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Research article

Generalizations of strongly hollow ideals and a corresponding topology

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Abstract: In this paper, we introduce and study the notions of M-strongly hollow and M-PS-hollow ideals where M is a module over a commutative ring R. These notions are generalizations of strongly hollow ideals. We investigate some properties and characterizations of M-strongly hollow (M-PS-hollow) ideals. Then we define and study a topology on the set of all M-PS-hollow ideals of a commutative ring R. We investigate when this topological space is irreducible, Noetherian, T_0, T_1 and spectral space.

Keywords: strongly hollow submodule; pseudo strongly hollow submodule; M-strongly hollow ideal; PSH-Zariski topology
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1. Introduction

Topologies induced by various types of algebraic structures have been attracted attention of many authors for a long time. For example, in [13, 23] the authors investigated some topologies related to monoid or induced by monoid actions. In [13]. Given a monoid S acting on a set X, all the subsets of X which are invariant with respect to the action constitute the family of the closed subsets of an Alexandroff topology on X. In [13], the authors proved that any Alexandroff topology may be obtained through a monoid action. Based on such a link between monoid actions and Alexandroff topologies, the authors established several topological properties for Alexandroff spaces in [13]. In [23], the authors studied the notion of weak ideal topology j^I on the topos $Act - S$ of all (right) representations of S, where S is a monoid and I is a left ideal of S. Also, some topologies related to groups or induced by groups were investigated by some authors. For example, in [15], the authors introduced and studied some canonical topologies induced by actions of topological groups on groups and rings. In [28], the author presented the relationship between ultrafilters and topologies on groups. He showed how ultrafilters are used in constructing topologies on groups with extremal properties and how topologies on groups serve in deriving algebraic results about ultrafilters (see [28]). Topologies related to rings and modules also have been attracted attention of many authors. These topologies have an important role in characterizing algebraic structures. For example, the Zariski topology on the spectrum of prime ideals of a ring is one of the main tools in commutative algebra and algebraic geometry (see [9]). Many of the topologies related to rings and modules were constructed by using a special class of ideals or submodules (see [1, 3, 4, 6, 11, 21]). In [1, 3, 6, 11] some topologies were defined and studied by using strongly irreducible submodules and ideals. In [2, 6], some topologies were constructed by using strongly hollow submodules from a lattice theoretical point of view.

Let R be a commutative ring with identity and M be an R-module. In this paper, we introduce and investigate M-strongly hollow ideals and M-PS-hollow ideals which are generalizations of strongly hollow ideals. We establish some properties and characterizations of these ideals and elements. We generalize most of the results in [24]. We also define and study a topology on the set of all M-PS-hollow ideals of R. We investigate when this topological space is irreducible, Noetherian, T_0, T_1 and spectral space.

After this introductory section, this paper is divided into three parts. In the second section, we recall some basic concepts which will be used in the sequel. In the third section, first we give the definitions of M-strongly hollow ideals, M-strongly hollow elements and M-PS-hollow ideals where M is an R-module. Then we give the relationships between strongly hollow ideals and M-strongly hollow ideals where M is a multiplication module (see Proposition 3.3). In propositions 3.8, 3.9 and 3.10 we obtain some results concerning maximal submodules of an module M under some conditions by using M-strongly hollow ideals and elements. Let $(R, \mathfrak{m})$ be a local ring and M be a non-zero multiplication R-module such that $\mathfrak{m}M$ is finitely generated. In Proposition 3.11, we give a necessary and sufficient condition for $\mathfrak{m}$ to be an M-strongly hollow ideal. Let R be a local ring, M be a multiplication R-module and a be an element of R such that $aM \neq (0)$. In Theorem 3.14, we give a necessary and sufficient condition for a to be an M-strongly hollow element of R. Let $R = R_1 \times ... \times R_n$, $M = M_1 \times ... \times M_n$ and I be an ideal of R where R_i is a ring and M_i is an R_i-module. In Proposition 3.15, we give a necessary and sufficient condition for I to be an M-strongly hollow ideal of R. Let I be an ideal of R and M be an R-module. We define the set T_I^M as $T_I^M := \{K : K \text{ is an ideal of } R \text{ and } IM \nsubseteq KM\}$ and we define the ideal Γ_I^M as $\Gamma_I^M := \sum_{K \in T_I^M} K$. For an element a of R, we write Γ_a^M instead of Γ_{Ra}^M. In Theorem 3.17, Propositions 3.18 and 3.20, we give some characterizations of M-strongly hollow ideals and elements by using Γ_I^M and Γ_a^M. Let M be an R-module and a be an element of R. We denote the ideal $(\Gamma_a^M M : aM)$ by L_a^M, i.e., $L_a^M := (\Gamma_a^M M : aM) = \{r \in R : raM \subseteq \Gamma_a^M M\}$. In Proposition 3.22, we prove that L_a^M is a maximal ideal of R and in Proposition 3.23, we show that the ring $R/ann_R(aM)$ is a local ring with unique maximal ideal $L_a^M/ann_R(aM)$. Let M be a finitely generated multiplication R-module such that $ann_R(M) = Re$ for some idempotent element e of R and let a be an M-strongly hollow element of R. In Theorem 3.24, we give some equivalent conditions for $(\Gamma_a^M M : aM)$ to be a prime ideal of R. Let M be an R-module. We will denote the set of all M-PS-hollow ideals of R by $PSH^M(R)$. In the fourth section we construct a topology on $PSH^M(R)$ which we call PSH-Zariski topology. Let Y be a subset of $PSH^M(R)$. In Theorem 4.5, we give a necessary and sufficient condition for Y to be an irreducible subset of $PSH^M(R)$. In Theorem 4.6, we determine irreducible closed subsets of $PSH^M(R)$ and we give a bijection from the set of irreducible components of $PSH^M(R)$ onto the set of maximal elements of $PSH^M(R)$ when $PSH^M(R)$ is a T_0-space. In Theorem 4.7, we determine some cases in which $PSH^M(R)$ is a Noetherian space. We also investigate $PSH^M(R)$ from the point of view of spectral spaces (see Corollaries 4.8 and 4.10). Finally, we examine $PSH^M(R)$ in terms of seperation axioms (see Propositions 4.9 and 4.11).

2. Recalls and basic notions

Throughout this paper all rings will be commutative with non-zero identity and all modules will be unital left modules. Unless otherwise stated R will denote a ring. For a submodule N of an R-module M, $(N :_R M)$ will denote the ideal $\{r \in R : rM \subseteq N\}$. If there is no ambiguity for the ring we will write $(N : M)$. The annihilator of M which is denoted by $ann_R(M)$ is $(0 :_R M)$. Also, $Max(M)$ and $J(M)$ will denote the set of all maximal submodules of M and the Jacobson radical of M, i.e., the intersection of all maximal submodules of M, respectively.

A proper submodule N of an R-module M is called irreducible if for any submodules L and K of M, $N = L \cap K$ implies either $N = L$ or $N = K$. An ideal I of a ring R is said to be irreducible if it is irreducible as a submodule of the R-module R. Strongly irreducible ideals and submodules are subclasses of these concepts and they were extensively studied in [2, 8, 11, 17, 19, 22]. A proper submodule N of an R-module M is called strongly irreducible if for any submodules L and K of M, $L \cap K \subseteq N$ implies either $L \subseteq N$ or $K \subseteq N$. An ideal I of a ring R is said to be strongly irreducible if it is strongly irreducible as a submodule of the R-module R.

The dual notion of irreducible submodules (ideals) are known as hollow submodules (ideals). Recall that a non-zero submodule N of an R-module M is called hollow in M if for any submodules K, L of M, $N = K + L$ implies either $N = K$ or $N = L$. The dual notion of strongly irreducible submodule was named as strongly hollow in [4]. They were extensively studied in [2, 4, 6, 24]. Following [4], a non-zero submodule N of an R-module M is called strongly hollow in M if for any submodules K, L of M, $N \subseteq K + L$ implies either $N \subseteq K$ or $N \subseteq L$. An ideal I of a ring R is said to be strongly hollow if it is strongly hollow as a submodule of the R-module R. A non-zero element a of R is called a strongly hollow element in R if the principal ideal (a) is a strongly hollow ideal of R [24].

In [5], Abuhlail and Hroub generalized the concept of strongly hollow submodule as follows. A submodule N of an R-module M is called pseudo strongly hollow (or PS-hollow for short) if, for any ideal I of R and any submodule L of M, $N \subseteq IM + L$ implies either $N \subseteq IM$ or $N \subseteq L$. The module M is called pseudo strongly hollow module (or PS-hollow module for short) if M is a PS-hollow submodule of itself [5].

3. M-strongly hollow and M-PS-hollow ideals

In this section, we introduce and study M-strongly hollow ideals, M-strongly hollow elements and M-PS-hollow ideals.

Definition 3.1. *Let M be an R-module and I be an ideal of R. We say that I is an M-strongly hollow (respectively, M-PS-hollow) ideal of R if IM is a strongly hollow (respectively PS-hollow) submodule of M. An element a of R is called an M-strongly hollow (respectively, M-PS-hollow) element if aM is a strongly hollow (respectively, M-PS-hollow) submodule of M.*

Recall that an R-module M is said to be multiplication if every submodule N of M is of the form $N = IM$ for some ideal I of R. It is well-known that M is a multiplication module if and only if every submodule N of M is of the form $N = (N : M)M$ (see [12]).

Note that if M is a multiplication R-module, then the concept of M-strongly hollow ideal coincides with the concept of M-PS-hollow ideal.

Notice that if we take $M = R$, then an ideal I of R is strongly hollow (respectively PS-hollow) if and only if I is R-strongly hollow (respectively R-PS-hollow).

Clearly, every M-strongly hollow ideal of a ring R is M-PS-hollow for any R-module M. But the converse is not true in general as the following example shows.

Example 3.2. *Consider the ring $R := \mathbb{Z}_{pq}$ where p and q are distinct prime numbers and the R-module $M := R[X]$. In [5, Example 2.12] it was shown that $(\overline{p})M$ and $(\overline{q})M$ are PS-hollow submodules of M while they are not strongly hollow. So $(\overline{p})$ and $(\overline{q})$ are M-PS-hollow ideals of R but they are not M-strongly hollow ideals of R.*

In the following proposition we give the relationships between strongly hollow ideals and M-strongly hollow ideals where M is a multiplication module.

Proposition 3.3. *Let M be a multiplication R-module and I be an ideal of R. Then the following hold.*

(1) *Suppose that M is finitely generated and faithful. Then I is a strongly hollow ideal of R if and only if I is an M-strongly hollow ideal of R.*

(2) *If M is finitely generated and I is a strongly hollow ideal of R, then I is an M-strongly hollow ideal of R.*

Proof. (1) This result can be easily proved by using [16, Theorem 3.1].

(2) Let $IM \subseteq JM + KM$ for some ideals J, K of M. Then $I \subseteq J + K + ann_R(M)$ by [25, Corollary of Theorem 9]. Since I is a strongly hollow ideal, we have either $I \subseteq J$ or $I \subseteq K + ann_R(M)$. This implies that $IM \subseteq JM$ or $IM \subseteq KM$. Thus I is an M-strongly hollow ideal of R. □

In the following example we show that the converse of Proposition 3.3-(2) is not true in general.

Example 3.4. *Consider the $\mathbb{Z}$-module $M := \mathbb{Z}_{30}$. Clearly, M is a finitely generated multiplication $\mathbb{Z}$-module. Consider the submodule $N := (\overline{6}) = (6\mathbb{Z})M$. In [5, Example 2.33] it was shown that N is a strongly hollow (PS-hollow) submodule of M. So $6\mathbb{Z}$ is an M-strongly hollow ideal of $\mathbb{Z}$. But $6\mathbb{Z}$ is not a strongly hollow ideal of $\mathbb{Z}$. Because $6\mathbb{Z} \subseteq 4\mathbb{Z} + 18\mathbb{Z} = 2\mathbb{Z}$ while $6\mathbb{Z} \not\subseteq 4\mathbb{Z}$ and $6\mathbb{Z} \not\subseteq 18\mathbb{Z}$.*

In the following proposition we give some basic properties of M-strongly hollow and M-PS-hollow ideals.

Proposition 3.5. *Let M be an R-module. Then the following hold.*

(1) *If I is an M-strongly hollow (respectively M-PS-hollow) ideal of R, then $(IM :_R M)$ is an M-strongly hollow (respectively, M-PS-hollow) ideal of R*

(2) *If $\{I_\lambda\}_{\lambda \in \Lambda}$ is a family of M-strongly hollow (respectively, M-PS-hollow) ideals of R with $I_\lambda M = N$ for each $\lambda \in \Lambda$, then $\sum_{\lambda \in \Lambda} I_\lambda$ is an M-strongly hollow (respectively, M-PS-hollow) ideal of R.*

(3) *If M is a uniserial R-module, then every ideal I of R with $IM \neq (0)$ is an M-strongly hollow (M-PS-hollow) ideal of R.*

(4) *If R is a uniserial ring and M is a multiplication R-module, then every ideal I of R with $IM \neq (0)$ is an M-strongly hollow ideal of R.*

(5) *If I is a finitely generated M-strongly hollow (respectively M-PS-hollow) ideal of R, then there exists an element $x \in I$ such that x is an M-strongly hollow (respectively M-PS-hollow) element of R.*

(6) *Let* $a_1,...,a_n$ *be* M*-strongly hollow (*M*-PS-hollow) elements of* R *such that* $a_iM \nsubseteq \sum_{i\neq j} a_jM$ *for each* $i \in \{1, ..., n\}$*. Then* $\sum_{j=1}^{n} a_jM = (a_1 + ...a_n)M$*.*

Proof. (1) This follows from the equality $(IM : M)M = IM$.

(2) Since $(\sum_{\lambda\in\Lambda} I_\lambda)M = \sum_{\lambda\in\Lambda}(I_\lambda M) = N$, we deduce that $\sum_{\lambda\in\Lambda} I_\lambda$ is an M-strongly hollow (respectively, M-PS-hollow) ideal of R.

(3) This follows from the fact that every non-zero submodule of a uniserial module is an M-strongly hollow (M-PS-hollow) submodule of M.

(4) Every multiplication module over a uniserial ring is a uniserial module. So the result follows from part (3).

(5) Let $I = \sum_{i=1}^{n} Rx_i$ where $x_i \in I$ and $n \in \mathbb{Z}^+$. Then we have $IM = (\sum_{i=1}^{n} Rx_i)M = \sum_{i=1}^{n}(Rx_i)M$. By assumption, there exists $j \in \{1, ..., n\}$ such that $IM \subseteq (Rx_j)M \subseteq \left(\sum_{i=1}^{n} Rx_i\right) M = IM$ and so $IM = (Rx_j)M$. Therefore, Rx_j is an M-strongly hollow (respectively M-PS-hollow) ideal of R.

(6) We use induction on n. If $n = 1$, then the result is evident. Suppose that $n \geq 2$ and the result is true for $n - 1$. Since $a_nM \nsubseteq \sum_{j=1}^{n-1} a_jM$ and $a_nM \subseteq (a_1 + ... + a_n)M + \sum_{j=1}^{n-1} a_jM$, we have $a_nM \subseteq (a_1 + ... + a_n)M$. Thus $(a_1 + ... + a_n)M = a_nM + (a_1 + ... + a_{n-1})M$. By the inductive hypothesis, $\sum_{j=1}^{n-1} a_jM = (a_1 + ... + a_{n-1})M$. Therefore, $\sum_{j=1}^{n} a_jM = (a_1 + ... + a_n)M$. □

The following proposition gives some further properties of M-strongly hollow (M-PS-hollow) ideals and it will be used in the sequel.

Proposition 3.6. *Let* M *be an* R*-module. Then the following hold.*

(1) *If* IM *is finitely generated and* I *is an* M*-strongly hollow (*M*-PS-hollow) ideal of* R*, then the set* $\Psi = \{JM : J \text{ is an ideal of } R \text{ such that } JM \subsetneq IM\}$ *has exactly one maximal element with respect to inclusion.*

(2) *Let* I *and* J *be two* M*-strongly hollow (respectively,* M*-PS-hollow) ideals of* R*. Then* $I + J$ *is an* M*-strongly hollow (respectively,* M*-PS-hollow) ideal of* R *if and only if either* $IM \subseteq JM$ *or* $JM \subseteq IM$*.*

Proof. (1) Clearly, $(0)M = (0) \in \Psi$ whence $\Psi \neq \emptyset$. Let $\{J_\alpha M\}_{\alpha\in A}$ be a chain in Ψ. Then $\cup_{\alpha\in A}(J_\alpha M) = \sum_{\alpha\in A}(J_\alpha M) = (\sum_{\alpha\in A} J_\alpha)M$ and $\cup_{\alpha\in A}(J_\alpha M) \subsetneq IM$ as IM is finitely generated. Therefore $\cup_{\alpha\in A}(J_\alpha M) \in \Psi$. By Zorn's Lemma, Ψ has at least one maximal element. Suppose that J_1M and J_2M be two maximal elements of Ψ where J_1, J_2 are ideals of R. Then $IM = J_1M + J_2M$ by the maximalities of J_1M and J_2M. Since IM is an M-strongly hollow (M-PS-hollow) submodule, we have either $IM \subseteq J_1M$ or $IM \subseteq J_2M$, a contradiction. Thus Ψ has exactly one maximal element with respect to inclusion.

(2) This proof is straightforward. □

Recall that a non-zero submodule N of an R-module M is called a second submodule if $IN = (0)$ or $IN = N$ for every ideal I of R (see [14], [27]). We use second submodules to give an example of M-strongly hollow ideal.

Example 3.7. *Let* M *be a multiplication* R*-module,* I *be an ideal of* R *and* IM *be a second submodule of* M *such that* $I^2M = IM$*. Then* I *is an* M*-strongly hollow ideal of* R*. To see this let* $IM \subseteq JM + KM$ *for some ideals* J*,* K *of* R*. Suppose that* $IM \nsubseteq JM$ *and* $IM \nsubseteq KM$*. Then* $IJM = IKM = (0)$*. Hence* $I^2M = IM \subseteq IJM + IKM = (0)$ *whence* $IM = (0)$*, a contradiction.*

In the following proposition we obtain a result concerning maximal submodules of an R-module M by using M-strongly hollow ideals.

Proposition 3.8. *Let M be an R-module, I be an ideal of R. If I is an M-strongly hollow ideal of R, then either $IM \subseteq J(M)$ or there exists exactly one maximal submodule of M not containing IM.*

Proof. Suppose that $IM \nsubseteq J(M)$. Then there exists a maximal submodule P of M such that $IM \nsubseteq P$. Let Q be another maximal submodule of M. Then $P + Q = M$ and so $IM = IP + IQ$. Since IM is a strongly hollow submodule, we have $IM \subseteq Q$. □

Recall that a module which has only one maximal submodule is said to be a local module [26]. In the following proposition we obtain a result concerning the number of maximal submodules of a finitely generated multiplication R-module M by using M-strongly hollow ideals.

Proposition 3.9. *Let M be a finitely generated multiplication R-module such that $M = I_1M + ... + I_nM$ where I_i is an M-strongly hollow ideal of R for each $i \in \{1, ..., n\}$. Then M has only a finite number of maximal submodules.*

Proof. If $I_jM = M$ for some $j \in \{1, ..., n\}$, then M is a local module and we are done. So we may assume that $I_iM \neq M$ for all $i \in \{1, ..., n\}$. Note that each I_iM is contained in at least one maximal submodule and for each i, either $I_iM \subseteq J(M)$ or there exists exactly one maximal submodule of M not containing I_iM. Hence M has at most n maximal submodule. □

Let M be an R-module. Recall that an element a of R is said to be a zero-divisor of M if there exists a non-zero element $m \in M$ such that $am = 0$. In the following proposition we deal with M-strongly hollow elements which are not zero-divisors of M.

Proposition 3.10. *Let M be a non-zero multiplication R-module. If R has an M-strongly hollow element which is not a zero-divisor of M, then M is a local module.*

Proof. M has a maximal submodule by [16, Theorem 2.5]. Let R have an M-strongly hollow element a which is not a zero-divisor of M. Suppose that P and Q are distinct maximal submodules of M. Then $P = pM$ and $Q = qM$ for some maximal ideals p, q of R by [16, Theorem 2.5]. We have $pM + qM = (p + q)M = M$. Since $p + q = R$, there exit $x \in p$, $y \in q$ such that $x + y = 1$. It follows that $aM \subseteq xaM + yaM$. Since aM is a strongly hollow submodule, either $aM \subseteq xaM$ or $aM \subseteq yaM$. Assume that $aM \subseteq xaM$. Let $m \in M$. Then $am = xam'$ for some $m' \in M$. It follows that $a(m - xm') = 0$. Since a is not a zero-divisor of M, we have $m = xm' \in pM = P$. This yields the contradiction that $M = P$. Similarly, if $aM \subseteq yM$, then we get the contradiction that $M = Q$. Thus M has only one maximal submodule. □

In the following proposition we give a necessary and sufficient condition for the maximal ideal of a local ring to be an M-strongly hollow ideal where M is a multiplication module satisfying an additional condition.

Proposition 3.11. *Let $(R, \mathfrak{m})$ be a local ring and M be a non-zero multiplication R-module such that $\mathfrak{m}M$ is finitely generated. Then $\mathfrak{m}$ is an M-strongly hollow ideal of R if and only if $(0) \neq \mathfrak{m}M = xM$ for some $x \in \mathfrak{m}$.*

Proof. First note that $\mathfrak{m}M$ is the only maximal submodule of M by [16, Theorem 2.5]. Every multiplication module over a local ring is cyclic by [12, Theorem 1]. So M is cyclic and hence finitely generated. We may assume that M is not simple.

Suppose that $\mathfrak{m}$ is an M-strongly hollow ideal of R. We have $\mathfrak{m} = \sum_{x_i \in \mathfrak{m}} Rx_i$ whence $\mathfrak{m}M = \sum_{x_i \in \mathfrak{m}}(Rx_iM)$. Since $\mathfrak{m}M$ is finitely generated, there exist $x_{i_1}, ..., x_{i_n} \in \mathfrak{m}$ such that $\mathfrak{m}M = \sum_{j=1}^{n} x_{i_j}M$ for some positive integer n. By assumption, $\mathfrak{m}M = x_{i_k}M$ for some $k \in \{1, ..., n\}$.

Conversely, suppose that $\mathfrak{m}M = xM$ for some $x \in \mathfrak{m}$ and $\mathfrak{m}M \subseteq IM + JM$ for some ideals I, J of R. If either $IM = M$ or $JM = M$, then we are done. So we may assume that $IM \neq M$ and $JM \neq M$. Since I, J are proper ideals and R is local, $I + J \subseteq \mathfrak{m}$ and so $(I + J)M = IM + JM \subseteq \mathfrak{m}M$. Thus $\mathfrak{m}M = xM = (I+J)M$. By [25, Corollary of Theorem 9], we have $\mathfrak{m} = Rx+ann_R(M) = I+J+ann_R(M)$. So $x = a + b + c$ for some $a \in I$, $b \in J$ and $c \in ann_R(M)$. Since I, $J \subseteq \mathfrak{m}$, there exist r_1, $r_2 \in R$ and s_1, $s_2 \in ann_R(M)$ such that $a = r_1x + s_1$ and $b = r_2x + s_2$. Thus $x = r_1x + r_2x + s$ for some $s \in ann_R(M)$ and so $x(1-(r_1+r_2)) = s$. If r_1, $r_2 \in \mathfrak{m}$, then $1-(r_1+r_2)$ is a unit and so $xM = 0$, a contradiction as M is not simple. Thus either $r_1 \notin \mathfrak{m}$ or $r_2 \notin \mathfrak{m}$. Assume that $r_1 \notin \mathfrak{m}$, then r_1 is a unit and so $x \in Ra + ann_R(M)$. Thus $xM \subseteq RaM \subseteq IM$. Similarly, if $r_2 \in \mathfrak{m}$, we get that $xM \subseteq JM$. Therefore $\mathfrak{m}$ is an M-strongly hollow ideal of R. □

The following lemma which will be used in the sequel gives a characterization of an M-strongly hollow element where M is a finitely generated multiplication R-module.

Lemma 3.12. *Let M be a finitely generated multiplication R-module and a be an element of R such that $aM \neq (0)$. Then the following are equivalent.*

(1) *a is an M-strongly hollow element of R.*

(2) *For every $y \in R$, if $aM \nsubseteq yM$, then there exist an $x \in R$ and $z \in ann_R(M)$ such that $a(1 - x) = xy + z$.*

Proof. (1) $\Longrightarrow$ (2) Suppose that $aM \nsubseteq yM$ for $y \in R$. Since $aM \subseteq (a + y)M + (-y)M$, we have $aM \subseteq (a + y)M$. By [25, Corollary of Theorem 9], $Ra \subseteq R(a + y) + ann_R(M)$. So $a = x(a + y) + z$ for some $x \in R$, $z \in ann_R(M)$. It follows that $a(1 - x) = xy + z$.

(2) $\Longrightarrow$ (1) Suppose that $aM \subseteq JM + KM$ for some ideals J, K of R. By [25, Corollary of Theorem 9], $Ra \subseteq J + K + ann_R(M)$. So $a = r + s + t$ for some $r \in J$, $s \in K$, $t \in ann_R(M)$. If $aM \nsubseteq rM = (-r)M$, then there exist $x \in R$ and $z \in ann_R(M)$ such that $a(1 - x) = x(-r) + z$. Thus $a = xa - xr + z = x(a - r) + z = x(s + t) + z$. It follows that $aM \subseteq xsM \subseteq KM$. If $aM \subseteq rM$, then clearly, $aM \subseteq JM$. Thus a is an M-strongly hollow element of R. □

We obtain the following proposition as an application of Lemma 3.12.

Proposition 3.13. *Let M be a finitely generated multiplication R-module and a be an M-strongly hollow element of R. If b is an element of R such that $aM \cap bM = (0)$, then $ann_R(aM) + ann_R(bM) = R$.*

Proof. Let b be an element of R such that $aM \cap bM = (0)$. Hence $aM \nsubseteq bM$. By Lemma 3.12, there exist $x \in R$ and $z \in ann_R(M)$ such that $a(1 - x) = xb + z$. Thus $a(1 - x)M = xbM \subseteq aM \cap bM = (0)$. Hence $1 - x \in ann_R(aM)$, $x \in ann_R(bM)$ and so $ann_R(aM) + ann_R(bM) = R$. □

Recall that a submodule N of an R-module M is called a waist submodule if it is comparable with every submodule of M [2]. In the following theorem we give a relationship between M-strongly hollow elements and waist submodules where R is a local ring and M is a multiplication R-module.

Theorem 3.14. *Let R be a local ring, M be a multiplication R-module and a be an element of R such that $aM \neq (0)$. Then a is an M-strongly hollow element of R if and only if aM is a waist submodule of M.*

Proof. Suppose that a is an M-strongly hollow element of R and N is a submodule of M such that $aM \nsubseteq N$. There exists an ideal J of R such that $N = JM$. Also, since R is a local ring, M is a cyclic and hence a finitely generated R-module by [12, Theorem 1]. Let $y \in J$. Then, $aM \nsubseteq yM$. By Lemma 3.12, there exist $x \in R$, $z \in ann_R(M)$ such that $a(1 - x) = xy + z$. Since R is a local ring and $aM \nsubseteq yM$, x must be unit.Thus $y \in Ra + ann_R(M)$ and so $yM \subseteq aM$. This shows that $N = JM \subseteq aM$. Hence aM is a waist submodule of M.

Conversely, suppose that aM is a non-zero waist submodule of M such that $aM \subseteq BM + CM$ for some ideals B, C of R. If $aM \nsubseteq BM$, then $BM \subseteq aM$. By [25, Corollary of Theorem 9], $B \subseteq Ra + ann_R(M)$ and $Ra \subseteq B + C + ann_R(M)$. Thus $a = r + s + z$ for some $r \in B$, $s \in C$, $z \in ann_R(M)$ and $r = aa' + t$ for some $a' \in R$ and $t \in ann_R(M)$. It follows that $a = aa' + s + k$ for some $k \in ann_R(M)$ and we have $a(1 - a') = s + k$. Since $aM \nsubseteq BM$ and R is a local ring, $1 - a'$ must be unit. Thus $aM \subseteq sM \subseteq CM$ and so a is an M-strongly hollow element of R. □

Let $R = R_1 \times ... \times R_n$, $M = M_1 \times ... \times M_n$ and I be an ideal of R where R_i is a ring and M_i is an R_i-module. The following proposition gives a characterization of M-strongly hollow ideals and M-strongly hollow elements of R.

Proposition 3.15. *Let $R = R_1 \times ... \times R_n$, $M = M_1 \times ... \times M_n$ and I be an ideal of R where R_i is a ring and M_i is an R_i-module. Then I is an M-strongly hollow ideal of R if and only if there exists $i \in \{1, ..., n\}$ such that the submodule IM is of the form $IM = 0 \times ...0 \times I_iM_i \times 0 \times ... \times 0$ where I_i is an M_i-strongly hollow ideal of R_i. In particular, an element $a = (a_1, ..., a_n)$ of R is M-strongly hollow if and only if there exists $i \in \{1, ..., n\}$ such that the submodule aM is of the form $aM = 0 \times ... \times 0 \times a_iM_i \times 0 \times ... \times 0$ where a_i is an M_i-strongly hollow element of R_i.*

Proof. Let I be an M-strongly hollow ideal of R. It is well-known that $I = I_1 \times ... \times I_n$ where I_j is an ideal of R_j for each $j \in \{1, ..., n\}$. For each j, put $I'_j = 0 \times ... \times I_j \times 0 \times ... \times 0$. Thus $I = I'_1 \times ... \times I'_n$ whence $IM = I'_1M + ... + I'_nM$. Since IM is a strongly hollow submodule, there exists $i \in \{1, ..., n\}$ such that $IM \subseteq I'_iM$ and so $IM = I'_iM$. Hence, $IM = 0 \times ... \times I_iM_i \times 0 \times ... \times 0$. Now we show that I_i is an M_i-strongly hollow ideal of R_i. Let $I_iM_i \subseteq K_i + L_i$ where K_i and L_i are submodules of M_i. Then $IM \subseteq (0 \times ... \times K_i \times 0 \times ... \times 0) + (0 \times ... \times L_i \times 0 \times ... \times 0)$. Thus, either $IM \subseteq 0 \times ... \times K_i \times 0 \times ... \times 0$ or $IM \subseteq 0 \times ... \times L_i \times 0 \times ... \times 0$. Hence, either $I_iM_i \subseteq K_i$ or $I_iM_i \subseteq L_i$. So I_i is an M_i-strongly hollow ideal of R_i.

Conversely, suppose that the submodule IM is of the form $IM = 0 \times ...0 \times I_iM_i \times 0 \times ... \times 0$ where I_i is an M_i-strongly hollow ideal of R_i for some $i \in \{1, ..., n\}$. Let $N = N_1 \times ... \times N_n$ and $L = L_1 \times ... \times L_n$ be two submodules of M such that $IM \subseteq N + L$. Then we have $I_iM_i \subseteq N_i + L_i$. Hence, either $I_iM_i \subseteq N_i$ or $I_iM_i \subseteq L_i$. So either $IM \subseteq N$ or $IM \subseteq L$. Thus I is an M-strongly hollow ideal of R. □

We obtain the following corollary by combining Proposition 3.15 and Theorem 3.14.

Corollary 3.16. *Let $R = R_1 \times ... \times R_n$, $M = M_1 \times ... \times M_n$ and $a = (a_1, ..., a_n)$ be an element of R where R_i is a local ring and M_i is a multiplication R_i-module for each i. Then a is an M-strongly hollow element of R if and only if there exists $i \in \{1, ..., n\}$ such that the submodule aM is of the form $aM = 0 \times ... \times 0 \times a_iM_i \times 0 \times ... \times 0$ where a_iM_i is a non-zero waist submodule of M_i.*

Let I be an ideal of R and M be an R-module. We define the set T_I^M as $T_I^M := \{K : K$ is an ideal of R and $IM \nsubseteq KM\}$ and we define the ideal Γ_I^M as $\Gamma_I^M := \sum_{K \in T_I^M} K$. For an element a of R, we write Γ_a^M

instead of Γ^M_{Ra}. We adopt these notations in the rest of the paper. Clearly if IM is a non-zero submodule, then $T^M_I \neq \emptyset$ as $(0) \in T^M_I$.

In the following theorem we give a characterization of an M-strongly hollow ideal I by using T^M_I where M is a multiplication R-module such that IM is a non-zero finitely generated submodule of M.

Theorem 3.17. *Let M be a multiplication R-module, I be an ideal of R such that IM is a non-zero finitely generated submodule of M. Then I is an M-strongly hollow ideal of R if and only if there exists an ideal J of R which is the greatest element of the set T^M_I, namely $J = \Gamma^M_I$.*

Proof. Let I be an M-strongly hollow ideal of R. Suppose that $IM \subseteq \Gamma^M_I M$. Then $IM \subseteq \sum_{K \in T^M_I} KM$. Since IM is finitely generated and strongly hollow, $IM \subseteq KM$ for some $K \in T^M_I$ which contradicts the definition of T^M_I. So $IM \nsubseteq \Gamma^M_I M$. By definition of Γ^M_I, Γ^M_I is the greatest element of T^M_I.

Now suppose that T^M_I has the greatest element, say J. Let $IM \subseteq I_1M + I_2M$ for some ideals I_1, I_2 of R. Suppose that $IM \nsubseteq I_iM$ for each $i \in \{1, 2\}$. Fix $i \in \{1, 2\}$. Set $T^M_i := \{K : K$ is an ideal of R such that $I_iM \subseteq KM$ and $IM \nsubseteq KM\}$. Then $T^M_i \neq \emptyset$ as $I_i \in T^M_i$. Since IM is finitely generated, T^M_i has a maximal element by Zorn's Lemma. Let A_i be a maximal element of T_i. Then $A_i \subseteq J$ whence $I_iM \subseteq A_iM \subseteq JM$. Thus $J \in T^M_i$. By the maximality of A_i, we have $A_i = J$. Thus $I_1M \subseteq JM$ and $I_2M \subseteq JM$ whence $IM \subseteq JM$, a contradiction. Therefore I is an M-strongly hollow ideal of R. □

In the following proposition we give a characterization of an M-strongly hollow element a by using Γ^M_a where M is a multiplication R-module such that aM is a non-zero finitely generated submodule of M.

Proposition 3.18. *Let M be a multiplication R-module, a be an element of R such that aM is a non-zero finitely generated submodule of M. Then a is an M-strongly hollow element of R if and only if $aM \nsubseteq \Gamma^M_a M$. In this case $\Gamma^M_a = \{r \in R : aM \nsubseteq rM\}$.*

Proof. Suppose that a is an M-strongly hollow element of R. Then $aM \nsubseteq \Gamma^M_a M$ by Theorem 3.17.

Suppose that $aM \nsubseteq \Gamma^M_a M$ and $aM \subseteq I_1M + I_2M$ for some ideals I_1, I_2 of R. If $aM \nsubseteq I_1M$ and $aM \nsubseteq I_2M$, then $I_1 \subseteq \Gamma^M_a$ and $I_2 \subseteq \Gamma^M_a$. Hence $aM \subseteq I_1M + I_2M \subseteq \Gamma^M_a M$ which is a contradiction. Thus a is an M-strongly hollow element of R.

Now, let $r \in \Gamma^M_a$. Then $r = x_1 + ... + x_n$ for some positive integer n and $x_i \in K_i$ $(1 \leq i \leq n)$ where $K_i \in T^M_{Ra}$, i.e., $aM \nsubseteq K_iM$. We have $rM \subseteq x_1M + ... + x_nM$. If $aM \subseteq rM$, then $aM \subseteq x_jM \subseteq K_jM$ for some $j \in \{1, ..., n\}$ as aM is a strongly hollow submodule. But this is a contradiction. So $aM \nsubseteq rM$ whence $\Gamma^M_a \subseteq \{r \in R : aM \nsubseteq rM\}$. The reverse inclusion is clear by definition. Thus $\Gamma^M_a = \{r \in R : aM \nsubseteq rM\}$. □

Corollary 3.19. *Let M be a multiplication R-module, a and b be two M-strongly hollow elements of R such that aM and bM are finitely generated submodules of M. Then $aM \subseteq bM$ if and only if $\Gamma^M_a \subseteq \Gamma^M_b$.*

Proof. Suppose that $aM \subseteq bM$ and $r \in \Gamma^M_a$. Then by Proposition 3.18, $aM \nsubseteq rM$ and this implies that $bM \nsubseteq rM$. So $b \in \Gamma^M_a$ by Proposition 3.18. Thus $\Gamma^M_a \subseteq \Gamma^M_b$.

Suppose that $\Gamma^M_a \subseteq \Gamma^M_b$. If $aM \nsubseteq bM$, then $b \in \Gamma^M_a \subseteq \Gamma^M_b$ by Proposition 3.18. But this contradicts with the fact that $b \notin \Gamma^M_b$. Thus $aM \subseteq bM$. □

In the following proposition we investigate when $Ra + Rb$ is an M-strongly hollow ideal where M is a multiplication R-module, a and b are M-strongly hollow elements.

Proposition 3.20. *Let M be a multiplication R-module, a and b be two M-strongly hollow elements of R such that aM and bM are finitely generated submodules of M. Then the following are equivalent.*

(1) *$Ra + Rb$ is an M-strongly hollow ideal of R.*

(2) *Either $\Gamma_a^M \subseteq \Gamma_b^M$ or $\Gamma_b^M \subseteq \Gamma_a^M$.*

(3) *Either $aM \subseteq bM$ or $bM \subseteq aM$.*

Proof. (1) $\Longrightarrow$ (2) Let $Ra + Rb$ be an M-strongly hollow ideal of R. Suppose on the contrary that $\Gamma_a^M \nsubseteq \Gamma_b^M$ or $\Gamma_b^M \nsubseteq \Gamma_a^M$. Thus there exist $x \in \Gamma_a^M \backslash \Gamma_b^M$ and $y \in \Gamma_b^M \backslash \Gamma_a^M$. By Proposition 3.18, we have $aM + bM = (Ra + Rb)M \subseteq xM + yM$. But $(Ra + Rb)M \nsubseteq xM$ and $(Ra + Rb)M \nsubseteq yM$ which is a contradiction.

(2) $\Longrightarrow$ (3) By Corollary 3.19.

(3) $\Longrightarrow$ (1) Clear. □

Let M be an R-module and a be an element of R. We denote the ideal $(\Gamma_a^M M : aM)$ by L_a^M, i.e., $L_a^M := (\Gamma_a^M M : aM) = \{r \in R : raM \subseteq \Gamma_a^M M\}$.

In the following proposition we investigate M-strongly hollow elements of quotient rings.

Proposition 3.21. *Let M be a multiplication R-module, a be an M-strongly hollow element of R such that aM is a finitely generated submodule of M and I be an ideal of R such that $aM \nsubseteq IM$. Then $a + I$ is an $\overline{M}$-strongly hollow element of R/I where $\overline{M} = M/IM$. In this case, $\Gamma_{a+I}^{\overline{M}} = \Gamma_a^M/I$ and $(L_a^M + I)/I = L_{a+I}^{\overline{M}}$.*

Proof. Since $aM \nsubseteq IM$, $I \subseteq \Gamma_a^M$ and $(a + I)(M/IM) \neq (\bar{0})$. Note that M/IM is a multiplication (R/I)-module. Let $(a + I)(M/IM) = (aM + IM)/IM \subseteq (J/I)(M/IM) + (K/I)(M/IM)$ for some ideals J, K of R containing I. Then $aM \subseteq JM + KM + IM$. Thus we have either $aM \subseteq JM$ or $aM \subseteq KM + IM$ and so either $(aM + IM)/IM = (a + I)(M/IM) \subseteq (JM + IM)/IM = (J/I)(M/IM)$ or $(a + I)(M/IM) = (aM + IM)/IM \subseteq (KM + IM)/IM = (K/I)(M/IM)$. Therefore, $a + I$ is an $\overline{M}$-strongly hollow element of R/I.

Now let $r + I \in \Gamma_{a+I}^{\overline{M}}$. Assume that $r \notin \Gamma_a^M$. Then $aM \subseteq rM$ by Proposition 3.18. Thus $(aM + IM)/IM = (a+I)(M/IM) \subseteq (rM+IM)/IM = (r+I)(M/IM)$ which contradicts $r+I \in \Gamma_{a+I}^{\overline{M}}$. Thus $r \in \Gamma_a^M$ and so $\Gamma_{a+I}^{\overline{M}} \subseteq \Gamma_a^M/I$. Since $aM \nsubseteq \Gamma_a^M M$, we have $(a + I)(M/IM) = (aM + IM)/IM \nsubseteq (\Gamma_a^M/I)(M/IM)$. So $\Gamma_a^M/I \subseteq \Gamma_{a+I}^{\overline{M}}$. Therefore, $\Gamma_a^M/I = \Gamma_{a+I}^{\overline{M}}$.

Now we show that $(L_a^M + I)/I = L_{a+I}^{\overline{M}}$. Let $r + I \in (L_a^M + I)/I$ where $r \in L_a^M$. Then $raM \subseteq \Gamma_a^M M$. Note that $L_{a+I}^{\overline{M}} = (\Gamma_{a+I}^{\overline{M}} \overline{M} : (a + I)\overline{M}) = \{r + I : (r + I)(a + I)\overline{M} \subseteq \Gamma_{a+I}^{\overline{M}} \overline{M}\} = \{r + I : (raM + IM)/IM \subseteq \left(\Gamma_a^M M\right)/IM\}$. Thus $raM \subseteq \Gamma_a^M M$ implies that $(raM + IM)/IM \subseteq \left(\Gamma_a^M M\right)/IM$ whence $r + I \in L_{a+I}^{\overline{M}}$. Therefore, $(L_a^M + I)/I \subseteq L_{a+I}^{\overline{M}}$. To show the reverse inclusion, take an element $r + I \in L_{a+I}^{\overline{M}}$. Then we have $(raM + IM)/IM \subseteq \left(\Gamma_a^M M\right)/IM$ whence $raM \subseteq \Gamma_a^M M$. This shows that $r \in (\Gamma_a^M M : aM) = L_a^M$. Thus $r + I \in (L_a^M + I)/I$ whence $L_{a+I}^{\overline{M}} \subseteq (L_a^M + I)/I$. Hence $L_{a+I}^{\overline{M}} = (L_a^M + I)/I$. □

By using Proposition 3.21, we show that L_a^M is a maximal ideal of R for an M-strongly hollow element a of R under some conditions.

Proposition 3.22. *Let M be a multiplication R-module and a be an M-strongly hollow element of R such that aM is finitely generated. Then L_a^M is a maximal ideal of R.*

Proof. Let a be an M-strongly hollow element of R. Then $aM \nsubseteq \Gamma_a^M M$ by Proposition 3.18. Thus $a + \Gamma_a^M$ is an $(M/\Gamma_a^M M)$-strongly hollow element of R/Γ_a^M by Proposition 3.21. Now we show that $(a + \Gamma_a^M)\overline{M}$ is a simple submodule of the (R/Γ_a^M)-module $\overline{M}$ where $\overline{M} = M/\Gamma_a^M M$. Let $x + \Gamma_a^M M$ be an element of $(a+\Gamma_a^M)\overline{M}$ such that $< x+\Gamma_a^M M >= (Rx+\Gamma_a^M M)/\Gamma_a^M M \subsetneq (a+\Gamma_a^M)\overline{M} = (aM+\Gamma_a^M M)/\Gamma_a^M M$. Then $aM \nsubseteq Rx$. Since M is multiplication, there exists an ideal I of R such that $Rx = IM$. So $aM \nsubseteq IM$ whence $I \subseteq \Gamma_a^M$. Thus $IM = Rx \subseteq \Gamma_a^M M$ whence $x \in \Gamma_a^M M$. Therefore, $< x + \Gamma_a^M M >= (\overline{0})$. This shows that $(a + \Gamma_a^M)\overline{M}$ is a simple submodule of $\overline{M}$. Hence $ann_{R/\Gamma_a^M}((a + \Gamma_a^M)\overline{M})$ is a maximal ideal of R/Γ_a^M. It is easy to see that $ann_{R/\Gamma_a^M}((a + \Gamma_a^M)\overline{M}) = (\Gamma_a^M M :_R aM)/\Gamma_a^M = L_a^M/\Gamma_a^M$. Thus L_a^M/Γ_a^M is a maximal ideal of R/Γ_a^M and so L_a^M is a maximal ideal of R. □

In the following proposition we show that the ring $R/ann_R(aM)$ is a local ring for an M-strongly hollow element a of R under some conditions.

Proposition 3.23. *Let M be a multiplication R-module and a be an M-strongly hollow element of R such that aM is finitely generated. Then the ring $R/ann_R(aM)$ is a local ring with unique maximal ideal $L_a^M/ann_R(aM)$.*

Proof. Since aM is a finitely generated strongly hollow submodule of M, it is a hollow module with $Max(aM) \neq \emptyset$. Hence $R/ann_R(aM)$ is a local ring by [10, Theorem 2.2]. Also, by Proposition 3.22, L_a^M is a maximal ideal of R. Since $ann_R(aM) \subseteq L_a^M$, we conclude that $R/ann_R(aM)$ is a local ring with the unique maximal ideal $L_a^M/ann_R(aM)$. □

Let M be a finitely generated multiplication R-module such that $ann_R(M) = Re$ for some idempotent element e of R and let a be an M-strongly hollow element of R. In the following theorem we give some equivalent conditions for $(\Gamma_a^M M : aM)$ to be a prime ideal of R.

Theorem 3.24. *Let M be a finitely generated multiplication R-module such that $ann_R(M) = Re$ for some idempotent element e of R and let a be an M-strongly hollow element of R. Then the following are equivalent.*

(1) $(\Gamma_a^M M :_R M)$ *is a prime ideal of* R.

(2) $a^2M \nsubseteq \Gamma_a^M M$.

(3) $aM = a^2M$.

(4) $aM \nsubseteq J(M)$.

(5) $(\Gamma_a^M M :_R M)$ *is a maximal ideal of* R.

(6) $(\Gamma_a^M M :_R M) = L_a^M$.

Proof. (1) $\Longrightarrow$ (2) By Proposition 3.18, we have $a \notin (\Gamma_a^M M :_R M)$. Thus $a^2 \notin (\Gamma_a^M M :_R M)$ whence $a^2M \nsubseteq \Gamma_a^M M$.

(2) $\Longrightarrow$ (3) Since $a^2M \nsubseteq \Gamma_a^M M$, we have $a^2 \notin \Gamma_a^M$. By Proposition 3.18, $aM \subseteq a^2M$ whence $aM = a^2M$.

(3) $\Longrightarrow$ (4) By [25, Theorem 11], M is a projective R-module. So $J(M) = J(R)M$ (see [26, page 113]). Since M is multiplication, $J(M) = (J(M) : M)M = J(R)M$. [25, Corollary of Theorem 9] implies that $(J(M) : M) = J(R) + ann_R(M)$. Suppose on the contrary that $aM \subseteq J(M)$. Then $a \in J(R) + ann_R(M)$ and so there exist $b \in J(R)$ and $c \in ann_R(M)$ such that $a = b + c$. Since $(b + c)M = bM$, we have $b^2M = bM$. By Nakayama's Lemma, $bM = (0)$ whence $b \in ann_R(M)$. This implies that $aM = (0)$, a contradiction. Thus $aM \nsubseteq J(M)$.

(4) $\Longrightarrow$ (5) Suppose $aM \nsubseteq J(M)$. By Proposition 3.8, there exists a unique maximal submodule Q of M not containing aM. Since M is multiplication, $Q = qM$ for some maximal ideal q of R by [16, Theorem 2.5]. Since $aM \nsubseteq qM$, we have $q \subseteq \Gamma_a^M$. Hence $\Gamma_a^M = q$ is a maximal ideal of R. Also, $\Gamma_a^M \subseteq (\Gamma_a^M M : M) \neq R$ implies that $(\Gamma_a^M M : M) = \Gamma_a^M$ is a maximal ideal of R.

(5) $\Longrightarrow$ (1) Clear.

(5) $\Longrightarrow$ (6) We have $(\Gamma_a^M M : M) \subseteq (\Gamma_a^M M : aM) = L_a^M \neq R$. Thus $(\Gamma_a^M M :_R M) = L_a^M$.

(6) $\Longrightarrow$ (1) This follows from Proposition 3.22. □

In the following proposition we investigate the behaviour of M-strongly hollow elements under localization where M is a finitely generated multiplication R-module.

Proposition 3.25. *Let M be a finitely generated multiplication R-module and a be an M-strongly hollow element of R. Then $\frac{a}{1}$ is an $(S^{-1}M)$-strongly hollow element of $S^{-1}R$ where $S = R \backslash L_a^M$. In this case, $\Gamma_{\frac{a}{1}}^{S^{-1}M} = S^{-1}\Gamma_a^M$ and $L_{\frac{a}{1}}^M = S^{-1}L_a^M$.*

Proof. If $(\frac{a}{1})(S^{-1}M) = S^{-1}(aM) = (0)$, then there exists $s \in R \backslash L_a^M$ such that $s(aM) = (0)$ as aM is finitely generated. This yields the contradiction that $s \in L_a^M$. Hence $(\frac{a}{1})(S^{-1}M) \neq (0)$. If $(\frac{a}{1})(S^{-1}M) = S^{-1}(aM) \subseteq (S^{-1}\Gamma_a^M)(S^{-1}M) = S^{-1}(\Gamma_a^M M)$, then $t(aM) \subseteq \Gamma_a^M M$ for some $t \in R \backslash L_a^M$ as aM is finitely generated. This yields the contradiction that $t \in L_a^M$. Thus $(\frac{a}{1})(S^{-1}M) \nsubseteq (S^{-1}\Gamma_a^M)(S^{-1}M)$. Now, if $S^{-1}I$ is an ideal of $S^{-1}R$ such that $(\frac{a}{1})(S^{-1}M) = S^{-1}(aM) \nsubseteq (S^{-1}I)(S^{-1}M) = S^{-1}(IM)$. Then $aM \nsubseteq IM$. Hence $I \subseteq \Gamma_a^M$ and so $S^{-1}I \subseteq S^{-1}\Gamma_a^M$. This shows that $S^{-1}\Gamma_a^M$ is the greatest element in the set of ideals $S^{-1}J$ such that $(\frac{a}{1})(S^{-1}M) \nsubseteq (S^{-1}J)(S^{-1}M)$. Thus $\frac{a}{1}$ is an $(S^{-1}M)$-strongly hollow element of $S^{-1}R$ by Theorem 3.17. The last part is clear. □

4. A topology on the set Of M-PS-hollow ideals

In [1, 3, 6, 11] some topologies were defined and studied by using strongly irreducible submodules and ideals. Also, in [2, 6], some topologies were constructed by using strongly hollow submodules from a lattice theoretical point of view. Motivated by this background, in this section, we define and study a topology on the set of M-PS-hollow ideals of a ring.

Let M be an R-module. We will denote the set of all M-PS-hollow ideals of R by $PSH^M(R)$. For each ideal I of R, we define the set $V_M^{psh}(I)$ as follows:

$$V_M^{psh}(I) := \{p \in PSH^M(R) : pM \subseteq IM\}$$

The following Lemma shows that the family $\{V_M^{psh}(I) : I \text{ is an ideal of } R\}$ satisfies the axioms of closed sets for a topology.

Lemma 4.1. *The following properties hold for an R-module M.*

(1) $V_M^{psh}(0) = \emptyset$ *and* $V_M^{psh}(R) = PSH^M(R)$.

(2) $V_M^{psh}(I) \cup V_M^{psh}(J) = V_M^{psh}(I + J)$ *for all ideals I, J of R.*

(3) $\cap_{\lambda\in\Lambda} V_M^{psh}(I_\lambda) = V_M^{psh}(\cap_{\lambda\in\Lambda}(I_\lambda M : M))$ *for a family of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ of R.*

Proof. (1) This is clear by definition.

(2) Let $p \in V_M^{psh}(I) \cup V_M^{psh}(J)$. Then $pM \subseteq IM$ or $pM \subseteq JM$. This implies that $pM \subseteq IM + JM = (I + J)M$. Thus $p \in V_M^{psh}(I + J)$ and so $V_M^{psh}(I) \cup V_M^{psh}(J) \subseteq V_M^{psh}(I + J)$. Now, let $p \in V_M^{psh}(I + J)$.

Then $pM \subseteq (I + J)M = IM + JM$. Since pM is an M-PS-hollow submodule, either $pM \subseteq IM$ or $pM \subseteq JM$. Therefore, $p \in V_M^{psh}(I) \cup V_M^{psh}(J)$ and so $V_M^{psh}(I + J) \subseteq V_M^{psh}(I) \cup V_M^{psh}(J)$. Thus $V_M^{psh}(I) \cup V_M^{psh}(J) = V_M^{psh}(I + J)$.

(3) Let $p \in \cap_{\lambda\in\Lambda} V_M^{psh}(I_\lambda)$. Then $pM \subseteq I_\lambda M$ for every $\lambda \in \Lambda$ whence $pM \subseteq \cap_{\lambda\in\Lambda}(I_\lambda M)$. It follows that $(pM : M) \subseteq (\cap_{\lambda\in\Lambda}(I_\lambda M) : M) = \cap_{\lambda\in\Lambda}(I_\lambda M : M)$ whence $pM = (pM : M)M \subseteq (\cap_{\lambda\in\Lambda}(I_\lambda M : M))\, M$. Thus $p \in V_M^{psh}(\cap_{\lambda\in\Lambda}(I_\lambda M : M))$ and so $\cap_{\lambda\in\Lambda} V_M^{psh}(I_\lambda) \subseteq V_M^{psh}(\cap_{\lambda\in\Lambda}(I_\lambda M : M))$. To see the reverse inclusion, take an element $p \in V_M^{psh}(\cap_{\lambda\in\Lambda}(I_\lambda M : M))$. Then $pM \subseteq (\cap_{\lambda\in\Lambda}(I_\lambda M : M))\, M \subseteq (I_\lambda M : M)M = I_\lambda M$ for every $\lambda \in \Lambda$. Thus $p \in V_M^{psh}(I_\lambda)$ for every $\lambda \in \Lambda$ and so $p \in \cap_{\lambda\in\Lambda} V_M^{psh}(I_\lambda)$. Therefore, $V_M^{psh}(\cap_{\lambda\in\Lambda}(I_\lambda M : M)) \subseteq \cap_{\lambda\in\Lambda} V_M^{psh}(I_\lambda)$. □

Now, we put $\zeta_M^{psh}(R) := \{V_M^{psh}(I) : I \text{ is an ideal of } R\}$. By Lemma 4.1, for any R-module M, there exists a topology, τ^{psh} say, on $PSH^M(R)$ having $\zeta_M^{psh}(R)$ as the family of all closed sets. The topology τ^{psh} is called PSH-Zariski topology on $PSH^M(R)$. Let $Y \subseteq PSH^M(R)$ for an R-module M. We will denote the sum of all elements in Y by $\Theta(Y)$ and the closure of Y in $PSH^M(R)$ with respect to PSH-Zariski topology by $Cl^{psh}(Y)$.

The proof of the following lemma is easily proved by using definitions. So it is left to the reader.

Lemma 4.2. *Let M be an R-module and I, J be ideals of R. Then the following hold.*

(1) *If $IM = JM$, then $V_M^{psh}(I) = V_M^{psh}(J)$. The converse is also true if I and J are M-PS-hollow ideals.*

(2) $V_M^{psh}(I) = V_M^{psh}((IM : M))$.

(3) *Let $Y \subseteq PSH^M(R)$. Then $Y \subseteq V_M^{psh}(I)$ if and only if $\Theta(Y)M \subseteq IM$.*

In the following proposition we determine the closure of a subset of $PSH^M(R)$ with respect to PSH-Zariski topology.

Proposition 4.3. *Let M be an R-module and $Y \subseteq PSH^M(R)$. Then the following hold.*

(1) *$Cl^{psh}(Y) = V_M^{psh}(\Theta(Y))$. In particular, $Cl^{psh}(\{p\}) = V_M^{psh}(p)$ for each $p \in PSH^M(R)$.*

(2) *If $\Theta(Y) = R$, then Y is dense in $PSH^M(R)$.*

Proof. (1) Let $p \in Y$. Then $p \subseteq \Theta(Y)$ whence $pM \subseteq \Theta(Y)M$ and we get that $p \in V_M^{psh}(\Theta(Y))$. This shows that $Y \subseteq V_M^{psh}(\Theta(Y))$. Now, let $V_M^{psh}(I)$ be any closed subset of $PSH^M(R)$ with $Y \subseteq V_M^{psh}(I)$ where I is an ideal of R. Then $\Theta(Y)M \subseteq IM$ by Lemma 4.2. Hence for every $p \in V_M^{psh}(\Theta(Y))$, $pM \subseteq \Theta(Y)M \subseteq IM$ whence $V_M^{psh}(\Theta(Y)) \subseteq V_M^{psh}(I)$. This shows that $V_M^{psh}(\Theta(Y))$ is the smallest closed subset of $PSH^M(R)$ that contains Y. So $Cl^{psh}(Y) = V_M^{psh}(\Theta(Y))$.

(2) This follows from part (1). □

Recall that a topological space X is said to be irreducible if for any decomposition $X = A_1 \cup A_2$ with closed subsets A_1, A_2 of X, we have $A_1 = X$ or $A_2 = X$. A subset A of X is said to be an irreducible subspace (subset) if it is irreducible as a subspace of X. In fact, $A \subseteq X$ is irreducible iff for any proper closed subsets B_1, B_2 of X, $A \subseteq B_1 \cup B_2$ implies $A \subseteq B_1$ or $A \subseteq B_2$. An irreducible component of a topological space X is a maximal irreducible subset of X.

Any singleton subset and its closure in a topological space are irreducible. So we obtain the following corollary applying Proposition 4.3-(1).

Corollary 4.4. *Let M be an R-module and J be an M-PS-hollow ideal of R. Then $V_M^{psh}(J)$ is an irreducible closed subset of $PSH^M(R)$.*

In the following theorem we give a necessary and sufficient condition for $Y \subseteq PSH^M(R)$ to be an irreducible subset of $PSH^M(R)$.

Theorem 4.5. *Let M be an R-module and Y be a subset of $PSH^M(R)$. Then $\Theta(Y)$ is an M-PS-hollow ideal of R if and only if Y is an irreducible subset of $PSH^M(R)$. In particular, $\Theta(PSH^M(R))$ is an M-PS-hollow ideal of R if and only if $PSH^M(R)$ is an irreducible topological space.*

Proof. Suppose that $\Theta(Y)$ is an M-PS-hollow ideal of R. Let $Y \subseteq Y_1 \cup Y_2$ where Y_1 and Y_2 are two closed subset of $PSH^M(R)$. Then $Y_1 = V_M^{psh}(I)$ and $Y_2 = V_M^{psh}(J)$ for some ideals I, J of R. Hence $Y \subseteq V_M^{psh}(I) \cup V_M^{psh}(J) = V_M^{psh}(I+J)$. By Lemma 4.2, $\Theta(Y)M \subseteq (I+J)M = IM + JM$. Since $\Theta(Y)M$ is a PS-hollow submodule of M, either $\Theta(Y)M \subseteq IM$ or $\Theta(Y)M \subseteq JM$. By Lemma 4.2, $Y \subseteq V_M^{psh}(I) = Y_1$ or $Y \subseteq V_M^{psh}(J) = Y_2$. This yields that Y is an irreducible subset of $PSH^M(R)$. Conversely, suppose that Y is an irreducible subset of $PSH^M(R)$. Let $\Theta(Y)M \subseteq IM+L$ where I is an ideal of R and L is a submodule of M. Suppose on the contrary that $\Theta(Y)M \nsubseteq IM$ and $\Theta(Y)M \nsubseteq L$. Then $\Theta(Y)M \nsubseteq (L : M)M$ and Lemma 4.2 implies that $Y \nsubseteq V_M^{psh}(I)$ and $Y \nsubseteq V_M^{psh}((L : M))$. Let $p \in Y$. Then $pM \subseteq \Theta(Y)M \subseteq IM + L$. Since pM is a PS-hollow submodule, we have either $pM \subseteq IM$ or $pM \subseteq L$. So either $pM \subseteq IM$ or $(pM : M)M = pM \subseteq (L : M)M$. Thus $p \in V_M^{psh}(I)$ or $p \in V_M^{psh}((L : M))$. This yields that $Y \subseteq V_M^{psh}(I) \cup V_M^{psh}((L : M))$ which contradicts with the irreducibility of Y. So $\Theta(Y)$ is an M-PS-hollow ideal of R. □

Let Y be a closed subset of a topological space. An element $y \in Y$ is called a *generic point* of Y if $Y = Cl(\{y\})$.

It is well-known that a topological space is a T_0-space if and only if the closures of distinct points are distinct. So a generic point of an irreducible closed subset Y of a topological space is unique if the topological space is a T_0-space.

In the following theorem we determine irreducible closed subsets of $PSH^M(R)$ and we give a bijection from the set of irreducible components of $PSH^M(R)$ onto the set of maximal elements of $PSH^M(R)$ when $PSH^M(R)$ is a T_0-space

Theorem 4.6. *Let M be an R-module and Y be a subset of $PSH^M(R)$.*

(1) *Y is an irreducible closed subset of $PSH^M(R)$ if and only if $Y = V_M^{psh}(p)$ for some M-PS-hollow ideal p of R. Thus every irreducible closed subset of $PSH^M(R)$ has a generic point.*

(2) *If $PSH^M(R)$ is a T_0-space, then the correspondence $V_M^{psh}(q) \longmapsto q$ is a bijection from the set of irreducible components of $PSH^M(R)$ onto the set of maximal elements of $PSH^M(R)$.*

Proof. (1) By Corollary 4.4, $Y = V_M^{psh}(q)$ is an irreducible closed subset of $PSH^M(R)$. Conversely, if Y is an irreducible closed subset of $PSH^M(R)$, then $Y = V_M^{psh}(I)$ for some ideal I of R. By Theorem 4.5, $\Theta(Y)$ is an M-PS-hollow ideal of R. Proposition 4.3 implies that $Y = V_M^{psh}(\Theta(Y))$ as desired.

(2) Let Y be an irreducible component of $PSH^M(R)$. By part (1), Y is a maximal element of the set $\{V_M^{psh}(I) : I \in PSH^M(R)\}$. So $Y = V_M^{psh}(p)$ for some $p \in PSH^M(R)$. If $q \in PSH^M(R)$ with $p \subseteq q$, then $V_M^{psh}(p) \subseteq V_M^{psh}(q)$. By the maximality of $V_M^{psh}(p)$, we get that $Cl^{psh}(\{p\}) = V_M^{psh}(p) = V_M^{psh}(q) = Cl^{psh}(\{q\})$. Since $PSH^M(R)$ is a T_0-space, $p = q$. Thus p is a maximal element of $PSH^M(R)$.

Now, let p be a maximal element of $PSH^M(R)$ with $V_M^{psh}(p) \subseteq V_M^{psh}(J)$ for some M-PS-hollow ideal J of R. Then $p \in V_M^{psh}(J)$ and so $p \subseteq (JM : M)$. Since $(JM : M)$ is an M-PS-hollow ideal of R, we get that $p = (JM : M)$ by the maximality of p. It follows by Lemma 4.2 that $V_M^{psh}(J) = V_M^{psh}((JM : M)) = V_M^{psh}(p)$. This shows that $V_M^{psh}(p)$ is an irreducible component of $PSH^M(R)$. □

Recall that a topological space X is said to be Noetherian if the open subsets of X satisfy ascending chain condition. This is equivalent to say that the closed subsets of X satisfy descending chain condition. It is well-known that if X is a Noetherian topological space, then every subspace of X is quasi-compact.

In the following theorem we determine some cases in which $PSH^M(R)$ is a Noetherian space.

Theorem 4.7. *Let M be an R-module. In each of the following cases $PSH^M(R)$ is a Noetherian topological space.*

(1) *R is an Artinian ring.*

(2) *M is an Artinian R-module.*

Proof. (1) Let $V_M^{psh}(I_1) \supseteq V_M^{psh}(I_2) \supseteq ...$ be a descending chain of closed subsets of $PSH^M(R)$ where I_i is an ideal of R for each $i \in \{1, 2...\}$. Then $\Theta(V_M^{psh}(I_1)) \supseteq \Theta(V_M^{psh}(I_2)) \supseteq ...$ is a descending chain of ideals of R. By assumption, there exists a positive integer k such that $\Theta(V_M^{psh}(I_k)) = \Theta(V_M^{psh}(I_{k+i}))$ for each $i \in \{1, 2, ...\}$. By Proposition 4.3, $V_M^{psh}(I_k) = V_M^{psh}(I_{k+i})$ for each $i \in \{1, 2, ...\}$. Hence $PSH^M(R)$ is a Noetherian space.

(2) Let $V_M^{psh}(I_1) \supseteq V_M^{psh}(I_2) \supseteq ...$ be a descending chain of closed subsets of $PSH^M(R)$ where I_i is an ideal of R for each $i \in \{1, 2...\}$. Then $\Theta(V_M^{psh}(I_1))M \supseteq \Theta(V_M^{psh}(I_2))M \supseteq ...$ is a descending chain of submodules of the Artinian module M. So there exists a positive integer k such that $\Theta(V_M^{psh}(I_k))M = \Theta(V_M^{psh}(I_{k+i}))M$ for each $i \in \{1, 2, ...\}$. By Lemma 4.2, $V_M^{psh}(\Theta(V_M^{psh}(I_k))) = V_M^{psh}(\Theta(V_M^{psh}(I_{k+i})))$ for each $i \in \{1, 2, ...\}$. Proposition 4.3 implies that $V_M^{psh}(I_k) = V_M^{psh}(I_{k+i})$ for each $i \in \{1, 2, ...\}$. Hence $PSH^M(R)$ is a Noetherian space. □

A topological space X is said to be a *spectral space* if X is homeomorphic to $Spec(S)$, with the Zariski topology, for some commutative ring S. Spectral spaces were characterized by Hochster [20, p. 52, Proposition 4] as the topological spaces X which satisfy the following conditions:

(a) X is a T_0-space;

(b) X is quasi-compact and has a basis of quasi-compact open subsets;

(c) The family of quasi-compact open subsets of X is closed under finite intersections;

(d) Every irreducible closed subset of X has a generic point.

It is well-known that a Noetherian space is spectral if and only if it is a T_0-space and every irreducible closed subset has a generic point. By using this fact, Theorem 4.6 and Theorem 4.7, we get the following corollary.

Corollary 4.8. *Let R be an Artinian ring or M be an Artinian R-module. Then $PSH^M(R)$ is a spectral space if and only if $PSH^M(R)$ is a T_0-space.*

Recall from [7] that an R-module M is called cancellation (resp. restricted cancellation) if $IM = JM$ (resp. $0 \neq IM = JM$) implies $I = J$ for all ideals I, J of R. In the following proposition we show that Then $PSH^M(R)$ is a T_0-space where M is a restricted cancellation R-module.

Proposition 4.9. *Let M be a restricted cancellation R-module. Then $PSH^M(R)$ is a T_0-space.*

Proof. It is well-known that a topological space is a T_0-space if and only if the closures of distinct points are distinct. Let $Cl(\{p\}) = Cl(\{q\})$ for $p, q \in PSH^M(R)$. Then $V_M^{psh}(q) = V_M^{psh}(p)$ by Proposition 4.3. By Lemma 4.2, we have $pM = qM$. Since M is restricted cancellation, $p = q$. Thus $PSH^M(R)$ is a T_0-space. □

Combining Corollary 4.8 and Proposition 4.9, we get the following corollary.

Corollary 4.10. *Let M be a restricted cancellation R-module. If R is an Artinian ring or M is an Artinian R-module, then $PSH^M(R)$ is a spectral space.*

Let M be an R-module. We will denote the set of all minimal elements of $PSH^M(R)$ by $Min(PSH^M(R))$.

Recall that a topological space is a T_1-space if and only if every singleton subset is closed. In the following proposition we determine when $PSH^M(R)$ is a T_1-space.

Proposition 4.11. *Let M be an R-module. Then $PSH^M(R)$ is a T_1-space if and only if $Min(PSH^M(R)) = PSH^M(R)$.*

Proof. Suppose that $PSH^M(R)$ is a T_1-space. Let $q \in PSH^M(R)$ with $p \subseteq q$ for some $p \in PSH^M(R)$. Then $pM \subseteq qM$ whence $p \in V_M^{psh}(q) = Cl^{psh}(\{q\}) = \{q\}$ by assumption and Proposition 4.3. Thus $p = q$ and so $Min(PSH^M(R)) = PSH^M(R)$.

Conversely, suppose that $Min(PSH^M(R)) = PSH^M(R)$. First note that $(IM : M) \in PSH^M(R)$ for every $I \in PSH^M(R)$. By assumption, we see that $I = (IM : M)$ for every $I \in PSH^M(R)$. Let $p \in PSH^M(R)$. By Proposition 4.3 it is sufficient to show that $V_M^{psh}(p) = \{p\}$. Let $q \in V_M^{psh}(p)$. Then $qM \subseteq pM$ whence $q = (qM : M) \subseteq (pM : M) = p$. We must have $q = p$ by assumption. Thus $V_M^{psh}(p) = \{p\}$ and so $PSH^M(R)$ is a T_1-space. □

Recall that a proper submodule N of an R-module M is said to be completely irreducible if $N = \cap_{i\in I} N_i$ where $\{N_i\}_{i\in I}$ is a family of submodules of M, then $N = N_i$ for some $i \in I$. Every submodule of M is an intersection of completely irreducible submodules of M (see [18]).

Finally, we find a base for PSH-Zariski topology when M is a faithful multiplication module.

Proposition 4.12. *Let M be a faithful multiplication R-module. Then the set $\{PSH^M(R)\backslash V_M^{psh}(q) : q$ is a completely irreducible ideal of $R\}$ is a base for PSH-Zariski topology.*

Proof. Let I be an ideal of R. Put $X := PSH^M(R)$. There exists a family of completely irreducible ideals $\{q_i\}_{i\in\Lambda}$ such that $I = \cap_{i\in\Lambda} q_i$. By using [16, Theorem 1.6], we have $X\backslash V_M^{psh}(I) = X\backslash V_M^{psh}(\cap_{i\in\Lambda} q_i) = X\backslash V_M^{psh}(((\cap_{i\in\Lambda} q_i)M : M)) = X\backslash V_M^{psh}((\cap_{i\in\Lambda}(q_iM) : M)) = X\backslash(\cap_{i\in\Lambda} V_M^{psh}(q_i)) = \cup_{i\in\Lambda}(X\backslash V_M^{psh}(q_i))$. This completes the proof. □

5. Conclusions

In this study, we introduced the notions of M-strongly hollow and M-PS-hollow ideals where M is a module over a commutative ring R. We investigated some properties and characterizations of M-strongly hollow (M-PS-hollow) ideals. Then we constructed a topology on the set of all M-PS-hollow ideals of a commutative ring R. We investigated when this topological space is irreducible, Noetherian, T_0, T_1 and spectral space.

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