AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(12): 13024–13040.
DOI: 10.3934/math.2021753
Received: 27 March 2021
Accepted: 07 September 2021
Published: 14 September 2021

Research article

Optimal bounds for the sine and hyperbolic tangent means by arithmetic and centroidal means in exponential type

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Abstract: In this paper, optimal bounds for the sine and hyperbolic tangent means by arithmetic and centroidal means in exponential type are established using the monotone form of L'Hospital's rule and the criterion for the monotonicity of the quotient of power series.

Keywords: bounds; Seiffert-like means; sine mean; hyperbolic tangent mean; arithmetic mean; centroidal mean
Mathematics Subject Classification: 26D15

1. Introduction

In 1998, Kahlig and Matkowski [1] proved in particular that every homogeneous bivariable mean $\mathbf{M}$ in $(0,\infty)$ can be represented in the form

$$\mathbf{M}(x,y) = \mathbf{A}(x,y) f_{\mathbf{M},\mathbf{A}}\left(\frac{x-y}{x+y}\right),$$

where $\mathbf{A}$ is the arithmetic mean and $f_{\mathbf{M},\mathbf{A}}\colon (-1,1) \longrightarrow (0,2)$ is a unique single variable function (with the graph laying in a set of a butterfly shape), called an $\mathbf{A}$-index of $\mathbf{M}$.

In this paper we consider Seiffert function $f:(0,1)\to\mathbb{R}$ which fulfils the following condition

$$\frac{t}{1+t} \leq f(t) \leq \frac{t}{1-t}.$$

According to the results of Witkowski [2] we introduce the mean $\mathbf{M}_f$ of the form

$$\mathbf{M}_f(x,y) = \begin{cases} \frac{|x-y|}{2f\left(\frac{|x-y|}{x+y}\right)} & x \neq y, \\ x & x = y. \end{cases} \tag{1.1}$$

In this paper a mean $\mathbf{M}_f : \mathbb{R}_+^2 \longrightarrow \mathbb{R}$ is the function that is symmetric, positively homogeneous and internal in sense [2]. Basic result of Witkowski is correspondence between a mean $\mathbf{M}_f$ and Seiffert function f of $\mathbf{M}_f$ is given by the following formula

$$f(t) = \frac{t}{\mathbf{M}_f(1-t, 1+t)}, \tag{1.2}$$

where

$$t = \frac{|x-y|}{x+y}. \tag{1.3}$$

Therefore, f and $\mathbf{M}_f$ form a one-to-one correspondence via (1.1) and (1.2). For this reason, in the following we can rewrite $f =: f_{\mathbf{M}}$.

Throughout this article, we say $x \neq y$, that is, $t \in (0,1)$. For convenience, we note that $\mathbf{M}_1 < \mathbf{M}_2$ means $\mathbf{M}_1(x,y) < \mathbf{M}_2(x,y)$ holds for two means $\mathbf{M}_1$ and $\mathbf{M}_2$ with $x \neq y$. Then there is a fact that the inequality $f_{\mathbf{M}_1}(t) > f_{\mathbf{M}_2}(t)$ holds if and only if $\mathbf{M}_1 < \mathbf{M}_2$. That is to say,

$$\frac{1}{f_{\mathbf{M}_1}} < \frac{1}{f_{\mathbf{M}_2}} \Longleftrightarrow \mathbf{M}_1 < \mathbf{M}_2. \tag{1.4}$$

The above relationship (1.4) inspires us to ask a question: Can we transform the means inequality problem into the reciprocal inequality problem of the corresponding Seiffert functions? Witkowski [2] answers this question from the perspective of one-to-one correspondence. We find that these two kinds of inequalities are equivalent in similar linear inequalities. We describe this result in Lemma 2.1 as a support of this paper.

As we know, the study of inequalities for mean values has always been a hot topic in the field of inequalities. For example, two common means can be used to define some new means. The recent success in this respect can be seen in references [3–8]. In [2], Witkowski introduced the following two new means, one called sine mean

$$\mathbf{M}_{\sin}(x,y) = \begin{cases} \frac{|x-y|}{2\sin\left(\frac{|x-y|}{x+y}\right)} & x \neq y \\ x & x = y \end{cases}, \tag{1.5}$$

and the other called hyperbolic tangent mean

$$\mathbf{M}_{\tanh}(x,y) = \begin{cases} \frac{|x-y|}{2\tanh\left(\frac{|x-y|}{x+y}\right)} & x \neq y \\ x & x = y \end{cases}. \tag{1.6}$$

Recently, Nowicka and Witkowski [9] determined various optimal bounds for the $\mathbf{M}_{\sin}(x,y)$ and $\mathbf{M}_{\tanh}(x,y)$ by the arithmetic mean $\mathbf{A}(x,y) = (x+y)/2$ and centroidal mean

$$\mathbf{C}_e(x,y) = \frac{2}{3}\frac{x^2+xy+y^2}{x+y}$$

as follows:

Proposition 1.1. The double inequality

$$(1-\alpha)\mathbf{A} + \alpha\mathbf{C}_e < \mathbf{M}_{\sin} < (1-\beta)\mathbf{A} + \beta\mathbf{C}_e$$

holds if and only if $\alpha \le 1/2$ and $\beta \ge (3/\sin 1) - 3 \approx 0.5652$.

Proposition 1.2. The double inequality

$$(1-\alpha)\mathbf{A} + \alpha\mathbf{C}_e < \mathbf{M}_{\tanh} < (1-\beta)\mathbf{A} + \beta\mathbf{C}_e$$

holds if and only if $\alpha \le (3/\tanh 1) - 3 \approx 0.9391$ and $\beta \ge 1$.

Proposition 1.3. The double inequality

$$(1-\alpha)\mathbf{C}_e^{-1} + \alpha\mathbf{A}^{-1} < \mathbf{M}_{\sin}^{-1} < (1-\beta)\mathbf{C}_e^{-1} + \beta\mathbf{A}^{-1}$$

holds if and only if $\alpha \le 4\sin 1 - 3 \approx 0.3659$ and $\beta \ge 1/2$.

Proposition 1.4. The double inequality

$$(1-\alpha)\mathbf{C}_e^{-1} + \alpha\mathbf{A}^{-1} < \mathbf{M}_{\tanh}^{-1} < (1-\beta)\mathbf{C}_e^{-1} + \beta\mathbf{A}^{-1}$$

holds if and only if $\alpha \le 0$ and $\beta \ge 4\tanh 1 - 3 \approx 0.0464$.

Proposition 1.5. The double inequality

$$(1-\alpha)\mathbf{A}^2 + \alpha\mathbf{C}_e^2 < \mathbf{M}_{\sin}^2 < (1-\beta)\mathbf{A}^2 + \beta\mathbf{C}_e^2$$

holds if and only if $\alpha \le 1/2$ and $\beta \ge \left(9\cot^2 1\right)/7 \approx 0.5301$.

Proposition 1.6. The double inequality

$$(1-\alpha)\mathbf{A}^2 + \alpha\mathbf{C}_e^2 < \mathbf{M}_{\tanh}^2 < (1-\beta)\mathbf{A}^2 + \beta\mathbf{C}_e^2$$

holds if and only if $\alpha \le \left(9\left(\coth^2 1 - 1\right)\right)/7 \approx 0.9309$ and $\beta \ge 1$.

Proposition 1.7. The double inequality

$$(1-\alpha)\mathbf{C}_e^{-2} + \alpha\mathbf{A}^{-2} < \mathbf{M}_{\sin}^{-2} < (1-\beta)\mathbf{C}_e^{-2} + \beta\mathbf{A}^{-2}$$

holds if and only if $\alpha \le \left(16\sin^2 1 - 9\right)/7 \approx 0.3327$ and $\beta \ge 1/2$.

Proposition 1.8. The double inequality

$$(1-\alpha)\mathbf{C}_e^{-2} + \alpha\mathbf{A}^{-2} < \mathbf{M}_{\tanh}^{-2} < (1-\beta)\mathbf{C}_e^{-2} + \beta\mathbf{A}^{-2}$$

holds if and only if $\alpha \le 0$ and $\beta \ge \left(16\tanh^2 1 - 9\right)/7 \approx 0.0401$.

In essence, the above results are how the two new means $\mathbf{M}_{\sin}$ and $\mathbf{M}_{\tanh}$ are expressed linearly, harmoniously, squarely, and harmoniously in square by the two classical means $\mathbf{C}_e(x,y)$ and $\mathbf{A}(x,y)$. In this paper, we study the following two-sided inequalities in exponential form for nonzero number $p \in \mathbb{R}$

$$(1-\alpha_p)\mathbf{A}^p + \alpha_p\mathbf{Ce}^p < \mathbf{M}_{\sin}^p < (1-\beta_p)\mathbf{A}^p + \beta_p\mathbf{Ce}^p, \tag{1.7}$$

$$(1-\lambda_p)\mathbf{A}^p + \lambda_p\mathbf{Ce}^p < \mathbf{M}_{\tanh}^p < (1-\mu_p)\mathbf{A}^p + \mu_p\mathbf{Ce}^p \tag{1.8}$$

in order to reach a broader conclusion including all the above properties. The main conclusions of this paper are as follows:

Theorem 1.1. Let $x, y > 0$, $x \neq y$, $p \neq 0$ and

$$p^{\clubsuit} = \frac{3\cos 2 + \sin 2 + 1}{3\sin 2 - \cos 2 - 3} \approx 4.588.$$

Then the following are considered.

(i) If $p \geq p^{\clubsuit}$, the double inequality

$$(1-\alpha_p)\mathbf{A}^p + \alpha_p \mathbf{Ce}^p < \mathbf{M}_{\sin}^p < (1-\beta_p)\mathbf{A}^p + \beta_p \mathbf{Ce}^p \tag{1.9}$$

holds if and only if $\alpha_p \leq 3^p (1 - \sin^p 1) / [(\sin^p 1)(4^p - 3^p)]$ and $\beta_p \geq 1/2$.

(ii) If $0 < p \leq 12/5$, the double inequality

$$(1-\alpha_p)\mathbf{A}^p + \alpha_p \mathbf{Ce}^p < \mathbf{M}_{\sin}^p < (1-\beta_p)\mathbf{A}^p + \beta_p \mathbf{Ce}^p \tag{1.10}$$

holds if and only if $\alpha_p \leq 1/2$ and $\beta_p \geq 3^p (1 - \sin^p 1) / [(\sin^p 1)(4^p - 3^p)]$.

(iii) If $p < 0$, the double inequality

$$(1-\beta_p)\mathbf{A}^p + \beta_p \mathbf{Ce}^p < \mathbf{M}_{\sin}^p < (1-\alpha_p)\mathbf{A}^p + \alpha_p \mathbf{Ce}^p \tag{1.11}$$

holds if and only if $\alpha_p \leq 1/2$ and $\beta_p \geq 3^p (1 - \sin^p 1) / [(\sin^p 1)(4^p - 3^p)]$.

Theorem 1.2. Let $x, y > 0$, $x \neq y$, $p \neq 0$ and

$$p^{*} = -\frac{16\cosh 2 - 3\cosh 4 + 4\sinh 2 + 3}{\cosh 4 - 12\sinh 2 + 15} \approx -3.4776.$$

Then the following are considered:

(i) If $p > 0$, the double inequality

$$(1-\lambda_p)\mathbf{A}^p + \lambda_p \mathbf{Ce}^p < \mathbf{M}_{\tanh}^p < (1-\mu_p)\mathbf{A}^p + \mu_p \mathbf{Ce}^p \tag{1.12}$$

holds if and only if $\lambda_p \leq ((\coth 1)^p - 1) / ((4/3)^p - 1)$ and $\mu_p \geq 1$.

(ii) If $p^* \leq p < 0$,

$$(1-\mu_p)\mathbf{A}^p + \mu_p \lambda_p \mathbf{Ce}^p < \mathbf{M}_{\tanh}^p < (1-\lambda_p)\mathbf{A}^p + \lambda_p \mathbf{Ce}^p \tag{1.13}$$

holds if and only if $\lambda_p \leq ((\coth 1)^p - 1) / ((4/3)^p - 1)$ and $\mu_p \geq 1$.

2. Lemmas

We first introduce a theoretical support of this paper.

Lemma 2.1. ([10]) Let $\mathbf{K}(x, y), \mathbf{R}(x, y)$, and $\mathbf{N}(x, y)$ be three means with two positive distinct parameters x and y; $f_{\mathbf{K}}(t)$, $f_{\mathbf{R}}(t)$, and $f_{\mathbf{N}}(t)$ be the corresponding Seiffert functions of the former, $\vartheta_1, \vartheta_2, \theta_1, \theta_2, p \in \mathbb{R}$, and $p \neq 0$. Then

$$\begin{aligned}\vartheta_1\mathbf{K}^p(x,y)+\vartheta_2\mathbf{N}^p(x,y) &\leq \mathbf{R}^p(x,y)\leq\theta_1\mathbf{K}^p(x,y)+\theta_2\mathbf{N}^p(x,y) \quad (2.1)\\ &\Longleftrightarrow \\ \frac{\vartheta_1}{f_{\mathbf{K}}^p(t)}+\frac{\vartheta_2}{f_{\mathbf{N}}^p(t)} &\leq \frac{1}{f_{\mathbf{R}}^p(t)}\leq\frac{\theta_1}{f_{\mathbf{K}}^p(t)}+\frac{\theta_2}{f_{\mathbf{N}}^p(t)}. \quad (2.2)\end{aligned}$$

It must be mentioned that the key steps to prove the above results are following:

$$\begin{aligned}\mathbf{M}_f(u,v) &= \mathbf{M}_f\left(\lambda\frac{2x}{x+y},\lambda\frac{2y}{x+y}\right)=\lambda\mathbf{M}_f\left(\frac{2x}{x+y},\frac{2y}{x+y}\right)=\lambda\mathbf{M}_f\left(1-t,1+t\right)\\ &= \lambda\frac{t}{f_{\mathbf{M}}(t)}, \quad (2.3)\end{aligned}$$

where

$$\begin{cases} u=\lambda\frac{2x}{x+y} \\ v=\lambda\frac{2y}{x+y}\end{cases}, \ 0<x<y,\ \lambda>0.$$

and $0<t<1$,

$$t=\frac{y-x}{x+y}.$$

In order to prove the main conclusions, we shall introduce some very suitable methods which are called the monotone form of L'Hospital's rule (see Lemma 2.2) and the criterion for the monotonicity of the quotient of power series (see Lemma 2.3).

Lemma 2.2. ([11, 12]) For $-\infty<a<b<\infty$, let $f,g:[a,b]\to\mathbb{R}$ be continuous functions that are differentiable on (a,b), with $f(a)=g(a)=0$ or $f(b)=g(b)=0$. Assume that $g'(t)\neq 0$ for each x in (a,b). If f'/g' is increasing (decreasing) on (a,b), then so is f/g.

Lemma 2.3. ([13]) Let a_n and b_n $(n=0,1,2,\cdots)$ be real numbers, and let the power series $A(x)=\sum_{n=0}^{\infty}a_nx^n$ and $B(x)=\sum_{n=0}^{\infty}b_nx^n$ be convergent for $|x|<R$ $(R\leq+\infty)$. If $b_n>0$ for $n=0,1,2,\cdots$, and if $\varepsilon_n=a_n/b_n$ is strictly increasing (or decreasing) for $n=0,1,2,\cdots$, then the function $A(x)/B(x)$ is strictly increasing (or decreasing) on $(0,R)$ $(R\leq+\infty)$.

Lemma 2.4. ([14, 15]) *Let B_{2n} be the even-indexed Bernoulli numbers. Then we have the following power series expansions*

$$\begin{aligned}\cot x &= \frac{1}{x}-\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|x^{2n-1},\ 0<|x|<\pi, \quad (2.4)\\ \frac{1}{\sin^2 x} &= \csc^2 x=-(\cot x)'=\frac{1}{x^2}+\sum_{n=1}^{\infty}\frac{2^{2n}(2n-1)}{(2n)!}|B_{2n}|x^{2n-2},\ 0<|x|<\pi. \quad (2.5)\end{aligned}$$

Lemma 2.5. ([16–20]) Let B_{2n} the even-indexed Bernoulli numbers, $n=1,2,\ldots$. Then

$$\frac{2^{2n-1}-1}{2^{2n+1}-1}\frac{(2n+2)(2n+1)}{\pi^2}<\frac{|B_{2n+2}|}{|B_{2n}|}<\frac{2^{2n}-1}{2^{2n+2}-1}\frac{(2n+2)(2n+1)}{\pi^2}.$$

Lemma 2.6. Let $l_1(t)$ be defined by

$$l_1(t) = \frac{s_1(t)}{r_1(t)},$$

where

$$\begin{aligned} s_1(t) &= \frac{6t^2 + 2t^4 - 12\sin^2 t - 2t^3\cos t\sin t + 6t\cos t\sin t}{\sin^2 t}, \\ r_1(t) &= \frac{8t^2\sin^2 t + 2t^4\sin^2 t - 6t^2 - 2t^4 - 6\sin^2 t + 12t\cos t\sin t}{\sin^2 t}. \end{aligned}$$

Then the double inequality

$$\frac{12}{5} < l_1(t) < p^{\clubsuit} = \frac{3\cos 2 + \sin 2 + 1}{3\sin 2 - \cos 2 - 3} \approx 4.588 \tag{2.6}$$

holds for all $t \in (0,1)$, where the constants $12/5$ and $(3\cos 2 + \sin 2 + 1)/(3\sin 2 - \cos 2 - 3) \approx 4.588$ are the best possible in (2.6).

Proof. Since

$$\frac{1}{l_1(t)} = \frac{r_1(t)}{s_1(t)},$$

and

$$\begin{aligned} r_1(t) &= \frac{8t^2\sin^2 t + 2t^4\sin^2 t - 6t^2 - 2t^4 - 6\sin^2 t + 12t\cos t\sin t}{\sin^2 t} \\ &= 8t^2 - 2t^4\frac{1}{\sin^2 t} - 6t^2\frac{1}{\sin^2 t} + 2t^4 + 12t\frac{\cos t}{\sin t} - 6 \\ &= 8t^2 - 2t^4\left[\frac{1}{t^2} + \sum_{n=1}^{\infty}\frac{2^{2n}(2n-1)}{(2n)!}|B_{2n}|t^{2n-2}\right] - 6t^2\left[\frac{1}{t^2} + \sum_{n=1}^{\infty}\frac{2^{2n}(2n-1)}{(2n)!}|B_{2n}|t^{2n-2}\right] \\ &\quad + 2t^4 + 12t\left[\frac{1}{t} - \sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|t^{2n-1}\right] - 6 \\ &= \frac{2}{3}t^4 - \sum_{n=3}^{\infty}\left[\frac{2^{2n-1}(2n-3)}{(2n-2)!}|B_{2n-2}| + \frac{6\cdot 2^{2n}(2n+1)}{(2n)!}|B_{2n}|\right]t^{2n} \\ &=: \sum_{n=2}^{\infty} a_n t^{2n}, \end{aligned}$$

where

$$\begin{aligned} a_2 &= \frac{2}{3}, \\ a_n &= -\left[\frac{2^{2n-1}(2n-3)}{(2n-2)!}|B_{2n-2}| + \frac{6\cdot 2^{2n}(2n+1)}{(2n)!}|B_{2n}|\right],\ n = 3, 4, \ldots, \end{aligned}$$

$$
\begin{aligned}
s_1(t) &= \frac{6t^2 + 2t^4 - 12\sin^2 t - 2t^3\cos t\sin t + 6t\cos t\sin t}{\sin^2 t} \\
&= 6t^2\frac{1}{\sin^2 t} + 2t^4\frac{1}{\sin^2 t} + 6t\frac{\cos t}{\sin t} - 2t^3\frac{\cos t}{\sin t} - 12 \\
&= 6t^2\left[\frac{1}{t^2} + \sum_{n=1}^{\infty}\frac{2^{2n}(2n-1)}{(2n)!}|B_{2n}|t^{2n-2}\right] + 2t^4\left[\frac{1}{t^2} + \sum_{n=1}^{\infty}\frac{2^{2n}(2n-1)}{(2n)!}|B_{2n}|t^{2n-2}\right] \\
&\quad +6t\left[\frac{1}{t} - \sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|t^{2n-1}\right] - 2t^3\left[\frac{1}{t} - \sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}|B_{2n}|t^{2n-1}\right] - 12 \\
&= \sum_{n=2}^{\infty}\frac{12\cdot 2^{2n}(n-1)}{(2n)!}|B_{2n}|t^{2n} + \sum_{n=1}^{\infty}\frac{4n\cdot 2^{2n}}{(2n)!}|B_{2n}|t^{2n+2} \\
&= \sum_{n=2}^{\infty}\frac{12\cdot 2^{2n}(n-1)}{(2n)!}|B_{2n}|t^{2n} + \sum_{n=2}^{\infty}\frac{(n-1)\cdot 2^{2n}}{(2n-2)!}|B_{2n-2}|t^{2n} \\
&= \sum_{n=2}^{\infty}\left[\frac{12\cdot 2^{2n}(n-1)}{(2n)!}|B_{2n}| + \frac{(n-1)\cdot 2^{2n}}{(2n-2)!}|B_{2n-2}|\right]t^{2n} \\
&= \frac{8}{5}t^4 + \sum_{n=3}^{\infty}\left[\frac{12\cdot 2^{2n}(n-1)}{(2n)!}|B_{2n}| + \frac{(n-1)\cdot 2^{2n}}{(2n-2)!}|B_{2n-2}|\right]t^{2n} \\
&=: \sum_{n=2}^{\infty} b_n t^{2n},
\end{aligned}
$$

where

$$
\begin{aligned}
b_2 &= \frac{8}{5}, \\
b_n &= \frac{12\cdot 2^{2n}(n-1)}{(2n)!}|B_{2n}| + \frac{(n-1)\cdot 2^{2n}}{(2n-2)!}|B_{2n-2}| > 0,\ n = 3, 4, \ldots.
\end{aligned}
$$

Setting

$$
q_n = \frac{a_n}{b_n},\ n = 2, 3, \ldots,
$$

we have

$$
\begin{aligned}
q_2 &= \frac{5}{12} = 0.416\,67, \\
q_n &= -\frac{\frac{2^{2n-1}(2n-3)}{(2n-2)!}|B_{2n-2}| + \frac{6\cdot 2^{2n}(2n+1)}{(2n)!}|B_{2n}|}{\frac{12\cdot 2^{2n}(n-1)}{(2n)!}|B_{2n}| + \frac{(n-1)\cdot 2^{2n}}{(2n-2)!}|B_{2n-2}|},\ n = 3, 4, \ldots.
\end{aligned}
$$

Here we prove that the sequence $\{q_n\}_{n\geq 2}$ decreases monotonously. Obviously, $q_2 > 0 > q_3$. We shall prove that for $n \geq 3$,

$$
\begin{aligned}
q_n &> q_{n+1} \Longleftrightarrow \\
-\frac{\frac{2^{2n-1}(2n-3)}{(2n-2)!}|B_{2n-2}| + \frac{6\cdot 2^{2n}(2n+1)}{(2n)!}|B_{2n}|}{\frac{12\cdot 2^{2n}(n-1)}{(2n)!}|B_{2n}| + \frac{(n-1)\cdot 2^{2n}}{(2n-2)!}|B_{2n-2}|} &> -\frac{\frac{2^{2n+1}(2n-1)}{(2n)!}|B_{2n}| + \frac{6\cdot 2^{2n+2}(2n+3)}{(2n+2)!}|B_{2n+2}|}{\frac{12\cdot 2^{2n+2}n}{(2n+2)!}|B_{2n+2}| + \frac{n\cdot 2^{2n+2}}{(2n)!}|B_{2n}|} \Longleftrightarrow \\
\frac{\frac{2^{2n-1}(2n-3)}{(2n-2)!}|B_{2n-2}| + \frac{6\cdot 2^{2n}(2n+1)}{(2n)!}|B_{2n}|}{\frac{12\cdot 2^{2n}(n-1)}{(2n)!}|B_{2n}| + \frac{(n-1)\cdot 2^{2n}}{(2n-2)!}|B_{2n-2}|} &< \frac{\frac{2^{2n+1}(2n-1)}{(2n)!}|B_{2n}| + \frac{6\cdot 2^{2n+2}(2n+3)}{(2n+2)!}|B_{2n+2}|}{\frac{12\cdot 2^{2n+2}n}{(2n+2)!}|B_{2n+2}| + \frac{n\cdot 2^{2n+2}}{(2n)!}|B_{2n}|},
\end{aligned}
$$

that is,

$$
\frac{2}{(2n)!\,(2n-2)!}\frac{|B_{2n-2}|}{|B_{2n}|} + \frac{24\,(4n-3)}{(2n-2)!\,(2n+2)!}\frac{|B_{2n+2}||B_{2n-2}|}{|B_{2n}||B_{2n}|} > \frac{24\,(4n-1)}{((2n)!)^2} + \frac{864}{(2n)!\,(2n+2)!}\frac{|B_{2n+2}|}{|B_{2n}|}. \tag{2.7}
$$

By Lemma 2.5 we have

$$
\begin{aligned}
&\frac{2}{(2n)!\,(2n-2)!}\frac{|B_{2n-2}|}{|B_{2n}|} + \frac{24\,(4n-3)}{(2n-2)!\,(2n+2)!}\frac{|B_{2n+2}||B_{2n-2}|}{|B_{2n}||B_{2n}|} \\
> &\frac{2}{(2n)!\,(2n-2)!}\frac{2^{2n}-1}{2^{2n-2}-1}\frac{\pi^2}{(2n)(2n-1)} \\
&+\frac{24\,(4n-3)}{(2n-2)!\,(2n+2)!}\frac{2^{2n-1}-1}{2^{2n+1}-1}\frac{(2n+2)(2n+1)}{\pi^2}\frac{2^{2n}-1}{2^{2n-2}-1}\frac{\pi^2}{(2n)(2n-1)} \\
= &\frac{2\pi^2}{(2n)!\,(2n)!}\frac{2^{2n}-1}{2^{2n-2}-1} + \frac{24\,(4n-3)}{(2n)!\,(2n)!}\frac{2^{2n-1}-1}{2^{2n+1}-1}\frac{2^{2n}-1}{2^{2n-2}-1},
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{24\,(4n-1)}{(2n)!^2} + \frac{864}{(2n)!\,(2n+2)!}\frac{|B_{2n+2}|}{|B_{2n}|} \\
< &\frac{24\,(4n-1)}{(2n)!^2} + \frac{864}{(2n)!\,(2n+2)!}\frac{2^{2n}-1}{2^{2n+2}-1}\frac{(2n+2)(2n+1)}{\pi^2} \\
= &\frac{24\,(4n-1)}{(2n)!^2} + \frac{864}{(2n)!\,(2n)!}\frac{2^{2n}-1}{2^{2n+2}-1}\frac{1}{\pi^2}.
\end{aligned}
$$

So we can complete the prove (2.7) when proving

$$
\frac{2\pi^2}{(2n)!\,(2n)!}\frac{2^{2n}-1}{2^{2n-2}-1} + \frac{24\,(4n-3)}{(2n)!\,(2n)!}\frac{2^{2n-1}-1}{2^{2n+1}-1}\frac{2^{2n}-1}{2^{2n-2}-1} > \frac{24\,(4n-1)}{((2n)!)^2} + \frac{864}{(2n)!\,(2n)!}\frac{2^{2n}-1}{2^{2n+2}-1}\frac{1}{\pi^2}
$$

or

$$
\frac{2\pi^2\left(2^{2n}-1\right)}{2^{2n-2}-1} + 24\,(4n-3)\,\frac{2^{2n-1}-1}{2^{2n+1}-1}\frac{2^{2n}-1}{2^{2n-2}-1} > 24\,(4n-1) + \frac{2^{2n}-1}{2^{2n+2}-1}\frac{864}{\pi^2}.
$$

In fact,

$$\begin{aligned}&\frac{2\pi^2\left(2^{2n}-1\right)}{2^{2n-2}-1}+24\left(4n-3\right)\frac{2^{2n-1}-1}{2^{2n+1}-1}\frac{2^{2n}-1}{2^{2n-2}-1}-\left[24\left(4n-1\right)+\frac{2^{2n}-1}{2^{2n+2}-1}\frac{864}{\pi^2}\right]\\ =: &\frac{8H(n)}{\pi^2\left(2^{2n+2}-1\right)\left(2^{2n}-4\right)\left(2^{2n+1}-1\right)},\end{aligned}$$

where

$$\begin{aligned}H(n) &= 8\cdot 2^{6n}\left(\pi+3\right)\left(\pi-3\right)\left(\pi^2+3\right)+2\cdot 2^{4n}\left(72\pi^2 n+60\pi^2-7\pi^4+594\right)\\ &\quad -2^{2n}\left(36\pi^2 n+123\pi^2-7\pi^4+1404\right)+\left(24\pi^2-\pi^4+432\right)>0\end{aligned}$$

for all $n\geq 3$.

So the sequence $\{q_n\}_{n\geq 2}$ decreases monotonously. By Lemma 2.3 we obtain that $r_1(t)/s_1(t)$ is decreasing on $(0,1)$, which means that the function $l_1(t)$ is increasing on $(0,1)$. In view of

$$\lim_{t\to 0^+} l_1(t)=\frac{12}{5} \text{ and } \lim_{t\to 1^-} l_1(t)=p^{\clubsuit}=\frac{3\cos 2+\sin 2+1}{3\sin 2-\cos 2-3}\approx 4.588,$$

the proof of this lemma is complete. □

Lemma 2.7. Let $l_2(t)$ be defined by

$$l_2(t)=2\cdot\frac{3\cosh 4t-12t^2\cosh 2t-4t^4\cosh 2t+2t^3\sinh 2t-6t\sinh 2t-3}{t^2\cosh 4t-3\cosh 4t+24t\sinh 2t-25t^2-8t^4+3}=:2\frac{B(t)}{A(t)},\ 0<t<\infty,$$

where

$$\begin{aligned}A(t) &= t^2\cosh 4t-3\cosh 4t+24t\sinh 2t-25t^2-8t^4+3,\\ B(t) &= 3\cosh 4t-12t^2\cosh 2t-4t^4\cosh 2t+2t^3\sinh 2t-6t\sinh 2t-3.\end{aligned}$$

Then $l_2(t)$ is strictly decreasing on $(0,\infty)$.

Proof. Let's take the power series expansions

$$\sinh kt=\sum_{n=0}^{\infty}\frac{k^{2n+1}}{(2n+1)!}t^{2n+1},\ \cosh kt=\sum_{n=0}^{\infty}\frac{k^{2n}}{(2n)!}t^{2n}$$

into $A(t)$ and $B(t)$, and get

$$A(t)=\sum_{n=2}^{\infty}c_n t^{2n+2},\ B(t)=\sum_{n=2}^{\infty}d_n t^{2n+2},$$

where

$$\begin{aligned}c_2 &= 0,\\ c_n &= \left[\frac{2\left(3n+2n^2-23\right)2^{2n}+48\left(2n+2\right)}{(2n+2)!}\right]2^{2n},\ n=3,4,\ldots,\end{aligned}$$

$$d_n = \left[\frac{48\cdot 2^{2n} - 8(n+1)\left(5n - n^2 + 2n^3 + 6\right)}{(2n+2)!}\right]2^{2n},\ n = 2,3,\ldots,$$

Setting

$$k_n = \frac{c_n}{d_n} = \frac{48(n+1) + 2^{2n}\left(3n + 2n^2 - 23\right)}{4\left(6\cdot 2^{2n} - 11n - 4n^2 - n^3 - 2n^4 - 6\right)},\ n = 2,3,\ldots,$$

Here we prove that the sequence $\{k_n\}_{n\geq 2}$ decreases monotonously. Obviously, $k_2 = 0 < k_3$. For $n \geq 3$,

$$\begin{aligned}
k_n &< k_{n+1} \\
&\Longleftrightarrow \\
&\frac{48(n+1) + 2^{2n}\left(3n + 2n^2 - 23\right)}{4\left(6\cdot 2^{2n} - 11n - 4n^2 - n^3 - 2n^4 - 6\right)} \\
&< \frac{48(n+2) + 2^{2n+2}\left(3(n+1) + 2(n+1)^2 - 23\right)}{4\left(6\cdot 2^{2n+2} - 11(n+1) - 4(n+1)^2 - (n+1)^3 - 2(n+1)^4 - 6\right)} \\
&\Longleftrightarrow \\
&\frac{48(n+1) + 2^{2n}\left(3n + 2n^2 - 23\right)}{6\cdot 2^{2n} - 11n - 4n^2 - n^3 - 2n^4 - 6} \\
&< \frac{48n + 96 + 2^{2n+2}\left(7n + 2n^2 - 18\right)}{6\cdot 2^{2n+2} - 30n - 19n^2 - 9n^3 - 2n^4 - 24}
\end{aligned}$$

follows from $\Delta(n) > 0$ for all $n \geq 2$, where

$$\begin{aligned}
\Delta(n) &= \left(48n + 96 + 2^{2n+2}\left(7n + 2n^2 - 18\right)\right)\left(6\cdot 2^{2n} - 11n - 4n^2 - n^3 - 2n^4 - 6\right) \\
&\quad - \left(48(n+1) + 2^{2n}\left(3n + 2n^2 - 23\right)\right)\left(6\cdot 2^{2n+2} - 30n - 19n^2 - 9n^3 - 2n^4 - 24\right) \\
&= 24\cdot 2^{4n}(4n+5) - 2^{2n}\left(858n + 367n^2 + 218n^3 - 103n^4 + 40n^5 + 12n^6 + 696\right) \\
&\quad + 1248n + 1440n^2 + 1056n^3 + 288n^4 + 576 \\
&=: 2^{2n}\left[j(n)2^{2n} - i(n)\right] + w(n)
\end{aligned}$$

with

$$\begin{aligned}
j(n) &= 24(4n+5), \\
i(n) &= 858n + 367n^2 + 218n^3 - 103n^4 + 40n^5 + 12n^6 + 696, \\
w(n) &= 1248n + 1440n^2 + 1056n^3 + 288n^4 + 576 > 0.
\end{aligned}$$

We have that $\Delta(2) = 5376 > 0$ and shall prove that

$$\begin{aligned}
j(n)2^{2n} - i(n) &> 0 \Longleftrightarrow \\
2^{2n} &> \frac{i(n)}{j(n)}
\end{aligned} \tag{2.8}$$

holds for all $n \geq 3$. Now we use mathematical induction to prove (2.8). When $n = 3$, the left-hand side and right-hand side of (2.8) are $2^6 = 64$ and $i(3)/j(3) = 941/17 \approx 55.353$, which implies (2.8) holds for $n=3$. Assuming that (2.8) holds for $n=m$, that is,

$$2^{2m} > \frac{i(m)}{j(m)}. \tag{2.9}$$

Next, we prove that (2.8) is valid for $n=m+1$. By (2.9) we have

$$2^{2(m+1)} = 4\cdot 2^{2m} > 4\frac{i(m)}{j(m)},$$

in order to complete the proof of (2.8) it suffices to show that

$$4\frac{i(m)}{j(m)} > \frac{i(m+1)}{j(m+1)} \Longleftrightarrow 4i(m)j(m+1) - i(m+1)j(m) > 0.$$

In fact,

$$\begin{aligned}
& 4i(m)j(m+1) - i(m+1)j(m) \\
= \; & 17\,280m^7 + 90\,720m^6 - 60\,000m^5 - 97\,176m^4 + 1169\,232m^3 + 2266\,104m^2 \\
& +3581\,136m + 2154\,816 \\
= \; & 146\,337\,408 + 234\,401\,616\,(m-3) + 189\,746\,328\,(m-3)^2 + 92\,580\,720\,(m-3)^3 \\
& +27\,579\,624\,(m-3)^4 + 4838\,880\,(m-3)^5 + 453\,600\,(m-3)^6 + 17\,280\,(m-3)^7 \\
> \; & 0
\end{aligned}$$

for $m \geq 3$ due to the coefficients of the power square of $(m-1)$ are positive.

By Lemma 2.3 we get that $A(t)/B(t)$ is strictly increasing on $(0,\infty)$. So the function $l_2(x)$ is strictly decreasing on $(0,\infty)$.

The proof of Lemma 2.7 is complete. □

3. Proofs of main results

Via (1.3) and (1.2) we can obtain

$$\begin{aligned}
f_{\mathbf{A}}(t) &= t, \\
f_{\mathbf{C}_e}(t) &= \frac{3t}{3+t^2}, \\
f_{\mathbf{M}_{\sin}}(t) &= \sin t, \\
f_{\mathbf{M}_{\tanh}}(t) &= \tanh t.
\end{aligned}$$

Then by Lemma 2.1 and (2.3) we have

$$\begin{aligned}
\alpha_p &< \frac{\mathbf{M}_{\sin}^p - \mathbf{A}^p}{\mathbf{Ce}^p - \mathbf{A}^p} < \beta_p \Longleftrightarrow \alpha_p < \frac{\left(\frac{1}{\sin t}\right)^p - \left(\frac{1}{t}\right)^p}{\left(\frac{3+t^2}{3t}\right)^p - \left(\frac{1}{t}\right)^p} < \beta_p, \\
\lambda_p &< \frac{\mathbf{M}_{\tanh}^p - \mathbf{A}^p}{\mathbf{Ce}^p - \mathbf{A}^p} < \mu_p \Longleftrightarrow \lambda_p < \frac{\left(\frac{1}{\tanh t}\right)^p - \left(\frac{1}{t}\right)^p}{\left(\frac{3+t^2}{3t}\right)^p - \left(\frac{1}{t}\right)^p} < \mu_p.
\end{aligned}$$

So we turn to the proof of the following two theorems.

Theorem 3.1. Let $t \in (0,1)$ and

$$p^{\clubsuit} = \frac{3\cos 2 + \sin 2 + 1}{3\sin 2 - \cos 2 - 3} \approx 4.588.$$

Then,

(i) if $p \geq p^{\clubsuit}$, the double inequality

$$\alpha_p < \frac{\left(\frac{1}{\sin t}\right)^p - \left(\frac{1}{t}\right)^p}{\left(\frac{3+t^2}{3t}\right)^p - \left(\frac{1}{t}\right)^p} < \beta_p \tag{3.1}$$

holds if and only if $\alpha_p \leq 3^p\left(1-\sin^p 1\right)/\left[\left(\sin^p 1\right)\left(4^p - 3^p\right)\right]$ and $\beta_p \geq 1/2$;

(ii) if $0 \neq p \leq 12/5 = 2.4$ and $p \neq 0$ the double inequality

$$\beta_p < \frac{\left(\frac{1}{\sin t}\right)^p - \left(\frac{1}{t}\right)^p}{\left(\frac{3+t^2}{3t}\right)^p - \left(\frac{1}{t}\right)^p} < \alpha_p \tag{3.2}$$

holds if and only if $\alpha_p \leq 1/2$ and $\beta \geq 3^p\left(1-\sin^p 1\right)/\left[\left(\sin^p 1\right)\left(4^p - 3^p\right)\right]$.

Theorem 3.2. Let $t \in (0,1)$ and

$$p^{*} = -\frac{16\cosh 2 - 3\cosh 4 + 4\sinh 2 + 3}{\cosh 4 - 12\sinh 2 + 15} \approx -3.4776.$$

If $0 \neq p \geq -3.4776$, the double inequality

$$\lambda_p < \frac{\left(\frac{1}{\tanh t}\right)^p - \left(\frac{1}{t}\right)^p}{\left(\frac{3+t^2}{3t}\right)^p - \left(\frac{1}{t}\right)^p} < \mu_p \tag{3.3}$$

holds if and only if $\lambda_p \leq \left((\coth 1)^p - 1\right)/\left((4/3)^p - 1\right)$ and $\mu_p \geq 1$.

3.1. The proof of Theorem 3.1

Let

$$\begin{aligned} F(t) &= \frac{\left(\frac{1}{\sin t}\right)^p - \left(\frac{1}{t}\right)^p}{\left(\frac{3+t^2}{3t}\right)^p - \left(\frac{1}{t}\right)^p} = \frac{\left(\frac{t}{\sin t}\right)^p - 1}{\left(\frac{3+t^2}{3}\right)^p - 1} \\ &=: \frac{f(t)}{g(t)} = \frac{f(t) - f(0^+)}{g(t) - g(0^+)}, \end{aligned}$$

where

$$\begin{aligned} f(t) &= \left(\frac{t}{\sin t}\right)^p - 1, \\ g(t) &= \left(\frac{3+t^2}{3}\right)^p - 1. \end{aligned}$$

Then

$$\begin{aligned} f'(t) &= \frac{p}{\sin^2 t}(\sin t - t\cos t)\left(\frac{t}{\sin t}\right)^{p-1}, \\ g'(t) &= \frac{2}{3}\left(\frac{1}{3}\right)^{p-1} pt\left(t^2+3\right)^{p-1}, \end{aligned}$$

$$\frac{f'(t)}{g'(t)} = \frac{3^p}{2}\frac{1}{t\sin^2 t}(\sin t - t\cos t)\left(\frac{t}{(t^2+3)\sin t}\right)^{p-1},$$

and

$$\begin{aligned} \left(\frac{f'(t)}{g'(t)}\right)' &= \frac{1}{4}\left(\frac{3t}{(\sin t)(t^2+3)}\right)^p \frac{r_1(t)}{t^3\sin^2 t}\left[\frac{s_1(t)}{r_1(t)} - p\right] \\ &=: \frac{1}{4}\left(\frac{3t}{(\sin t)(t^2+3)}\right)^p \frac{r_1(t)}{t^3\sin^2 t}\left[l_1(t) - p\right], \end{aligned}$$

where the three functions $s_1(t)$, $r_1(t)$, and $l_1(t)$ are shown in Lemma 2.6.

By Lemma 2.6 we can obtain the following results:

(a) When $p \geq \max_{t\in(0,1)} l_1(t) =: p^{\clubsuit} = (3\cos 2 + \sin 2 + 1)/(3\sin 2 - \cos 2 - 3) \approx 4.588$,

$$\left(\frac{f'(t)}{g'(t)}\right)' \leq 0 \Longrightarrow \frac{f'(t)}{g'(t)} \text{ is decreasing on } (0,1),$$

this leads to $F(t) = f(t)/g(t)$ is decreasing on $(0,1)$ by Lemma 2.1. In view of

$$F(0^+) = \frac{1}{2},\ F(1^-) = \frac{3^p(1-\sin^p 1)}{(\sin^p 1)(4^p - 3^p)}, \tag{3.4}$$

we have that (3.1) holds.

(b) When $0 \neq p \leq 12/5 = \min_{t\in(0,1)} l_1(t)$,

$$\left(\frac{f'(t)}{g'(t)}\right)' \geq 0 \Longrightarrow \frac{f'(t)}{g'(t)} \text{ is increasing on } (0,1),$$

this leads to $F(t) = f(t)/g(t)$ is increasing on $(0,1)$ by Lemma 2.2. In view of (3.4) we have that (3.2) holds.

The proof of Theorem 3.1 is complete.

3.2. The proof of Theorem 3.2

Let

$$\begin{aligned} G(t) &= \frac{\left(\frac{1}{\tanh t}\right)^p - \left(\frac{1}{t}\right)^p}{\left(\frac{3+t^2}{3t}\right)^p - \left(\frac{1}{t}\right)^p} = \frac{\left(\frac{t}{\tanh t}\right)^p - 1}{\left(\frac{3+t^2}{3}\right)^p - 1} \\ &=: \frac{u(t)}{v(t)} = \frac{u(t)-u(0^+)}{v(t)-v(0^+)}. \end{aligned}$$

Then

$$\begin{aligned} u'(t) &= \frac{p}{\tanh^2 t}\left(\frac{t}{\tanh t}\right)^{p-1}\left(t\tanh^2 t+\tanh t-t\right), \\ v'(t) &= \frac{2}{3}pt\left(\frac{t^2+3}{3}\right)^{p-1}, \end{aligned}$$

$$\frac{u'(t)}{v'(t)}=\frac{3}{2}\frac{t\tanh^2 t+\tanh t-t}{t\tanh^2 t}\left[\frac{3t}{(t^2+3)\tanh t}\right]^{p-1},$$

and

$$\begin{aligned} \left(\frac{u'(t)}{v'(t)}\right)' &= -\frac{1}{16}\left[\frac{3t\cosh t}{(3+t^2)\sinh t}\right]^p\frac{A(t)}{t^3\cosh^2 t\sinh^2 t}\left[p+\frac{2B(t)}{A(t)}\right] \\ &=: -\frac{1}{16}\left[\frac{3t\cosh t}{(3+t^2)\sinh t}\right]^p\frac{A(t)}{t^3\cosh^2 t\sinh^2 t}\left[p+l_2(t)\right], \end{aligned}$$

where the three functions $A(t)$, $B(t)$, and $l_2(t)$ are shown in Lemma 2.7. By Lemma 2.7 we see that $l_2(x)$ is strictly decreasing on $(0,1)$. Since

$$\begin{aligned} \lim_{t\to 0^+} l_2(t) &= \infty, \\ \lim_{t\to 1^-} l_2(t) &= \frac{16\cosh 2-3\cosh 4+4\sinh 2+3}{\cosh 4-12\sinh 2+15} =: p_\# \approx 3.4776, \end{aligned}$$

we obtain the following result:

When $p\geq \max_{t\in(0,1)}\{-l_2(t)\}=-p_\# =: p^* \approx -3.4776$,

$$\left(\frac{u'(t)}{v'(t)}\right)'\leq 0 \Longrightarrow \frac{u'(t)}{v'(t)} \text{ is decreasing on } (0,1),$$

this leads to $G(t)=u(t)/v(t)$ is decreasing on $(0,1)$ by Lemma 2.2. Since

$$G(0^+)=1, G(1^-)=\frac{\left(\frac{\cosh 1}{\sinh 1}\right)^p-1}{\left(\frac{4}{3}\right)^p-1},$$

we have

$$G(1^-)<G(t)<G(0^+),$$

which completes the proof of Theorem 3.2.

4. Corollaries of main results and remarks

Remark 4.1. Letting $p=1,-1,2,-2$ in Theorems 1.1 and 1.2 respectively, one can obtain Propositions 1.1–1.8.

From Theorems 1.1 and 1.2, we can also get the following important conclusions:

Corollary 4.1. Let $x, y > 0$, $x \neq y$, and

$$\begin{aligned} p^{\clubsuit} &= \frac{3\cos 2 + \sin 2 + 1}{3\sin 2 - \cos 2 - 3} \approx 4.588, \\ \alpha &= \frac{3^{p^{\clubsuit}}\left(1 - \sin^{p^{\clubsuit}} 1\right)}{\left(\sin^{p^{\clubsuit}} 1\right)(4^{p^{\clubsuit}} - 3^{p^{\clubsuit}})} \approx 0.440\,25, \\ \beta &= \frac{1}{2}. \end{aligned}$$

Then the double inequality

$$(1-\alpha)\mathbf{A}^{p^{\clubsuit}} + \alpha\mathbf{Ce}^{p^{\clubsuit}} < \mathbf{M}_{\sin}^{p^{\clubsuit}} < (1-\beta)\mathbf{A}^{p^{\clubsuit}} + \beta\mathbf{Ce}^{p^{\clubsuit}} \tag{4.1}$$

holds, where the constants α and β are the best possible in (4.1).

Corollary 4.2. Let $x, y > 0$, $x \neq y$, and

$$\begin{aligned} \theta &= \frac{1}{2}, \\ \vartheta &= \frac{3^{12/5}\left(1 - \sin^{12/5} 1\right)}{\left(\sin^{12/5} 1\right)(4^{12/5} - 3^{12/5})} \approx 0.516\,03. \end{aligned}$$

Then the double inequality

$$(1-\theta)\mathbf{A}^{12/5} + \theta\mathbf{Ce}^{12/5} < \mathbf{M}_{\sin}^{12/5} < (1-\vartheta)\mathbf{A}^{12/5} + \vartheta\mathbf{Ce}^{12/5} \tag{4.2}$$

holds, where the constants θ and ϑ are the best possible in (4.2).

Corollary 4.3. Let $x, y > 0$, $x \neq y$, and

$$\begin{aligned} p^* &= -\frac{16\cosh 2 - 3\cosh 4 + 4\sinh 2 + 3}{\cosh 4 - 12\sinh 2 + 15} \approx -3.477\,6, \\ \lambda &= \frac{(\coth 1)^{p^*} - 1}{(4/3)^{p^*} - 1} \approx 0.968\,13, \\ \mu &= 1. \end{aligned}$$

Then the double inequality

$$(1-\mu)\mathbf{A}^{p^*} + \mu\mathbf{Ce}^{p^*} < \mathbf{M}_{\tanh}^{p^*} < (1-\lambda)\mathbf{A}^{p^*} + \lambda\mathbf{Ce}^{p^*} \tag{4.3}$$

holds, where the constants λ and μ are the best possible in (4.3).

5. Conclusions

In this paper, we have studied exponential type inequalities for $\mathbf{M}_{\sin}$ and $\mathbf{M}_{\tanh}$ in term of $\mathbf{A}$ and $\mathbf{Ce}$ for nonzero number $p \in \mathbb{R}$:

$$\begin{aligned} (1-\alpha_p)\mathbf{A}^p + \alpha_p\mathbf{Ce}^p &< \mathbf{M}_{\sin}^p < (1-\beta_p)\mathbf{A}^p + \beta_p\mathbf{Ce}^p, \\ (1-\lambda_p)\mathbf{A}^p + \lambda_p\mathbf{Ce}^p &< \mathbf{M}_{\tanh}^p < (1-\mu_p)\mathbf{A}^p + \mu_p\mathbf{Ce}^p, \end{aligned}$$

obtained a lot of interesting conclusions which include the ones of the previous similar literature. In fact, we can consider similar inequalities for dual means of the two means $\mathbf{M}_{\sin}$ and $\mathbf{M}_{\tanh}$, and we can replace $\mathbf{A}$ and $\mathbf{Ce}$ by other famous means. Therefore, the content of this research is very extensive.

Acknowledgements

The authors are grateful to editor and anonymous referees for their careful corrections to and valuable comments on the original version of this paper.

The first author was supported by the National Natural Science Foundation of China (no. 61772025). The second author was supported in part by the Serbian Ministry of Education, Science and Technological Development, under projects ON 174032 and III 44006.

Conflict of interest

The authors declare that they have no conflict of interest.

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