AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(11): 12878–12893.
DOI: 10.3934/math.2021743
Received: 21 July 2021
Accepted: 01 September 2021
Published: 09 September 2021

Research article

On [p,q]-order of growth of solutions of linear differential equations in the unit disc

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Abstract: The $[p,q]$-order of growth of solutions of the following linear differential equations $(**)$ is investigated,

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, (**)$$

where $A_i(z)$ are analytic functions in the unit disc, $i = 0, 1, ..., k-1$. Some estimations of $[p,q]$-order of growth of solutions of the equation $(**)$ are obtained when $A_j(z)$ dominate the others coefficients near a point on the boundary of the unit disc, which is generalization of previous results from S. Hamouda.

Keywords: linear differential equations; unit disc; [p,q]-order; bounded point
Mathematics Subject Classification: 34M10, 30D35

1. Introduction and main results

For the following complex linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.1}$$

where $A_i(z)$ are analytic in the unit disc $\Delta = \{z : |z| < 1\}$, $i = 0, 1, ..., k-1$, $k \geq 2$. For the properties of solutions of Eq (1.1), including growth, zero distribution and function space properties, many results have been obtained, for example [9, 10, 14, 16, 20, 22, 23, 25] and therein references. In addition, the differential equations have wide applications in various science discipline, for example engineering, predator-prey equations, population growth and decay, Newton's law of cooling and so on, see [1, 5, 24, 28, 29] and therein references. This paper is mainly concerned with the growth of solution of the Eq (1.1). It has been widely noted that Bernal [4] firstly introduced the idea of iterated order to express the fast growth of solutions of (1.1). Since then, the iterated order of solutions of (1.1) is very interesting topic in the unit disc Δ, many results concerning iterated order of solutions of (1.1) in the unit disc are obtained, see [3, 6, 7, 18, 26]. In order to better estimate the fast growth of solutions of (1.1), $[p,q]$-order was introduced, and many results on $[p,q]$-order of solutions of (1.1) have been found by different researchers in Δ. For example, see [2, 17, 19, 21, 27].

To state our results, firstly, we assume that readers are familiar with the fundamental results and the standard notation of the Nevanlinna value distribution theory in the unit disc (see [12, 13]). Secondly, we introduce some definitions, for all $r \in [0,1)$, $\exp_1 r = e^r$ and $\exp_{n+1} r = \exp(\exp_n r)$, $n \in N$, and for all $r \in (0,1)$, $\log_1 r = \log_r$ and $\log_{n+1} r = \log(\log_n r)$, $n \in N$. We also denote $\exp_0 r = r = \log_0 r$, $\exp_{-1} r = \log_1 r$.

Definition 1. Let f be a meromorphic function in Δ, then the iterated n-order of f is defined by

$$\sigma_n(f) = \overline{\lim_{r\to 1^-}} \frac{\log_n^+ T(r,f)}{-\log(1-r)},$$

where $\log_1^+ = \log^+ x = max\{\log x, 0\}$, $\log_{n+1}^+ x = \log^+(\log_n^+ x)$. If f is an analytic in Δ, then its iterated n-order is defined by

$$\sigma_{M,n}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_{n+1}^+ M(r,f)}{-\log(1-r)},$$

where $M(r,f) = \max\limits_{|z|=r} |f(z)|$. For $n = 1, \sigma_{M,1}(f) = \sigma_M(f)$, $\sigma_1(f) = \sigma(f)$.

Suppose p and q are integers satisfying $p \geq q \geq 1$. Then $[p,q]$-order is defined as follows.

Definition 2. Let f be a meromorphic function in Δ, then the $[p,q]$-order of f is defined by

$$\sigma_{[p,q]}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_p^+ T(r,f)}{-\log_q(1-r)}.$$

If f is analytic in Δ, then its $[p,q]$-order is defined by

$$\sigma_{M,[p,q]}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_{p+1}^+ M(r,f)}{-\log_q(1-r)}.$$

Remark 1. [2] Let p and q be integers such that $p \geq q \geq 1$, and f be an analytic function in Δ. The following two statements holds:

(i) If $p = q$, then $\sigma_{[p,q]}(f) \leq \sigma_{M,[p,q]}(f) \leq \sigma_{[p,q]}(f) + 1$.

(ii) If $p > q$, then $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f)$.

Recently, Hamouda [11] considered the fast growing solutions of Eq (1.1) by using a new idea which A_0 dominates the other coefficients near a point on the boundary of the unit disc, and obtained some results which improve and generalize results of Heittokangas et al. [15]. The following two results are proved.

Theorem 1.1. *[11] Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in Δ. If there exists $\omega_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to ω_0 such that for any constant $\mu > 0$,*

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\sum\limits_{j=1}^{k-1} |A_j(z)| + 1}{|A_0(z)|(1-|z|)^{\mu}} = 0,$$

then every nontrivial solution $f(z)$ of (1.1) is of infinite order.

Theorem 1.2. *[11] Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in Δ. If there exists $\omega_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to ω_0 such that*

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\sum_{j=1}^{k-1} |A_j(z)| + 1}{|A_0(z)|} \exp_n \left\{ \frac{\lambda}{(1-|z|)^{\mu}} \right\} = 0,$$

where $n \geq 1$ is an integer, $\lambda > 0$ is a real constant, then every nontrivial solution $f(z)$ of (1.1) satisfies $\sigma_n(f) = \infty$ and $\sigma_{n+1}(f) \geq \mu$.

A natural question is that how to character the $[p, q]$-order of growth of solutions of (1.1) under the similar conditions of Hamouda. Here, we study the problem and get the following results.

Theorem 1.3. *Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in Δ. If there exists $\omega_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to ω_0 satisfying*

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\sum_{i=1}^{k-1} |A_i(z)| + 1}{|A_0(z)|} \exp_p \left\{ \frac{\lambda}{(1-|z|)^{\mu}} \right\} < 1, \tag{1.2}$$

where $\lambda > 0$ and $\mu > 0$ are two real constants, then every nontrivial solution $f(z)$ of (1.1) satisfies $\sigma_{[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) \geq \mu$.

Corollary 1. *Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in Δ. If there exists $\omega_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to ω_0 such that for some constants $0 \leq \beta < \alpha$ and any given $\varepsilon(0 < \varepsilon < \infty)$, for $z \in \gamma$ and $|z| \to 1^-$, we have*

$$|A_0| \geq \exp_p \left\{ \frac{\alpha}{(1-|z|)^{\mu-\varepsilon}} \right\}$$

and

$$|A_i| \leq \exp_p \left\{ \frac{\beta}{(1-|z|)^{\mu-\varepsilon}} \right\}, i = 1, \cdots, k-1,$$

then every nontrivial solution $f(z)$ of (1.1) satisfies $\sigma_{[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) \geq \mu$.

In fact, by the assumptions and taking $0 < \lambda < \alpha - \beta$, we can get

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\sum_{i=1}^{k-1} |A_i(z)| + 1}{|A_0(z)|} \exp_p \left\{ \frac{\lambda}{(1-|z|)^{\mu}} \right\} < 1.$$

Thus Corollary 1 is obtained by applying Theorem 1.3.

Theorem 1.4. *Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions in Δ satisfying*

$$\max \left\{ \sigma_{M,[p,q]}(A_i) : i = 1, \cdots, k-1 \right\} \leq \sigma_{M,[p,q]}(A_0) = \mu,$$

where $\mu \in (0, \infty)$*. If there exists* $\omega_0 \in \partial\Delta$ *and a curve* $\gamma \subset \Delta$ *tending to* ω_0 *such that*

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i=1}^{k-1} e^{T(r,A_i)}}{e^{T(r,A_0)}} \exp_p\left\{\frac{\lambda}{(1-|z|)^{\mu}}\right\} < 1, \tag{1.3}$$

where $\lambda > 0$ *is a real constant, then every nontrivial solution* $f(z)$ *of (1.1) satisfies* $\sigma_{[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p,q]}(A_0)$*.*

Corollary 2. *Let* $A_0(z), \ldots, A_{k-1}(z)$ *be analytic functions in* Δ *satifying*

$$\max\left\{\sigma_{M,[p,q]}(A_i) : i = 1, \cdots, k-1\right\} \le \sigma_{M,[p,q]}(A_0) = \mu,$$

where $\mu \in (0, \infty)$*. If there exists* $\omega_0 \in \partial\Delta$ *and a curve* $\gamma \subset \Delta$ *tending to* ω_0 *such that for some constants* $0 \le \beta < \alpha$ *and any given* $\varepsilon(0 < \varepsilon < \infty)$*, for* $z \in \gamma$ *and* $|z| \to 1^-$*, we have*

$$T(r, A_0) \ge \exp_{p-1}\left\{\frac{\alpha}{(1-|z|)^{\mu-\varepsilon}}\right\}$$

and

$$T(r, A_i) \le \exp_{p-1}\left\{\frac{\beta}{(1-|z|)^{\mu-\varepsilon}}\right\}, i = 1, \cdots, k-1,$$

then every nontrivial solution $f(z)$ *of (1.1) satisfies* $\sigma_{[p,q]}(f) = \infty$ *and* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p,q]}(A_0)$*.*

In fact, for any given $\varepsilon(0 < \varepsilon < \mu)$, taking $\lambda < \alpha$, for $|z| \to 1^-$ and $z \in \gamma$, we have

$$\exp_{p-1}\left\{\frac{\lambda}{(1-|z|)^{\mu-\varepsilon}}\right\} < o\left(\exp_{p-1}\left\{\frac{\alpha}{(1-|z|)^{\mu-\varepsilon}}\right\}\right)$$

and

$$\exp_{p-1}\left\{\frac{\beta}{(1-|z|)^{\mu-\varepsilon}}\right\} < o\left(\exp_{p-1}\left\{\frac{\alpha}{(1-|z|)^{\mu-\varepsilon}}\right\}\right).$$

From the condition of Corollary 2, we have

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod_{i=1}^{k-1} e^{T(r,A_i)}}{e^{T(r,A_0)}} \exp_p\left\{\frac{\lambda}{(1-|z|)^{\mu-\varepsilon}}\right\} = 0 < 1, i = 0, 1, ..., k-1.$$

Thus Corollary 2 is obtained by applying Theorem 1.4.

In Theorems 1.3 and 1.4, the coefficient $A_0(z)$ is the dominant coefficient. A nature question: How to character $[p, q]$-order of growth of solutions of Eq (1.1) when $A_s(z)$ dominates the other coefficients near a point on the boundary of the unit disc. Next, we study the growth of the solution of (1.1) when $A_s(z)$ is the dominant coefficient, and get the following results.

Theorem 1.5. *Let $A_0(z), \dots, A_{k-1}(z)$ be analytic functions in Δ. If there exists $\omega_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to ω_0 such that for any $\mu > 0$, we have*

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod\limits_{i\neq s} e^{T(r,A_i)}}{e^{T(r,A_s)}(1-|z|)^{\mu}} < 1, \tag{1.4}$$

then every nontrivial solution $f(z)$ of (1.1), in which $f^{(n)}(z)$ just has finite many zeros for all $n < s(n = 0, ..., s-1)$, is of infinite order.

Theorem 1.6. *Let $A_0(z), \dots, A_{k-1}(z)$ be analytic functions in Δ and there exists $\mu \in (0, \infty)$ satisfying*

$$\max\left\{\sigma_{M,[p,q]}(A_i) : i = 0, 1, \cdots, s-1, s+1, \cdots, k-1\right\} \leq \sigma_{M,[p,q]}(A_s) = \mu.$$

If there exists $\omega_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to ω_0 such that

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod\limits_{i} e^{T(r,A_i)}}{e^{T(r,A_s)}} \exp_p\left\{\frac{\lambda}{(1-|z|)^{\mu}}\right\} < 1, \tag{1.5}$$

where $\lambda > 0$ is a real constant. Then, every nontrivial solution $f(z)$ of (1.1), in which $f^{(n)}(z)$ just has finite many zeros for all $n < s(n = 0, ..., s-1)$, satisfies $\sigma_{[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) = \sigma_{M,[p,q]}(A_s)$.

By Theorem 1.6 we can easily obtain the following Corollary 3.

Corollary 3. *Let $A_0(z), \dots, A_{k-1}(z)$ be analytic functions in Δ and there exists $\mu \in (0, \infty)$ satisfying*

$$\max\left\{\sigma_{M,[p,q]}(A_i) : i = 0, \cdots, s-1, s+1, \cdots, k-1\right\} \leq \sigma_{M,[p,q]}(A_s) = \mu.$$

If there exists $\omega_0 \in \partial\Delta$ and a curve $\gamma \subset \Delta$ tending to ω_0 such that for some constants $0 \leq \beta < \alpha$ and any given $\varepsilon(0 < \varepsilon < \infty)$, for $z \in \gamma$ and $|z| \to 1^-$, we have

$$T(r, A_s) \geq \exp_{p-1}\left\{\frac{\alpha}{(1-|z|)^{\mu-\varepsilon}}\right\}$$

and

$$T(r, A_i) \leq \exp_{p-1}\left\{\frac{\beta}{(1-|z|)^{\mu-\varepsilon}}\right\}, i = 0, \cdots, s-1, s+1, \cdots, k-1,$$

then, every nontrivial solution $f(z)$ of (1.1), in which $f^{(n)}(z)$ just has finite many zeros for all $n < s(n = 0, ..., s-1)$, satisfies $\sigma_{[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) = \sigma_{M,[p,q]}(A_s)$.

Remark 2. The recent papers of Long et al. [20, 22] and Sun et al. [25] discussed the function space properties of solutions of differential equation. Long et al. obtained analytic solutions of Eq (1.1) where $k = 2$ belong to *α-Bloch* space and Morrey space and Sun et al. obtained analytic solutions of the nonlinear equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = A_k(z)$$

belong to in H^{∞}_{Φ} or B^{α}. However, the results of this paper is about growth of solutions of Eq (1.1). Obviously, The results of recent papers of Long et al. and Sun et al. and the results of this paper are not directly related to each other, but they are both properties of solutions of Eq (1.1).

2. Some Lemmas

Lemma 2.1. *[8] Let k and j be integers satisfying $k > j \geq 0$, and let $\varepsilon > 0$ and $d \in (0,1)$. If $f(z)$ is a meromorphic in Δ such that $f^{(j)}(z)$ does not vanish identically, then*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left(\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max\left\{\log\frac{1}{1-|z|}, T\left(s(|z|), f\right)\right\}\right)^{k-j}, |z| \notin E,$$

where $E \subset [0,1)$ with finite logarithmic measure $\int_E \frac{dr}{1-r} < \infty$ and $s(|z|) = 1 - d(1-|z|)$. Moreover, if $\sigma_1(f) < \infty$, then

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \left(\frac{1}{1-|z|}\right)^{(k-j)(\sigma_1(f)+2+\varepsilon)}, |z| \notin E,$$

while if $\sigma_n(f) < \infty$ for $n \geq 2$, then

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq \exp_{n-1}\left\{\left(\frac{1}{1-|z|}\right)^{\sigma_n(f)+\varepsilon}\right\}, |z| \notin E.$$

Lemma 2.2. *Let $f(z) : \Delta \to R$ be a analytic or meromorphic function in Δ. If there exists a point ω_0 on the boundary $\partial\Delta = \{z : |z| = 1\}$ and a curve $\gamma \subset \Delta$ tending to ω_0 such that*

$$\lim_{\substack{z\to\omega_0\\ z\in\gamma}} f(z) < 1,$$

then there exists a set $E \subset [0,1)$ with infinite logarithmic measure $\int_E \frac{dr}{1-r} = \infty$ such that for all $|z| \in E$, we have $f(z) < 1$.

Proof. Let $\lim_{\substack{z\to\omega_0\\ z\in\gamma}} |f(z)| = a$, $0 \leq a < 1$. By definition, for $\varepsilon = 1 - a$, there exists $\delta > 0$ such that for all $z \in \gamma$ and $0 < |z - \omega_0| < \delta$, we have $|f(z)| < a + \varepsilon = 1$. Let $E = \{|z| : z \in \gamma \text{ and } 0 < |z - \omega_0| < \delta\}$. We have

$$\int_E \frac{dr}{1-r} = \int_{1-\delta}^{1} \frac{dr}{1-r} = +\infty.$$

□

Lemma 2.3. *[13] Let $f(z)$ be a meromorphic function in Δ, and let $k \geq 1$ be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f),$$

where $S(r,f) = O\left(\log^+ T(r,f) + \log\left(\frac{1}{1-r}\right)\right)$, possibly outside a set $E \subset [0,1)$ with $\int_E \frac{dr}{1-r} < \infty$. If $f(z)$ is of finite order, then

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\log\left(\frac{1}{1-r}\right)\right).$$

Lemma 2.4. *[17] Let p and q be integers such that $p \geq q \geq 1$, and let $k \geq 1$ be an integer and $f(z)$ be a meromorphic function in Δ satisfies $\sigma_{[p,q]}(f) = \sigma < +\infty$. Then for any $\varepsilon > 0$ and for all $r \in E \subset [0,1)$, we have*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O\left(\exp_{p-q-1}\left\{\frac{1}{1-r}\right\}^{\sigma+\varepsilon}\right),$$

where E has finite logarithmic measure.

Lemma 2.5. *[27] Let p and q be integers such that $p \geq q \geq 1$. If the coefficient $A_0(z)$, $A_1(z)$, $\ldots$, $A_{k-1}(z)$ are analytic functions in Δ, then all solutions $f(z)$ of (1.1) satisfies $\sigma_{[p+1,q]}(f) \leq \max\left\{\sigma_{M,[p,q]}(A_j) : j = 0, \ldots, k-1\right\}$.*

3. Proofs of Theorems 1.3 and 1.4

Proof of Theorem1.3 . Suppose that f is a nontrivial solution of (1.1) with $\sigma_{[p,q]}(f) = \sigma < \infty$. From Lemma 2.1, for any given $\varepsilon > 0$, there exists a set $E_1 \subset [0, 1)$ with $\int_{E_1} \frac{dr}{1-r} < \infty$ such that for all $z \in \Delta$ satisfying $|z| \notin E_1$, we have that if $p = q = 1$, $\sigma_{[1,1]}(f) = \sigma$, then

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \left(\frac{1}{1-|z|}\right)^{j(\sigma+2+\varepsilon)}, j = 1, ..., k. \tag{3.1}$$

If $p \geq q > 1$, then $\sigma_p(f) < \sigma_{[p,q]}(f) = \sigma$,

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \exp_{p-1}\left\{\left(\frac{1}{1-|z|}\right)^{\sigma+\varepsilon}\right\}, j = 1, ..., k. \tag{3.2}$$

It follows from (1.1) that

$$|A_0(z)| \leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \ldots |A_1(z)|\left|\frac{f'(z)}{f(z)}\right|. \tag{3.3}$$

Combining (3.1), (3.2) and (3.3), we get

$$|A_0(z)| \leq (\sum_{i=1}^{k-1} |A_i(z)| + 1)\left(\frac{1}{1-|z|}\right)^{j(\sigma+2+\varepsilon)}, \tag{3.4}$$

$$|A_0(z)| \leq (\sum_{i=1}^{k-1} |A_i(z)| + 1)\exp_{p-1}\left\{\left(\frac{1}{1-|z|}\right)^{\sigma+\varepsilon}\right\}. \tag{3.5}$$

By the assumption (1.2) and Lemma 2.2, for any $\mu > 0$, there exists a set $E_2 \subset [0, 1)$ with $\int_{E_2} \frac{dr}{1-r} = \infty$ such that for all $z \in \Delta$ satisfying $|z| \in E_2$, we obtain

$$\frac{\sum_{i=1}^{k-1} |A_i(z)| + 1}{|A_0(z)|} \exp_p\left\{\frac{\lambda}{(1-|z|)^{\mu}}\right\} < 1.$$

Hence, for any $z \in \Delta$ satisfying $|z| \in E_2$,

$$|A_0(z)| > (\sum_{i=1}^{k-1} |A_i(z)| + 1)\exp_p\left\{\frac{\lambda}{(1-|z|)^{\mu}}\right\}. \tag{3.6}$$

Obviously, (3.6) contradicts with (3.4) and (3.5) for all $z \in \{z \in \Delta : |z| \in E_2 \setminus E_1\}$, where $E_2 \setminus E_1$ is of infinite logarithmic measure. So, $\sigma_{[p,q]}(f) = \infty$.

From Lemma 2.1 and $\sigma_{[p,q]}(f) = \infty$, there exists a set $E_3 \subset [0, 1)$ with $\int_{E_3} \frac{dr}{1-r} < \infty$ such that for all $z \in \Delta$ satisfying $|z| \notin E_3$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq \left(\frac{1}{1-|z|}\right)^{j(2+\varepsilon)} [T(s(|z|), f)]^j, \tag{3.7}$$

where $s(|z|) = 1 - d(1 - |z|), d \in (0, 1)$.

By (3.3) and (3.7), we get

$$|A_0(z)| \leq (\sum_{i=1}^{k-1} |A_i(z)| + 1) \left(\frac{1}{1-|z|}\right)^{j(2+\varepsilon)} [T(s(|z|), f)]^j. \tag{3.8}$$

Thus, combining (3.8) and (3.6), for all $z \in \Delta$ satisfying $|z| \in E_2 \setminus E_3$, we have

$$\exp_p\left\{\frac{\lambda}{(1-|z|)^{\mu}}\right\} \leq \left(\frac{1}{1-|z|}\right)^{j(2+\varepsilon)} [T(s(|z|), f)]^j. \tag{3.9}$$

Now, let $s(|z|) = R$, we have $1 - |z| = \frac{1}{d}(1 - R)$ and $\int_{E_3} \frac{dR}{1-R} < \infty$. Thus, (3.9) can be written as

$$\exp_p\left\{\frac{\lambda d^{\mu}}{(1-R)^{\mu}}\right\} \left(\frac{1-R}{d}\right)^{j(2+\varepsilon)} \leq [T(R, f)]^j, \tag{3.10}$$

where $R \in d(E_2 \setminus E_3) + 1 - d$. Obviously, $\{d(E_2 \setminus E_3) + 1 - d\}$ is of infinite logarithmic measure. Then by (3.10), we can get

$$\sigma_{[p+1,q]}(f) = \varlimsup_{R\to 1^-} \frac{\log^+_{p+1} T(R, f)}{-\log_q(1-R)} \geq \mu.$$

□

Proof of Theorem 1.4 . Suppose that f is a nontrivial solution of (1.1) with $\sigma_{[p,q]}(f) = \sigma < \infty$. If $p = q = 1$, from Lemma 2.3, for any $\varepsilon > 0$, for all r outside a set $E_4 \subset [0, 1)$ with $\int_{E_4} \frac{dR}{1-r} < \infty$, we have

$$m\left(r, \frac{f^{(j)}}{f}\right) = O\left(\log\left(\frac{1}{1-r}\right)\right), j = 1, ..., k. \tag{3.11}$$

By (1.1), we have

$$-A_0(z) = \frac{f^{(k)}(z)}{f(z)} + A_{k-1}(z)\frac{f^{(k-1)}(z)}{f(z)} + ... + A_1(z)\frac{f^{'}(z)}{f(z)},$$

thus

$$m(r, A_0) \leq \sum_{i=1}^{k-1} m(r, A_i) + \sum_{j=1}^{k} m\left(r, \frac{f^{(j)}}{f}\right) + O(1). \tag{3.12}$$

By (3.11) and (3.12), we have

$$T(r,A_0)=m(r,A_0)\le\sum_{i=1}^{k-1}T(r,A_i)+O\left(\log\left(\frac{1}{1-r}\right)\right),|z|\notin E_4. \tag{3.13}$$

If $p\ge q>1$, from Lemma 2.4, for any $\varepsilon>0$, for all r outside a set $E_5\subset[0,1)$ with $\int_{E_5}\frac{dR}{1-r}<\infty$, we have

$$m\left(r,\frac{f^{(j)}}{f}\right)=O\left(\exp_{p-q-1}\left(\frac{1}{1-r}\right)^{\sigma+\varepsilon}\right),j=1,...,k. \tag{3.14}$$

By (3.12) and (3.14), for all $|z|=r\notin E_5$,

$$T(r,A_0)=m(r,A_0)\le\sum_{i=1}^{k-1}T(r,A_i)+O\left(\exp_{p-q-1}\left(\frac{1}{1-r}\right)^{\sigma+\varepsilon}\right). \tag{3.15}$$

By the assumption (1.3) and Lemma 2.2, for any $\mu>0$, there exists a set $E_6\subset[0,1)$ with $\int_{E_6}\frac{dr}{1-r}=\infty$ such that for all $|z|\in E_6$, we have

$$\frac{\prod_{i=1}^{k-1}e^{T(r,A_i)}}{e^{T(r,A_0)}}\exp_p\left\{\frac{\lambda}{(1-|z|)^{\mu}}\right\}<1.$$

It yields that

$$T\left(r,A_0\right)-\sum_{i=1}^{k-1}T\left(r,A_i\right)>\exp_{p-1}\left\{\frac{\lambda}{(1-|z|)^{\mu}}\right\},|z|\in E_6. \tag{3.16}$$

It is easy to see (3.16) contradicts with (3.15) in $|z|\in E_6\setminus E_5$ or contradicts with (3.13) in $|z|\in E_6\setminus E_4$. Therefore, $\sigma_{[p,q]}(f)=\infty$.

Next, by Lemma 2.3, we have

$$m\left(r,\frac{f^{(j)}}{f}\right)\le O(\log^+T(r,f)+\log\frac{1}{1-r}), \tag{3.17}$$

possibly outside a set $E_7\subset[0,1)$ with $\int_{E_7}\frac{dr}{1-r}<\infty$. By (3.12) and (3.17), we have

$$T\left(r,A_0\right)\le\sum_{i=1}^{k-1}T\left(r,A_i\right)+O(\log^+T(r,f)+\log\frac{1}{1-r}),|z|\notin E_7. \tag{3.18}$$

Combining (3.16) and (3.18), we have

$$\exp_{p-1}\left\{\frac{\lambda}{(1-|z|)^{\mu}}\right\}\le O\left(\log^+T(r,f)+\log\left(\frac{1}{1-r}\right)\right),|z|\in E_6\setminus E_7. \tag{3.19}$$

Obviously, $E_6\setminus E_7$ is of infinite logarithmic measure. Then by (3.19) we can get

$$\sigma_{[p+1,q]}(f)=\varlimsup_{r\to1^-}\frac{\log^+_{p+1}T(r,f)}{-\log_q(1-r)}\ge\mu.$$

By Lemma 2.5 and the assumption of Theorem 1.4, we have $\sigma_{[p+1,q]}(f)\le\sigma_{M,[p,q]}(A_0)=\mu$. Therefore, $\sigma_{[p+1,q]}(f)=\sigma_{M,[p,q]}(A_0)=\mu$. The proof of Theorem 1.4 is completed. □

4. Proofs of Theorems 1.5 and 1.6

Proof of Theorem 1.5 . Suppose that f is a nontrivial solution of (1.1) with $\sigma(f) = \sigma < \infty$. Taking $s+1 \le j \le k$ and by the first fundamental theorem of Nevanlinna, we have

$$m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) \le m\left(r, \frac{f^{(j)}}{f}\right) + m\left(r, \frac{f}{f^{(s)}}\right).$$

Because f just has finite many zeros, by the definition of the counting function, we get

$$N\left(r, \frac{f^{(s)}}{f}\right) = O(1).$$

So,

$$\begin{aligned} m\left(r, \frac{f}{f^{(s)}}\right) &\le T\left(r, \frac{f}{f^{(s)}}\right) \\ &= T\left(r, \frac{f^{(s)}}{f}\right) + O(1) \\ &= m\left(r, \frac{f^{(s)}}{f}\right) + N\left(r, \frac{f^{(s)}}{f}\right) + O(1) \\ &\le m\left(r, \frac{f^{(s)}}{f}\right) + O(1). \end{aligned}$$

Thus, from (3.11),

$$m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) \le O\left(\log\left(\frac{1}{1-r}\right)\right), |z| \notin E_4, \tag{4.1}$$

where E_4 with finite logarithmic measure. Taking $0 \le j \le s-1$, by the first fundamental theorem of Nevanlinna, we can get

$$\begin{aligned} T\left(r, \frac{f^{(j)}}{f^{(s)}}\right) &= T\left(r, \frac{f^{(s)}}{f^{(j)}}\right) + O(1) \\ &= m\left(r, \frac{f^{(s)}}{f^{(j)}}\right) + N\left(r, \frac{f^{(s)}}{f^{(j)}}\right) + O(1). \end{aligned}$$

Because $f^{(j)}$ just has finite many zeros when $0 \le j \le s-1$, by the definition of the counting function, we get

$$N\left(r, \frac{f^{(s)}}{f^{(j)}}\right) = O(1).$$

So,

$$m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) \le T\left(r, \frac{f^{(j)}}{f^{(s)}}\right) = m\left(r, \frac{f^{(s)}}{f^{(j)}}\right) + O(1). \tag{4.2}$$

From (1.1),

$$-A_s = \frac{f^{(k)}}{f^{(s)}} + A_{k-1}\frac{f^{(k-1)}}{f^{(s)}} + ... + A_{s+1}\frac{f^{(s+1)}}{f^{(s)}} + A_{s-1}\frac{f^{(s-1)}}{f^{(s)}} + ... + A_0\frac{f}{f^{(s)}}.$$

It follows that

$$m\left(r, A_s\right) \le \sum_{i\ne s} m\left(r, A_i\right) + \sum_{s+1\le j\le k} m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) + \sum_{0\le j\le s-1} m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) + O(1). \tag{4.3}$$

Combining (4.1), (4.2) in (4.3), we have

$$T(r, A_s) = m(r, A_s) \le \sum_{i\ne s} T\left(r, A_i\right) + O\left(\log\left(\frac{1}{1-r}\right)\right), |z| \notin E_4. \tag{4.4}$$

By the assumption (1.4) and Lemma 2.2, for any $\mu > 0$, there exists a set $E_8 \subset [0, 1)$ with $\int_{E_8} \frac{dr}{1-r} = \infty$ such that for all $|z| \in E_8$, we have

$$\frac{\prod\limits_{i\ne s} e^{T(r,A_i)}}{e^{T(r,A_s)}(1-|z|)^{\mu}} < 1.$$

It yields that for any $\mu > 0$, $|z| \in E_8$,

$$\sum_{i\ne s} T\left(r, A_i\right) - T\left(r, A_s\right) + \mu\log\left(\frac{1}{1-|z|}\right) < 0. \tag{4.5}$$

By (4.4) and (4.5), for any $|z| \in E_8 \setminus E_4$, we have

$$\mu\log\left(\frac{1}{1-|z|}\right) < O\left(\log\left(\frac{1}{1-r}\right)\right).$$

Obviously, $E_8 \setminus E_4$ is of infinite logarithmic measure. It is easy to see this is a contradicts, thus $\sigma(f) = \infty$. □

Proof of Theorem 1.6. Suppose that f is a nontrivial solution of (1.1) with finite order $\sigma_{[p,q]}(f) = \sigma < \infty$. If $p = q = 1$, from Lemma 2.1, for any $\varepsilon > 0$, there exists a set $E_9 \subset [0, 1)$ with $\int_{E_9} \frac{dr}{1-r} < \infty$ such that $s + 1 \le j \le k$ and $r \in [0, 1) \setminus E_9$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \le O\left(\frac{1}{1-r}\right)^{(\sigma+2+\varepsilon)(j-s)}.$$

Therefore,

$$m\left(r, \frac{f^{(j)}(z)}{f^{(s)}(z)}\right) \le O\left(\log^+\left(\frac{1}{1-r}\right)\right), r \to 1^-. \tag{4.6}$$

For $0 \le j \le s-1$, by (4.2), (4.3) and (4.6), we have

$$T(r, A_s) = m(r, A_s) \le \sum_{i\ne s} T\left(r, A_i\right) + O\left(\log^+\left(\frac{1}{1-r}\right)\right). \tag{4.7}$$

If $p \geq q > 1$, from Lemma 2.4 and for $j = 1, ..., k$,

$$m\left(r, \frac{f^{(j)}}{f}\right) = O\left(\exp_{p-q-1}\left(\frac{1}{1-r}\right)^{\sigma+\varepsilon}\right),$$

holds for all r outside a set $E_{10} \subset [0, 1)$ with $\int_{E_{10}} \frac{dr}{1-r} < \infty$. Note that $s + 1 \leq j \leq k$, f just has finite many zeros, by the definition of the counting function, we get

$$N\left(r, \frac{f^{(s)}}{f}\right) = O(1).$$

So,

$$\begin{aligned} m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) &\leq m\left(r, \frac{f^{(j)}}{f}\right) + m\left(r, \frac{f}{f^{(s)}}\right) \\ &\leq O\left(\exp_{p-q-1}\left(\frac{1}{1-r}\right)^{\sigma+\varepsilon}\right) + T\left(r, \frac{f^{(s)}}{f}\right) \\ &\leq O\left(\exp_{p-q-1}\left(\frac{1}{1-r}\right)^{\sigma+\varepsilon}\right) + T\left(r, \frac{f}{f^{(s)}}\right) \\ &\leq O\left(\exp_{p-q-1}\left(\frac{1}{1-r}\right)^{\sigma+\varepsilon}\right) + m\left(r, \frac{f^{(s)}}{f}\right) \\ &\leq O\left(\exp_{p-q-1}\left(\frac{1}{1-r}\right)^{\sigma+\varepsilon}\right) \end{aligned} \tag{4.8}$$

For $0 \leq j \leq s - 1$, by (4.2), (4.3) and (4.8), for any $|z| \notin E_{10}$, we have

$$T(r, A_s) \leq \sum_{i \neq s} T(r, A_i) + O\left(\exp_{p-q-1}\left(\frac{1}{1-r}\right)^{\sigma+\varepsilon}\right). \tag{4.9}$$

By the assumption (1.5) and Lemma 2.2, for any $\mu > 0$, there exists a set $E_{11} \subset [0, 1)$ with $\int_{E_{11}} \frac{dr}{1-r} = \infty$ such that for all $|z| \in E_{11}$, we have

$$\frac{\prod_{i \neq s} e^{T(r,A_i)}}{e^{T(r,A_s)}} \exp_p\left\{\frac{\lambda}{(1-|z|)^{\mu}}\right\} < 1, i = 0, ..., k-1.$$

It yields that

$$T(r, A_s) - \sum_{i \neq s} T(r, A_i) > \exp_{p-1}\left\{\frac{\lambda}{(1-|z|)^{\mu}}\right\}, |z| = r \in E_{11}. \tag{4.10}$$

It is easy to see (4.10) contradicts with (4.9) in $|z| \in E_{11} \setminus E_{10}$ and contradicts with (4.7) in $|z| \in E_{11} \setminus E_9$. Therefore, $\sigma_{[p,q]}(f) = \infty$.

Next from Lemma 2.1, for any $\varepsilon > 0$, there exists a set $E_{12} \subset [0, 1)$ with $\int_{E_{12}} \frac{dr}{1-r} < \infty$ such that $s + 1 \leq j \leq k$ and $|z| \in [0, 1) \setminus E_{12}$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \leq \left(\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \max\left\{\log\frac{1}{1-|z|}, T(s(|z|), f)\right\}\right)^{j-s},$$

where $s(|z|) = 1 - d(1-|z|), d \in (0,1)$. Since $\lim\limits_{r\to 1^-} \frac{s(r)}{r} = 1$, then let $r \to 1^-$, we have $\log \frac{1}{1-r} > 1$, and $T(s(|z|), f) = T(r, f)$. Thus,

$$\begin{aligned} m\left(r, \frac{f^{(j)}}{f^{(s)}}\right) &\leq \log^+ \left(\left(\frac{1}{1-r}\right)^{2+\varepsilon} \left\{\log \frac{1}{1-r} + T(s(r), f)\right\}\right)^{j-s} \\ &\leq (j-s) \log^+ \left(\left(\frac{1}{1-r}\right)^{2+\varepsilon} \left\{\log \frac{1}{1-r} + T(r, f)\right\}\right) \\ &\leq O(\log^+ T(r, f) + \log^+ \frac{1}{1-r}), r \to 1^-. \end{aligned} \tag{4.11}$$

For $0 \leq j \leq s-1$, combining (4.2), (4.3), (4.10) and (4.11), we have

$$\exp_{p-1}\left\{\frac{\lambda}{(1-|z|)^\mu}\right\} \leq O\left(\log^+ T(r, f) + \log^+ \left(\frac{1}{1-r}\right)\right), |z| \in E_{11} \setminus E_{12}. \tag{4.12}$$

Obviously, $E_{11} \setminus E_{12}$ is of infinite logarithmic measure. Then by (4.12) we can get

$$\sigma_{[p+1,q]}(f) = \overline{\lim_{r\to 1^-}} \frac{\log^+_{p+1} T(r, f)}{-\log_q(1-r)} \geq \mu.$$

By Lemma 2.5 and the assumption of Theorem 1.6, we have $\sigma_{[p+1,q]}(f) \leq \sigma_{M,[p,q]}(A_s) = \mu$. So, $\sigma_{[p+1,q]}(f) = \sigma_{M,[p,q]}(A_s) = \mu$. The proof of Theorem 1.6 is completed. □

5. Conclusions

Many results on $[p, q]$-order of solutions of (1.1) have been found by different researchers in Δ, in this paper the difference is that we discussed the [p, q]-order of growth of solutions of linear differential Eq (1.1) which $A_j(z)$ dominate the others coefficients near a point on the boundary of the unit disc.

(1) Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions satisfying

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\sum\limits_{i=1}^{k-1} |A_i(z)| + 1}{|A_0(z)|} \exp_p\left\{\frac{\lambda}{(1-|z|)^\mu}\right\} < 1,$$

then every nontrivial solution $f(z)$ of (1.1) satisfies $\sigma_{[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) \geq \mu$ by Theorem 1.3, and

$$\max\left\{\sigma_{M,[p,q]}(A_i) : i = 1, \cdots, k-1\right\} \leq \sigma_{M,[p,q]}(A_0) = \mu,$$

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod\limits_{i=1}^{k-1} e^{T(r,A_i)}}{e^{T(r,A_0)}} \exp_p\left\{\frac{\lambda}{(1-|z|)^\mu}\right\} < 1,$$

then every nontrivial solution $f(z)$ of (1.1) satisfies $\sigma_{[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) = \sigma_{M,[p,q]}(A_0)$ by Theorem 1.4 .

(2) At the same time, we considered $j = s$ in Theorems 1.5 and 1.6 which add an essential condition for every nontrivial solution $f(z)$ of (1.1) , where $s = 1, 2, ..., k$. Let $A_0(z), \ldots, A_{k-1}(z)$ be analytic functions satisfying

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod\limits_{i\neq s} e^{T(r,A_i)}}{e^{T(r,A_s)}(1-|z|)^{\mu}} < 1,$$

then every nontrivial solution $f(z)$ of (1.1), in which $f^{(n)}(z)$ just has finite many zeros for all $n < s(n = 0, ..., s-1)$, is of infinite order by Theorem 1.5,
and

$$\max\left\{\sigma_{M,[p,q]}(A_i) : i = 0, 1, \cdots, s-1, s+1, \cdots, k-1\right\} \leq \sigma_{M,[p,q]}(A_s) = \mu.$$

$$\lim_{\substack{|z|\to 1^- \\ z\in\gamma}} \frac{\prod\limits_{i} e^{T(r,A_i)}}{e^{T(r,A_s)}} \exp_p\left\{\frac{\lambda}{(1-|z|)^{\mu}}\right\} < 1,$$

then, every nontrivial solution $f(z)$ of (1.1), in which $f^{(n)}(z)$ just has finite many zeros for all $n < s(n = 0, ..., s-1)$, satisfies $\sigma_{[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) = \sigma_{M,[p,q]}(A_s)$ by Theorem 1.6.

Acknowledgments

This research work is supported by the National Natural Science Foundation of China (Grant No. 11861023), and the Foundation of Science and Technology project of Guizhou Province of China (Grant No. [2018]5769-05).

Conflict of interest

The authors declare no conflicts of interest in this paper.

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