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Research article

A novel stability analysis for the Darboux problem of partial differential equations via fixed point theory

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Abstract: We present Ulam-Hyers-Rassias (UHR) stability results for the Darboux problem of partial differential equations (DPPDEs). We employ some fixed point theorem (FPT) as the main tool in the analysis. In this manner, our results are considered as some generalized version of several earlier outcomes.

Keywords: Ulam stability; Darboux problem; partial differential equation; fixed points
Mathematics Subject Classification: 45N05, 34G20, 47H10

1. Introduction

Ulam's stability problem, also referred to as Ulam-Hyers problem, started as a tool to find the error people usually face when they replace solutions of functional equations by functions that satisfy them only approximately [10, 14]. It is widely used for investigating the stability of many kinds of (difference, integral, differential, fractional differential, partial differential) equations. The Ulam's stability problem is due to the well-known Polish mathematician S. M. Ulam who presented a list (one of which is the stability) of open problems during a conference held in Wisconsin University in the fall term of 1940. The stability problem posed by Ulam [14] concerning group homomorphism can be stated as:
If G_1^* is some group and (G_1^{**}, χ) a metric group. Assume $\varepsilon_1 > 0$, does there exist $\delta_1 > 0$ such that if $f_1 : G_1^* \to G_1^{**}$ fulfilling

$$\chi(f_1(x_1x_2), f_1(x_1)f_1(x_2)) < \delta_1$$

for every $x_1, x_2 \in G_1^*$, then a homomorphism $g_1 : G_1^* \to G_1^{**}$ exists satisfying

$$\chi(f_1(x_1), g_1(x_1)) < \varepsilon_1$$

for all $x_1 \in G_1^*$?

Many mathematicians have interacted with the question of Ulam and introduced interesting solutions in many settings. In particular, in 1941, D. H. Hyers introduced an affirmative answer to the question of Ulam in case of Banach spaces. Since then, the stability problem is called UH and sometimes Hyers-Ulam stability problem. Afterwards, Rassias in 1978 [29] introduced a general interesting form of Hyers's result. The result obtained by Rassias and concerning the well-known Cauchy equation ($f(x+y) = f(x) + f(y)$) takes the form [29]:

Theorem 1. *Assume that B_1 and B_1^* are Banach spaces, suppose a continuous mapping $h : \mathbb{R} \to B_1^*$ from $\mathbb{R}$ into B_1^*. Suppose that there is $\omega \geq 0$, $\vartheta \in [0, 1)$ with*

$$\|h(\varepsilon^* + \varepsilon^{**}) - h(\varepsilon^*) - h(\varepsilon^{**})\| \leq \omega(\|\varepsilon^*\|^\vartheta + \|\varepsilon^{**}\|^\vartheta), \ \varepsilon^*, \varepsilon^{**} \in B_1 \setminus \{0\}. \tag{1.1}$$

Then a unique solution $P : B_1 \to B_1^*$ of the Cauchy equation exists with

$$\|h(\varepsilon) - P(\varepsilon)\| \leq \frac{2\omega\|\varepsilon\|^\vartheta}{|2 - 2^\vartheta|}, \ \varepsilon \in B_1 \setminus \{0\}. \tag{1.2}$$

The theorem above introduced by Rassias see [29] is nowadays known as the UHR stability.

For the past three decades, the stability issue of differential equations has been a focus of scientific investigations by many mathematicians that can be briefly stated as follows. In 1993, Obloza [25] pioneered the stability issue of differential equations in the sense of UH [26]. Five years latter, in particular in 1998, Alsina and Ger [3] studied the UH stability of the ordinary differential equation $y'(t) = y(t)$. They end up with the estimation $|y(t) - y_0(t)| \leq 3\epsilon$, where $y_0(t)$ is some solution of the differential equation. In 2002, an extension of the results presented by Alsina and Ger has been introduced by Takahasi et al., where they investigated the stability of the equation $g'(\varsigma) = \lambda g(\varsigma)$ in Banach spaces. In 2003, Miura et al. generalized the work of Alsina and Ger to higher order differential equations [23, 24].

Following such interesting results, many articles devoted to this subject have been introduced [5, 6, 8, 9, 15, 28]. In 2010, Jung employed an approach based on FPT to study the stability of the differential equation $x_2' = h(x_1, x_2)$ in UHR sense [19]. It should be remarked that Jung in [19] generalized the work of Alsina and Ger to the nonlinear case. In 2012, Bojor [7] improved the result of Jung in [19] and used some different assumptions to study the stability of the equation

$$h'(\xi) + m(\xi)h(\xi) = r(\xi).$$

In 2015, The authors in [40] modified the approach of Jung in [19] for the functional differential equation

$$z'(x_1) = H(x_1, z(x_1), z(x_1 - \tau)),$$

for some nonnegative τ. In [17], the stability of the following nonlinear differential equation

$$Z^{(n)}(x_1) = G(x_1, Z(x_1), Z'(x_1), \cdots, Z^{(n-1)}(x_1)),$$

has been investigated using some FPT. In 2016, the authors investigated UH stability of Euler's differential equation [27]. In 2017, the authors introduced Ulam stability results for differential equations on time scales [36].

The stability issue of partial differential equation (PDEs) has been investigated by many mathematicians using different tools (see the articles [1, 2, 15, 20, 21, 28, 37] and the references therein). Our contribution can be seen as some generalized version of the results in [1,4]. The rest of the article is organized as follows. In the next Section we recall some preliminaries, in Section 3 we present the stability results in UHR sense, and we use Section 5 to conclude our work.

2. Preliminaries

From now on, $\mathbb{R}$ is used to denote real numbers set, $\mathbb{C}$ to denote the complex numbers set, and we fix an interval $J := [a_1, a_1 + T_1] \times [a_2, a_2 + T_2]$ for some reals $a_i, T_i, i = 1, 2$ with $T_i > 0$.

Definition 1. *If $\sigma : S \times S \to [0, \infty]$ is some mapping. The mapping σ is said to be a generalized metric on a nonempty set S iff σ fulfills:*

G1 $\sigma(r_1, r_2) = 0$ *iff* $r_1 = r_2$;

G2 $\sigma(r_1, r_2) = \sigma(r_2, r_1)$ *for all* $r_1, r_2 \in S$;

G3 $\sigma(r_1, r_3) \le \sigma(r_1, r_2) + \sigma(r_2, r_3)$ *for all* $r_1, r_2, r_3 \in S$.

Theorem 2. *[12] If (Z, γ) is a generalized complete metric space. Assume that $\Gamma : Z \to Z$ is an operator which is strictly contractive with some Lipschitz constant $L < 1$. If there is a nonnegative integer k such that $\gamma(\Gamma^{k+1} y, \Gamma^k y) < \infty$ for some $y \in Z$, then*

(a) $\lim_{n \longrightarrow +\infty} \Gamma^n y = y^*$ *with* $\Gamma(y^*) = y^*$;

(b) y^* *is the unique fixed point of* Γ *in* $Z^* := \{y_1 \in Z : \gamma(\Gamma^k y, y_1) < \infty\}$;

(c) *If* $y_1 \in Z^*$, *then* $\gamma(y_1, y^*) \le \frac{1}{1-L} \gamma(\Gamma y_1, y_1)$.

The current article is devoted to study the stability of the following PDE:

$$\frac{\partial^2 u(\omega_1, \omega_2)}{\partial \omega_1 \partial \omega_2} = f(\omega_1, \omega_2, u(\omega_1, \omega_2)) \tag{2.1}$$

for all $(\omega_1, \omega_2) \in J$ satisfying the initial conditions

$$\begin{cases} u(\omega_1, a_2) = \varphi(\omega_1), & \text{if } \omega_1 \in [a_1, a_1 + T_1] \\ u(a_1, \omega_2) = \psi(\omega_2), & \text{if } \omega_2 \in [a_2, a_2 + T_2] \\ \varphi(a_1) = \psi(a_2). \end{cases}$$

The function $f : J \times \mathbb{R} \to \mathbb{R}$ is continuous and $\varphi : [a_1, a_1 + T_1] \to \mathbb{R}$, $\psi : [a_2, a_2 + T_2] \to \mathbb{R}$ are given absolutely continuous functions. Equation (2.1) is equivalent to the integral equation (I.E.)

$$u(\omega_1, \omega_2) = h(\omega_1, \omega_2) + \int_{a_1}^{\omega_1} \int_{a_2}^{\omega_2} f(t, s, u(t, s)) ds dt,$$

where

$$h(\omega_1, \omega_2) = \varphi(\omega_1) + \psi(\omega_2) - \varphi(a_1).$$

Let us denote the space $\mathbb{E}$ as follows

$$\mathbb{E} = C\Big(J, \mathbb{R}\Big).$$

Define a metric d in the following way

$$d(\vartheta_1, \vartheta_2) := \inf\left\{K \in [0, \infty] : \frac{|\vartheta_1(\omega_1, \omega_2) - \vartheta_2(\omega_1, \omega_2)|}{e^{M^q(\omega_2 - a_2)} e^{M^p(\omega_1 - a_1)}} \leq K\zeta(\omega_1, \omega_2), \forall(\omega_1, \omega_2) \in J\right\}, \tag{2.2}$$

where $\zeta \in C\Big(J, (0, \infty)\Big)$ and M, q and p are some positive constants with $q + p = 1$. Then the space $(\mathbb{E}, d)$ is a complete generalized metric space.

3. Stability results

This section is used to show our main results. In other words, we prove that under certain conditions, functions that satisfy (2.1) approximately (in some sense) are close (in some way) to the solutions of (2.1). We have done this in UHR sense.

Theorem 3. *Assume that f satisfies*

$$|f(\omega_1, \omega_2, u_1) - f(\omega_1, \omega_2, u_2)| \leq L|u_1 - u_2|,$$

for all $(\omega_1, \omega_2) \in J, u_i \in \mathbb{R}, i = 1, 2$ and for some $L > 0$. If an absolutely continuous function $V : J \to \mathbb{R}$ satisfies

$$\Big|\frac{\partial^2 V(\omega_1, \omega_2)}{\partial\omega_1\partial\omega_2} - f(\omega_1, \omega_2, V(\omega_1, \omega_2))\Big| \leq \epsilon\zeta(\omega_1, \omega_2), \tag{3.1}$$

for some continuous, positive, nondecreasing function $\zeta(\omega_1, \omega_2)$ in both ω_1 and ω_2 and $\epsilon > 0$, then there is a unique solution U_0 of (2.1) such that

$$|V(\omega_1, \omega_2) - U_0(\omega_1, \omega_2)| \leq \epsilon(\frac{L + \delta}{\delta})M_1M_2e^{(L+\delta)^pT_1 + (L+\delta)^qT_2}\zeta(\omega_1, \omega_2), \forall(\omega_1, \omega_2) \in J,$$

for any positive constants δ, p and q with $p + q = 1$, where $M_1 = \sup_{s\in[a_1, a_1+T_1]}\Big(\frac{s - a_1}{e^{(L+\delta)^p(s-a_1)}}\Big)$ and $M_2 = \sup_{s\in[a_2, a_2+T_2]}\Big(\frac{s - a_2}{e^{(L+\delta)^q(s-a_2)}}\Big)$.

Proof. Let consider the space $(\mathbb{E}, \tilde{d})$ where the metric on $\mathbb{E}$ is defined as in the following manner

$$\tilde{d}(\vartheta_1, \vartheta_2) = \inf\left\{\gamma \in [0, \infty] : \frac{|\vartheta_1(\omega_1, \omega_2) - \vartheta_2(\omega_1, \omega_2)|}{e^{(L+\delta)^q(\omega_2 - a_2)} e^{(L+\delta)^p(\omega_1 - a_1)}} \leq \gamma\zeta(\omega_1, \omega_2), \forall(\omega_1, \omega_2) \in J\right\}.$$

Now, define the operator $\mathcal{A} : \mathbb{E} \to \mathbb{E}$ such that

$$(\mathcal{A}u)(\omega_1, \omega_2) := V(a_1, \omega_2) + V(\omega_1, a_2) - V(a_1, a_2) + \int_{a_1}^{\omega_1}\int_{a_2}^{\omega_2} f(s_1, s_2, u(s_1, s_2))ds_2ds_1, \ \forall(\omega_1, \omega_2) \in J.$$

We have $\mathcal{A}u \in \mathbb{E}$ and $\tilde{d}(\mathcal{A}u_0, u_0) < \infty$, $\forall u_0 \in \mathbb{E}$.

Also, we have that $\tilde{d}(\mathcal{A}u_0, u) < \infty \ \forall u_0, u \in \mathbb{E}$, then $\{u \in \mathbb{E} : \tilde{d}(u_0, u) < \infty\} = \mathbb{E} \ \forall u_0 \in \mathbb{E}$.
Now, we show that $\mathcal{A}$ is strictly contractive. For this purpose, we take any $u_1, u_2 \in \mathbb{E}$ and we see that

$$\begin{aligned}
\Big|(\mathcal{A}u_1)(\omega_1,\omega_2) - (\mathcal{A}u_2)(\omega_1,\omega_2)\Big| &\leq \Big|\int_{a_1}^{\omega_1}\int_{a_2}^{\omega_2}\Big\{f(s_1,s_2,u_1(s_1,s_2)) - f(s_1,s_2,u_2(s_1,s_2))\Big\}ds_2ds_1\Big| \\
&\leq \int_{a_1}^{\omega_1}\int_{a_2}^{\omega_2}\Big|f(s_1,s_2,u_1(s_1,s_2)) - f(s_1,s_2,u_2(s_1,s_2))\Big|ds_2ds_1 \\
&\leq L\tilde{d}(u_1,u_2)\int_{a_1}^{\omega_1}\int_{a_2}^{\omega_2}\zeta(s_1,s_2)e^{(L+\delta)^q(s_2-a_2)}e^{(L+\delta)^p(s_1-a_1)}ds_2ds_1 \\
&\leq L\tilde{d}(u_1,u_2)\zeta(\omega_1,\omega_2)\int_{a_1}^{\omega_1}\int_{a_2}^{\omega_2}e^{(L+\delta)^q(s_2-a_2)}e^{(L+\delta)^p(s_1-a_1)}ds_2ds_1 \\
&\leq \frac{L}{L+\delta}\tilde{d}(u_1,u_2)\zeta(\omega_1,\omega_2)e^{(L+\delta)^q(\omega_2-a_2)}e^{(L+\delta)^p(\omega_1-a_1)}, \ \forall(\omega_1,\omega_2) \in I.
\end{aligned}$$

So that

$$\tilde{d}(\mathcal{A}u_1, \mathcal{A}u_2) \leq \frac{L}{L+\delta}\tilde{d}(u_1,u_2)$$

then $\mathcal{A}$ is strictly contractive. Now, we get from (3.1)

$$\begin{aligned}
|V(\omega_1,\omega_2) - \mathcal{A}V(\omega_1,\omega_2)| &\leq \epsilon\int_{a_1}^{\omega_1}\int_{a_2}^{\omega_2}\zeta(s_1,s_2)ds_2ds_1 \\
&\leq \epsilon\zeta(\omega_1,\omega_2)(\omega_1-a_1)(\omega_2-a_2), \ (\omega_1,\omega_2) \in J,
\end{aligned}$$

then

$$\frac{|V(\omega_1,\omega_2) - \mathcal{A}V(\omega_1,\omega_2)|}{e^{(L+\delta)^p(\omega_1-a_1)}e^{(L+\delta)^q(\omega_2-a_2)}} \leq \epsilon\zeta(\omega_1,\omega_2)M_1M_2, \ \forall(\omega_1,\omega_2) \in J,$$

so that

$$\tilde{d}(V, \mathcal{A}V) \leq \epsilon M_1M_2.$$

By employing Theorem 2, we find that there is a solution U_0 of (2.1) satisfying

$$\tilde{d}(V, U_0) \leq \frac{L+\delta}{\delta}\tilde{d}(\mathcal{A}V, V) \leq \epsilon\frac{L+\delta}{\delta}M_1M_2,$$

so that

$$|V(\omega_1,\omega_2) - U_0(\omega_1,\omega_2)| \leq \epsilon\frac{L+\delta}{\delta}M_1M_2e^{(L+\delta)^pT_1+(L+\delta)^qT_2}\zeta(\omega_1,\omega_2),$$

for all $(\omega_1,\omega_2) \in J$. □

Remark 1. *Note that, if we consider* $\zeta(\tau_1,\tau_2) = 1$*, we get the Ulam stability of* (2.1).

Remark 2. *It should be noted that in our analysis, we used a FPT as the basic tool unlike the case in [20] where the authors used the Gronwall Lemma (see Lemma 3.1 in [20]), see also the interesting results [21, 22, 35].*

Remark 3. *Note that in [37], the authors obtained interesting stability results in the same sense of our interests but using the Pachpatte's inequality (see Theorem 3.4 in [37]).*

4. An example

Example 1. *We consider Eq.* (2.1) *for* $a_1 = a_2 = 0$, $T_1 = T_2 = 2$, $\varphi(v_1) = v_1^2 + 1$, $\psi(v_2) = e^{v_2}$ *and* $f(v_1, v_2, r) = {v_1}^2{v_2}^3\cos(r)$.
We have

$$|{v_1}^2{v_2}^3\cos(r_1) - {v_1}^2{v_2}^3\cos(r_2)| \leq 32|r_1 - r_2|,\ \forall\ (v_1, v_2) \in [0,2] \times [0,2],\ r_1, r_2 \in \mathbb{R}.$$

Then $L = 32$.
Suppose that V *satisfies*

$$|\frac{\partial^2 V(\omega_1, \omega_2)}{\partial\omega_1\partial\omega_2} - {\omega_1}^2{\omega_2}^3\cos\left(V(\omega_1, \omega_2)\right)| \leq 0.1(\omega_1 + \omega_2 + 1), \tag{4.1}$$

for all $(\omega_1, \omega_2) \in [0,2] \times [0,2]$.
Here, $\epsilon = 0.1$ *and* $\zeta(\omega_1, \omega_2) = \omega_1 + \omega_2 + 1$. *It follows from Theorem 3 that there is a solution* U_0 *of the equation and* $\eta > 0$ *such that*

$$|V(\omega_1, \omega_2) - U_0(\omega_1, \omega_2)| \leq 0.1\eta(\omega_1 + \omega_2 + 1),\ \forall\ (\omega_1, \omega_2) \in [0,2] \times [0,2].$$

5. Conclusions

It is recognized that a generally applicable general approach to finding analytical solutions is not available for most partial differential equation's problems. In this work, we used a version of Banach FPT to prove that under certain conditions, functions that satisfy some DPPDEs approximately, are close in some sense to the exact solutions of such problems. In other words, we present stability results for some DPPDEs in UHR sense.

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Conflict of interest

Authors declare that no conflict of interest in this manuscript.

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