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*Research article*

# Harmonic and subharmonic solutions of quadratic Liénard type systems with sublinearity

**Chunmei Song**[1], **Qihuai Liu**[1,2,*] **and Guirong Jiang**[1,2]

[1] School of Mathematics and Computing Science, Guangxi Colleges and Universities Key Laboratory of Data Analysis and Computation, Guilin University of Electronic Technology, Guilin 541004, China

[2] Center for Applied Mathematics of Guangxi (Guangxi Normal University), Guilin 541001, China

* **Correspondence:** Email: qhuailiu@gmail.com.

**Abstract:** In this paper, we prove the existence of harmonic solutions and infinitely many subharmonic solutions of dissipative second order sublinear differential equations named quadratic Liénard type systems. The method of the proof is based on the Poincaré-Birkhoff twist theorem.

**Keywords:** Poincaré-Birkhoff twist theorem; sublinear; harmonic solutions; subharmonic solutions
**Mathematics Subject Classification:** 34A12, 34C25

## 1. Introduction

Generally speaking, the equations of dissipative systems usually contain the first derivative term, which reflects the energy loss of the system. Second order dissipative differential equations arise widely in various research fields such as celestial mechanics, fluid mechanics [1, 2], relativistic mechanics [3, 4], engineering [5, 6].

The purpose of this paper is to study a class of special dissipative second order differential equations, named a quadratic Liénard type systems in [7],

$$\ddot{x} + f(x)(\dot{x})^2 + g(x) = p(t), \tag{1.1}$$

where $p(t)$ is a continuous periodic function (let $T > 0$ be its minimum period), and $f(x), g(x)$ are local Lipschitz continuous functions. Equation (1.1) models one-dimensional oscillator studied at the classical and also at the quantum level [8].

Equation (1.1) is a special case of the form

$$\ddot{u} + f(u)(\dot{u})^2 = \tilde{g}(t, u),$$

which arises from nonlinear elastic mechanics [9]. Take $F(u)$ such that $F'(u) = f(u)$, then multiplying by $e^{F(u)}$, the latter equation is written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{F(u)}\dot{u}\right] = \mathrm{e}^{F(u)}\tilde{g}(t,u),$$

which in turn is just

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}[\Psi(u)] = \mathrm{e}^{F(u)}\tilde{g}(t,u),$$

where $\Psi'(u) = \mathrm{e}^{F(u)}$. Now, the change $x = \Psi(u)$ leads to the desired newtonian formulation $\ddot{x} = \tilde{f}(t,x)$. For the nonlinear vibrations of a radially forced thick-walled hollow sphere made of an elastic, homogeneous, isotropic, and incompressible material, Del Pino and Manásevich [10] have shown that $\tilde{f}$ has the strong singularity at the origin and the superlinearity at infinite by an asymptotic analysis. Then by Poincaré-Birkhoff twist theorem, they have prove the existence of infinitely many periodic solutions. Another concrete example of this equation is the Rayleigh-Plesset equation

$$\rho x\ddot{x} + \frac{3}{2}(\dot{x})^2 = p_B(t) - p_\infty(t).$$

We can refer to [11, 12] for the related development.

When Eq (1.1) has no dissipative term, it is just a Duffing equation

$$\ddot{x} + g(x) = p(t).$$

Under various conditions such as superlinearity, sublinearity and semilinearity, the existence and multiplicity of periodic solutions has been established, see [13–15] for instance. We can refer to [16–19] for more research on the problem of periodic solutions. When the system has nonconvex potentials or periodic nonlinearities, the existence of subharmonic solutions has been established in [20,21], where the authors exploited different techniques by using critical point theory. Moreover, with a similar approach based on the Poincaré-Birkhoff fixed point theorem, the existence of subharmonic solutions for different second-order differential systems has been widely studied by Zanolin and his collaborators, see [22–25].

When the forced term $p$ vanishes, Eq (1.1) is an autonomous equation

$$\ddot{x} + f(x)(\dot{x})^2 + g(x) = 0. \tag{1.2}$$

Owing to the second term $f(x)(\dot{x})^2$ of (1.1), equation is not conservative in usual phase space $(x, \dot{x})$. However, in the generalized coordinates $(x, p)$, Eq (1.1) can be transformed into a Hamiltonian system whose energy is conserved; see [26] for details.

In case of

$$f(x) = \frac{\lambda x}{1 - \lambda x^2}, \quad g(x) = \frac{\alpha x^2}{1 - \lambda x^2},$$

Eq (1.2) is a one-dimensional Mathews-Lakshmanan (ML) oscillator with real constant parameters $\lambda, \alpha$ (see [27, 28]). The ML oscillator can be regarded as the zero-dimensional version of a scalar nonpolynomial field equation or as a velocity dependent potential oscillator. The kinetic term in its Hamiltonian function features a position-dependent mass term that produces a variable "spring constant" of the oscillator. As a consequence, the classical Euler-Lagrange equation associated with the model admits simple, sinusoidal solutions [29]. The ML oscillator exhibits simple harmonic periodic solutions but with amplitude dependent frequency,

$$x(t) = A\cos(\Omega t + \delta), \quad \Omega = \frac{\alpha}{\sqrt{1 - \lambda A^2}},$$

where $A$ is the amplitude and $\delta$ is the initial phase. We can refer to [28] for details.

When the functions $f(x)$ and $g(x)$ are of class $C^1$, a sufficient condition for the monotonicity of the period $T$ or for the isochronicity of the origin $O$ has been established for Eq (1.2) in [30] by Sabatini. In the analytic case of $f(x)$ and $g(x)$, Chouikha has given a necessary and sufficient condition for the isochronicity of the origin $O$; see [31]. We can refer to [32, 33] for the development. Moreover, a complete classification of the Lie point symmetry groups is given in [7] for Eq (1.2).

Recently, Atslega [34] provides some conditions on the functions $f(x)$ and $g(x)$ which ensure the existence of solutions of Eq (1.2) with the Neumann boundary conditions

$$\dot{x}(0) = 0, \quad \dot{x}(1) = 0.$$

A natural problem is, if the periodic external force $p(t)$ is added, how about the existence and multiplicity of solutions for Eq (1.1) under the periodic boundary value condition

$$x(0) = x(mT), \quad \dot{x}(0) = \dot{x}(mT).$$

Comparing with the case considered by Del Pino and Manásevich [10], we do not give any assumption of singularity.

In this paper, we consider the sublinear case, and need the following conditions:

($H_1$) $g(x)$ satisfies the sublinear condition

$$\lim_{|x|\to\infty} \frac{g(x)}{x} = 0.$$

($H_2$) $F(x)$ and $xf(x)$ are bounded on $(-\infty, +\infty)$, where $F(x)$ is a prime function of $f(x)$;
($H_3$) for all $x \neq 0$, there is $xf(x) < 0$.
($H_4$) $\lim_{x\to+\infty} g(x) = +\infty, \lim_{x\to-\infty} g(x) = -\infty.$

We remark that condition ($H_4$) implies the sign condition of $g$, that is, there exist $d > 0$ such that $xg(x) > 0$, for all $|x| > d$. Without loss of generality, in view of assumption $(H_2)$ we assume that $0 \leq F(x) \leq a$ with a positive constant $a > 0$. Moreover, in condition $(H_2)$, if $f(x)$ is a monotone function, then the boundedness of $xf(x)$ holds naturally. In fact, the boundedness of $F$ implies that infinite integral

$$\int_0^{+\infty} f(x)\mathrm{d}x, \quad \int_{-\infty}^{0} f(x)\mathrm{d}x$$

are all convergent, and the monotonicity of $f$ yields the limit $\lim_{x\to\infty} xf(x) = 0$.

The condition $(H_3)$ is only a sign condition of the functions $f(x)$ and $g(x)$. A simple example which satisfies all conditions above is given in the following

$$\ddot{x} - \frac{x}{1 + x^4}(\dot{x})^2 + x^{1/3} = \sin t.$$

In order to state our main results, we recall some definitions form [35]. Assume $x(t)$ is a periodic solution of Eq (1.1) with its minimum positive period $T_0$. If $T_0 = T$, we call $x(t)$ is a harmonic solution; if $T_0 = mT$ with some positive integer $m \geq 2$, we call $x(t)$ is a $m$-order subharmonic solution.

Now we give our main results as follows.

**Theorem 1.1.** *Assume that conditions* $(H_1)$–$(H_4)$ *hold, then there exists a sufficiently large positive integer* $m_0$ *such that, for each positive integer* $m \geq m_0$*, Eq (1.1) possesses at least two distinct m-order subharmonic solution.*

**Theorem 1.2.** *Assume that conditions* $(H_1)$–$(H_4)$ *hold, then Eq (1.1) has at least one harmonic solution.*

Theorem 1.1 implies that Eq (1.1) possesses infinitely many subharmonic solutions. The proof of Theorem 1.1 is based on the Poincaré-Birkhoff twist theorem, and the main difficulty is how to verify the twist property of the Poincaré map corresponding to Eq (1.1). Compared with the superlinear case [36], when $g(x)$ satisfies the sublinear condition, the rotation speed of the solutions of (1.2) around the origin is very slow outside a sufficiently large disk.

The paper is organized as follows. Preliminaries and some lemmas which are useful for proving our theorems are stated in Section 2. We will prove Theorem 1.1 and 1.2 in Section 3.

## 2. Preliminaries and some lemmas

Firstly, Eq (1.1) is a special case of Benoulli type equation with $n = -1$ and $p(t) \equiv 0$ (see [26]), whose characteristic equation is

$$\frac{\mathrm{d}\dot{x}}{\mathrm{d}x} = -f(x)\dot{x} - \frac{g(x)}{\dot{x}},$$

where we regard $t$ as an independent parameter variable. Then we have the first integral of (1.2)

$$I(x, \dot{x}) = (\dot{x})^2 \mathrm{e}^{2F(x)} + 2\int_0^x [g(s)]\mathrm{e}^{2F(s)}\mathrm{d}s.$$

In the following, we introduce the generalized coordinates and the generalized momentum. The generalized coordinates $x$ is used to describe a conservative mechanical system whose configuration is completely specified by the value of a certain single variable $x$. The variable need not represent a displacement; it might, for example, be an angle, or even the reading on a dial forming part of the system. The generalized momentum $y$ is a function of both the generalized velocity and generalized coordinates; see [37, p.31 and p.374] or [26].

Let

$$y = 2\dot{x}\mathrm{e}^{2F(x)} + \varphi(x),$$

then the Hamiltonian function corresponding to (1.1) is given by

$$H(x, y) = \frac{1}{4}\mathrm{e}^{-2F(x)}y^2 + 2\int_0^x g(s)\mathrm{e}^{2F(s)}\mathrm{d}s - 2\int_0^x \mathrm{e}^{2F(s)}p(t)\mathrm{d}s.$$

Here, we take $\varphi(x) \equiv 0$. For simplification, we rewrite the Hamiltonian function in the following form

$$H(t, x, y) = \frac{1}{4}\mathrm{e}^{-2F(x)}y^2 + V(x) + P(t, x), \tag{2.1}$$

where

$$V(x) = \int_0^x 2\mathrm{e}^{2F(s)} g(s)\mathrm{d}s, \quad P(t,x) = -\int_0^x 2\mathrm{e}^{2F(s)} p(t)\mathrm{d}s.$$

The corresponding Hamiltonian system is defined by

$$\begin{cases} \dot{x} = \frac{1}{2}\mathrm{e}^{-2F(x)}y, \\ \dot{y} = \frac{1}{2}\mathrm{e}^{-2F(x)}f(x)y^2 - 2\mathrm{e}^{2F(x)}\left(g(x) - p(t)\right). \end{cases} \tag{2.2}$$

**Lemma 2.1.** *The existence of periodic solutions of Eq (1.1) is equivalent to the existence of periodic solutions of Hamiltonian system (2.2).*

*Proof.* Equation (1.1) is equivalent to the following plane differential system

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -f(x)v^2 - g(x) + p(t). \end{cases} \tag{2.3}$$

By introducing a global differential homeomorphism

$$x = x, \quad y = 2\mathrm{e}^{2F(x)}v,$$

system (1.1) is transformed into (2.2). Thus, we complete the proof. □

**Lemma 2.2.** *Assume that conditions $(H_2)$–$(H_4)$ hold, then the solutions of Eq (2.2) are well defined on $(-\infty, +\infty)$.*

*Proof.* Assume that $(x(t; x_0, y_0), y(t; x_0, y_0))$ is a solution of Eq (2.2) with the initial value condition $(x(t_0), y(t_0)) = (x_0, y_0)$, which is defined on the maximum existence interval $(\alpha, \beta)$. Let

$$\mathcal{V}(t) = \frac{1}{4}e^{-2F(x(t;x_0,y_0))}y^2(t; x_0, y_0) + V(x(t; x_0, y_0)),$$

then we have

$$\frac{\mathrm{d}\mathcal{V}(t)}{\mathrm{d}t} = -\frac{\partial P(t,x)}{\partial x}\dot{x} = 2e^{2F(x)}p(t)(\frac{1}{2}e^{-2F(x)}y) = yp(t).$$

From the conditions $(H_2)$ and $(H_3)$, we have

$$\begin{aligned} \left|\frac{\mathrm{d}\mathcal{V}(t)}{\mathrm{d}t}\right| &= |yp(t)| \le \frac{1}{2}y^2(t) + |p|_\infty^2 \le \frac{1}{2}e^{2F(a)-2F(x(t))}y^2(t) + |p|_\infty^2 \\ &\le C_1\left[\frac{1}{4}e^{-2F(x(t))}y^2(t) + V(x(t))\right] + C_2 \\ &\le C_1\mathcal{V}(t) + C_2, \end{aligned} \tag{2.4}$$

where $C_1, C_2$ are positive constants, since $V(x) \to +\infty$ as $x \to \infty$.

If $\beta < +\infty$, by the continuation of solutions, we have

$$\lim_{t\to\beta^-}\left(|x(t)| + |y(t)|\right) = +\infty.$$

In view of condition $(H_4)$, it follows that $\lim\limits_{t\to\beta^-}\mathcal{V}(t)=+\infty$. By using the Grownwall inequality for (2.4), for all $t\in[t_0,\beta)$, we know that

$$\mathcal{V}(t)\le -\frac{C_2}{C_1}+e^{C_1(t-t_0)}\mathcal{V}(t_0)+\frac{C_2}{C_1}e^{C_1(t-t_0)},$$

which implies $\mathcal{V}(t)$ is bounded on $[t_0,\beta)$ since $\beta$ is finite. Therefore, we obtain a contradiction and $\beta=+\infty$.

If $\alpha>-\infty$, with the same argument we obtain

$$\mathcal{V}(t)\le -\frac{C_2}{C_1}+e^{C_1(t_0-t)}\mathcal{V}(t_0)+\frac{C_2}{C_1}e^{C_1(t_0-t)},$$

which also is a contradiction.

Thus the proof of the lemma is completed. □

Let $(x(t;x_0,y_0),y(t;x_0,y_0))$ be the solution of Eq (2.2) with the initial value $(x(t_0;x_0,y_0),y(t_0;x_0,y_0))=(x_0,y_0)$. Let

$$x_1=x(T;x_0,y_0),\quad y_1=y(T;x_0,y_0).$$

By Lemma 2.2, the Poincaré mapping of (2.2)

$$\mathcal{P}:(x_0,y_0)\mapsto(x_1,y_1)$$

is well defined. Moreover, it is a area-preserving smooth homeomorphism in the phase plane since (2.2) is a Hamiltonian system.

With the same argument as [38, 39], we have the elastic property for system (2.2).

**Lemma 2.3.** *For any constants $d>0$ and $L>0$, there is a sufficiently large constant $c>d$, such that the solution $(x(t;x_0,y_0),y(t;x_0,y_0))$ of system (2.2) satisfies the inequality*

$$x^2(t;x_0,y_0)+y^2(t;x_0,y_0)\ge d^2,\quad t\in[t_0,t_0+L], \tag{2.5}$$

*if the initial value $(x_0,y_0)$ satisfies the condition*

$$x_0^2+y_0^2\ge c^2.$$

If the solution does not pass through the origin, we can use the polar coordinates to represent the solution. Now we transform system (2.2) into the polar coordinates, that is,

$$\begin{cases}\dot r=\dfrac{1}{2}r\sin\theta\cos\theta\mathrm{e}^{-2F(r\cos\theta)}+\dfrac{1}{2}r^2\sin^3\theta\mathrm{e}^{-2F(r\cos\theta)}f(r\cos\theta)\\ \qquad -2\sin\theta\mathrm{e}^{2F(r\cos\theta)}(g(r\cos\theta)-p(t)),\\ \dot\theta=\dfrac{1}{2}r\sin^2\theta\cos\theta e^{-2F(r\cos\theta)}f(r\cos\theta)\\ \qquad -\dfrac{2\cos\theta}{r}\mathrm{e}^{2F(r\cos\theta)}(g(r\cos\theta)-p(t))-\dfrac{1}{2}\sin^2\theta\mathrm{e}^{-2F(r\cos\theta)}.\end{cases} \tag{2.6}$$

From Lemma 2.3, if $r_0 > c$, the solution can be written in the form of polar coordinates

$$r(t) = r(t; r_0, \theta_0) > d, \quad \theta(t) = \theta(t; r_0, \theta_0), \quad t \in [0, T], \tag{2.7}$$

which satisfies the initial value $r(0) = r_0, \theta(0) = \theta_0$. Furthermore, we have

$$r(t; r_0, \theta_0 + 2\pi) = r(t; r_0, \theta_0), \quad \theta(t; r_0, \theta_0 + 2\pi) = \theta(t; r_0, \theta_0) + 2\pi.$$

Then the function $r(t; r_0, \theta_0)$ is the period of $2\pi$ for $\theta_0$, and the function $\theta(t; r_0, \theta_0)$ is $2\pi$-appreciation with respect to $\theta_0$.

When $r_0 > c$, we can get the Poincaré mapping $\mathcal{P}$ in the polar coordinate form

$$r_1 = r_0 + R(r_0, \theta_0), \quad \theta_1 = \theta_0 + \Theta(r_0, \theta_0),$$

where

$$R(r_0, \theta_0) = r(T; r_0, \theta_0) - r_0, \quad \Theta(r_0, \theta_0) = \theta(T; r_0, \theta_0) - \theta_0.$$

**Lemma 2.4.** *Assume that conditions* $(H_3)$ *and* $(H_4)$ *hold, then there exists a sufficiently large disc*

$$D_d = \{(x, y) |\, x^2 + y^2 \le d^2\}$$

*such that the trajectory of (2.6) outside the disc* $D_d$ *rotates clockwise around the origin* $O$.

*Proof.* By the second equality of (2.6), we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{2}\sin^2\theta e^{-2F(x)}(xf(x) - 1) - 2\cos^2\theta e^{2F(x)}\frac{(g(x) - p(t))}{x}.$$

From conditions $(H_3)$ and $(H_4)$, there exists a positive constant $N$ such that

$$\frac{g(x) - p(t)}{x} > 0, \quad |x| \ge N.$$

Moreover, from $(H_3)$ we have $xf(x) < 0$, for all $|x| \ge N$. Therefore, when $|x| \ge N$, we have

$$\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} < 0.$$

When $|x| \le N$, we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{2}\sin^2\theta e^{-2F(x)}(xf(x) - 1) - 2e^{2F(x)}\frac{(g(x) - p(t))}{x^2 + y^2}x.$$

Notice that if $d$ is large enough, for $|x| \le N$, we have $\theta \in [\pi/4, 3\pi/4] \cup [5\pi/4, 7\pi/4]$ so that $\sin^2\theta$ is uniformly positive. On the other hand, for any $\varepsilon > 0$, we can take a sufficiently large $d$, we obtain

$$\left|\frac{(g(x) - p(t))}{x^2 + y^2}x\right| \le \varepsilon.$$

Consequently, we have

$$\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} < 0, \quad t \in [0, T]. \tag{2.8}$$

□

By combining Lemma 2.3 and Lemma 2.4, we know that if the initial value $(\theta_0, r_0)$ such that $r_0 > c$, then for $t \in [0, L]$, the solution of (2.6) rotates clockwise around the origin $O$ and is always outside the disc $D_d$.

## 3. Harmonic and subharmonic solutions of (1.1)

Denote by $\Delta\tau(r_0,\theta_0)$ the time for which the solution $(\theta(t;r_0,\theta_0), r(t;r_0,\theta_0))$ of system (2.2) rotates one around the origin.

**Lemma 3.1.** *Assume that conditions* $(H_1)$–$(H_4)$ *hold, we have*

$$\lim_{r_0\to+\infty}\Delta\tau(r_0,\theta_0)=+\infty.$$

*Proof.* Taking two large enough constants $A>0$ and $B>0$ such that

$$\frac{A}{B}<\varepsilon\ll 1,$$

we consider the following regions respectively:

$$D_1=\{(x,y)\in\mathbb{R}^2: |x|\le A, y>B\},$$

$$D_2=\{(x,y)\in\mathbb{R}^2: x\ge A, |y|<\infty\},$$

$$D_3=\{(x,y)\in\mathbb{R}^2: |x|\le A, y<-B\},$$

$$D_4=\{(x,y)\in\mathbb{R}^2: x\le -A, |y|<\infty\},$$

and $D=D_1\cup D_2\cup D_3\cup D_4\subset D_d$, where $D_d$ is defined in Lemma 2.4. According to Lemma 2.3 and Lemma 2.4, when $r_0$ is sufficiently large, the solution of (2.6) rotates clockwise around the origin $O$ and is always outside the region $D$. Let $[t_1,t_2]$, $[t_2,t_3]$ be the time intervals for the solution staying at $D_1$, $D_2$, respectively, and so on.

From the first equality of Eq (2.2), we obtain

$$t_2-t_1=\int_{-A}^{A}\frac{2e^{2F(x)}}{y}\mathrm{d}x<\frac{4Ae^{2a}}{B}.$$

Similarly, we have

$$t_4-t_3=\int_{A}^{-A}\frac{2e^{2F(x)}}{y}\mathrm{d}x<\frac{4Ae^{2a}}{B}.$$

From the second equality of (2.6), we obtain

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}t}=&\frac{1}{2}r\sin^2\theta\cos\theta\mathrm{e}^{-2F(r\cos\theta)}f(r\cos\theta)\\&-2\cos\theta\mathrm{e}^{2F(r\cos\theta)}\frac{(g(r\cos\theta)-p(t))}{r}-\frac{1}{2}\sin^2\theta\mathrm{e}^{-2F(r\cos\theta)}.\end{aligned}$$

Let

$$\mathcal{F}=-\frac{\mathrm{d}\theta}{\mathrm{d}t}.$$

Then we have

$$\Delta\tau(r_0,\theta_0)=\int_0^{2\pi}\frac{1}{\mathcal{F}}\mathrm{d}\theta.$$

By condition $(H_1)$, for any positive constant $\delta \ll 1$, we can take $B \gg 1$ such that

$$0 < \frac{g(x) - p(t)}{x} < \delta, \quad |x| \geq B.$$

We take $A \gg B$ to ensure that $\theta(t) \in [-\pi/4, \pi/4]$ for all $t \in [t_2, t_3]$. Moreover, $xf(x)$ is bounded on $x \in (-\infty, +\infty)$. Then, for $K > 0$ large enough such that $xf(x) \geq -K$, we have

$$\begin{aligned} t_3 - t_2 = \int_{\theta(t_3)}^{\theta(t_2)} \frac{\mathrm{d}\theta}{H} \geq \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\mathrm{d}\theta}{\frac{1}{2}\sin^2\theta + \frac{1}{2}K\sin^2\theta + 2e^{2a}\delta\cos^2\theta} \\ = \frac{\pi}{\sqrt{(1+K)e^{2a}\delta}}. \end{aligned}$$

Similarly, we have

$$t_5 - t_4 \geq \frac{\pi}{\sqrt{(1+K)e^{2a}\delta}}.$$

Consequently, if $0 < \delta \ll 1$, then

$$\begin{aligned} \Delta\tau(r_0, \theta_0) =& (t_2 - t_1) + (t_3 - t_2) + (t_4 - t_3) + (t_5 - t_4) \\ \geq& \frac{2\pi}{\sqrt{(1+K)e^{2a}\delta}} \gg 1. \end{aligned}$$

Thus we complete the proof. □

In the proof of Lemma 3.1, it seems that the estimates of time intervals $[t_1, t_2]$ and $[t_3, t_4]$ are not needed. However, it has shown that the main part of $\Delta\tau(r_0, \theta_0)$ is the solution staying at $D_2$ and $D_4$.

### *3.1. Proof of Theorem 1.2*

***Proof of Theorem 1.2***. By using Lemma 3.1, we conclude that the time required for the solution rotating clockwisely one around the origin is sufficiently large. Therefore, when $|z_0| = a$ is sufficiently large, $\theta(t, z_0)$ satisfies the condition

$$-2\pi < \theta(T, z_0) - \theta(0, z_0) < 0. \tag{3.1}$$

Then the Poincaré mapping $\mathcal{P}$ of Eq (2.2) satisfies the conditions of the Poincaré-Bohl fixed point theorem on the disk $D_a$. Therefore, $\mathcal{P}$ has at least one fixed point $\zeta_0$, then $z = z(t, \zeta_0)$ is the harmonic solution of Eq (2.2).

Thus we complete the proof. □

### *3.2. Proof of Theorem 1.1*

According to Theorem 1.2, Eq (2.2) has at least one harmonic solution $z = z_0(t) = (x_0(t), y_0(t))$. Let

$$x = u + x_0(t), y = v + y_0(t).$$

Substituting into Eq (2.2), we have

$$\begin{cases} \dot{u} = \dfrac{1}{2}v\mathrm{e}^{-2F(u+x_0(t))} + \dfrac{1}{2}y_0(t)\left[\mathrm{e}^{-2F(u+x_0(t))} - \mathrm{e}^{-2F(x_0(t))}\right], \\ \dot{v} = \dfrac{1}{2}\mathrm{e}^{-2F(u+x_0(t))}f(u+x_0(t))(y_0(t)+v)^2 - 2\mathrm{e}^{2F(u+x_0(t))}g(u+x_0(t)) \\ \qquad +2p(t)\left[\mathrm{e}^{2F(u+x_0(t))} - \mathrm{e}^{2F(x_0(t))}\right] + 2\mathrm{e}^{2F(x_0(t))}g(x_0(t)) \\ \qquad -\frac{1}{2}\mathrm{e}^{-2F(x_0(t))}f(x_0(t))y_0^2(t). \end{cases} \tag{3.2}$$

Obviously, Eq (3.2) has a trivial solution $(u, v) = (0, 0)$. From the uniqueness of the solution, if $(u_0, v_0) \neq 0$, we have $(u(t; u_0, v_0), v(t; u_0, v_0)) \neq (0, 0)$ for all $t \in \mathbb{R}$. Therefore, it can be written in the form of polar coordinates

$$\Lambda : \qquad u(t) = \rho(t)cos\varphi(t),\ v(t) = \rho(t)sin\varphi(t),$$

where $\rho(t) > 0$ and $\varphi(t)$ are continuous functions of $t$.

**Lemma 3.2.** *Let $t_1 > 0$ and assume that the angle $\varphi$ of $\Lambda$ satisfies the condition*

$$\varphi(t_1) - \varphi(0) < -2N\pi. \tag{3.3}$$

*Then for any $t_2 > t_1$, we have*

$$\varphi(t_2) - \varphi(0) < -2N\pi + \pi. \tag{3.4}$$

*Proof.* From the first equality of Eq (3.2), when $\Lambda$ intersects with the positive half axis of the $v$-axis, that is, $u = 0$ and $v > 0$, then

$$\dot{u}(t) = \frac{1}{2}v\mathrm{e}^{-2F(x_0(t))} > 0.$$

When $\Lambda$ intersects with the negative half axis of the $v$-axis, that is, $u = 0$ and $v < 0$, then $\dot{u}(t) < 0$. Therefore, when the trajectory $\Lambda$ intersects with the $v$-axis, it crosses in a clockwise direction. That is, when $\Lambda$ goes from the positive (or negative) $v$-axis to the negative (or positive) $v$-axis, the angle $\varphi$ of $\Lambda$ gets an increment of $-\pi$. And when $\Lambda$ is in the right (or left) half plane, no matter how active, the increment of the angle will not exceed $+\pi$. Therefore, when $t_1 \leq t \leq t_2$, we have

$$\varphi(t_2) - \varphi(t_1) < \pi.$$

By using inequality (3.3), we get

$$\varphi(t_2) - \varphi(0) = (\varphi(t_1) - \varphi(0)) + (\varphi(t_2) - \varphi(t_1)) < -2N\pi + \pi.$$

Therefore,the proof of Lemma 3.2 is now completed. □

Let

$$x = x(t) = u(t) + x_0(t), y = y(t) = v(t) + y_0(t)$$

be the solution of Eq (2.2) on the $(x, y)$ plane. The trajectory on the plane of $(x, y)$ is also denoted as $\Lambda$.

Consider the moving point

$$P(t) = (x(t), y(t)) \in \Lambda, Q(t) = (x_0(t), y_0(t)) \in \Lambda_0\ (t \in R),$$

and the triangle $\Delta_{OPQ}$, where O is the origin on the plane $(x, y)$ and Q is the origin on the plane $(u, v)$. Let the constant $c_0$ satisfy

$$c_0 > \sup_{0\le t\le T} \sqrt{x_0^2(t) + y_0^2(t)}. \tag{3.5}$$

That is, the closed orbit $\Lambda_0$ is within the disk $D_{c_0}(O)$. Then when the poin $P(t)$ is outside the disk $D_{c_0}(O)$ (i.e. $r(t, z_0) = |P(t)| > c_0$), the angle $\Delta_{OPQ}$ at the vertex $P$ of the triangle $\sigma(t)$ is an acute angle. Therefore, using $|\theta(t, \zeta_0) - \varphi(t)| = \sigma(t)$, we can derive

$$|\theta(\tau_2, \zeta_0) - \theta(\tau_1, \zeta_0)| < |\varphi(\tau_2) - \varphi(\tau_1)| + \frac{\pi}{2}\ (\forall \tau_2 > \tau_1 > 0). \tag{3.6}$$

**Lemma 3.3.** *Let* $0 < t_1 < mT$. *Suppose the argument* $\theta(t, z_0)$ *of the trajectory* $\Lambda$ *satisfies*

$$\theta(t_1, z_0) - \theta(0, z_0) < -2N\pi. \tag{3.7}$$

*Then*

$$\theta(mT, z_0) - \theta(0, z_0) < -(2N - \frac{3}{2})\pi. \tag{3.8}$$

*Proof.* From Lemma 3.2, for any $t_2 > t_1$, we have

$$\varphi(t_2) - \varphi(t_1) < \pi.$$

According to inequalities (3.6) and (3.7), there is

$$\begin{aligned}\theta(t_2, z_0) - \theta(0, z_0) \le& |\theta(t_2, z_0) - \theta(t_1, z_0)| + |\theta(t_1, z_0) - \theta(0, z_0)|\\ \le& |\varphi(t_2) - \varphi(t_1)| + \frac{\pi}{2} + (-2N\pi) < -(2N - \frac{3}{2})\pi.\end{aligned}$$

Take $t_2 = mT$, then we complete the proof of Lemma 3.3. □

Now we began to prove Theorem 1.1 as follows.

***Proof of Theorem 1.1***. The proof follows the method of Fonda, Manásevich and Zanolin (see [25]), and we also can refer to the book [35]. The proof is essentially the same as the one in [25], however, for the sake of the integrity of the article, we repeat this process in our setting.

First of all, according to Lemma 2.4, we know that there exists a constant $c_0 > 0$ such that, for any trajectory of Eq (2.2), namely,

$$\Gamma : z = (x(t), y(t)) = (r(t, z_0)\cos\theta(t, z_0), r(t, z_0)\sin\theta(t, z_0)),$$

we have

$$\theta'(\mathrm{t}) < 0, \text{ if } r(t, z_0) \ge c_0. \tag{3.9}$$

Assume that inequality (3.5) holds, where $(x_0(t), y_0(t))$ is a harmonic solution of Eq (2.2).

Secondly, for an arbitrarily large $t_1 > 0$, there is a sufficiently large $d_0 > c_0$ such that when $0 \le t \le t_1$, we get

$$r(t, z_0) > c_0, \theta'(\mathrm{t}) < 0, \text{ if } |z_0| \ge d_0.$$

Thus

$$\theta(t_1, z_0) - \theta(0, z_0) < -2N\pi, \tag{3.10}$$

where the integer $N \geq 0$.

Next we have to prove that the constant $N$ of inequality (3.10) can be arbitrarily large, if $t_1$ is sufficiently large.

In fact, assume by contradiction that there is a constant $K > 0$ such that

$$\theta(t_1, z_0) - \theta(0, z_0) > -K(t_1 \gg 1). \tag{3.11}$$

There include two situations. If $t_1 \to +\infty$, $r(t_1, z_0) \to +\infty$, then we obtain

$$\lim_{t\to\infty} \theta(t, z_0) = \theta^* > -\infty,$$

that is, the trajectory $\Gamma_{z_0}$ takes the ray $\theta = \theta^*$ as the asymptote. From the second equality of Eq (2.6), we have

$$\begin{aligned}\frac{d\theta}{dt} =& \frac{1}{2}\sin^2\theta e^{-2F(x)} x f(x) - 2\cos^2\theta e^{2F(x)} \frac{(g(x) - p(t))}{x} \\ &- \frac{1}{2}\sin^2\theta e^{-2F(x)}.\end{aligned}$$

Thus we have $\theta^* = k\pi$, where $k$ is a certain integer. That is, the trajectory $\Gamma_{z_0}$ takes the positive $x$-axis (or negative $x$-axis) as the asymptotes. Let the positive $x$ axis be the asymptote. Therefore, the tangent slope of the trajectory $\Gamma_{z_0}$ has a limit

$$\lim_{x\to\infty} \frac{dy}{dx} = \lim_{x\to\infty} \frac{\frac{1}{2}e^{-2F(x)} f(x) y^2 - 2e^{2F(x)} (g(x) - p(t))}{\frac{1}{2}e^{-2F(x)} y} = 0.$$

which yields that, when $x \to +\infty$, we have $y \to 0$. It contradicts with the condition $(H_4)$, and the above situation cannot happen.

We consider the other case. Suppose $c_0 > 0$, which is given by (3.5). Then there is $t_1 \gg 1$ such that $r(t_1, z_0) = c_0$. Since $|z_0|$ can be sufficiently large, there exists a trajectory $\Gamma^*$ starting from the circle $|z| = c_0$ with a negative direction asymptote, which is similar to the first case. Then we can also deduce contradictions. This proves that inequality (3.11) does not hold.

Therefore, the constant $N > 0$ in the inequality (3.10) can be arbitrarily large, if $t_1$ is sufficiently large.

Then, take an appropriately large constant $a_0 > c_0$, so that the initial value of the trajectory $\Gamma$ which satisfies $|z(0)| = a_0$ has the following properties:

$(P_1)$ For a given prime number $Q \geq 2$, there is $t_1 > 0$ such that

$$|z(t, z_0)| > c_0 \ (0 \leq t < t_1),$$

$$|z(t_1, z_0)| = c_0\ ,\ \theta(t_1, z_0) - \theta(0, z_0) < -(2Q + 2)\pi.$$

or

$(P_2)$ For any sufficiently large $t_1 > 0$, such that

$$|z(t, z_0)| > c_0 \ (0 \leq t < t_1),$$

$$\theta(t_1, z_0) - \theta(0, z_0) < -(2Q + 2)\pi.$$

Let

$$E = \{t_1 > 0 : \text{the property } (P_1) \text{ holds}\}.$$

Then when $E \neq \emptyset$, we have an upper bound

$$t_1^* = \sup_{t_1 \in E} t_1.$$

And when $E = \emptyset$ (that is, for any $t_1 > 0$, the property $(P_2)$ holds). Let $t_1^* = 0$ and take the integer

$$m_0 = \max\{2, t_1^*\}.$$

Then when $m \geq m_0$, from Lemma 3.3, we obtain

$$\theta(mT, z_0) - \theta(0, z_0) < -(2(Q + 1) - \frac{3}{2})\pi < -(2Q + \frac{1}{2})\pi, |z(0)| = a_0. \tag{3.12}$$

On the other hand, from Lemma 3.1, there is a sufficiently large constant $b_m > 0 (b_m > a_0)$ such that

$$-2\pi < \theta(mT, z_0) - \theta(0, z_0) < 0, |z(0)| = b_m. \tag{3.13}$$

Considering the annular domain

$$\mathcal{A}_m : a_0^2 \leq x^2 + y^2 \leq b_m^2,$$

we denote the $m$ iterations of the Poincaré-Birkhoff of Eq (2.2) as $\mathcal{P}^m$. Obviously, the composition of the Poincaré maps for Hamiltonian Systems is an area-preserving homeomorphism. And by (3.12) and (3.13), it is twisted on the annular domain $\mathcal{A}_m$. Therefore, according to the Poincaré-Birkhoff twist theorem, $\mathcal{P}^m$ has at least two fixed points $\zeta_m^{(k)} (k = 1, 2)$ in $\mathcal{A}_m$, and satisfies the conditions

$$\theta(mT, \zeta_m^{(k)}) = -2Q\pi, (k = 1, 2).$$

where $Q \geq 2$ is a prime number. Obviously, $z = z_m(t, \zeta_m^{(k)})$ is the $mT$ periodic solution of Eq (2.2).

With the same argument in [25, 35], owing to the fact that $Q$ is prime and $Q \geq 2$, we know that $mT$ is the minimum period of $z = z_m(t, \zeta_m^{(k)})$. Thus we complete the proof of Theorem 1.1. □

## 4. Conclusions

Based on the Poincaré-Birkhoff twist theorem, we have proved the existence of harmonic solutions and infinitely many subharmonic solutions of dissipative second order sublinear differential equations named quadratic Liénard type systems.

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**Conflict of interest**

The authors declare that they have no conflicts of interest.

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