Mathematics

Research article

Existence and uniqueness results for a nonlinear singular fractional differential equation of order $\sigma \in (1, 2)$

Sinan Serkan Bilgici and Müfit ŞAN[*]

Department of Mathematics, Çankırı Karatekin University, Faculty of Sicence, Uluyazı Campus, 18100, Çankırı, Turkey

* **Correspondence:** Email: mufitsan@karatekin.edu.tr; Tel: +903782189500.

Abstract: The first objective of this article is to discuss the local existence of the solution to an initial value problem involving a non-linear differential equation in the sense of Riemann-Liouville fractional derivative of order $\sigma \in (1, 2)$, when the nonlinear term has a singularity at zero of its independent argument. Hereafter, by using some tools of Lebesgue spaces such as Hölder inequality, we obtain Nagumo-type, Krasnoselskii-Krein-type and Osgood-type uniqueness theorems for the problem.

Keywords: fractional differential equations; Riemann-Liouville derivative; existence and uniqueness; Hölder inequality; Lebesgue spaces
Mathematics Subject Classification: 34A08, 37C25, 34A12, 74G20, 26A33

1. Introduction

The importance of existence and uniqueness theorems for initial value and boundary value problems (IVP and BVP) involving the classical derivative operator is indisputable because, without them, one cannot understand modeled systems correctly and make predictions how they behave. Recently, with the popularity of fractional derivative operators such as Riemann-Liouville (R-L), Caputo (C), etc., the equations involving these operators have begun to be studied in detail (See, [1–6]). However, such a generalization leads to some difficulties and differences especially in R-L case. For instance, unlike the initial value problems involving the classical derivative, the existence of continuous solution to some IVPs in the sense of R-L derivative strictly depends on the initial values and smoothness conditions on nonlinear functions in right-hand side of equations in IVPs. For example, as shown in [7], the following initial value problem for R-L derivative of order $\sigma \in (0, 1)$

$$\begin{cases} D^\sigma \omega(x) = f(x, \omega(x)), & x > 0 \\ \omega(0) = b \neq 0 \end{cases} \tag{1.1}$$

has no continuous solution when f is continuous on $[0, T] \times \mathbb{R}$. Therefore, this equation was investigated when subjected to different initial values (See, Theorem 3.3 in [1]). By considering the aforementioned remark, we investigate the existence and uniqueness of solutions to the following problem

$$\begin{cases} D^{\sigma} \omega(x) = f(x, \omega(x), D^{\sigma-1} \omega(x)), & x > 0 \\ \omega(0) = 0, & D^{\sigma-1} \omega(x)|_{x=0} = b, \end{cases} \tag{1.2}$$

where $\sigma \in (1, 2)$, $b \in \mathbb{R}$, D^{σ} represents the Riemann-Liouville fractional derivative of order σ, which is given by

$$D^{\sigma} \omega(x) = \frac{1}{\Gamma(2 - \sigma)} \frac{d^2}{dx^2} \int_0^x \frac{\omega(t)}{(x - t)^{\sigma-1}} dt,$$

and f fulfills the following condition:

(C1)   Let $f(x, t_1, t_2) \in C((0, T] \times \mathbb{R} \times \mathbb{R})$ and $x^{\sigma-1} f(x, t_1, t_2) \in C([0, T] \times \mathbb{R} \times \mathbb{R})$,

where $C(X)$ represents the class of continuous functions defined on the subspace X of $\mathbb{R}^3$.

The equation in (1.2) was first considered by Yoruk et. al. [8], when the second initial value is also homogeneous ($b = 0$) and the right-hand side function is continuous on $[0, T] \times \mathbb{R} \times \mathbb{R}$. They gave Krasnoselskii-Krein, Roger and Kooi-type uniqueness results. As seen from condition (C1), the equation in (1.2) has a singularity at $x = 0$. Such singular equations involving R-L and Caputo derivatives were recently investigated in [7, 9–12] and they proved local or global existence theorem for initial value problems involving singular equations. In the investigation of existence of solutions to these problems, converting the problem into the Volterra-type integral equation is one of the referenced tools. These integral equations have weakly and double singularities since the corresponding fractional differential equations are singular. For this reason, some new techniques or lemmas had to be developed to reveal the existence and uniqueness of solutions to integral equations (See for example [10, 11]). In this paper, we also encounter a Volterra-type integral equations having a single singularity when we transform the problem (1.2) into a integral equation. For the uniqueness of solutions to the integral equation we need to generalize the definitions and methods previously given in [7, 8, 13] and use the tools of Lebesgue spaces such as Hölder inequality. These uniqueness theorems are the type of Nagumo-type, Krasnoselskii-Krein and Osgood which are well-known in the literature (See, [13–16]) Moreover, we give a Peano-type existence theorem for this problem as well.

2. Preliminaries

We first give definition of the space of functions where we investigate the existence of solutions to the problem (1.2). The space is defined below as in [17] :

Theorem 2.1. *The space of continuous functions defined on $[0, T]$, whose R-L fractional derivative of order $\sigma - 1$, $1 < \sigma < 2$ are continuous on $[0, T]$ is a Banach space when endowed with the following norm:*

$$\|\omega\|_{\sigma-1} = \|\omega\|_{\infty} + \|D^{\sigma-1} \omega\|_{\infty},$$

where $\|.\|_{\infty}$ is the supremum norm defined on the class of continuous functions. This space will be denoted by $C^{\sigma-1}([0, T])$.

According to this space, we define R-L integral and R-L derivative of higher order [11]. The lower terminal points of integrals in their formulas will be taken as zero.

Definition 2.1. *The R-L integral of order $\sigma > 0$ of a function $\omega(x) \in C^0[0, T]$ is defined for all $x \in [0, T]$ by*

$$I^\sigma \omega(x) := \frac{1}{\Gamma(\sigma)} \int_0^x \frac{\omega(t)}{(x-t)^{1-\sigma}} dt. \tag{2.1}$$

Definition 2.2. *For $1 < \sigma < 2$ and $\omega(x) \in C^{\sigma-1}[0, T]$ with $D^{\sigma-1}\omega(x) \in C^1(0, T] \cap L^1[0, T]$ the R-L fractional derivative $D^\beta \omega$ is defined for all $x \in (0, T]$ by*

$$D^\sigma \omega(x) = \frac{1}{\Gamma(2-\sigma)} \frac{d^2}{dx^2} \int_0^x \frac{\omega(t)}{(x-t)^{\sigma-1}} dt. \tag{2.2}$$

The local existence of solutions to problem (1.2) will be proved with the aid of Schauder fixed point theorem [18]:

Theorem 2.2. *Let C be a closed, bounded, convex subset of a Banach space $X := \{u : I \to \mathbb{R} \text{ continuous} : I \subset \mathbb{R} \text{ closed and bounded interval}\}$. If operator $\mathcal{S} : C \to C$ is continuous and, if $\mathcal{S}(C)$ is an equicontinuous set on I, then $\mathcal{S}$ has at least one fixed point in C.*

3. Main results

The one of mathematical tools used for showing the existence and uniqueness of the desired type of solution to a given initial or boundary value problem is first to convert them into an integral equation. One investigates the existence and uniqueness of the solution to the integral equation instead of the associated problem. Here, we follow this way by taking the aid of the lemma given below:

Lemma 3.1. *Under condition (C1), if $\omega \in C^{\sigma-1}[0, T]$ is a solution of problem (1.2), then $\omega \in C^{\sigma-1}[0, T]$ is a solution of the following integral equation*

$$\omega(x) = \frac{b}{\Gamma(\sigma)} x^{\sigma-1} + \frac{1}{\Gamma(\sigma)} \int_0^x \frac{f(t, \omega(t), D^{\sigma-1}\omega(t))}{(x-t)^{1-\sigma}} dt \tag{3.1}$$

and, vice versa.

Proof. We assume that $\omega \in C^{\sigma-1}[0, T]$ is a solution of problem (1.2). By condition (C1), we have $f(x, \omega(x), D^{\sigma-1}\omega(x))$ is continuous on $(0, T]$ and $x^{\sigma-1}f(x, \omega(x), D^{\sigma-1}\omega(x))$ is continuous on $[0, T]$. It means that $f(x, \omega(x), D^{\sigma-1}\omega(x))$ is integrable, i.e $f(x, \omega(x), D^{\sigma-1}\omega(x)) \in C(0, T] \cap L^1[0, T]$. Then, by integrating the both sides of the equation in (1.2) and using the relation $ID^\sigma = IDD^{\sigma-1}$

$$D^{\sigma-1}\omega(x) = D^{\sigma-1}\omega(0) + If(x, \omega(x), D^{\sigma-1}\omega(x)) \in C[0, T]$$

is obtained. From here, by integration of the both sides of last equation and by use of $ID^{\sigma-1} = IDI^{2-\sigma}$ we have

$$I^{2-\sigma}\omega(x) = bx + I^2 f(x, \omega(x), D^{\sigma-1}\omega(x)) \in C^1[0, T] \tag{3.2}$$

where we used $D^{\sigma-1}\omega(0) = b$ and $I^{2-\sigma}\omega(0) = 0$ since $\omega(0) = 0$.

If the operator $I^{\sigma-1}$ is applied to the both sides of (3.2), then by semigroup and commutative property of R-L derivative we get

$$I\omega(x) = b\frac{x^\sigma}{\Gamma(1+\sigma)} + I[I^\sigma f(x, \omega(x), D^{\sigma-1}\omega(x))] \tag{3.3}$$

for all $x \in [0, T]$. Differentiating the both sides of (3.3), we have

$$\omega(x) = b\frac{x^{\sigma-1}}{\Gamma(\sigma)} + I^\sigma f(x, \omega(x), D^{\sigma-1}\omega(x))$$

which is the equivalent to the integral Eq (3.1).

Now we suppose that $\omega \in C^{\sigma-1}[0, T]$ is a solution of integral Eq (3.1), and let us show that ω is a solution of the problem (1.2). If D^σ is applied to the both sides of (3.1), and then, if

$$D^\sigma I^\sigma \omega(x) = \omega(x) \text{ for all } \omega \in C^{\sigma-1}[0, T]$$

is used, then one can observe that $\omega \in C^{\sigma-1}[0, T]$ satisfies the equation in (1.2). Moreover, let us prove that $\omega \in C^{\sigma-1}[0, T]$ also fulfills initial value conditions. By change of variables and condition (C1) we have

$$\begin{aligned}
\omega(0) = \lim_{x \to 0^+} \omega(x) &= \frac{b}{\Gamma(\sigma)}x^{\sigma-1} + \frac{1}{\Gamma(\sigma)}\lim_{x \to 0^+}\int_0^x \frac{f(t, \omega(t), D^{\sigma-1}\omega(t))}{(x-t)^{1-\sigma}}dt \\
&= \frac{1}{\Gamma(\sigma)}\lim_{x \to 0^+}\int_0^x \frac{t^{\sigma-1}f(t, \omega(t), D^{\sigma-1}\omega(t))}{t^{\sigma-1}(x-t)^{1-\sigma}}dt \\
&= \frac{1}{\Gamma(\sigma)}\lim_{x \to 0^+} x\int_0^1 \frac{(x\tau)^{\sigma-1}f(x\tau, \omega(x\tau), D^{\sigma-1}\omega(x\tau))}{\tau^{\sigma-1}(1-\tau)^{1-\sigma}}d\tau = 0,
\end{aligned} \tag{3.4}$$

since the integral is finite. Thus ω satisfies the first initial condition in (1.2).

Now let us show that ω provides the second initial condition in (1.2). If $D^{\sigma-1}$ is applied to both sides of (3.1), and if the relation $D^{\sigma-1}I^\sigma h(x) = Ih(x)$ is used, then we can first get

$$D^{\sigma-1}\omega(x) = b + \int_0^x f(t, \omega(t), D^{\sigma-1}\omega(t))dt. \tag{3.5}$$

From here, by passing the limit as $x \to 0^+$, we then obtain

$$\begin{aligned}
D^{\sigma-1}\omega(0) &= b + \lim_{x \to 0^+}\int_0^x f(t, \omega(t), D^{\sigma-1}\omega(t))dt \\
&= b + \lim_{x \to 0^+}\int_0^x \frac{1}{t^{\sigma-1}}t^{\sigma-1}f(t, \omega(t), D^{\sigma-1}\omega(t))dt \\
&= b + \lim_{x \to 0^+} x^{2-\sigma}\int_0^1 \frac{1}{\tau^{\sigma-1}}(x\tau)^{\sigma-1}f(x\tau, \omega(x\tau), D^{\sigma-1}\omega(x\tau))d\tau = b,
\end{aligned} \tag{3.6}$$

since $2 - \sigma > 0$ and the integral is finite due to the continuity of $t^{\sigma-1}f(t, \omega(t), D^{\sigma-1}\omega(t))$ on $[0, T]$. Consequently, it has been shown that any solution of (3.1) provides the problem (1.2) if condition (C1) is assumed to be satisfied. □

Theorem 3.1 (Existence). *Let condition (C1) be satisfied, and assume that there exist positive real numbers r_1, r_2 and $\mathcal{M}$ such that $\left|x^{\sigma-1}f(x,\omega,v)\right| \leq \mathcal{M}$ for all $(x,\omega,v) \in I = [0,T] \times [-r_1,r_1] \times [b-r_2, b+r_2]$. Then problem (1.2) admits at least one solution in $C^{\sigma-1}[0,T_0]$, where*

$$T_0 = \begin{cases} T & \text{if} \quad T < \frac{r}{C(b,\sigma,\mathcal{M})} \\ \frac{r}{C(b,\sigma,\mathcal{M})} & \text{if} \quad T \geq \frac{r}{C(b,\sigma,\mathcal{M})} \geq 1 \\ \left[\frac{r}{C(b,\sigma,\mathcal{M})}\right]^{\sigma-1}, & \text{if} \quad T \geq \frac{r}{C(b,\sigma,\mathcal{M})}, \quad 1 \geq \frac{r}{C(b,\sigma,\mathcal{M})} \quad \text{and} \quad 1 < \sigma \leq 1.5 \\ \left[\frac{r}{C(b,\sigma,\mathcal{M})}\right]^{2-\sigma}, & \text{if} \quad T \geq \frac{r}{C(b,\sigma,\mathcal{M})}, \quad 1 \geq \frac{r}{C(b,\sigma,\mathcal{M})} \quad \text{and} \quad 1.5 < \sigma < 2, \end{cases} \tag{3.7}$$

and

$$r = r_1 + r_2 \quad \text{and} \quad C(b,\sigma,\mathcal{M}) = \left[\frac{|b|}{\Gamma(\sigma)} + \mathcal{M}\left(\frac{1+\Gamma(3-\sigma)}{2-\sigma}\right)\right]. \tag{3.8}$$

Proof. As it is known from Lemma 3.1, solutions of problem (1.2) are solutions of integral equation (3.1) as well. Moreover, the fixed points of the opeator $\mathcal{S} : C^{\sigma-1}[0,T_0] \rightarrow C^{\sigma-1}[0,T_0]$ defined by

$$\mathcal{S}\omega(x) = \frac{b}{\Gamma(\sigma)}x^{\sigma-1} + \int_0^x \frac{f(t,\omega(t),D^{\sigma-1}\omega(t))}{(x-t)^{1-\sigma}}dt \tag{3.9}$$

interfere with solutions of the integral equation. For this reason, it is sufficient to prove that operator $\mathcal{S}$ admits at least one fixed point. For this, it will be verified that operator $\mathcal{S}$ satisfies the hypotheses of Schauder fixed-point theorem. Let us start with showing the following inclusion to be valid:

$$\mathcal{S}(B_r) \subset B_r$$

where

$$B_r = \{\omega \in C^{\sigma-1}[0,T_0] : \|\omega\|_\infty + \|D^{\sigma-1}\omega - b\|_\infty \leq r\}$$

is a closed compact subset of $C^{\sigma-1}[0,T_0]$. Accordingly to the norm on $C^{\sigma-1}[0,T_0]$, upper bounds of $\|\mathcal{S}\omega(x)\|_\infty$ and $\left\|D^{\sigma-1}\mathcal{S}\omega(x) - b\right\|_\infty$ can be determined as follows:

$$|\mathcal{S}\omega(x)| \leq \frac{|b|}{\Gamma(\sigma)}x^{\sigma-1} + \frac{1}{\Gamma(\sigma)}\int_0^x \frac{\left|t^{\sigma-1}f(t,\omega(t),D^{\sigma-1}\omega(t)\right|}{t^{\sigma-1}(x-t)^{1-\sigma}}dt$$

$$\leq \frac{|b|}{\Gamma(\sigma)}x^{\sigma-1} + \frac{\mathcal{M}}{\Gamma(\sigma)}\int_0^1 \frac{x}{\tau^{\sigma-1}(1-\tau)^{1-\sigma}}d\tau \leq \frac{|b|}{\Gamma(\sigma)}x^{\sigma-1} + \Gamma(2-\sigma)\mathcal{M}x \tag{3.10}$$

and

$$\left|D^{\sigma-1}\mathcal{S}\omega(x) - b\right| \leq \int_0^x \frac{\left|t^{\sigma-1}f(t,\omega(t),D^{\sigma-1}\omega(t)\right|}{t^{\sigma-1}}dt \leq \mathcal{M}x^{2-\sigma}\int_0^1 \tau^{1-\sigma}d\tau = \frac{\mathcal{M}x^{2-\sigma}}{2-\sigma}. \tag{3.11}$$

From (3.10)) and (3.11),

$$|\mathcal{S}\omega(x)| + \left|D^{\sigma-1}\mathcal{S}\omega(x) - b\right| \leq \frac{|b|}{\Gamma(\sigma)}x^{\sigma-1} + \Gamma(2-\sigma)\mathcal{M}x + \frac{\mathcal{M}x^{2-\sigma}}{2-\sigma} \tag{3.12}$$

is obtained. Taking supremum over $[0, T_0]$ for a $T_0 > 0$ for the right hand-side of the above equation,

$$|\mathcal{S}\omega(x)| + \left|D^{\sigma-1}\mathcal{S}\omega(x) - b\right| \leq C(b, \sigma, \mathcal{M})T_0^{\alpha} \tag{3.13}$$

can be written, where $\alpha \in \Omega = \{\sigma - 1, 1, 2 - \sigma\}$. α depends on values of $b, \mathcal{M}, \sigma, r$. To determine T_0 and α, let

$$C(b, \sigma, \mathcal{M})T_0^{\alpha} = r.$$

If $T_0^{\alpha} = \frac{r}{C(b,\sigma,\mathcal{M})} < 1$, then it is observed that $T_0 < 1$ for any $\alpha \in \Omega$. If $T_0^{\alpha} = \frac{r}{C(b,\sigma,\mathcal{M})} \geq 1$, it must be $T_0 \geq 1$ for any $\alpha \in \Omega$. Thus,

$$\sup_{x \in [0, T_0]} \left[|\mathcal{S}\omega(x)| + \left|D^{\sigma-1}\mathcal{S}\omega(x) - b\right|\right] \leq C(b, \sigma, \mathcal{M})T_0^{\alpha} = r, \tag{3.14}$$

where

$$T_0 := \left[\frac{r}{C(b, \sigma, \mathcal{M})}\right]^{1/\alpha}$$

and

$$\alpha = \begin{cases} 1 & \text{if} \quad \frac{r}{C(b,\sigma,\mathcal{M})} \geq 1 \\ \sigma - 1 & \text{if} \quad \frac{r}{C(b,\sigma,\mathcal{M})} < 1 \quad \text{and} \quad 1 < \sigma \leq 1.5 \\ 2 - \sigma & \text{if} \quad \frac{r}{C(b,\sigma,\mathcal{M})} < 1 \quad \text{and} \quad 1.5 \leq \sigma < 2. \end{cases} \tag{3.15}$$

As a result, for all cases we obtain

$$\|\mathcal{S}\omega\|_{\infty} + \|D^{\sigma-1}\mathcal{S}\omega - b\|_{\infty} \leq r,$$

which is the desired.

Now, let us prove the equicontinuity of $\mathcal{S}(B_r) \subset C^{\sigma-1}[0, T_0]$. Since the composition of uniformly continuous functions is so as well, the function $x^{\sigma-1}f(x, \omega(x), D^{\sigma-1}\omega(x))$ is uniformly continuous on $[0, T_0]$. Because for any $\omega \in B_r$, both $\omega(x)$ and $D^{\sigma-1}\omega(x)$ and $x^{\sigma-1}f(x, \omega, v)$ are uniformly continuous on I, respectively. Therefore, for given any $\epsilon > 0$, one can find a $\delta = \delta(\epsilon) > 0$ so that for all $x_1, x_2 \in [0, T_0]$ with $|x_1 - x_2| < \delta$ it is

$$\left|x_1^{\sigma-1}f(x_1, \omega(x_1), D^{\sigma-1}\omega(x)) - x_2^{\sigma-1}f(x_2, \omega(x_2), D^{\sigma-1}\omega(x_2))\right| < K\epsilon,$$

where $K = \max\left(\frac{1}{T_0\Gamma(2-\sigma)}, \frac{2-\sigma}{T_0^{2-\sigma}}\right)$. It follows that

$$\left|\mathcal{S}\omega(x_1) - \mathcal{S}\omega(x_2)\right| + \left|D^{\sigma-1}\mathcal{S}\omega(x_1) - D^{\sigma-1}\mathcal{S}\omega(x_2)\right|$$
$$\leq \int_0^1 \frac{|h(\eta x_1) - h(\eta x_2)|}{\Gamma(\sigma)\eta^{1-\sigma}(1-\eta)^{\sigma-1}}x\,d\eta + \int_0^1 \frac{|h(\eta x_1) - h(\eta x_2)|}{\eta^{1-\sigma}}x^{2-\sigma}\,d\eta$$
$$< T_0\Gamma(2-\sigma)K\epsilon + \frac{T_0^{2-\sigma}}{2-\sigma}K\epsilon = \epsilon,$$

where $h(x) = x^{\sigma-1}f\left(x, \omega(x), D^{\sigma-1}\omega(x)\right)$. This implies that $\mathcal{S}(B_r)$ is an equicontinuous set of $C^{\sigma-1}[0, T_0]$.

Finally, the continuity of $\mathcal{S}$ on B_r will be proven. Assume that $\{\omega_k\}_{k=1}^{\infty} \subset B_r$ is a sequence with $\omega_k \xrightarrow{C^{\sigma}[0,T_0]} \omega$ as $k \to \infty$. Then, one can easily conclude that ω_k and $D^{\sigma-1}\omega_k(t)$ converges uniformly to ω and $D^{\sigma-1}\omega(t)$, respectively. With these and the uniform continuity of $x^{\sigma-1}f(x,\omega,v)$ on $I = [0,T] \times [-r_1, r_1] \times [b - r_2, b + r_2]$, it leads to

$$
\begin{aligned}
\|\mathcal{S}\omega_k - \mathcal{S}\omega\|_{\sigma-1} = {} & \sup_{x \in [0,T_0]} \left| \frac{1}{\Gamma(\sigma)} \int_0^x \frac{\left[f\left(t, \omega_k(t), D^{\sigma-1}\omega_k(t)\right) - f\left(t, \omega(t), D^{\sigma-1}\omega(t)\right)\right]}{(x-t)^{1-\sigma}} dt \right| \\
& + \sup_{x \in [0,T_0]} \left| \int_0^x \left[f\left(t, \omega_k(t), D^{\sigma-1}\omega_k(t)\right) - f\left(t, \omega(t), D^{\sigma-1}\omega(t)\right)\right] dt \right| \\
\leq {} & \sup_{\eta x \in [0,T_0]} \int_0^1 \frac{(\eta x)^{\sigma-1} \left| f\left(\eta x, \omega_k(\eta x), D^{\sigma-1}\omega_k(\eta x)\right) - f\left(\eta x, \omega(\eta x), D^{\sigma-1}\omega(\eta x)\right)\right|}{\Gamma(\sigma)\eta^{\sigma-1}(1-\eta)^{1-\sigma}} x d\eta \\
& + \sup_{\eta x \in [0,T_0]} \int_0^1 \frac{(\eta x)^{\sigma-1} \left| f\left(\eta x, \omega_k(\eta x), D^{\sigma-1}\omega_k(\eta x)\right) - f\left(\eta x, \omega(\eta x), D^{\sigma-1}\omega(\eta x)\right)\right|}{\eta^{\sigma-1}} x^{2-\sigma} d\eta \\
\to {} & 0 \quad \text{as} \quad k \to \infty.
\end{aligned}
$$

In conclusion, since hypotheses of Theorem 2.2 are fulfilled, it implies that operator $\mathcal{S}$ admits at least one fixed point in $C^{\sigma-1}[0, T_0]$, which is a solution of problem (1.2) as well. □

The mean value theorem for R-L derivative of order $\sigma \in (0, 1)$ was correctly given by [7]. Now, its counterparts for order $\sigma \in (1, 2)$ is given as follows:

Lemma 3.2. *Let $\sigma \in (1, 2)$ and $\omega \in C^{\sigma-1}([0,T])$. Then, there is a function $\mu : [0,T] \to [0,T]$ with $0 < \mu(x) < x$ so that*

$$
\omega(x) = D^{\sigma-1}\omega(0)\frac{x^{\sigma-1}}{\Gamma(\sigma)} + \Gamma(2-\sigma)x(\mu(x))^{\sigma-1}D^{\sigma-1}\omega(\mu(x)),
$$

is satisfied.

The lemma can be proved by following the way used in [7] and so we omit it here. With the aid of this lemma we can obtain the Nagumo-type uniqueness:

Theorem 3.2. *(Nagumo type uniqueness) Let $1 < \sigma < 2$, $0 < T < \infty$ and let condition (**C1**) be satisfied. Moreover, assume that there exists a positive real number $L \leq \frac{2-\sigma}{\max\left(T, T^{2-\sigma}\right)(1+\Gamma(3-\sigma))}$ such that the inequality*

$$
x^{\sigma-1}|f(x, \omega_1, v_1) - f(x, \omega_2, v_2)| \leq L(|\omega_1 - \omega_2| + |v_1 - v_2|) \tag{3.16}
$$

is fulfilled for all $x \in [0,T]$ and for all $\omega i, v i \in \mathbb{R}$ with $i = 1, 2$. Then, (1.2) has at most one solution in the space of $C^{\sigma-1}([0, T_0])$.

Proof. We have just showed the existence of the solution to problem (1.2) in the previous theorem. For the uniqueness, we first assume that (1.2) admits two different solutions such as ω_1 and ω_2 in the space of $C^{\sigma-1}([0, T_0])$. Let us define a function $\Phi(x)$ to be in the form

$$
\Phi(x) := \begin{cases} |\omega_1(x) - \omega_2(x)| + \left|D^{\sigma-1}\omega_1(x) - D^{\sigma-1}\omega_2(x)\right|, & x > 0 \\ 0 \quad , & x = 0. \end{cases}
$$

Since $\omega_1, \omega_2 \in C^{\sigma-1}([0, T])$, the continuity of $\Phi(x)$ on $x \in (0, T_0]$ can obviously be seen. For its continuity at $x = 0$,

$$
\begin{aligned}
0 \leq \lim_{x \to 0^+} \Phi(x) = {} & \lim_{x \to 0^+} \frac{1}{\Gamma(\sigma)} \left| \int_0^x \frac{f\left(t, \omega_1(t), D^{\sigma-1}\omega_1(t)\right) - f\left(t, \omega_2(t), D^{\sigma-1}\omega_2(t)\right)}{(x-t)^{1-\sigma}} dt \right| \\
& + \lim_{x \to 0^+} \left| \int_0^x f\left(t, \omega_1(t), D^{\sigma-1}\omega_1(t)\right) - f\left(t, \omega_2(t), D^{\sigma-1}\omega_2(t)\right) dt \right| \\
\leq {} & \int_0^1 \frac{\lim_{x \to 0^+} x \left| H(x\eta, \omega_1(x\eta)) - H(x\eta, \omega_2(x\eta)) \right|}{\eta^{\sigma-1}(1-\eta)^{1-\sigma}} d\eta \\
& + \int_0^1 \frac{\lim_{x \to 0^+} x^{2-\sigma} \left| H(x\eta, \omega_1(x\eta)) - H(x\eta, \omega_2(x\eta)) \right|}{\eta^{\sigma-1}} d\eta = 0,
\end{aligned}
$$

where $H(x, \omega(x)) = x^{\sigma-1} f\left(x, \omega(x), D^{\sigma-1}\omega(x)\right)$ and we made the change of variable $t = x\eta$ and used condition (C1), respectively. Consequently, $\lim_{x \to 0^+} \Phi(x) = 0 = \Phi(0)$.

The fact that $\Phi(x) \geq 0$ on $[0, T]$ allows us to choose a point $x_0 \in (0, T]$ so that

$$
0 < \Phi(x_0) = |\omega_1(x_0) - \omega_2(x_0)| + \left| D^{\sigma-1}\omega_1(x_0) - D^{\sigma-1}\omega_2(x_0) \right|.
$$

By using the mean value theorem given by Lemma 3.2

$$
\begin{aligned}
|\omega_1(x_0) - \omega_2(x_0)| &= \Gamma(2-\sigma)x_0 \left| x_1^{\sigma-1} D^{\sigma-1}(\omega_1 - \omega_2)(x_1) \right| \\
&= \Gamma(2-\sigma)x_0 x_1^{\sigma-1} \left| f\left(x_1, \omega_1(x_1), D^{\sigma-1}\omega_1(x_1)\right) - f\left(x_1, \omega_2(x_1), D^{\sigma-1}\omega_2(x_1)\right) \right| \quad (3.17)
\end{aligned}
$$

is obtained for $x_1 \in (0, x_0)$.

Secondly, for the estimation of $\left| D^{\sigma-1}\omega_1(x_0) - D^{\sigma-1}\omega_2(x_0) \right|$, we have from the well-known integral mean theorem for the classical calculus

$$
\begin{aligned}
\left| D^{\sigma-1}\omega_1(x_0) - D^{\sigma-1}\omega_2(x_0) \right| &= \int_0^{x_0} \frac{t^{\sigma-1} \left| f(t, \omega_1(t), D^{\sigma-1}\omega_1(t)) - f(t, \omega_2(t), D^{\sigma-1}\omega_2(t)) \right|}{t^{\sigma-1}} dt \\
&= \frac{x_0^{2-\sigma}}{2-\sigma} x_2^{\sigma-1} \left| f\left(x_2, \omega_1(x_2), D^{\sigma-1}\omega_1(x_2)\right) - f\left(x_2, \omega_2(x_2), D^{\sigma-1}\omega_2(x_2)\right) \right|, \quad (3.18)
\end{aligned}
$$

where $x_2 \in (0, x_0)$.

We assign x_3 as one of the points x_1 and x_2 so that $|H(x_3, \omega_1(x_3)) - H(x_3, \omega_2(x_3))| := \max\left(|H(x_1, \omega_1(x_1)) - H(x_1, \omega_2(x_1))|, |H(x_2, \omega_1(x_2)) - H(x_2, \omega_2(x_2))|\right)$.

Thus, from (3.17) and (3.18), we have

$$
\begin{aligned}
0 < \Phi(x_0) &\leq \left(\Gamma(2-\sigma)x_0 + \frac{x_0^{2-\sigma}}{2-\sigma} \right) |H(x_3, \omega_1(x_3)) - H(x_3, \omega_2(x_3))| \\
&\leq \max\left(T, T^{2-\sigma}\right) \left(\frac{1 + \Gamma(3-\sigma)}{2-\sigma} \right) x_3^{\sigma-1} |f(x_3, \omega_1(x_3)) - f(x_3, \omega_2(x_3))| \\
&\leq x_3^{\sigma-1} \left(|\omega_1(x_3) - \omega_2(x_3)| + \left| D^{\sigma-1}\omega_1(x_3) - D^{\sigma-1}\omega_2(x_3) \right| \right) = \Phi(x_3)
\end{aligned}
$$

since $L \leq \frac{2-\sigma}{\max\left(T,T^{2-\sigma}\right)(1+\Gamma(3-\sigma))}$. Repeating the same procedure for the point x_3, it enables us to find some points $x_6 \in (0, x_3)$ so that $0 < \Phi(x_0) \leq \Phi(x_3) \leq \Phi(x_6)$. Continuing in the same way, the sequence $\{x_{3n}\}_{n=1}^{\infty} \subset [0, x_0)$ can be constructed so that $x_{3n} \to 0$ and

$$0 < \Phi(x_0) \leq \Phi(x_3) \leq \Phi(x_6) \leq ... \leq \Phi(x_{3n}) \leq ... \tag{3.19}$$

However, the fact that $\Phi(x)$ is continuous at $x = 0$ and $x_{3n} \to 0$ leads to $\Phi(x_{3n}) \to \Phi(0) = 0$, and this contradicts with (3.19). Consequently, IVP (1.2) possesses a unique solution. $\qquad \square$

Theorem 3.3. *(Krasnoselskii-Krein type uniqueness) Let $1 < \sigma < 2$ and $T_0^* = \min\{T_0, 1\}$, where T_0 is defined by (3.7). Let condition (C1) be fulfilled. Furthermore, suppose that there exist a $L > 0$ and an $\alpha \in (0, 1)$ so that*

$$x^{1-\alpha(\sigma-1)} |f(x, \omega_1, v_1) - f(x, \omega_2, v_2)| \leq \Gamma(\sigma) \frac{\alpha(\sigma-1) + L}{2} (|\omega_1 - \omega_2| + |v_1 - v_2|) \tag{3.20}$$

holds for all $x \in [0, T]$ and for all $\omega_i, v_i \in \mathbb{R}$ with $i = 1, 2$, and that there exist $C > 0$ satisfying $(1 - \sigma)(1 - \alpha) - L(1 - \alpha) + 1 > 0$ such that

$$x^{\sigma-1} |f(x, \omega_1, v_1) - f(x, \omega_2, v_2)| \leq C \left(|\omega_1 - \omega_2|^\alpha + x^{\alpha(\sigma-1)} |v_1 - v_2|^\alpha \right) \tag{3.21}$$

holds for all $x \in [0, T]$ and for all $\omega_i, v_i \in \mathbb{R}$ with $i = 1, 2$. Then, problem (1.2) has a unique solution in the space of $C^{\sigma-1}\left([0, T_0^*]\right)$.

Proof. As claimed in the previous theorem, we first assume that problem (1.2) has two different solutions such as $\omega_1(x)$ and $\omega_2(x)$ in $C^{\sigma-1}\left([0, T_0^*]\right)$. However, by contradiction, we will show that they are indeed equal. For this, let us first define $\Phi_1(x) = |\omega_1(x) - \omega_2(x)|$ and $\Phi_2(x) = \left|D^{\sigma-1}\omega_1(x) - D^{\sigma-1}\omega_2(x)\right|$ and try to find estimates for each functions by using condition (C1) and inequality (3.21). Hence, we first have

$$\begin{aligned}
\Phi_1(x) &\leq \frac{1}{\Gamma(\sigma)} \int_0^x \frac{\left|f\left(t, \omega_1(t), D^{\sigma-1}\omega_1(t)\right) - f\left(t, \omega_2(t), D^{\sigma-1}\omega_2(t)\right)\right|}{(x-t)^{1-\sigma}} dt \\
&\leq \frac{1}{\Gamma(\sigma)} \int_0^x \frac{t^{\sigma-1}\left|f\left(t, \omega_1(t), D^{\sigma-1}\omega_1(t)\right) - f\left(t, \omega_2(t), D^{\sigma-1}\omega_2(t)\right)\right|}{t^{\sigma-1}(x-t)^{1-\sigma}} dt \\
&\leq \frac{C}{\Gamma(\sigma)} \int_0^x \frac{\left[\Phi_1^\alpha(x) + t^{\alpha(\sigma-1)}\Phi_2^\alpha(x)\right]}{t^{\sigma-1}(x-t)^{1-\sigma}} dt \\
&\leq \frac{C}{\Gamma(\sigma)} \left(\int_0^x \left(\frac{1}{t^{\sigma-1}(x-t)^{1-\sigma}}\right)^q dt\right)^{1/q} \left(\int_0^x \left[\Phi_1^\alpha(t) + t^{\alpha(\sigma-1)}\Phi_2^\alpha(t)\right]^p dt\right)^{1/p} \\
&\leq \frac{C\Gamma(1 + (1-\sigma)q)\Gamma(1 + (\sigma-1)q))}{\Gamma(\sigma)} x^{1/q}\Omega^{1/p}(x)
\end{aligned}$$

where we used Hölder inequality with $q > 1$ satisfying $(1 - \sigma)q + 1 > 0$ and $p = q/(q-1)$, and $\Omega(x)$ is defined by

$$\Omega(x) = \int_0^x \left[\Phi_1^\alpha(t) + t^{\alpha(\sigma-1)}\Phi_2^\alpha(t)\right]^p dt.$$

From here, we have the following estimation

$$\Phi_1^p(x) \le Cx^{p/q}\Omega(x), \tag{3.22}$$

where C is not specified here and throughout the proof. In addition to this, the upper bound for $\Phi_2(x)$ can be found as follows:

$$
\begin{aligned}
\Phi_2(x) &\le \int_0^x \left| f\left(t, \omega_1(t), D^{\sigma-1}\omega_1(t)\right) - f\left(t, \omega_2(t), D^{\sigma-1}\omega_2(t)\right) \right| dt \\
&\le \int_0^x \frac{t^{\sigma-1} \left| f\left(t, \omega_1(t), D^{\sigma-1}\omega_1(t)\right) - f\left(t, \omega_2(t), D^{\sigma-1}\omega_2(t)\right) \right|}{t^{\sigma-1}} dt \\
&\le \frac{C}{\Gamma(\sigma)} \int_0^x \frac{\left[\Phi_1^\alpha(x) + t^{\alpha(\sigma-1)}\Phi_2^\alpha(x)\right]}{t^{\sigma-1}(x-t)^{1-\sigma}} dt \\
&\le C\left(\int_0^x \left(\frac{1}{t^{\sigma-1}}\right)^q dt\right)^{1/q} \left(\int_0^x \left[\Phi_1^\alpha(t) + t^{\alpha(\sigma-1)}\Phi_2^\alpha(t)\right]^p dt\right)^{1/p} \\
&\le \frac{C\Gamma(1+(1-\sigma)q)}{\Gamma(2+(\sigma-1)q))} x^{(1+q(1-\sigma))/q}\Omega^{1/p}(x).
\end{aligned}
$$

From here,

$$\Phi_2^p(x) \le Cx^{(1+q(1-\sigma))p/q}\Omega(x) \tag{3.23}$$

is then obtained. By using estimations in (3.22) and (3.23) in the derivative of $\Omega(x)$ we have

$$
\begin{aligned}
\Omega'(x) &= \left[\Phi_1^\alpha(x) + t^{\alpha(\sigma-1)}\Phi_2^\alpha(x)\right]^p \le 2^{p-1}\left[(\Phi_1^\alpha(x))^p + t^{p\alpha(\sigma-1)}(\Phi_2^\alpha(x))^p\right] \\
&= 2^{p-1}\left[\left(\Phi_1^p(x)\right)^\alpha + x^{p\alpha(\sigma-1)}\left(\Phi_2^p(x)\right)^\alpha\right] \\
&\le 2^{p-1}\left[C^\alpha x^{\alpha p/q}\Omega^\alpha(x) + C^\alpha x^{p\alpha(\sigma-1)} x^{(1+q(1-\sigma))\alpha p/q}\Omega^\alpha(x)\right] \le Cx^{\alpha p/q}\Omega^\alpha(x). \tag{3.24}
\end{aligned}
$$

If we multiply the both sides of the above inequality with $(1-\alpha)\Omega^{-\alpha}(x)$,

$$(1-\alpha)\Omega^{-\alpha}(x)\Omega'(x) = \frac{d}{dx}\left[\Omega^{1-\alpha}(x)\right] \le Cx^{\alpha p/q}.$$

is then obtained. Integrating the both sides of the inequality over $[0, x]$, we get

$$\Omega^{1-\alpha}(x) \le Cx^{(\alpha p+q)/q},$$

since $\Omega(0) = 0$. Consequently, this leads to the following estimation on $\Omega(x)$

$$\Omega(x) \le Cx^{(\alpha p+q)/(1-\alpha)q}. \tag{3.25}$$

By considering (3.22) and (3.23) together with (3.25), one can conclude that

$$\Phi_1^p(x) \le Cx^{p/q}\Omega(x) \le Cx^{p/q}x^{(\alpha p+q)/(1-\alpha)pq} = Cx^{p+q/(1-\alpha)q}.$$

or

$$\Phi_1(x) \le C x^{(p+q)/(1-\alpha)pq} = x^{1/(1-\alpha)}, \tag{3.26}$$

and

$$\Phi_2^p(x) \le C x^{p(1+q(1-\sigma))/q} \Omega(x) \le C x^{p(1+q(1-\sigma))/q} x^{(\alpha p+q)/(1-\alpha)pq} = C x^{\frac{(1-\alpha)(1-\sigma)pq+p+q}{(1-\alpha)q}}.$$

or

$$\Phi_2(x) \le C x^{(1-\sigma)+\frac{p+q}{(1-\alpha)pq}} = x^{(1-\sigma)+\frac{1}{(1-\alpha)}}, \tag{3.27}$$

since $\frac{p+q}{pq} = 1$.

Let us now define
$$\Psi(x) = x^{-L} \max\{\Phi_1(x), \Phi_2(x)\},$$

where $L(1-\alpha) < 1 + (1-\sigma)(1-\alpha)$. If $\Phi_1(x) = \max\{\Phi_1(x), \Phi_2(x)\}$, then from (3.26) we get

$$0 \le \Psi(x) \le x^{\frac{1}{1-\alpha}-L},$$

or in the case of $\Phi_2(x) = \max\{\Phi_1(x), \Phi_2(x)\}$, by the inequality (3.27) we have the following

$$0 \le \Psi(x) \le x^{(1-\sigma)+\frac{1}{1-\alpha}-L} = x^{\frac{(1-\sigma)(1-\alpha)-L(1-\alpha)+1}{1-\alpha}}.$$

In both cases, $\Psi(x)$ is continuous on $[0, T_0^*]$ and $\Psi(0) = 0$. Let us now show that $\Psi(x) \equiv 0$ on $[0, T_0^*]$. For this, suppose otherwise and let $\Psi(x) \not\equiv 0$. This means $\Psi(x) > 0$, and from its continuity one can say that there exists a point $x_1 \in [0, T_0^*]$ so that $\Psi(x)$ takes its maximum value at that point. Thus, let

$$M = \Psi(x_1) = \max_{x \in [0, T_0^*]} \Psi(x).$$

By assuming $\Psi(x) = x^{-L} \Phi_1(x)$,

$$M = \Psi(x_1) = x_1^{-L} \Phi_1(x_1) \tag{3.28}$$

$$\le \frac{x_1^{-L}}{\Gamma(\sigma)} \int_0^{x_1} \frac{t^{1-\alpha(\sigma-1)} \left| f\left(t, \omega_1(t), D^{\sigma-1}\omega_1(t)\right) - f\left(t, \omega_2(t), D^{\sigma-1}\omega_2(t)\right) \right|}{t^{1-\alpha(\sigma-1)} (x-t)^{1-\sigma}} dt$$

$$\le \frac{(\alpha(\sigma-1)+L)x_1^{-L}}{2} \int_0^{x_1} (x_1-t)^{\sigma-1} t^{-1+\alpha(\sigma-1)} [\Phi_1(t) + \Phi_2(t)] dt$$

$$\le M(\alpha(\sigma-1)+L)x_1^{-L+\sigma-1} \int_0^{x_1} t^{-1+\alpha(\sigma-1)+L} dt$$

$$< M x_1^{(\alpha+1)(\sigma-1)} \le M$$

is obtained for $t \in [0, x_1)$. However, we get to a contradiction.

On the other hand, when $\Psi(x) = x^{-L} \Phi_2(x)$, we get

$$M = \Psi(x_1) = x_1^{-L} \Phi_2(x_1) \tag{3.29}$$

$$\leq x_1^{-L} \int_0^{x_1} \frac{t^{1-\alpha(\sigma-1)} \left| f\left(t, \omega_1(t), D^{\sigma-1}\omega_1(t)\right) - f\left(t, \omega_2(t), D^{\sigma-1}\omega_2(t)\right) \right|}{t^{1-\alpha(\sigma-1)}} dt$$

$$\leq \Gamma(\sigma) \frac{(\alpha(\sigma-1) + L)x_1^{-L}}{2} \int_0^{x_1} t^{-1+\alpha(\sigma-1)+L} \left[\Phi_1(t) + \Phi_2(t)\right] dt$$

$$< M x_1^{\alpha(\sigma-1)} \leq M,$$

which is contraction as well.

Consequently, $\Psi(x)$ vanishes identically on $[0, T_0^*]$. This gives us the uniqueness of solutions to the considered problem. $\qquad\square$

Theorem 3.4 (Osgood-type uniqueness). *Let $1 < \sigma < 2$, and T_0 be defined by (3.7), and let condition (C1) be satisfied. Furthermore, suppose that the equality*

$$x^{\sigma-1} |f(x, \omega_1, v_1) - f(x, \omega_2, v_2)| \leq C \left(g\left(|\omega_1 - \omega_2|^p + |v_1 - v_2|^p\right)\right)^{1/p} \qquad (3.30)$$

is fulfilled for all $x \in [0, T]$ and for all $\omega_i, v_i \in \mathbb{R}$ with $i = 1, 2$, where $p > 1$ is conjugate of $q > 1$ satisfying $1 + (1 - \sigma)q > 0$ and

$$C^q \geq 2 \max\left(b\Gamma^q(\sigma)[T_0\Gamma(1 + (1 - \sigma)q)\Gamma(1 + (\sigma - 1)q)]^{-1}, (1 + (1 - \sigma)q)T_0^{-1-q(1-\sigma)}\right).$$

Moreover, assume that g is a continuous, non-negative and non-decreasing function in $[0, \infty)$ so that $g(0) = 0$ and it satisfies

$$\lim_{\epsilon \to 0^+} \int_\epsilon^\gamma \frac{du}{g(u)} = \infty \qquad (3.31)$$

for any $\gamma \in \mathbb{R}$. Then, (1.2) has a unique solution in the space $C^{\sigma-1}([0, T_0])$.

Proof. As made in previously given uniqueness theorems, we assume that there exist two different solutions such as $\omega_1(x)$ and $\omega_2(x)$ to problem (1.2) in $C^{\sigma-1}\left([0, T_0^*]\right)$. Moreover, let $\Phi_1(x) = |\omega_1(x) - \omega_2(x)|$ and $\Phi_2(x) = \left|D^{\sigma-1}\omega_1(x) - D^{\sigma-1}\omega_2(x)\right|$. At first, we get the estimation on $\Phi_1(x)$ as follows:

$$\Phi_1(x) \leq \frac{1}{\Gamma(\sigma)} \int_0^x \frac{t^{\sigma-1} \left| f\left(t, \omega_1(t), D^{\sigma-1}\omega_1(t)\right) - f\left(t, \omega_2(t), D^{\sigma-1}\omega_2(t)\right) \right|}{t^{\sigma-1}(x-t)^{1-\sigma}} dt$$

$$\leq \frac{C}{\Gamma(\sigma)} \int_0^x \frac{\left[g\left(|\omega_1(t) - \omega_2(t)|^p + \left|D^{\sigma-1}\omega_1(t) - D^{\sigma-1}\omega_2(t)\right|^p\right)\right]^{1/p}}{t^{\sigma-1}(x-t)^{1-\sigma}} dt$$

$$\leq \frac{C}{\Gamma(\sigma)} \left(\int_0^x \left(\frac{1}{t^{\sigma-1}(x-t)^{1-\sigma}}\right)^q dt\right)^{1/q} \left(\int_0^x g\left(\Phi_1^p(t) + \Phi_2^p(t)\right) dt\right)^{1/p}$$

$$\leq C \left[\frac{\Gamma(1 + (1-\sigma)q)\Gamma(1 + (\sigma-1)q)}{\Gamma^q(\sigma)}\right]^{1/q} x^{1/q} \left(\int_0^x g\left(\Phi_1^p(t) + \Phi_2^p(t)\right) dt\right)^{1/p}$$

$$\leq \frac{1}{2^{1/p}} \left(\int_0^x g\left(\Phi_1^p(t) + \Phi_2^p(t)\right) dt\right)^{1/p} \qquad (3.32)$$

where we used the inequality (3.30), Hölder inequality and the assumption on C, respectively. From here, it follows that

$$\Phi_1^p(x) \le \frac{1}{2} \int_0^x g\left(\Phi_1^p(t) + \Phi_2^p(t)\right) dt. \tag{3.33}$$

Similarly to above, we have

$$\Phi_2(x) \le \int_0^x \frac{t^{\sigma-1} \left| f\left(t, \omega_1(t), D^{\sigma-1}\omega_1(t)\right) - f\left(t, \omega_2(t), D^{\sigma-1}\omega_2(t)\right)\right|}{t^{\sigma-1}} dt$$

$$\le C \int_0^x \frac{\left[g\left(|\omega_1(t) - \omega_2(t)|^p + \left|D^{\sigma-1}\omega_1(t) - D^{\sigma-1}\omega_2(t)\right|^p\right)\right]^{1/p}}{t^{\sigma-1}} dt$$

$$\le C \left(\int_0^x \left(\frac{1}{t^{\sigma-1}}\right)^q dt\right)^{1/q} \left(\int_0^x g\left(\Phi_1^p(t) + \Phi_2^p(t)\right) dt\right)^{1/p}$$

$$\le C \left[\frac{1}{(1 + (1-\sigma)q)}\right]^{1/q} x^{(1+q(1-\sigma))/q} \left(\int_0^x g\left(\Phi_1^p(t) + \Phi_2^p(t)\right) dt\right)^{1/p}$$

$$\le \frac{1}{2^{1/p}} \left(\int_0^x g\left(\Phi_1^p(t) + \Phi_2^p(t)\right) dt\right)^{1/p}. \tag{3.34}$$

This leads to

$$\Phi_2^p(x) \le \frac{1}{2} \int_0^x g\left(\Phi_1^p(t) + \Phi_2^p(t)\right) dt. \tag{3.35}$$

Now, set

$$\Psi(x) := \max_{0 \le t \le x} \left[\Phi_1^p(x) + \Phi_2^p(x)\right],$$

and assume that $\Psi(x) > 0$ for $x \in (0, T_0]$. We will show that it can not be possible under assumptions.

From the definition of $\Psi(x)$ one easily conclude that for each $x \in [0, T_0]$, it is $\Phi_1^p(x) + \Phi_2^p(x) \le \Psi(x)$ and there exists a $x_1 \le x$ so that $\Psi(x) = \Phi_1^p(x_1) + \Phi_2^p(x_1)$. Then, from estimations (3.33)–(3.35) and from the fact that g is non-decreasing function

$$\Psi(x) = \Phi_1^p(x_1) + \Phi_2^p(x_1) \le \int_0^{x_1} g\left(\Phi_1^p(t) + \Phi_2^p(t)\right) \le \int_0^x g\left(\Psi(t)\right) dt := \Psi_*(x) \tag{3.36}$$

is then obtained. It can be seen that $\Psi(x) \le \Psi_*(x)$. Moreover, we have

$$\frac{d}{dx}\Psi_*(x) = g\left(\Psi(x)\right) \le g\left(\Psi_*(x)\right)$$

for all $x \in [0, T_0]$. From this fact, for sufficiently small $\delta > 0$, we have

$$\int_\delta^{T_0} \frac{\Psi_*'(x)}{g\left(\Psi_*(x)\right)} dx \le T_0 - \delta.$$

Furthermore, by changing variables $u = \Psi_*(x)$ in the above integral and by using the continuity of $\Psi_*(x)$ and $\Psi_*(0) = 0$, we have

$$\int_\epsilon^\gamma \frac{du}{g(u)} \le T_0 - \delta.$$

for sufficiently small $\epsilon > 0$ with $\epsilon = \Psi_*(\delta)$ and for $\gamma = \Psi_*(T_0)$. However, this contradicts with the assumption on g given in (3.31). Consequently, $\Psi(x) = 0$ for $x \in [0, T_0]$, i.e. $\omega_1 = \omega_2$. $\square$

Remark 3.1. *It must be pointed out that, as noted in Theorem 1.4.3 in [13], the condition that function g(u) is non-decreasing can be dropped.*

Example 3.1. *Let us consider the following problem*

$$D^{\frac{4}{3}}\omega(x) = \begin{cases} x^{-\frac{1}{3}}\left[-\omega\ln\omega - (D^{\frac{1}{3}}\omega)\ln(D^{\frac{1}{3}}\omega)\right], & if \quad 0 < \omega, D^{\frac{1}{3}}\omega \le 1/e \\ 0, & if \quad \omega = 0 \quad or \quad D^{\frac{1}{3}}\omega = 0, \end{cases} \tag{3.37}$$

with initial conditions $\omega(0) = 0$ and $D^{\sigma-1}\omega(x)|_{x=0} = 1$. Let $T = 1$ in Theorem 3.1. Then, $\mathcal{M} = \max_{(x,u,v)\in[0,1]\times[0,1/e]\times[0,1/e]} \left|x^{\frac{1}{3}}f(x,\omega,v)\right| \cong \frac{2}{e}$, $C(b,\sigma,\mathcal{M}) = C(1,\frac{4}{3},\frac{2}{e}) \cong 3.2$ and $T_0 = \frac{r}{C(b,\sigma,\mathcal{M})} \cong 0.22$. Hence, problem (3.37) has a solution in $C^{\frac{1}{3}}([0, 0.22])$. Now, we investigate the uniqueness of the solution to the problem in $C^{\frac{1}{3}}([0, 0.22])$. Let the function g in the previous theorem defined by

$$g(u) = \begin{cases} 0 & if \quad u = 0 \\ -\sqrt{u}\ln\sqrt{u} & if \quad 0 < u \le 1/e^2, \\ 1/e & if \quad u \ge 1/e^2, \end{cases} \tag{3.38}$$

It is obvious that g is positive for $u > 0$. Since $g'(u) = -\frac{1}{2\sqrt{u}}\left(1 + \ln(\sqrt{u})\right) > 0$ for $0 < u \le 1/e^2$, it is non-decreasing. Also,

$$\lim_{\epsilon\to0^+} \int_\epsilon^\gamma \frac{du}{g(u)} = \infty \tag{3.39}$$

for any $\gamma \in \mathbb{R}$, which can be seen by considering the inequality $\frac{1}{g(u)} = -\frac{1}{\sqrt{u}\ln\sqrt{u}} \le \frac{1}{u}$ in the neighborhood of $u = 0$ and the divergence of the integral $\lim_{\epsilon\to0^+} \int_\epsilon^\gamma \frac{du}{u}$ and by applying comparison test. Moreover, by using the concavity of the nonlinear function f with respect to second and third variables, we have

$$\begin{aligned} x^{1/3}\left|f(x,\omega_1,v_1) - f(x,\omega_2,v_2)\right| &\le |-\omega_1\ln\omega_1 - v_1\ln v_1 - (-\omega_2\ln\omega_2 - v_2\ln v_2)| \\ &\le -|\omega_1\ln\omega_1 - \omega_2\ln\omega_2| - |v_1\ln v_1 - v_2\ln v_2| \\ &\le -|\omega_1 - \omega_2||\ln(\omega_1 - \omega_2)| - |v_1 - v_2||\ln(v_1 - v_2)| \\ &\le -2\sqrt{|\omega_1 - \omega_2|^2 + |v_1 - v_2|^2}\ln\sqrt{|\omega_1 - \omega_2|^2 + |v_1 - v_2|^2} \\ &\le C\left(g\left(|\omega_1 - \omega_2|^2 + |v_1 - v_2|^2\right)\right)^{1/2} \end{aligned} \tag{3.40}$$

where $v_i = D^{\frac{1}{3}}\omega_i$ for $i = 1, 2$, $p = q = 2$ and

$$C \ge 1.725 \cong \sqrt{2}\max\left(\frac{\Gamma(4/3)}{0.22 \times \Gamma(1/3)\Gamma(5/3)}, \frac{1}{3 \times 0.22^{1/3}}\right).$$

Hence, assumptions of Theorem 3.4 are satisfied. So, the problem has a unique solution in $C^{\frac{1}{3}}([0, 0.22])$.

4. Conclusions

In this research, we gave some sufficient conditions for the existence and uniqueness of a problem involving a nonlinear differential equations in the sense of R-L derivative when the right-hand side function has a discontinuity at zero. We presented an example associated with two theorems. Considering the literature, these results can be generalized and improved. Besides, one can obtain another uniqueness results for this problem as well.

Conflict of interest

There is no conflict of interest.

References

1. A. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, Vol. 204, Elsevier Science Limited, 2006.

2. K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, A Wiley-Interscience Publication, John Wiley Sons Inc., New York, 1993.

3. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

4. S. G. Samko, A. A. A Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, Switzerland, 1993.

5. S. Hristova, A. Dobreva, Existence, continuous dependence and finite time stability for Riemann-Liouville fractional differential equations with a constant delay, *AIMS Math.* **5** (2020), 3809–3824.

6. J. M. Shen, S. Rashid, M. A. Noor, R. Ashraf, Y. M. Chu, Certain novel estimates within fractional calculus theory on time scales, *AIMS Math.*, **5** (2020), 6073–6086.

7. M. Şan, U. Sert, Some analysis on a fractional differential equation with a right-hand side which has a discontinuity at zero, *Hacet. J. Math. Stat.*, **49** (2020), 1718–1725.

8. F. Yoruk, T. G. Bhaskar, R. P. Agarwal, New uniqueness results for fractional differential equations, *Appl. Anal.*, **2** (2013), 259–269.

9. C. Li, S. Sarwar, Existence and continuation of solutions for Caputo type fractional differential equations, *Electron. J. Differ. Equ.*, **2016** (2016), 1–14.

10. M. Şan, Complex variable approach to the analysis of a fractional differential equation in the real line, *C. R. Math.*, **356** (2018), 293–300.

11. J. R. L. Webb, Weakly singular Gronwall inequalities and applications to fractional differential equations, *J. Math. Anal. Appl.*, **471** (2019), 692–711.

12. J. R. L. Webb, Initial value problems for Caputo fractional equations with singular nonlinearities, *Electron. J. Differ. Equ.*, **2019** (2019), 1–32.

13. R. P. Agarwal, V. Lakshmikantham, Uniqueness and nonuniqueness criteria for ordinary differential equations, (Vol. 6), World Scientific, 1993.

14. K. Diethelm, The mean value theorems and a Nagumo-type uniqueness theorem for Caputo's fractional calculus, *Fract. Calc. Appl. Anal.*, **2** (2012), 304–313.

15. V. Lakshmikantham, S. Leela, Nagumo-type uniqueness result for fractional differential equations, *Nonlinear Anal. Theory Methods Appl.*, **7-8** (2009), 2886–2889.

16. V. Lakshmikantham, S. Leela, Krasnoselskii–Krein-type uniqueness result for fractional differential equations, *Nonlinear Anal. Theory Methods Appl.*, **7-8** (2009), 3421–3424.

17. D. Delbosco, Fractional calculus and function spaces, *J. Fract. Calc.*, **6** (1994), 45–53.

18. E. Zeidler, *Nonlinear Functional Analysis and its Applications, I: Fixed-point Theorems*, New York: Springer-Verlag, 1985.