Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(11): 12929–12951.
DOI:10.3934/math.2021748
Received: 28 June 2021
Accepted: 06 September 2021
Published: 13 September 2021

Research article

Existence and concentration of positive solutions for a p-fractional Choquard equation

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Abstract: In this work, we study the existence, multiplicity and concentration behavior of positive solutions for the following problem involving the fractional p-Laplacian

$$\varepsilon^{ps}(-\Delta)_p^s u + V(x)|u|^{p-2}u = \varepsilon^{\mu-N}(\frac{1}{|x|^\mu} * K|u|^q)K(x)|u|^{q-2}u \text{ in } \mathbb{R}^N,$$

where $0 < s < 1 < p < \infty$, $N > ps$, $0 < \mu < ps$, $p < q < \frac{p_s^*}{2}(2 - \frac{\mu}{N})$, $(-\Delta)_p^s$ is the fractional p-Laplacian and $\varepsilon > 0$ is a small parameter. Under certain conditions on V and K, we prove the existence of a positive ground state solution and express the location of concentration in terms of the potential functions V and K. In particular, we relate the number of solutions with the topology of the set where V attains its global minimum and K attains its global maximum.

Keywords: fractional p-Laplacian; fractional Choquard equations; variational methods
Mathematics Subject Classification: 35A15, 35B38, 35J60

1. Introduction

In this paper, we consider the following nonlinear equation governed by the fractional p-Laplacian

$$\varepsilon^{ps}(-\Delta)_p^s u + V(x)|u|^{p-2}u = \varepsilon^{\mu-N}(\frac{1}{|x|^\mu} * K|u|^q)K(x)|u|^{q-2}u \text{ in } \mathbb{R}^N, \quad (1.1)$$

where $\varepsilon > 0$ is a small parameter, $N > ps$, $s \in (0, 1)$, $1 < p < \infty$, $p < q < \frac{p_s^*}{2}(2 - \frac{\mu}{N})$ and V, K are positive functions. $(-\Delta)_p^s$ denotes the fractional p-Laplacian defined for all $u : \mathbb{R}^N \to \mathbb{R}$ smooth enough by

$$(-\Delta)_p^s u(x) = P.V. \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} dy,$$

the *P.V.* stands for the Cauchy principle value (see [10]).

In the case $s = 1$, $p = 2$ and $K(x) \equiv 1$, the Eq (1.1) boils down to the Choquard equation

$$-\varepsilon^2 \Delta u + V(x)u = \varepsilon^{\mu-N}(\frac{1}{|x|^\mu} * |u|^q)|u|^{q-2}u \ \text{ in } \mathbb{R}^N. \tag{1.2}$$

When $N = 3$, $\mu = 1$ and $q = 2$, the Eq (1.2) has appeared in the several context of quantum physics, such as the description of a Polaron at rest [20] and the model of an electron trapped in its own hole [13] and the coupling of the Schrödinger equation under a classical Newtonian gravitational potential [12]. The pioneering mathematical research goes back to Lieb [13] and Lions [14]. The existence and qualitative of solutions of equation like (1.2) have been extensively studied by variational methods, see for example [16–19, 29] and their references. For the existence of semi-classical solutions to Choquard equation (1.2) were studied in some papers. In [28], Wei and Winter constructed a family of solutions which concentrate to the non-degenerate critical points of the potential V. Moroz and Schaftingen [18] proved the existence of solutions concentrating around the local minimum of V by a nonlocal penalization method. See [3] for the existence and multiplicity for a generalized quasilinear Choquard equation.

In recent years, a great attention has been given to problems driven by the fractional Laplacian. One of the reasons for this comes from the fact that this operator appears in several applications in different subjects, such as crystal dislocation, thin obstacle problems, optimization and finance, anomalous diffusion and many others, we can see [10, 25]. Recently, d'Avenia, Siciliano and Squassina [21] considered the existence, regularity, symmetry as well as decay properties of the following fractional Choquard equation

$$(-\Delta)^s u + au = \varepsilon^{\mu-N}(\frac{1}{|x|^\mu} * |u|^q)|u|^{q-2}u \ \text{ in } \mathbb{R}^N. \tag{1.3}$$

Shen, Gao and Yang [23] obtained the existence of ground states of (1.3) with the nonlinearity satisfies the generaal Beresty-Lions type assumptions. Zhang and Wu [31] studied the existence of nodal solutions of (1.3). Chen and Liu [5] studied (1.3) with nonconstant linear potential and proved the existence of ground states without any symmetry property. Ambrosio [1] investigated the multiplicity and concentration of positive solutions for a fractional Choquard equation with general nonlinearity. In [7], Chen, Li and Yang obtained the multiplicity and concentration of nontrivial nonnegative solutions for a fractional Choquard equation with critical exponent. For other existence results we refer to [4, 11, 24, 32] and the references therein.

To the best of our knowledge, there are few results about fractional Choquard equation like (1.3). Belchior et al. [8] investigated the following equation

$$(-\Delta)_p^s u + A|u|^{p-2}u = (\frac{1}{|x|^\mu} * F(u))f(u) \ \text{ in } \mathbb{R}^N, \tag{1.4}$$

where F is the primitive of f and A is a positive constant. They showed the existence of ground states and asymptotic of the solutions for (1.4). In [2], Ambrosio studied the following problem

$$\varepsilon^{ps}(-\Delta)_p^s u + V(x)|u|^{p-2}u = \varepsilon^{\mu-N}(\frac{1}{|x|^\mu} * F(u))f(u) \ \text{ in } \mathbb{R}^N.$$

He proved solutions concentrating around global minimum of the potential V.

Recently, Wang et al. [27] applied a kind of structure introduced by Ding and Liu [9] to study the existence and concentration of positive solutions for semilinear Schrödinger-Poisson system. The similar results for the fractional Schrödinger-Poisson system, we can see [30]. Alves and Yang [3] considered the generalized quasilinear Choquard equation

$$-\varepsilon^p \Delta_p u + V(x)|u|^{p-2}u = \varepsilon^{\mu-N}\left(\int_{\mathbb{R}^N} \frac{Q(y)F(u(y))}{|x-y|^\mu}\right) Q(x)f(u) \ \text{ in } \mathbb{R}^N,$$

where $1 < p < N$, V and Q are two continuous functions satisfy the structure of [9], they established concentration behavior for the Choquard equation. It is quite natural to ask how the potentials will affect the existence and concentration of solutions for (1.1). In this paper, we shall give an affirmative answers for this question.

Motivated by the above papers, we will establish the existence, multiplicity and concentration of positive solutions for Eq (1.1). To gain further insight into the effect of potential functions V and K on the concentration process, we give the following assumptions introduced by [9]. Set

$$\theta = \min_{x\in\mathbb{R}^N} V(x), \quad \mathcal{V} = \{x \in \mathbb{R}^N : V(x) = \theta\}, \quad V_\infty = \liminf_{|x|\to\infty} V(x),$$

$$\kappa = \max_{x\in\mathbb{R}^N} K(x), \quad \mathcal{K} = \{x \in \mathbb{R}^N : K(x) = \kappa\}, \quad K_\infty = \limsup_{|x|\to\infty} K(x).$$

We assume that V and K satisfy:
(H_1) $V, K \in L^\infty(\mathbb{R}^N)$ are uniformly continuous and $\theta > 0$, $\inf_{x\in\mathbb{R}^N} K(x) > 0$.
(H_2) $\theta < V_\infty < \infty$ and there exist $R > 0, x_* \in \mathcal{V}$ such that

$$K(x_*) \geq K(x) \ \text{ for all } \ |x| \geq R.$$

(H_3) $\kappa > K_\infty \geq \inf_{x\in\mathbb{R}^N} K(x)$ and there exist $R > 0, x^* \in \mathcal{K}$ such that

$$V(x^*) \leq V(x) \ \text{ for all } \ |x| \geq R.$$

From (H_2), we may assume that $K(x_*) = \max_{x\in\mathcal{V}} K(x)$. Set

$$\Omega_V = \{x \in \mathcal{V} : K(x) = K(x_*)\} \cup \{x \notin \mathcal{V} : K(x) > K(x_*)\}.$$

From (H_3), we may assume that $V(x^*) = \min_{x\in\mathcal{K}} V(x)$. Set

$$\Omega_K = \{x \in \mathcal{K} : V(x) = V(x^*)\} \cup \{x \notin \mathcal{K} : V(x) < V(x^*)\}.$$

Clearly, Ω_V and Ω_K are bounded sets. Moreover, if $\mathcal{V} \cap \mathcal{K} \neq \emptyset$, then $\Omega_V = \Omega_K = \mathcal{V} \cap \mathcal{K}$. In particular, $\Omega_V = \mathcal{V}$ if $K(x)$ is a constant, and $\Omega_K = \mathcal{K}$ if $V(x)$ is a constant.

We now state our main results.

Theorem 1.1. *Assume that (H_1) and (H_2) hold, then for all small $\varepsilon > 0$, (1.1) has a positive ground state solution u_ε, and there exists a maximum point x_ε, such that up to a subsequence, $x_\varepsilon \to x_0$ as*

$\varepsilon \to 0$, $\lim_{\varepsilon\to 0} dist(x_\varepsilon, \Omega_V) = 0$, *and* $v_\varepsilon(x) = u_\varepsilon(\varepsilon x + x_\varepsilon)$ *converges in* $W^{s,p}(\mathbb{R}^N)$ *to a ground state solution of*

$$(-\Delta)^s_p u + V(x_0)|u|^{p-2}u = K^2(x_0)(\int_{\mathbb{R}^N} \frac{|u(y)|^q}{|x-y|^\mu}dy)|u|^{q-2}u,\ x \in \mathbb{R}^N.$$

In particular, if $\mathcal{V} \cap \mathcal{K} \neq \emptyset$, *then* $\lim_{\varepsilon\to 0} dist(x_\varepsilon, \mathcal{V} \cap \mathcal{K}) = 0$, *and up to a subsequence,* v_ε *converges in* $W^{s,p}(\mathbb{R}^N)$ *to a ground state solution of*

$$(-\Delta)^s_p u + \theta|u|^{p-2}u = \kappa^2(\int_{\mathbb{R}^N} \frac{|u(y)|^q}{|x-y|^\mu}dy)|u|^{q-2}u,\ x \in \mathbb{R}^N.$$

If (H_1) *and* (H_3) *hold, and we replace* Ω_V *by* Ω_K, *then all the conclusions remain true.*

Let $\mathcal{V} \cap \mathcal{K} \neq \emptyset$. Now we denote $\Lambda = \mathcal{V} \cap \mathcal{K}$. It is easy to check that Λ is compact. For any $\delta > 0$, set $\Lambda_\delta = \{x \in \mathbb{R}^N : dist(x, \Lambda) \leq \delta\}$.

Theorem 1.2. *Assume that* (H_1) *and* (H_2) *or* (H_3) *hold, then for all small* $\varepsilon > 0$, *problem (1.1) has at least* $cat_{\Lambda_\delta}(\Lambda)$ *solutions, if* x_ε *its global maximum, up to a subsequence, such that* $\lim_{\varepsilon\to 0} dist(x_\varepsilon, \Lambda) = 0$, u_ε *converges in* $W^{s,p}(\mathbb{R}^N)$ *to a ground state solution of*

$$(-\Delta)^s_p u + \theta|u|^{p-2}u = \kappa^2(\int_{\mathbb{R}^N} \frac{|u(y)|^q}{|x-y|^\mu}dy)|u|^{q-2}u,\ x \in \mathbb{R}^N.$$

Note that our main results are also new for the case $p = 2$. Our main theorem improves the result in [2,8] with both linear potential V and nonlinear potential K of the concentration behavior of positive solutions. There are some difficulties in such a problem. The first one is that there would presumably be competition between the V and K: each would try to attract ground states to their minimum and maximum points, respectively. The second one, the operator $(-\Delta)^s_p$ and the convolution term are all nonlocal operators, make our analysis more complicated with respect to [3], so we need more accurate estimates.

The plan of this paper is the following: In Section 2, we give some preliminary results which will be used later. In Section 3, we show some compactness lemmas of the functional associated to our problem. In Section 4, we consider the existence of ground states of case of (1.1) and the concentration phenomenon. In the final section, we prove Theorem 1.2.

In this paper, we will use the following notations:

The notations $C, C_1, C_2 \cdots$ are positive (possibly different) constants.

$B_r(z_0)$ denotes the ball in $\mathbb{R}^N$ centered at z_0 with radius r.

$o_n(1)$ and $o_\varepsilon(1)$ denotes the vanishing quantities as $n \to \infty$ and $\varepsilon \to 0$.

We will use $\|\cdot\|_q$ for the norm in $L^q(\mathbb{R}^N)$, $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$.

2. Preliminaries

In this section, we recall some known results for the readers convenience and the later use. First, we will give some useful facts for the fractional order Sobolev spaces. Let $0 < s < 1 < p < \infty$ be real numbers, the homogeneous fractional Sobolev space $\mathcal{D}^{s,p}(\mathbb{R}^N)$ as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the Gagliardo seminorm

$$[u]^p_{s,p} = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}dxdy.$$

The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy < \infty\},$$

equipped with the norm

$$\|u\|^p = [u]^p_{s,p} + \|u\|^p_p.$$

It is easy to see that the embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for any $r \in [p, p^*_s]$, and compactly in $L^r_{loc}(\mathbb{R}^N)$ for any $r \in [p, p^*_s)$.

Making the change of variable $x \mapsto \varepsilon x$, Eq (1.1) becomes

$$(-\Delta)^s_p u + V_\varepsilon(x)|u|^{p-2}u = (\int_{\mathbb{R}^N} \frac{K_\varepsilon(y)|u(y)|^q}{|x - y|^\mu} dy)K_\varepsilon(x)|u|^{q-2}u, \ x \in \mathbb{R}^N, \tag{2.1}$$

where $V_\varepsilon(x) = V(\varepsilon x)$ and $K_\varepsilon(x) = K(\varepsilon x)$. Eqs (1.1) and (2.1) are equivalent, we shall thereafter focus on Eq (2.1). For any $\varepsilon > 0$, let $E_\varepsilon = \{u \in W^{s,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V_\varepsilon(x)|u|^p dx < \infty\}$ be the Sobolev space endowed with the norm

$$\|u\|^p_\varepsilon = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V_\varepsilon(x)|u|^p dx.$$

By the assumption of V, we see that $\|\cdot\|_\varepsilon$ and $\|\cdot\|$ are equivalent norms for $\varepsilon > 0$. Define the energy functional associated with (2.1) by

$$I_\varepsilon(u) = \frac{1}{p}\|u\|^p_\varepsilon - \frac{1}{2q}\int_{\mathbb{R}^{2N}} \frac{K_\varepsilon(y)|u(y)|^q K_\varepsilon(x)|u(x)|^q}{|x - y|^\mu} dxdy.$$

Note that $p < q < \frac{p^*_s}{2}(2 - \frac{\mu}{N})$, by the Hardy-Littlewood-Sobolev inequality ([15]) and the boundedness of K, we have

$$\int_{\mathbb{R}^{2N}} \frac{K_\varepsilon(y)|u(y)|^q K_\varepsilon(x)|u(x)|^q}{|x - y|^\mu} dxdy \le C_1 \left(\int_{\mathbb{R}^N} |u|^{\frac{2N\mu}{2N-\mu}} dx\right)^{\frac{2N-\mu}{N}} \le C_2\|u\|^{2q}_\varepsilon. \tag{2.2}$$

Therefore, the functional I_ε is well defined on E_ε and belongs to $C^1(E_\varepsilon, \mathbb{R})$.

Define the solution manifold of (2.1) by

$$\mathcal{N}_\varepsilon = \left\{u \in E_\varepsilon \setminus \{0\} : \|u\|^p_\varepsilon = \int_{\mathbb{R}^{2N}} \frac{K_\varepsilon(y)|u(y)|^q K_\varepsilon(x)|u(x)|^q}{|x - y|^\mu} dxdy\right\}.$$

For any $u \in \mathcal{N}_\varepsilon$, by (2.2) we have

$$\|u\|_\varepsilon \ge r^*, \tag{2.3}$$

for some $r^* > 0$.

The ground energy associated with (2.1) is defined as

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} I_\varepsilon(u).$$

The following vanishing lemma is a version of the Concentration-compactness principle of P. L. Lions. We can see (Lemma 2.1 of [2]).

Lemma 2.1. *Let $N > ps$. Assume that $\{u_n\}$ is bounded in $W^{s,p}(\mathbb{R}^N)$ and it satisfies*

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}|u_n(x)|^p dx = 0,$$

for some $R > 0$. Then $u_n \to 0$ strongly in $L^r(\mathbb{R}^N)$ for every $r \in (p, p_s^*)$.

From (2.2) and $q > p$, it follows that I_ε satisfies the geometry of the mountain pass (see [26]). Hence, there is a sequence $\{u_n\} \subset E_\varepsilon$ such that

$$I_\varepsilon(u_n) \to c_\varepsilon^* \text{ and } I_\varepsilon'(u_n) \to 0, \tag{2.4}$$

where c_ε^* is the mountain pass level given by

$$c_\varepsilon^* = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I_\varepsilon(\gamma(t)) > 0,$$

and $\Gamma = \{\gamma \in C^1([0,1], E_\varepsilon) : \gamma(0) = 0, I_\varepsilon(\gamma(1)) < 0\}$.

We observe that for any $u \in E_\varepsilon \setminus \{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\varepsilon$, and the maximum of the function $g(t) = I_\varepsilon(tu)$ for $t \geq 0$ is achieved at $t = t_u$. By a standard arguments, we have

$$c_\varepsilon = c_\varepsilon^* = \inf_{u\in E_\varepsilon\setminus\{0\}}\max_{t\geq 0} I_\varepsilon(tu).$$

For any $a, b > 0$, consider the limit problem

$$(-\Delta)_p^s u + a|u|^{p-2}u = b^2\Big(\int_{\mathbb{R}^N}\frac{|u(y)|^q}{|x-y|^\mu}dy\Big)|u|^{q-2}u, \quad x \in \mathbb{R}^N. \tag{2.5}$$

Solutions of (2.5) are critical points of the functional defined by

$$I_{ab}(u) = \frac{1}{p}[u]_{s,p}^p + \frac{a}{p}\int_{\mathbb{R}^N}|u|^p dx - \frac{b^2}{2q}\int_{\mathbb{R}^{2N}}\frac{|u(y)|^q|u(x)|^q}{|x-y|^\mu}dxdy.$$

Define the solution manifold of (2.5) by

$$\mathcal{M}_{ab} = \left\{u \in W^{s,p}(\mathbb{R}^N) \setminus \{0\} : \langle I_{ab}'(u), u\rangle = 0\right\}.$$

The ground energy associated with (2.5) is defined as $c_{ab} = \inf_{u\in\mathcal{M}_{ab}} I_{ab}(u)$. It is easy to check that

$$c_{ab} = \inf_{u\in W^{s,p}(\mathbb{R}^N)\setminus\{0\}}\max_{t\geq 0} I_{ab}(tu).$$

By [8], we known that (2.5) has a positive ground state solution ω, that is $c_{ab} = I_{ab}(\omega)$.

Lemma 2.2. *Let $a_1, a_2 > 0$ and $b_1, b_2 > 0$, with $a_1 \leq a_2$ and $b_1 \geq b_2$. Then $c_{a_1b_1} \leq c_{a_2b_2}$. In particular, if one of inequalities is strict, then $c_{a_1b_1} < c_{a_2b_2}$.*

Proof. Let $u \in \mathcal{M}_{a_2b_2}$ be a ground state solution of (2.5) with coefficients a_2, b_2 such that

$$c_{a_2b_2} = I_{a_2b_2}(u) = \max_{t\geq 0} I_{a_2b_2}(tu). \tag{2.6}$$

It is easy to check that there exists $t_0 > 0$ such that $t_0u \in \mathcal{M}_{a_1b_1}$. Then we get

$$I_{a_1b_1}(t_0u) = \max_{t\geq 0} I_{a_1b_1}(tu). \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$\begin{aligned}
c_{a_2b_2} &= I_{a_2b_2}(u) \geq I_{a_2b_2}(t_0u) \\
&= I_{a_1b_1}(t_0u) + \frac{t_0^p}{p}(a_2 - a_1)\int_{\mathbb{R}^N} |u|^p dx \\
&\quad + \frac{t_0^{2q}}{2q}(b_1^2 - b_2^2)\int_{\mathbb{R}^{2N}} \frac{|u(y)|^q |u(x)|^q}{|x-y|^\mu} dxdy \\
&\geq I_{a_1b_1}(t_0u) \geq \inf_{v\in\mathcal{M}_{a_1b_1}} I_{a_1b_1}(v) = c_{a_1b_1}.
\end{aligned}$$

The proof is completed. □

3. A compactness condition

In this section we will show some compactness results for the functional I_ε.

Lemma 3.1. *$\{u_n\} \subset E_\varepsilon$ is a $(PS)_c$ sequence for I_ε with $u_n \rightharpoonup 0$ weakly in E_ε. If $u_n \nrightarrow 0$ in E_ε, the $c \geq c_\infty := c_{V_\infty K_\infty}$.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence for I_ε, by (2.4), we have

$$c + 1 + \|u_n\|_\varepsilon \geq I_\varepsilon(u_n) - \frac{1}{2q}\langle I'_\varepsilon(u_n), u_n\rangle = (\frac{1}{p} - \frac{1}{2q})\|u_n\|_\varepsilon^p \tag{3.1}$$

for n large enough. Therefore $\{u_n\}$ is bounded in E_ε.

For each n, there is a unique $t_n > 0$ such that $t_nu_n \in \mathcal{M}_{V_\infty K_\infty}$. We now show that the sequence $\{t_n\}$ satisfies $\limsup_{n\to\infty} t_n \leq 1$. By contradiction we assume that there exist $\sigma > 0$ and a subsequence (still denoted by $\{t_n\}$) such that $t_n \geq 1 + \sigma$ for all n. From the boundedness of $\{u_n\}$, we have $\langle I'_\varepsilon(u_n), u_n\rangle = o_n(1)$. That is

$$[u_n]_{s,p}^p + \int_{\mathbb{R}^N} V_\varepsilon(x)|u_n|^p dx = \int_{\mathbb{R}^{2N}} \frac{K_\varepsilon(y)|u_n(y)|^q K_\varepsilon(x)|u_n(x)|^q}{|x-y|^\mu} dxdy + o_n(1). \tag{3.2}$$

Since $t_nu_n \in \mathcal{M}_{V_\infty K_\infty}$, we obtain

$$t_n^p([u_n]_{s,p}^p + \int_{\mathbb{R}^N} V_\infty |u_n|^p dx) = t_n^{2q} K_\infty^2 \int_{\mathbb{R}^{2N}} \frac{|u_n(y)|^q |u_n(x)|^q}{|x-y|^\mu} dxdy. \tag{3.3}$$

We deduce from (3.2) and (3.3) that

$$\int_{\mathbb{R}^N} (V_\infty - V_\varepsilon(x))|u_n|^p dx = \int_{\mathbb{R}^{2N}} \frac{(t_n^{2q-p} K_\infty^2 - K_\varepsilon(y)K_\varepsilon(x))|u_n(y)|^q |u_n(x)|^q}{|x-y|^\mu} dxdy. \tag{3.4}$$

By the definition of V_∞ and K_∞, for any $\nu > 0$, there exist a constant $\rho > 0$ sufficiently large such that for $|x| > \rho$,

$$V(x) > V_\infty - \nu, \quad K(x) < K_\infty + \nu. \tag{3.5}$$

Since $\{u_n\}$ is bounded and $u_n \rightharpoonup 0$ in E_ε, by (3.5) we have

$$\begin{aligned}\int_{\mathbb{R}^N}(V_\infty - V_\varepsilon(x))|u_n|^p dx &= \int_{|x|\le\frac{\rho}{\varepsilon}}(V_\infty - V_\varepsilon(x))|u_n|^p dx \\ &\quad + \int_{|x|>\frac{\rho}{\varepsilon}}(V_\infty - V_\varepsilon(x))|u_n|^p dx \\ &\le o_n(1) + C\nu. \end{aligned}\tag{3.6}$$

On the other hand,

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \frac{K_\varepsilon(y)|u_n(y)|^q K_\varepsilon(x)|u_n(x)|^q}{|x-y|^\mu}dxdy \\ &= \int_{|x|>\frac{\rho}{\varepsilon}}\int_{|y|>\frac{\rho}{\varepsilon}} \frac{K_\varepsilon(y)|u_n(y)|^q K_\varepsilon(x)|u_n(x)|^q}{|x-y|^\mu}dxdy \\ &+ \int_{|x|>\frac{\rho}{\varepsilon}}\int_{|y|\le\frac{\rho}{\varepsilon}} \frac{K_\varepsilon(y)|u_n(y)|^q K_\varepsilon(x)|u_n(x)|^q}{|x-y|^\mu}dxdy \\ &+ \int_{|x|\le\frac{\rho}{\varepsilon}}\int_{|y|>\frac{\rho}{\varepsilon}} \frac{K_\varepsilon(y)|u_n(y)|^q K_\varepsilon(x)|u_n(x)|^q}{|x-y|^\mu}dxdy \\ &+ \int_{|x|\le\frac{\rho}{\varepsilon}}\int_{|y|\le\frac{\rho}{\varepsilon}} \frac{K_\varepsilon(y)|u_n(y)|^q K_\varepsilon(x)|u_n(x)|^q}{|x-y|^\mu}dxdy \\ &= I + II + III + IV. \end{aligned}\tag{3.7}$$

By (2.2), (3.5) and the boundedness of $\{u_n\}$, we obtain

$$\begin{aligned} I &< (K_\infty+\nu)^2 \int_{|x|>\frac{\rho}{\varepsilon}}\int_{|y|>\frac{\rho}{\varepsilon}} \frac{|u_n(y)|^q|u_n(x)|^q}{|x-y|^\mu}dxdy \\ &\le (K_\infty+\nu)^2 \int_{\mathbb{R}^{2N}} \frac{|u_n(y)|^q|u_n(x)|^q}{|x-y|^\mu}dxdy \\ &\le K_\infty^2 \int_{\mathbb{R}^{2N}} \frac{|u_n(y)|^q|u_n(x)|^q}{|x-y|^\mu}dxdy + C\nu + C\nu^2. \end{aligned}\tag{3.8}$$

From the boundedness of $K(x)$ and $\{u_n\}$, there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} \frac{K_\varepsilon(y)|u_n(y)|^q}{|x-y|^\mu}dxdy \le C. \tag{3.9}$$

By (3.9) and $u_n \rightharpoonup 0$, we have

$$II \le \int_{|y|\le\frac{\rho}{\varepsilon}}\int_{\mathbb{R}^N} \frac{K(\varepsilon y)|u_n(y)|^q K(\varepsilon x)|u_n(x)|^q}{|x-y|^\mu}dxdy$$

$$\leq \quad C \int_{|y| \leq \frac{\rho}{\varepsilon}} |u_n|^q dy = o_n(1). \tag{3.10}$$

Similarly, we have

$$III = o_n(1) \quad \text{and} \quad IV = o_n(1). \tag{3.11}$$

By (3.7), (3.8), (3.10) and (3.11), we deduce that

$$\begin{aligned} \int_{\mathbb{R}^{2N}} \frac{K_\varepsilon(y)|u_n(y)|^q K_\varepsilon(x)|u_n(x)|^q}{|x-y|^\mu} dxdy \quad &\leq \quad K_\infty^2 \int_{\mathbb{R}^{2N}} \frac{|u_n(y)|^q |u_n(x)|^q}{|x-y|^\mu} dxdy \\ &+ C\nu + C\nu^2 + o_n(1). \end{aligned} \tag{3.12}$$

Combining (3.6), (3.12) with (3.4), we obtain

$$K_\infty^2 (t_n^{2q-p} - 1) \int_{\mathbb{R}^{2N}} \frac{|u_n(y)|^q |u_n(x)|^q}{|x-y|^\mu} dxdy \leq C\nu + C\nu^2 + o_n(1). \tag{3.13}$$

From $u_n \nrightarrow 0$ in E_ε, there exists a sequence $\{z_n\} \subset \mathbb{R}^N$ and constant $R, \beta > 0$ such that

$$\liminf_{n \to \infty} \int_{B_R(z_n)} |u_n|^p dx \geq \beta > 0. \tag{3.14}$$

Indeed, if (3.14) does not true, Lemma 2.1 implies that $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for every $r \in (p, p_s^*)$. It follows from (2.2) that

$$\int_{\mathbb{R}^{2N}} \frac{K_\varepsilon(y)|u_n(y)|^q K_\varepsilon(x)|u_n(x)|^q}{|x-y|^\mu} dxdy = o_n(1).$$

This and (3.2) implies $\|u_n\|_\varepsilon \to 0$ as $n \to \infty$, which contradicts to $u_n \nrightarrow 0$ in E_ε.

Now we set $v_n(x) = u_n(x + z_n)$. We known that $\{v_n\}$ is bounded. Then there exists $v \in W^{s,p}(\mathbb{R}^N)$ such that $v_n \rightharpoonup v$ weakly in $W^{s,p}(\mathbb{R}^N)$. By (3.14), we see that $v \neq 0$. Hence, there is a set $\Omega \subset \mathbb{R}^N$ with $|\Omega| > 0$ such that $v(x) > 0$ in Ω. Then from (3.13) and $t_n \geq 1 + \sigma$, we have

$$0 < K_\infty^2 ((1+\sigma)^{2q-p} - 1) \int_{\mathbb{R}^{2N}} \frac{|v_n(y)|^q |v_n(x)|^q}{|x-y|^\mu} dxdy \leq C\nu + C\nu^2 + o_n(1).$$

Taking limit in the above inequality and by Fatou's lemma, we get

$$0 < K_\infty^2 ((1+\sigma)^{2q-p} - 1) \int_{\mathbb{R}^{2N}} \frac{|v(y)|^q |v(x)|^q}{|x-y|^\mu} dxdy \leq C\nu + C\nu^2,$$

for any $\nu > 0$. It's a contradiction. Therefore, $\limsup_{n\to\infty} t_n \leq 1$.

We next consider the following two cases:

Case 1. $\limsup_{n\to\infty} t_n = 1$. We assume that there exists a subsequence, still denoted by $\{t_n\}$ such that $\lim_{n\to\infty} t_n = 1$. Recalling $t_n u_n \in \mathcal{M}_{V_\infty K_\infty}$, then

$$\begin{aligned} c + o_n(1) = I_\varepsilon(u_n) \quad &= \quad I_\varepsilon(u_n) - I_{V_\infty K_\infty}(t_n u_n) + I_{V_\infty K_\infty}(t_n u_n) \\ &\geq \quad c_\infty + I_\varepsilon(u_n) - I_{V_\infty K_\infty}(t_n u_n). \end{aligned} \tag{3.15}$$

We observe that

$$\begin{aligned} I_\varepsilon(u_n) - I_{V_\infty K_\infty}(t_n u_n) &= \frac{1-t_n^p}{p}[u_n]_{s,p}^p + \frac{1}{p}\int_{\mathbb{R}^N}(V_\varepsilon(x) - t_n^p V_\infty)|u_n|^p dx \\ &+ \frac{1}{2q}\int_{\mathbb{R}^{2N}} \frac{(K_\infty^2 t_n^{2q} - K_\varepsilon(y)K_\varepsilon(x))|u_n(y)|^q|u_n(x)|^q}{|x-y|^\mu} dxdy. \end{aligned} \tag{3.16}$$

From the boundedness of $\{u_n\}$, $\lim_{n\to\infty} t_n = 1$, $u_n \rightharpoonup 0$ in E_ε and (3.5), one has

$$\frac{1-t_n^p}{p}[u_n]_{s,p}^p = o_n(1) \tag{3.17}$$

and

$$\int_{\mathbb{R}^N}(V_\varepsilon(x) - t_n^p V_\infty)|u_n|^p dx \geq o_n(1) - C\nu. \tag{3.18}$$

By (3.12), we have

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} \frac{(K_\infty^2 t_n^{2q} - K_\varepsilon(y)K_\varepsilon(x))|u_n(y)|^q|u_n(x)|^q}{|x-y|^\mu} dxdy \\ &\geq (t_n^{2q} - 1)K_\infty^2 \int_{\mathbb{R}^{2N}} \frac{|u_n(y)|^q|u_n(x)|^q}{|x-y|^\mu} dxdy - o_n(1) - C\nu - C\nu^2 \\ &= o_n(1) - C\nu - C\nu^2. \end{aligned} \tag{3.19}$$

It follows from (3.15)–(3.19) that

$$c + o_n(1) \geq c_\infty + o_n(1) - C\nu - C\nu^2.$$

Letting $n \to \infty$ and $\nu \to 0$, we get $c \geq c_\infty$.
Case 2. $\limsup_{n\to\infty} t_n = t_0 < 1$. In this case, without loss of generality, we assume that $t_n < 1$ for all n. Recalling that $t_n u_n \in \mathcal{M}_{V_\infty K_\infty}$, then by $I'_\varepsilon(u_n) \to 0$, (3.6) and the boundedness of $\{u_n\}$, we have

$$\begin{aligned} c_\infty &\leq I_{V_\infty K_\infty}(t_n u_n) = I_{V_\infty K_\infty}(t_n u_n) - \frac{1}{2q}\langle I'_{V_\infty K_\infty}(t_n u_n) t_n u_n\rangle \\ &= (\frac{1}{p} - \frac{1}{2q}) t_n^p \left([u_n]_{s,p}^p + \int_{\mathbb{R}^N} V_\infty |u_n|^p dx\right) \\ &< (\frac{1}{p} - \frac{1}{2q}) \left([u_n]_{s,p}^p + \int_{\mathbb{R}^N} V_\infty |u_n|^p dx\right) \\ &= I_\varepsilon(u_n) - \frac{1}{2q}\langle I'_\varepsilon(u_n), u_n\rangle + (\frac{1}{p} - \frac{1}{2q}) \int_{\mathbb{R}^N}(V_\infty - V_\varepsilon(x))|u_n|^p dx + o_n(1) \\ &\leq I_\varepsilon(u_n) + o_n(1) + C\nu. \end{aligned}$$

Letting $n \to \infty$ and $\nu \to 0$, we get $c_\infty \leq c$. The proof is completed. □

Lemma 3.2. *The functional I_ε satisfies $(PS)_c$ condition with $c < c_\infty$.*

Proof. Let $\{u_n\} \subset E_\varepsilon$ be a sequence such that $I_\varepsilon(u_n) \to c$ and $I'_\varepsilon(u_n) \to 0$ as $n \to \infty$. By (3.1) we have that $\{u_n\}$ is bounded in E_ε. Then, up to a subsequence, there exists $u \in E_\varepsilon$ such that

$$\begin{cases} u_n \rightharpoonup u, & \text{weakly in } E_\varepsilon, \\ u_n \to u, & \text{strongly in } L^r_{loc}(\mathbb{R}^N), \ \ p \le r < p_s^*, \\ u_n \to u, & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{3.20}$$

By (3.20), $p < q < \frac{p_s^*}{2}(2 - \frac{\mu}{N})$, and Hardy-Littlewood-Sobolev inequality, we obtain that

$$\int_{\mathbb{R}^N} \frac{K_\varepsilon(y)|u_n(y)|^q}{|x-y|^\mu} dy \rightharpoonup \int_{\mathbb{R}^N} \frac{K_\varepsilon(y)|u(y)|^q}{|x-y|^\mu} dy \quad \text{in } L^{\frac{2N}{\mu}}(\mathbb{R}^N).$$

Then, for any $\phi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^{2N}} \frac{K_\varepsilon(y)|u_n(y)|^q K_\varepsilon(x)|u_n(x)|^{q-2}u_n(x)\phi}{|x-y|^\mu} dxdy = \int_{\mathbb{R}^{2N}} \frac{K_\varepsilon(y)|u(y)|^q K_\varepsilon(x)|u(x)|^{q-2}u(x)\phi}{|x-y|^\mu} dxdy + o_n(1).$$

Then, we have $I'_\varepsilon(u) = 0$. Set $w_n = u_n - u$. By Brezis-Lieb lemma, we have

$$\|w_n\|_\varepsilon^p = \|u_n\|_\varepsilon^p - \|u\|_\varepsilon^p + o_n(1). \tag{3.21}$$

From a Brezis-Lieb lemma for the nonlocal term of the functional ([17]), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} (\frac{1}{|x|^\mu} * K_\varepsilon|w_n|^q) K_\varepsilon(x)|w_n|^q dx \ &= \ \int_{\mathbb{R}^N} (\frac{1}{|x|^\mu} * K_\varepsilon|u_n|^q) K_\varepsilon(x)|u_n|^q dx \\ &\quad - \int_{\mathbb{R}^N} (\frac{1}{|x|^\mu} * K_\varepsilon|u|^q) K_\varepsilon(x)|u|^q dx + o_n(1). \end{aligned} \tag{3.22}$$

It follows from (3.21) and (3.22) that

$$I_\varepsilon(w_n) = I_\varepsilon(u_n) - I_\varepsilon(u) + o_n(1) = c - I_\varepsilon(u) + o_n(1)$$

and $I'_\varepsilon(w_n) \to 0$ as $n \to \infty$. Since $I'_\varepsilon(u) = 0$, we get

$$I_\varepsilon(u) = I_\varepsilon(u) - \frac{1}{2q}\langle I'_\varepsilon(u), u\rangle = (\frac{1}{p} - \frac{1}{2q})\|u\|_\varepsilon^p \ge 0.$$

Hence, $I_\varepsilon(w_n) \to c - I_\varepsilon(u) < c_\infty$. By Lemma 3.1, $w_n \to 0$ in E_ε. Then $u_n \to u$ in E_ε. The proof is completed. □

Lemma 3.3. *Let $\{u_n\}$ be a $(PS)_c$ sequence restricted in $\mathcal{N}_\varepsilon$ and assume that $c < c_\infty$. Then $\{u_n\}$ has a convergent subsequence in E_ε.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence for I_ε on $\mathcal{N}_\varepsilon$ at level c, namely

$$I_\varepsilon(u_n) \to c \ \text{ and } \ I'_\varepsilon|_{\mathcal{N}_\varepsilon}(u_n) \to 0.$$

It's easy to check that $\{u_n\}$ is bounded in E_ε. We assume that

$$I'_\varepsilon(u_n) = o_n(1) + \lambda_n g'(u_n),$$

where $g(u) = \langle I'_\varepsilon(u), u\rangle$, and

$$\langle g'(u), u\rangle = p\|u\|_\varepsilon^p - 2q\int_{\mathbb{R}^{2N}} \frac{K_\varepsilon(y)|u(y)|^q K_\varepsilon(x)|u(x)|^q}{|x-y|^\mu}dxdy. \tag{3.23}$$

Since $\{u_n\}$ is bounded, we have

$$0 = g(u_n) = \langle I'_\varepsilon(u_n), u_n\rangle = o_n(1) + \lambda_n\langle g'(u_n), u_n\rangle. \tag{3.24}$$

Since $u_n \in \mathcal{N}_\varepsilon$, by (2.3) and (3.23) we have

$$\langle g'(u_n), u_n\rangle = (p-2q)\|u_n\|_\varepsilon^p \le (p-2q)r^*,$$

where r^* is defined in (2.3). Then,

$$|\lambda_n\langle g'(u_n), u_n\rangle| \ge |\lambda_n|(2q-p)r^*.$$

Thus $\lambda_n \to 0$ and $I'_\varepsilon(u_n) \to 0$ as $n \to \infty$. Therefore, $\{u_n\}$ is a $(PS)_c$ sequence for I_ε in E_ε. By Lemma 3.2, $\{u_n\}$ has a convergent subsequence. □

4. Proof of Theorem 1.1

We only give the details proof under the assumptions (H$_1$) and (H$_2$). The arguments of (H$_3$) is similar. Under the assumption (H$_2$), we may suppose that $x_* = 0 \in \mathcal{V}$ or $x_* = 0 \in \mathcal{V}\cap\mathcal{K}$ if $\mathcal{V}\cap\mathcal{K} \neq \emptyset$. Then

$$\theta = V(0) \quad \text{and} \quad \alpha := K(0) \ge K(x) \text{ for all } |x| \ge R.$$

Lemma 4.1. $\limsup_{\varepsilon\to 0} c_\varepsilon \le c_{\theta\alpha}$. *In particular, if* $\mathcal{V}\cap\mathcal{K} \neq \emptyset$, *then* $\limsup_{\varepsilon\to 0} c_\varepsilon = c_{\theta\kappa}$.

Proof. Let $w \in \mathcal{M}_{\theta\alpha}$ be such that

$$c_{\theta\alpha} = I_{\theta\alpha}(w) = \max_{t\ge 0} I_{\theta\alpha}(tw).$$

Then there exists a unique $t_\varepsilon > 0$ such that $t_\varepsilon w \in \mathcal{N}_\varepsilon$. Thus

$$c_\varepsilon \le I_\varepsilon(t_\varepsilon w) = \max_{t\ge 0} I_\varepsilon(tw). \tag{4.1}$$

Observe that

$$\begin{aligned} I_\varepsilon(t_\varepsilon w) &= I_{\theta\alpha}(t_\varepsilon w) + \frac{t_\varepsilon^p}{p}\int_{\mathbb{R}^N}(V_\varepsilon(x)-\theta)|w|^p dx \\ &\quad + \frac{t_\varepsilon^{2q}}{2q}\int_{\mathbb{R}^{2N}} \frac{(\alpha^2 - K_\varepsilon(y)K_\varepsilon(x))|w(y)|^q|w(x)|^q}{|x-y|^\mu}dxdy. \end{aligned} \tag{4.2}$$

Since $t_\varepsilon w \in \mathcal{N}_\varepsilon$, by the boundedness of $K(x)$, we get that there exist $T_2 > T_1 > 0$ such that $T_1 \le t_\varepsilon < T_2$. We may assume that $t_\varepsilon \to t_0$ as $\varepsilon \to 0$. Then by the boundedness of V, K, and the Lebesgue's theorem, we have

$$\frac{t_\varepsilon^p}{p}\int_{\mathbb{R}^N}(V_\varepsilon(x)-\theta)|w|^p dx = o_\varepsilon(1),$$

and

$$\frac{t_\varepsilon^{2q}}{2q}\int_{\mathbb{R}^{2N}}\frac{(\alpha^2 - K_\varepsilon(y)K_\varepsilon(x))|w(y)|^q|w(x)|^q}{|x-y|^\mu}dxdy = o_\varepsilon(1),$$

Thus, by (4.2) we obtain

$$I_\varepsilon(tw) = I_{\theta\alpha}(t_0 w) + o_\varepsilon(1).$$

It follows from (4.1) that

$$c_\varepsilon \;\;\leq\;\; I_{\theta\alpha}(t_0 w) + o_\varepsilon(1) \leq \max_{t\geq 0} I_{\theta\alpha}(tw) = I_{\theta\alpha}(w) = c_{\theta\alpha}.$$

The proof is completed. □

Proposition 1. *Assume that (H_1) and (H_2) hold. Then for any $\varepsilon > 0$ small enough, problem (2.1) has a positive ground state solution.*

Proof. Let $\{u_n\}$ denotes the (PS) sequence for I_ε given in (2.4). Recall that $\theta < V_\infty$ and $\alpha \geq K_\infty$. It follows from Lemma 2.2 that $c_{\theta\alpha} < c_\infty$. Then, by Lemmas 3.2 and 4.1, we obtain I_ε satisfies the $(PS)_{c_\varepsilon}$ condition for $\varepsilon > 0$ small enough. Hence, by mountain pass lemma we have problem (2.1) has a nontrivial ground state solution u_ε. We note that all the calculations above can be repeated by word by word, replacing I_ε^+ with the functional

$$I_\varepsilon^+(u) = \frac{1}{p}\|u\|_\varepsilon^p - \frac{1}{2q}\int_{\mathbb{R}^{2N}}\frac{K_\varepsilon(y)|u^+(y)|^q K_\varepsilon(x)|u^+(x)|^q}{|x-y|^\mu}dxdy.$$

Then we get a ground state solution u_ε of the equation

$$(-\Delta)_p^s u + V_\varepsilon(x)|u|^{p-2}u = \left(\int_{\mathbb{R}^N}\frac{K_\varepsilon(y)|u^+(y)|^q}{|x-y|^\mu}dy\right)K_\varepsilon(x)|u^+|^{q-2}u^+,\ x \in \mathbb{R}^N.$$

Taking u_ε^- as a test function in above equation, we have

$$\int_{\mathbb{R}^{2N}}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|^p}{|x-y|^{N+ps}}(u_\varepsilon(x)-u_\varepsilon(y))(u_\varepsilon^-(x)-u_\varepsilon^-(y))dxdy + \int_{\mathbb{R}^N}V_\varepsilon(x)|u_\varepsilon^-(x)|^p dx = 0. \tag{4.3}$$

For any $p \geq 1$, we have

$$|u_\varepsilon(x)-u_\varepsilon(y)|^{p-2}(u_\varepsilon(x)-u_\varepsilon(y))(u_\varepsilon^-(x)-u_\varepsilon^-(y)) \geq |u_\varepsilon^-(x)-u_\varepsilon^-(y)|^p. \tag{4.4}$$

Combining (4.3) with (4.4) yields

$$\|u_\varepsilon^-\|_\varepsilon^p = [u_\varepsilon^-]_{s,p}^p + \int_{\mathbb{R}^N}V_\varepsilon(x)|u_\varepsilon^-(x)|^p dx \leq 0.$$

Thus, we have $u_\varepsilon^-(x) \equiv 0$ and $u_\varepsilon \geq 0$. It follows from the maximum principle ([22]) that $u_\varepsilon > 0$ in $\mathbb{R}^N$. □

Lemma 4.2. *Let u_{ε_n} be a solution of (2.1) given in Proposition 1. Then, there exists a sequence $\{z_{\varepsilon_n}\} \subset \mathbb{R}^N$ with $\varepsilon_n z_{\varepsilon_n} \to z_0 \in \Omega_V$ such that $v_{\varepsilon_n} = u_{\varepsilon_n}(x+z_{\varepsilon_n})$ converges strongly in $W^{s,p}(\mathbb{R}^N)$ to a ground state solution of*

$$(-\Delta)_p^s u + V(z_0)|u|^{p-2}u = K^2(z_0)(\int_{\mathbb{R}^N} \frac{|u(y)|^q}{|x-y|^\mu}dy)|u|^{q-2}u,\ x \in \mathbb{R}^N.$$

In particular, if $\mathcal{V} \cap \mathcal{K} \neq \emptyset$, then $z_0 \in \mathcal{V} \cap \mathcal{K}$, and up to a subsequence, v_{ε_n} converges in $W^{s,p}(\mathbb{R}^N)$ to a ground state solution of

$$(-\Delta)_p^s u + \theta|u|^{p-2}u = \kappa^2(\int_{\mathbb{R}^N} \frac{|u(y)|^q}{|x-y|^\mu}dy)|u|^{q-2}u,\ x \in \mathbb{R}^N.$$

Proof. Let $\varepsilon_n \to 0$ as $n \to \infty$, $u_n := u_{\varepsilon_n} \in \mathcal{N}_{\varepsilon_n}$ be a solution of (2.1). Then $I_{\varepsilon_n}(u_n) = c_{\varepsilon_n}$ and $I'_{\varepsilon_n}(u_n) = 0$. It is easy to check that $\{u_n\}$ is bounded. Then, there exist $R^*, \beta > 0$ and a sequence $\{z_n\} \subset \mathbb{R}^N$ such that

$$\liminf_{n\to\infty} \int_{B_{R^*}(z_n)} |u_n|^p dx \geq \beta > 0. \tag{4.5}$$

Now we set $v_n = u_n(x + z_n)$. Then v_n is a solution of the following equation

$$(-\Delta)_p^s u + V_n(x)|u|^{p-2}u = (\int_{\mathbb{R}^N} \frac{K_n(y)|u(y)|^q}{|x-y|^\mu}dy)K_n(x)|u|^{q-2}u,\ x \in \mathbb{R}^N, \tag{4.6}$$

with the energy

$$\begin{aligned} J_{\varepsilon_n}(v_n) &= \frac{1}{p}([v_n]_{s,p}^p + \int_{\mathbb{R}^N} V_n(x)|v_n|^p dx) \\ &\quad -\frac{1}{2q}\int_{\mathbb{R}^{2N}} \frac{K_n(y)|v_n(y)|^q K_n(x)|v_n(x)|^q}{|x-y|^\mu}dxdy \\ &= I_{\varepsilon_n}(u_n) = c_{\varepsilon_n}, \end{aligned}$$

where $V_n(x) = V(\varepsilon_n x + \varepsilon_n z_n)$ and $K_n(x) = K(\varepsilon_n x + \varepsilon_n z_n)$. We see that $\{v_n\}$ is bounded, then there exists $v \in W^{s,p}(\mathbb{R}^N)$ satisfying, after passing to a subsequence if necessary

$$\begin{cases} v_n \rightharpoonup v, & \text{weakly in } W^{s,p}(\mathbb{R}^N), \\ v_n \to v, & \text{strongly in } L^r_{loc}(\mathbb{R}^N),\ p \leq r < p_s^*, \\ v_n \to v, & \text{a.e. in } \mathbb{R}^N. \end{cases} \tag{4.7}$$

It follows from (4.5) that $v \neq 0$.

We next show that $\{\varepsilon_n z_n\}$ is bounded. Assume by contradiction that $|\varepsilon_n z_n| \to \infty$ as $n \to \infty$. By the boundedness of V and K, we may assume that

$$V(\varepsilon_n z_n) \to V^\infty \quad \text{and} \quad K(\varepsilon_n z_n) \to K^\infty. \tag{4.8}$$

By the definition of V_∞ and K_∞, we have that

$$V^\infty \geq V_\infty > \theta, \quad \alpha \geq K^\infty. \tag{4.9}$$

Since V and K are uniformly continuous, by (4.8) one has

$$|V_n(x) - V^\infty| \le |V_n(x) - V(\varepsilon_n z_n)| + |V(\varepsilon_n z_n) - V^\infty| = o_n(1),$$

and

$$|K_n(x) - K^\infty| \le |K_n(x) - K(\varepsilon_n z_n)| + |K(\varepsilon_n z_n) - K^\infty| = o_n(1),$$

uniformly on bounded sets of $\mathbb{R}^N$. Then we have

$$V_n \to V^\infty \quad \text{and} \quad K_n \to K^\infty, \tag{4.10}$$

as $n \to \infty$ uniformly on bounded sets of $\mathbb{R}^N$. From (4.7) and (4.10), for each $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} V_n(x)|v_n|^{p-2} v_n \varphi dx = \int_{\mathbb{R}^N} V^\infty |v|^{p-2} v \varphi dx + o_n(1), \tag{4.11}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^{p-2}}{|x-y|^{N+ps}} (v_n(x) - v_n(y))(\varphi(x) - \varphi(y)) dxdy \\
&= \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^{p-2}}{|x-y|^{N+ps}} (v(x) - v(y))(\varphi(x) - \varphi(y)) dxdy + o_n(1).
\end{aligned} \tag{4.12}$$

Moreover, by (4.7), (4.10) and Hardy-Littlewood-Sobolev inequality, we infer that

$$\int_{\mathbb{R}^N} \frac{K_n(y)|v_n(y)|^q}{|x-y|^\mu} dy \rightharpoonup \int_{\mathbb{R}^N} \frac{K^\infty |v(y)|^q}{|x-y|^\mu} dy \ \text{ in } \ L^{\frac{2N}{\mu}}(\mathbb{R}^N),$$

and

$$K_n(x)|v_n|^q v_n \to K^\infty |v|^{q-2} v \ \text{ in } \ L^r(\mathbb{R}^N), \quad r \in [1, \frac{p_s^*}{q-1}).$$

Then, for each $\varphi \in C_0^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^{2N}} \frac{K_n(y)|v_n(y)|^q K_n(x)|v_n(x)|^{q-2} v_n \varphi}{|x-y|^\mu} dxdy = \int_{\mathbb{R}^{2N}} \frac{(K^\infty)^2 |v(y)|^q |v(x)|^{q-2} v \varphi}{|x-y|^\mu} dxdy + o_n(1). \tag{4.13}$$

Since v_n satisfies (4.6), by (4.11)–(4.13), for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}
0 &= \lim_{n\to\infty} \Big(\int_{\mathbb{R}^{2N}} \frac{|v_n(x) - v_n(y)|^{p-2}}{|x-y|^{N+ps}} (v_n(x) - v_n(y))(\varphi(x) - \varphi(y)) dxdy \\
&\quad + \int_{\mathbb{R}^N} V_n(x)|v_n|^{p-2} v_n \varphi dx - \int_{\mathbb{R}^{2N}} \frac{K_n(y)|v_n(y)|^q K_n(x)|v_n(x)|^{q-2} v_n \varphi}{|x-y|^\mu} dxdy \Big) \\
&= \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^{p-2}}{|x-y|^{N+ps}} (v(x) - v(y))(\varphi(x) - \varphi(y)) dxdy + \int_{\mathbb{R}^N} V^\infty |v|^{p-2} v \varphi dx \\
&\quad - \int_{\mathbb{R}^{2N}} \frac{(K^\infty)^2 |v(y)|^q |v(x)|^{q-2} v \varphi}{|x-y|^\mu} dxdy,
\end{aligned}$$

which implies v is a solution of

$$(-\Delta)_p^s u + V^\infty |u|^{p-2}u = (K^\infty)^2 (\int_{\mathbb{R}^N} \frac{|u(y)|^q}{|x-y|^\mu} dy)|u|^{q-2}u, \ x \in \mathbb{R}^N.$$

By (4.9) and Lemma 2.2, we have

$$I_{V^\infty K^\infty}(v) \geq c_{V^\infty K^\infty} > c_{\theta\alpha}. \tag{4.14}$$

Using Fatou's lemma and Lemma 4.1, we get

$$\begin{aligned} c_{V^\infty K^\infty} &\leq I_{V^\infty K^\infty}(v) = I_{V^\infty K^\infty}(v) - \frac{1}{2q}\langle I'_{V^\infty K^\infty}(v), v\rangle \\ &= (\frac{1}{p} - \frac{1}{2q})(\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V^\infty |v|^p dx) \\ &\leq \liminf_{n\to\infty} (\frac{1}{p} - \frac{1}{2q})(\int_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V_n(x)|v|^p dx) \\ &= \liminf_{n\to\infty} (J_{\varepsilon_n}(v_n) - \frac{1}{2q}\langle J'_{\varepsilon_n}(v_n), v_n\rangle) \\ &= \liminf_{n\to\infty} I_{\varepsilon_n}(u_n) = \liminf_{n\to\infty} c_{\varepsilon_n} \\ &\leq \limsup_{n\to\infty} c_{\varepsilon_n} \leq c_{\theta\alpha}, \end{aligned} \tag{4.15}$$

which contradicts to (4.14). Hence, $\{\varepsilon_n z_n\}$ is bounded.

Passing to a subsequence, still denoted by $\{\varepsilon_n z_n\}$, we may assume that $\varepsilon_n z_n \to z_0$ as $n \to \infty$. Then $V_n(x) \to V(z_0)$ and $K_n(x) \to K(z_0)$ as $n \to \infty$. Hence, v is a solution of

$$(-\Delta)_p^s u + V(z_0)|u|^{p-2}u = K^2(z_0)(\int_{\mathbb{R}^N} \frac{|u(y)|^q}{|x-y|^\mu} dy)|u|^{q-2}u, \ x \in \mathbb{R}^N. \tag{4.16}$$

Next, we claim that $z_0 \in \Omega_V$. Suppose by contradiction that $z_0 \notin \Omega_V$, then by (H_2) and Lemma 2.2, we have $c_{V(z_0)K(z_0)} > c_{\theta\alpha}$. It follows from Lemma 4.1 and the proof of (4.15) that

$$\limsup_{n\to\infty} c_{\varepsilon_n} \leq c_{\theta\alpha} < c_{V(z_0)K(z_0)} \leq \liminf_{n\to\infty} c_{\varepsilon_n},$$

which is absurd. Hence, $z_0 \in \Omega_V$, and then $\lim_{n\to\infty} dist(\varepsilon_n z_n, \Omega_V) = 0$. Moreover, v is a ground state solution of (4.16). In particular, if $\mathcal{V} \cap \mathcal{K} \neq \emptyset$, we see that $\lim_{n\to\infty} dist(\varepsilon_n z_n, \mathcal{V} \cap \mathcal{K}) = 0$ and v is a ground state solution of

$$(-\Delta)_p^s u + \theta|u|^{p-2}u = \kappa^2(\int_{\mathbb{R}^N} \frac{|u(y)|^q}{|x-y|^\mu} dy)|u|^{q-2}u, \ x \in \mathbb{R}^N. \tag{4.17}$$

Finally, we shall prove that $v_n \to v$ in $W^{s,p}(\mathbb{R}^N)$. Since v is a ground state solution of (4.16), by Fatou's lemma, we have

$$c_{V(z_0)K(z_0)} = I_{V(z_0)K(z_0)}(v) = I_{V(z_0)K(z_0)}(v) - \frac{1}{2q}\langle I'_{V(z_0)K(z_0)}(v), v\rangle$$

$$
\begin{aligned}
&= \left(\frac{1}{p}-\frac{1}{2q}\right)\left(\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}dxdy + \int_{\mathbb{R}^N} V(z_0)|v|^p dx\right) \\
&\leq \liminf_{n\to\infty}\left(\frac{1}{p}-\frac{1}{2q}\right)\left(\int_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+ps}}dxdy + \int_{\mathbb{R}^N} V_n(x)|v_n|^p dx\right) \\
&= \liminf_{n\to\infty}\left(J_{\varepsilon_n}(v_n) - \frac{1}{2q}\langle J'_{\varepsilon_n}(v_n), v_n\rangle\right) \\
&= \liminf_{n\to\infty} c_{\varepsilon_n} \leq \limsup_{n\to\infty} c_{\varepsilon_n} \leq c_{V(z_0)K(z_0)}.
\end{aligned}
$$

Then

$$
\lim_{n\to\infty}\left([v_n]_{s,p}^p + \int_{\mathbb{R}^N} V_n(x)|v_n|^p dx\right) = \int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}dxdy + \int_{\mathbb{R}^N} V(z_0)|v|^p dx.
$$

Thus by $V_n(x) \to V(z_0)$ and Brezis-Lieb lemma, we get $v_n \to v$ in $W^{s,p}(\mathbb{R}^N)$.
The proof is completed. □

Lemma 4.3. *$v_n \in L^\infty(\mathbb{R}^N)$ and there exists $C > 0$ such that $|v_n|_\infty \leq C$ for all n. Furthermore, $\lim_{|x|\to\infty} v_n(x) = 0$ uniformly in n, where v_n is given in Lemma 4.2.*

Proof. Given $T > 0$ and $\beta > 0$. For each n, we denote $v_{T,n} = v_n - (v_n - T)^+$ and $g(v_n) = v_n v_{T,n}^{p\beta} \in W^{s,p}(\mathbb{R}^N)$. Taking $g(v_n)$ as the test function in (4.6), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^{2N}} \frac{K_n(y)|v_n(y)|^q K_n(x)|v_n(x)|^{q-2}v_n g(v_n)}{|x-y|^\mu}dxdy \\
&= \int_{\mathbb{R}^{2N}} \frac{|v_n(x)-v_n(y)|^{p-2}}{|x-y|^{N+ps}}(v_n(x)-v_n(y))(g(v_n)(x)-g(v_n)(y))dxdy \\
&+ \int_{\mathbb{R}^N} V_n(x)|v_n|^{p-2}v_n g(v_n)dx. \qquad (4.18)
\end{aligned}
$$

By the boundedness of K and $\{v_n\}$, we have for each n,

$$
\int_{\mathbb{R}^N} \frac{K_n(y)|v_n(y)|^q}{|x-y|^\mu}dxdy \leq C.
$$

Therefore,

$$
\int_{\mathbb{R}^{2N}} \frac{K_n(y)|v_n(y)|^q K_n(x)|v_n(x)|^{q-2}v_n g(v_n)}{|x-y|^\mu}dxdy \leq C\int_{\mathbb{R}^N} |v_n|^q v_{T,n}^{p\beta}dx. \qquad (4.19)
$$

Set

$$
G(t) = \int_0^t (g'(\tau))^{\frac{1}{p}}d\tau.
$$

We have for any $a, b \in \mathbb{R}$,

$$
|G(a)-G(b)|^p \leq |a-b|^{p-2}(a-b)(g(a)-g(b)).
$$

It follows from (4.18) and (4.19) that

$$\int_{\mathbb{R}^{2N}} \frac{|G(v_n(x)) - G(v_n(y))|^p}{|x-y|^{N+ps}} dxdy \leq C \int_{\mathbb{R}^N} |v_n|^q v_{T,n}^{p\beta} dx. \tag{4.20}$$

By the definition of g, we can get $G(v_n) \geq \frac{1}{\beta+1} v_n v_{T,n}^{\beta}$. From (4.20) and the Sobolev inequality we obtain

$$\left(\int_{\mathbb{R}^N} |v_n v_{T,n}^{\beta}|^{p_s^*} dx \right)^{\frac{p}{p_s^*}} \leq C(\beta+1)^p \int_{\mathbb{R}^N} |v_n|^q v_{T,n}^{p\beta} dx. \tag{4.21}$$

Choosing $T_0 > 1$, by the definition of $v_{T,n}$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |v_n|^q v_{T,n}^{p\beta} dx &= \int_{\{v_n \leq T_0\}} |v_n|^q v_{T,n}^{p\beta} dx + \int_{\{v_n > T_0\}} |v_n|^q v_{T,n}^{p\beta} dx \\ &\leq T_0^{p\beta} \int_{\mathbb{R}^N} |v_n|^q dx + \int_{\{v_n > T_0\}} |v_n|^{p_s^*} v_{T,n}^{p\beta} dx \\ &\leq \left(\int_{\{v_n > T_0\}} |v_n|^{p_s^*} dx \right)^{\frac{p_s^*-p}{p_s^*}} \left(\int_{\mathbb{R}^N} |v_n^p v_{T,n}^{p\beta}|^{\frac{p_s^*}{p}} dx \right)^{\frac{p}{p_s^*}} \\ &\quad + T_0^{p\beta} \int_{\mathbb{R}^N} |v_n|^q dx. \end{aligned} \tag{4.22}$$

Since $v_n \to v$ in $W^{s,p}(\mathbb{R}^N)$, for T_0 large enough, we conclude that

$$\left(\int_{\{v_n > T_0\}} |v_n|^{p_s^*} dx \right)^{\frac{p_s^*-p}{p_s^*}} \leq \frac{1}{2C(\beta+1)^p}. \tag{4.23}$$

By (4.22), (4.23) and the boundedness of $\{v_n\}$, we get

$$\int_{\mathbb{R}^N} |v_n|^q v_{T,n}^{p\beta} dx \leq C T_0^{p\beta} \int_{\mathbb{R}^N} |v_n|^q dx \leq C.$$

Letting $T \to \infty$, by Fatou's lemma, we conclude that $v_n \in L^{q+p\beta}(\mathbb{R}^N)$, and (4.21) implies

$$\left(\int_{\mathbb{R}^N} |v_n|^{(\beta+1)p_s^*} dx \right)^{\frac{p}{p_s^*}} \leq C \int_{\mathbb{R}^N} |v_n|^{q+p\beta} dx.$$

Now, from an iterative procedure, in a finite number of steps, we can show that $v_n \in L^\infty(\mathbb{R}^N)$ and there exists $C > 0$ such that $\|v_n\|_\infty \leq C$. Moreover, by $v_n \to v$ in $W^{s,p}(\mathbb{R}^N)$, we infer that $\lim_{|x|\to\infty} v_n(x) = 0$ uniformly in n.
The proof is completed. □

Proof of Theorem 1.1. From proposition 1, problem (2.1) has a positive ground state solution u_ε. Then the function $w_\varepsilon(x) = u_\varepsilon(\frac{x}{\varepsilon})$ is a positive ground state solution of (1.1). Now we show the concentration of the maximum points. Let u_{ε_n} be a solution of (2.1). By Lemma 4.2, we have that $v_{\varepsilon_n} = u_{\varepsilon_n}(x + z_{\varepsilon_n})$ is a solution of the problem (4.6). Furthermore, $v_{\varepsilon_n} \to v$ in $W^{s,p}(\mathbb{R}^N)$ and $\varepsilon_n z_{\varepsilon_n} \to z_0 \in \Omega_V$. Furthermore, from Lemma 4.3 we have $v_n \in L^\infty(\mathbb{R}^N)$ for all n. It follows from (4.5) that

$$\beta \leq \int_{B_{R^*}(0)} |v_{\varepsilon_n}|^p dx \leq |B_{R^*}(0)| |v_{\varepsilon_n}|_\infty^p.$$

This implies there exist $\iota > 0$ such that

$$\|v_{\varepsilon_n}\|_\infty \geq \iota > 0. \tag{4.24}$$

Set p_n be a global maximum of v_n, by Lemma 4.3 and (4.24), we have that $p_n \in B_R(0)$ for some $R > 0$. Hence, the global maximum of u_{ε_n} given by $y_{\varepsilon_n} = p_n + z_{\varepsilon_n}$. Since $\{p_n\}$ is bounded and $\varepsilon_n z_{\varepsilon_n} \to z_0$, we have $\varepsilon_n y_{\varepsilon_n} \to z_0$, thus the continuity of V and K gives $\lim_{n\to\infty} V(\varepsilon_n z_{\varepsilon_n}) = V(z_0)$ and $\lim_{n\to\infty} K(\varepsilon_n z_{\varepsilon_n}) = K(z_0)$.
The proof is completed. □

5. Proof of Theorem 1.2

Let $\delta > 0$ be fixed. Define a smooth cut-off function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\eta(t) = 1$ if $0 \leq t \leq \frac{\delta}{2}$, $\eta(t) = 0$ if $t \geq \delta$. For any $\xi \in \Lambda$, define

$$W_{\varepsilon,\xi}(x) = \eta(|\varepsilon x - \xi|)\omega(\frac{\varepsilon x - \xi}{\varepsilon}),$$

where $\omega(x)$ is the positive ground state solution of (4.17). By definition, $W_{\varepsilon,\xi}$ has compact support for any $\xi \in \Lambda$, and hence it belongs to E_ε. It's easy to check that there exists a unique $t_\varepsilon > 0$ such that $t_\varepsilon W_{\varepsilon,\xi} \in \mathcal{N}_\varepsilon$. Now, we define the map $\phi_\varepsilon : N \to \mathcal{N}_\varepsilon$ by $\phi_\varepsilon(\xi) = t_\varepsilon W_{\varepsilon,\xi}$.

Lemma 5.1. *Uniformly in $\xi \in \Lambda$, we have*

$$\lim_{\varepsilon\to 0} I_\varepsilon(\phi_\varepsilon(\xi)) = c_{\theta\kappa}.$$

Proof. Let $\xi \in \Lambda$. By the definition of $\phi_\varepsilon(\xi)$ and a simple change of variable, one has

$$\begin{aligned}&\int_{\mathbb{R}^{2N}} \frac{|\eta(\varepsilon|x|)\omega(x) - \eta(\varepsilon|y|)\omega(y)|^p}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V(\varepsilon x + \xi)|\eta(\varepsilon|x|)\omega|^p dx \\ &= t_\varepsilon^{2q-p} \int_{\mathbb{R}^{2N}} \frac{K(\varepsilon y + \xi)|\eta(\varepsilon|y|)\omega(y)|^q K(\varepsilon x + \xi)|\eta(\varepsilon|x|)\omega(x)|^q}{|x-y|^\mu} dxdy.\end{aligned} \tag{5.1}$$

By Lemma 2.2 of [2] and Lebesgue's theorem, we get

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^{2N}} \frac{|\eta(\varepsilon|x|)\omega(x) - \eta(\varepsilon|y|)\omega(y)|^p}{|x-y|^{N+ps}} dxdy = \int_{\mathbb{R}^{2N}} \frac{|\omega(x) - \omega(y)|^p}{|x-y|^{N+ps}} dxdy, \tag{5.2}$$

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} V(\varepsilon x + \xi)|\eta(\varepsilon|x|)\omega|^p dx = \theta \int_{\mathbb{R}^N} |\omega|^p dx, \tag{5.3}$$

and

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^{2N}} \frac{K(\varepsilon y + \xi)|\eta(\varepsilon|y|)\omega(y)|^q K(\varepsilon x + \xi)|\eta(\varepsilon|x|)\omega(x)|^q}{|x-y|^\mu} dxdy = \kappa^2 \int_{\mathbb{R}^{2N}} \frac{|\omega(y)|^q |\omega(x)|^q}{|x-y|^\mu} dxdy. \tag{5.4}$$

On the other hand, since ω is a ground state solution of (4.17), we have

$$\int_{\mathbb{R}^{2N}} \frac{|\omega(x) - \omega(y)|^p}{|x-y|^{N+ps}} dxdy + \theta \int_{\mathbb{R}^N} |\omega|^p dx = \kappa^2 \int_{\mathbb{R}^{2N}} \frac{|\omega(y)|^q |\omega(x)|^q}{|x-y|^\mu} dxdy. \tag{5.5}$$

Then, by (5.1)–(5.5), we deduce that $\lim_{\varepsilon\to 0} t_\varepsilon = 1$. At this point, by the same change of variable as before and (5.2), (5.3), we have

$$\begin{aligned} I_\varepsilon(\phi_\varepsilon(\xi)) &= (\frac{1}{p}-\frac{1}{2q})t_\varepsilon^p(\int_{\mathbb{R}^{2N}} \frac{|\eta(\varepsilon|x|)\omega(x)-\eta(\varepsilon|y|)\omega(y)|^p}{|x-y|^{N+ps}}dxdy \\ &\quad + \int_{\mathbb{R}^N} V(\varepsilon x+\xi)|\eta(\varepsilon|x|)\omega|^p dx) \\ &= (\frac{1}{p}-\frac{1}{2q})(\int_{\mathbb{R}^{2N}} \frac{|\omega(x)-\omega(y)|^p}{|x-y|^{N+ps}}dxdy + \theta\int_{\mathbb{R}^N}|\omega|^p dx) + o_\varepsilon(1) \\ &= c_{\theta\kappa} + o_\varepsilon(1). \end{aligned}$$

Moreover, the limit is uniformly in ξ.
The proof is completed. □

Let $R>0$ be such that $\Lambda_\delta \subset B_R(0)$. Define $\chi : \mathbb{R}^N \to \mathbb{R}^N$ by $\chi(x) = x$ for $x \in B_R(0)$ and $\chi(x) = \frac{Rx}{|x|}$ for $x \in \mathbb{R}^N \backslash B_R(0)$. Now we define $\rho_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ by

$$\rho_\varepsilon(u) = \frac{\int_{\mathbb{R}^N}\chi(\varepsilon x)|u|^p dx}{\int_{\mathbb{R}^N}|u|^p dx}.$$

By the definition of χ and the Lebesgue's theorm, we have that

$$\begin{aligned} \rho_\varepsilon(\phi_\varepsilon(\xi)) &= \frac{\int_{\mathbb{R}^N}\chi(\varepsilon x)|\eta(|\varepsilon x-\xi|)\omega(\frac{\varepsilon x-\xi}{\varepsilon})|^p dx}{\int_{\mathbb{R}^N}|\eta(|\varepsilon x-\xi|)\omega(\frac{\varepsilon x-\xi}{\varepsilon})|^p dx} \\ &= \xi + \frac{\int_{\mathbb{R}^N}(\chi(\varepsilon x+\xi)-\xi)|\eta(|\varepsilon x|)\omega(x)|^p dx}{\int_{\mathbb{R}^N}|\eta(|\varepsilon x|)\omega(x)|^p dx} \\ &= \xi + o_\varepsilon(1), \end{aligned} \tag{5.6}$$

uniformly for $\xi \in \Lambda_\delta$.

Let $f(\varepsilon)$ be any positive function tending to 0 as $\varepsilon \to 0$. Set

$$\overline{\mathcal{N}}_\varepsilon = \{u \in \mathcal{N}_\varepsilon : I_\varepsilon(u) \le c_{\theta\kappa} + f(\varepsilon)\}.$$

By Lemma 5.1, we see that $\overline{\mathcal{N}}_\varepsilon \neq \emptyset$ for $\varepsilon > 0$ small enough.

Lemma 5.2.

$$\lim_{\varepsilon\to 0}\sup_{u\in\overline{\mathcal{N}}_\varepsilon}\inf_{\xi\in\Lambda_\delta}|\rho_\varepsilon(u)-\xi| = 0.$$

Proof. Let $\varepsilon_n \to 0$ as $n\to\infty$, by the definition, there exists a sequence $\{u_n\} \subset \overline{\mathcal{N}}_{\varepsilon_n}$ such that

$$\inf_{\xi\in\Lambda_\delta}|\rho_{\varepsilon_n}(u_n)-\xi| = \sup_{u\in\overline{\mathcal{N}}_\varepsilon}\inf_{\xi\in\Lambda_\delta}|\rho_\varepsilon(u)-\xi| + o_n(1).$$

So it suffices to find a sequence $\{\xi_n\} \subset \Lambda_\delta$ satisfying

$$|\rho_{\varepsilon_n}(u_n)-\xi_n| = o_n(1). \tag{5.7}$$

Since $\{u_n\} \subset \overline{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, by Lemma 2.2 we have that

$$c_{\theta\kappa} \leq c_{\varepsilon_n} \leq I_{\varepsilon_n}(u_n) \leq c_{\theta\kappa} + f(\varepsilon_n),$$

this implies $I_{\varepsilon_n}(u_n) \to c_{\theta\kappa}$ as $n \to \infty$. By the proof of Lemma 4.2, we obtain that there exists a sequence $\{\overline{\xi_n}\} \subset \mathbb{R}^N$ such that $\varepsilon_n\overline{\xi_n} \to \overline{z_0} \in \Lambda$ and $v_n(x) = u_n(x+\overline{\xi_n})$ converges strongly in $W^{s,p}(\mathbb{R}^N)$ to v, a positive ground state of (4.17). Set $\xi_n = \varepsilon_n\overline{\xi_n}$. Then by Lebesgue's theorem,

$$\begin{aligned}\rho_{\varepsilon_n}(u_n) &= \frac{\int_{\mathbb{R}^N} \chi(\varepsilon_n x + \xi_n)|v_n(x)|^p dx}{\int_{\mathbb{R}^N} |v_n(x)|^p dx} \\ &= \xi_n + \frac{\int_{\mathbb{R}^N} (\chi(\varepsilon_n x + \xi_n) - \xi_n)|v_n(x)|^p dx}{\int_{\mathbb{R}^N} |v_n(x)|^p dx} \\ &= \xi_n + o_n(1).\end{aligned}$$

Thus (5.7) holds.
The proof is completed. □

Proof of Theorem 1.2. For a fixed $\delta > 0$, by Lemma 5.1 there exists $\varepsilon_\delta > 0$ such that $I_\varepsilon(\phi_\varepsilon(\xi)) \leq c_{\theta\kappa} + f(\varepsilon)$ for any $\varepsilon \in (0, \varepsilon_\delta)$ and $\xi \in \Lambda$. Using Lemma 5.2, we have $dist(\rho_\varepsilon(u), \Lambda_\delta) < \frac{\delta}{2}$ for such ε and $u \in \overline{\mathcal{N}}_\varepsilon$. It follows that the map $\rho_\varepsilon \circ \phi_\varepsilon : \Lambda \to \Lambda_\delta$ is well defined. Then, by (5.6) the map $\rho_\varepsilon \circ \phi_\varepsilon$ is homotopic to the inclusion map: $Id : \Lambda \to \Lambda_\delta$. Applying homotopic and by the same arguments of [6], we obtain that $cat_{\overline{\mathcal{N}_\varepsilon}}(\overline{\mathcal{N}_\varepsilon}) \geq cat_{\Lambda_\delta}(\Lambda)$. By the definition of $\overline{\mathcal{N}_\varepsilon}$ and choosing ε_δ small, from Lemma 3.3, we have that I_ε satisfies the (PS) condition in $\overline{\mathcal{N}_\varepsilon}$. Therefore, standard Ljusternik-Schnirelmann theory implies that I_ε has at least $cat_{\overline{\mathcal{N}_\varepsilon}}(\overline{\mathcal{N}_\varepsilon})$ critical points on $\mathcal{N}_\varepsilon$. It is easy to check that I_ε has at least $cat_{\Lambda_\delta}(\Lambda)$ critical points in E_ε. The concentration behavior of these solutions as $\varepsilon \to 0$ are similar as in the proof of Theorem 1.1. □

Acknowledgments

The authors would like to express sincere thanks to the anonymous referees for their carefully reading this paper and valuable useful comments. This work was supported by NSFC (No.11601234, 11571176), Natural Science Foundation of Jiangsu Province of China for Young Scholar (No.BK20160571).

Conflict of interest

The author declares no conflicts of interest in this paper.

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