n(u_n^{(j)}) \\ &= \lim_{j\to\infty} \left(\Phi(u^{(j)}) - \frac{1}{2}\Phi'(u^{(j)})u^{(j)}\right) \\ &= \lim_{j\to\infty} \Phi(u^{(j)}) = c.\end{aligned}$$

Hence v is a nontrivial critical point of Φ with $\Phi(v) = c$, and Theorem 1.1 is proved. Moreover, one can follow the same way as in the proof of Theorem 6.1 in [28] to prove that the solution obtained above decays exponentially at infinity, that is, there exist constants $C > 0$ and $\gamma > 0$ such that $|u_n| \leq Ce^{-\gamma|n|}$ for $n \in \mathbb{Z}^m$, we omit it here. The proof is complete. □

Proof of Theorem 1.2. The proof is similar to that of Theorem 1.1 and is omitted. □

4. Conclusions

This study set out to prove the existence of ground state solitons of the discrete nonlinear Schrödinger equation in m dimensional lattices under weaker conditions on g_n. In general, therefore, it seems that our results woulds be applied to more general situations. The present study sharply extends and improves many existing ones in the literature. We hope that this work will bring a new perspective for researchers.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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