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Research article

Certain subclass of analytic functions with respect to symmetric points associated with conic region

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Abstract: The purpose of this paper is to introduce and study a new subclass of analytic functions with respect to symmetric points associated to a conic region impacted by Janowski functions. Also, the study has been extended to quantum calculus by replacing the ordinary derivative with a q-derivative in the defined function class. Interesting results such as initial coefficients of inverse functions and Fekete-Szegö inequalities are obtained for the defined function classes. Several applications, known or new of the main results are also presented.

Keywords: analytic functions; Bazilevič functions; starlike and convex functions; subordination; Fekete-Szegö problem; coefficient inequalities; q-calculus; Jackson's q-derivative operator
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1. Introduction

We let $\mathcal{A}$ to denote the usual class of analytic functions having a Taylor's series expansion of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{U} = \{z : |z| < 1\}). \tag{1.1}$$

Let $0 \leq \eta < 1$, $\mathcal{S}^*(\eta)$ and $\mathcal{C}(\eta)$ symbolize the classes of starlike functions of order η and convex functions of order η respectively. In $\mathcal{A}$, we classify the collection $\mathcal{P}$ of functions $p \in \mathcal{A}$ with $p(0) = 1$ and $Re\, p(z) > 0$. The class of functions in $\mathcal{P}$ is not univalent. However, if the family of functions in $\mathcal{P}$

are single valued then the set $\mathcal{P}$ is normal and compact [14, p. 136]. Babalola [4] introduced the class of functions $\mathcal{L}_\lambda(\eta)$ so called *λ-pseudo-starlike functions of order η* as follows: A function $f \in \mathcal{A}$ is said to be in $\mathcal{L}_\lambda(\eta)$, with $0 \leq \eta < 1$, $\lambda \geq 1$, if and only if it satisfies the inequality

$$\operatorname{Re} \frac{z\,(f'(z))^\lambda}{f(z)} > \eta, \; z \in \mathbb{U}.$$

Let f and g be analytic in $\mathbb{U}$. Then we say that the function f is *subordinate* to g in $\mathbb{U}$, if there exists a Schwarz function w in $\mathbb{U}$ such that $|w(z)| < |z|$ and $f(z) = g(w(z))$, denoted by $f \prec g$. If g is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

Using the concept of subordination for holomorphic functions, Ma and Minda [12] introduced the classes

$$\mathcal{S}^*(\psi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \psi(z) \right\} \quad \text{and} \quad C(\psi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \psi(z) \right\},$$

where $\psi \in \mathcal{P}$ with $\psi'(0) > 0$ maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to real axis. By choosing ψ to map unit disc on to some specific regions like parabolas, cardioid, lemniscate of Bernoulli, booth lemniscate in the right-half of the complex plane, various interesting subclasses of starlike and convex functions can be obtained.

For arbitrary fixed numbers C, D, $-1 < C \leq 1$, $-1 \leq D < C$, we denote by $\mathcal{P}(C, D)$ the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ analytic in the unit disc and $p(z) \in \mathcal{P}(C, D)$ if and only if

$$p(z) = \frac{1 + Cw(z)}{1 + Dw(z)},$$

where $w(z)$ is the Schwarz function. Geometrically, $p(z) \in \mathcal{P}(C, D)$ if and only if $p(0) = 1$ and $p(\mathbb{U})$ lies inside an open disc centred with center $\frac{1-CD}{1-D^2}$ on the real axis having radius $\frac{C-D}{1-D^2}$ with diameter end points $p_1(-1) = \frac{1-C}{1-D}$ and $p_1(1) = \frac{1+C}{1+D}$. On observing that $w(z) = \frac{p(z)-1}{p(z)+1}$ for $p(z) \in \mathcal{P}$, we have $S(z) \in \mathcal{P}(C, D)$ if and only if for some $p(z) \in \mathcal{P}$

$$S(z) = \frac{(1 + C)p(z) + 1 - C}{(1 + D)p(z) + 1 - D}. \tag{1.2}$$

For detailed studying on the class of Janowski functions, we refer to [8]. For $-1 \leq D < C \leq 1$ we denote by $\mathcal{S}^*(C, D)$ and by $C(C, D)$ the class of Janowski starlike functions and Janowski convex functions, defined by

$$\mathcal{S}^*(C, D) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Cz}{1 + Dz}, -1 \leq D < C \leq 1 \right\}$$

and

$$C(C, D) := \left\{ f \in \mathcal{A} : 1 + \frac{zf'(z)}{f(z)} \prec \frac{1 + Cz}{1 + Dz}, -1 \leq D < C \leq 1 \right\},$$

respectively.

The function $\hat{p}_{\eta,\sigma}(z)$ plays the role of an extremal functions those related to these conic domain $\mathcal{D}_k = \left\{u + iv : \ u > \eta \sqrt{(u-1)^2 + v^2}\right\}$ and is given by

$$\hat{p}_{\eta,\sigma}(z) = \begin{cases} \frac{1+(1-2\sigma)z}{1-z}, & \text{if } \eta = 0, \\ 1 + \frac{2(1-\sigma)}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & \text{if } \eta = 1, \\ 1 + \frac{2(1-\sigma)}{1-\eta^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos\eta\right) arc\tanh\sqrt{z}\right], & \text{if } 0 < \eta < 1, \\ 1 + \frac{2(1-\sigma)}{1-\eta^2}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{t}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right) + \frac{1}{\eta^2-1}, & \text{if } \eta > 1, \end{cases} \tag{1.3}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t \in (0, 1)$ and t is chosen such that $\eta = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, with $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$. Clearly, $\hat{p}_{\eta,\sigma}(z)$ is in $\mathcal{P}$ with the expansion of the form

$$\hat{p}_{\eta,\sigma}(z) = 1 + \delta_1 z + \delta_2 z^2 + \cdots \quad (\delta_j = p_j(\eta, \sigma),\ j = 1, 2, 3, \cdots), \tag{1.4}$$

we get

$$\delta_1 = \begin{cases} \frac{8(1-\sigma)(\arccos\eta)^2}{\pi^2(1-\eta^2)}, & \text{if } 0 \leq \eta < 1, \\ \frac{8(1-\sigma)}{\pi^2}, & \text{if } \eta = 1, \\ \frac{\pi^2(1-\sigma)}{4\sqrt{t}(\eta^2-1)R^2(t)(1+t)}, & \text{if } \eta > 1. \end{cases} \tag{1.5}$$

Noor in [15, 16] replaced $p(z)$ in (1.2) with $\hat{p}_{\eta,\sigma}(z)$ and studied the impact of Janowski function on conic regions.

For $f \in \mathcal{A}$ given by (1.1) and $0 < q < 1$, the Jackson's q-derivative operator or q-difference operator for a function $f \in \mathcal{A}$ is defined by (see [1,2])

$$\mathfrak{D}_q f(z) := \begin{cases} f'(0), & \text{if } z = 0, \\ \dfrac{f(z) - f(qz)}{(1-q)z}, & \text{if } z \neq 0. \end{cases} \tag{1.6}$$

From (1.6), if f has the power series expansion (1.1) we can easily see that $\mathfrak{D}_q f(z) = 1 + \sum\limits_{k=2}^{\infty} [k]_q a_k z^{k-1}$, for $z \neq 0$, where the q-integer number $[k]_q$ is defined by

$$[k]_q := \frac{1-q^k}{1-q}$$

and note that $\lim\limits_{q\to1^-} \mathfrak{D}_q f(z) = f'(z)$. Throughout this paper, we let denote

$$([k]_q)_n := [k]_q[k+1]_q[k+2]_q \ldots [k+n-1]_q.$$

The q-Jackson integral is defined by (see [7])

$$I_q\left[f(z)\right] := \int_0^z f(t)d_q t = z(1-q)\sum_{k=0}^{\infty} q^k f(zq^k)$$

provided the q-series converges. Further observe that

$$D_q I_q f(z) = f(z) \qquad \text{and} \qquad I_q D_q f(z) = f(z) - f(0),$$

where the second equality holds if f is continuous at $z = 0$.

For the function $f \in \mathcal{A}$ given by (1.1) and $h \in \mathcal{A}$ of the form $h(z) = z + \sum_{k=2}^{\infty} \Theta_k z^k$, the Hadamard product (or convolution) of these two functions is defined by

$$\mathcal{H}(z) := (f * h)(z) := z + \sum_{k=2}^{\infty} a_k \Theta_k z^k, \ z \in \mathbb{U}. \tag{1.7}$$

Throughout our present discussion, to avoid repetition, we will assume that $-1 \leq D < C \leq 1$ and $\Theta_k \neq 0$ may real or complex numbers.

Motivated by the definition of λ-pseudo-starlike functions, we now introduce the following class of functions:

Definition 1. For $s,\ t \in \mathbb{C}$, with $s \neq t$, $|t| \leq 1$, $\lambda \geq 1$, $0 \leq \alpha \leq 1$, $\beta \geq 0$ and $\mathcal{H} = f * h$ defined as in (1.7), we say that the function f belongs to the class $\mathcal{K}_{\lambda}^{\beta}(\alpha;\ s, t;\ \psi;\ h;\ C, D)$ if it satisfies the subordination condition

$$\frac{[(s-t)z]^{1-\beta}\,[\mathcal{H}'(z) + \alpha z \mathcal{H}''(z)]^{\lambda}}{[(1-\alpha)\,[\mathcal{H}(sz) - \mathcal{H}(tz)] + \alpha z\,[s\mathcal{H}'(sz) - t\mathcal{H}'(tz)]]^{1-\beta}} \prec \frac{(C+1)\psi(z) - (C-1)}{(D+1)\psi(z) - (D-1)}, \tag{1.8}$$

where "$\prec$" denotes subordination, $\psi \in \mathcal{P}$ and ψ which has a power series expansion of the form

$$\psi(z) = 1 + L_1 z + L_2 z^2 + L_3 z^3 + \cdots, \ z \in \mathbb{U}, \ L_1 \neq 0. \tag{1.9}$$

Remark 1. Here we list some special cases of the class $\mathcal{K}_{\lambda}^{\beta}(\alpha;\ s, t;\ \psi;\ h;\ C, D)$:

(i) If we replace $h(z) = z + \sum_{k=2}^{\infty} z^n$, $s = \lambda = 1$, $\beta = 0$ and $\psi(z) = \hat{p}_{\eta,\sigma}(z)$ in $\mathcal{K}_{\lambda}^{\beta}(\alpha;\ s, t;\ \psi;\ h;\ C, D)$, where $\hat{p}_{\eta,\sigma}(z)$ is defined as in (1.3), we can get the classes $\eta - \mathcal{US}[C, D, \sigma, t]$ and $\eta - \mathcal{US}[C, D, \sigma, t]$ defined by Arif et al. in [3] by choosing $\alpha = 0$ and $\alpha = 1$ respectively.

(ii) If we replace $h(z) = z + \sum_{k=2}^{\infty} z^n$, $s = \lambda = 1$, $t = \beta = 0$ and $\psi(z) = \hat{p}_{\eta,0}(z)$ in $\mathcal{K}_{\lambda}^{\beta}(\alpha;\ s, t;\ \psi;\ h;\ C, D)$, where $\hat{p}_{\eta,0}(z)$ is defined as in (1.3), we can get the classes $\eta - \mathcal{ST}[C, D]$ and $\eta - \mathcal{UC}[C, D]$ defined by Noor and Malik in [16, Definition 1.3 and Definition 1.4] by choosing $\alpha = 0$ and $\alpha = 1$ respectively.

Several other well-known classes can be obtained as special cases of $\mathcal{K}_{\lambda}^{\beta}(\alpha;\ s, t;\ \psi;\ h;\ C, D)$, refer to [3] and the references provided therein.

The study of geometric function theory in dual with quantum calculus was initiated by Srivastava [20]. For recent developments pertaining to this duality theory, refer to [27] and references provided therein. A number of families of q-extensions of analytic functions in the open unit disk $\mathbb{U}$ have been defined by means of basic (or q-)calculus and considered from many distinctive prospectives and viewpoints. Many authors, generalize and study certain subclasses of analytic functions involving q-derivative operators and settle characteristic equations for these presumably new classes and study numerous coefficient inequalities and also carry out appropriate connections with those in multiple other concerning works on this subject. The study of conic regions impacted by Janowski function involving q-derivative was dealt in detail by Srivastava et al. [21–26, 28], also see [11, 29, 30] and references cited therein.

Using the q-derivative, we now define the following.

Definition 2. For $s,\ t \in \mathbb{C}$, with $s \neq t$, $|t| \leq 1$, $\lambda \geq 1$, $0 \leq \alpha \leq 1$, $\beta \geq 0$ and $\mathcal{H} = f * h$ defined as in (1.7), we say that the function f belongs to the class $\mathcal{L}_{\lambda}^{\beta}(\alpha;\ s,t;\ \psi;\ h;\ C,D)$ if it satisfies the subordination condition

$$\frac{[(s-t)z]^{1-\beta}\,[\mathfrak{D}_q\mathcal{H}(z)+\alpha qz\mathfrak{D}_q^2\mathcal{H}(z)]^{\lambda}}{\left\{(1-\alpha)\,[\mathcal{H}(sz)-\mathcal{H}(tz)]+\alpha z\left[s\mathfrak{D}_q\mathcal{H}(sz)-t\mathfrak{D}_q\mathcal{H}(tz)\right]\right\}^{1-\beta}} \prec \frac{(C+1)\psi(z)-(C-1)}{(D+1)\psi(z)-(D-1)}, \tag{1.10}$$

where "$\prec$" denotes subordination, $\psi \in \mathcal{P}$ and ψ which has a power series expansion of the form (1.9).

Remark 2. Here we list some special cases of the class $\mathcal{L}_{\lambda}^{\beta}(\alpha;\ s,t;\ \psi;\ h;\ C,D)$:

(i) If we replace $h(z) = z + \sum_{k=2}^{\infty} z^n$, $C = s = \lambda = 1$, $\alpha = \beta = 0$, $D = t = -1$ and $\psi(z) = \hat{p}_{\eta,0}(z)$ in $\mathcal{L}_{\lambda}^{\beta}(\alpha;\ s,t;\ \psi;\ h;\ C,D)$, where $\hat{p}_{\eta,\sigma}(z)$ is defined as in (1.3), we can get the following class $\mathcal{S}_s(\hat{p}_{\eta})$ defined by

$$\mathcal{S}_s(\hat{p}_k) := \left\{ f \in \mathcal{A} :\ \frac{2z[\mathfrak{D}_q f(z)]}{f(z)-f(-z)} \prec \hat{p}_{\eta,0}(z) \right\}.$$

The class $\mathcal{S}_s(\hat{p}_{\eta})$ was defined by Olatunji and Dutta in [17, Definition 1.1].

(ii) If we replace $h(z) = z + \sum_{k=2}^{\infty} z^n$, $C = s = \lambda = 1$, $\beta = 0$, $D = t = -1$, $q \to 1^-$ and $\psi(z) = \hat{p}_{\eta,0}(z)$ in $\mathcal{L}_{\lambda}^{\beta}(\alpha;\ s,t;\ \psi;\ h;\ C,D)$, where $\hat{p}_{\eta,\sigma}(z)$ is defined as in (1.3), then we get the class $M_s(\alpha,\hat{p}_{\eta})$ recently studied by Kavitha and Dhanalakshmi [9, Definition 1.1].

We call by $Q\mathcal{L}_{\lambda}^{\beta}(\alpha;\ s,t;\ h;\ C,D)$ if $\psi(z)$ is replaced with $\dfrac{1+z}{1-qz}$, $q \in (0,\ 1)$ in (1.10). Remark that, by the definition of the subordination, a function $\mathcal{H} \in \mathcal{A}$ is said to be in $Q\mathcal{L}_{\lambda}^{\beta}(\alpha;\ s,t;\ h;\ C,D)$ if and only if there exists a function w analytic in $\mathbb{U}$, with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that

$$\begin{aligned} &\frac{[(s-t)z]^{1-\beta}\,[\mathfrak{D}_q\mathcal{H}(z)+\alpha qz\mathfrak{D}_q^2\mathcal{H}(z)]^{\lambda}}{\left\{(1-\alpha)\,[\mathcal{H}(sz)-\mathcal{H}(tz)]+\alpha z\left[s\mathfrak{D}_q\mathcal{H}(sz)-t\mathfrak{D}_q\mathcal{H}(tz)\right]\right\}^{1-\beta}} \\ =\ &\frac{(C+1)w(z)+2+(C-1)qw(z)}{(D+1)w(z)+2+(D-1)qw(z)}, \end{aligned} \tag{1.11}$$

where $q \in (0,\ 1)$. Class closely related to $Q\mathcal{L}_{\lambda}^{\beta}(\alpha;\ s,t;\ h;\ C,D)$ was studied by Srivastava et al. (see [22, Definition 8]).

2. Preliminaries

In this section we state the results that would be used to establish our main results which can be found in the standard text on univalent function theory.

Lemma 1. *[6, p. 56] If the function $f \in \mathcal{A}$ given by* (1.1) *and g given by*

$$g(w) = w + \sum_{k=2}^{\infty} b_k w^k \tag{2.1}$$

is inverse function, then the coefficients b_k, for $k \geq 2$, are given by

$$b_k = \frac{(-1)^{k+1}}{k!} \begin{vmatrix} ka_2 & 1 & 0 & \dots & 0 \\ 2ka_3 & (k+1)a_2 & 2 & \dots & 0 \\ 3ka_4 & (2k+1)a_3 & (k+2)a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & (k-2) \\ (k-1)ka_k & [k(k-2)+1]\, a_{k-1} & [k(k-3)+2]\, a_{k-2} & \dots & (2k-2)a_2 \end{vmatrix}. \tag{2.2}$$

Remark 3. The elements of the above determinant (2.2) are given by

$$\Lambda_{ij} = \begin{cases} [(i-j+1)k + j - 1]\, a_{i-j+2}, & \text{if } i+1 \geq j, \\ 0, & \text{if } i+1 < j. \end{cases}$$

Lemma 2. *[18, p. 41] If $p(z) = 1 + \sum\limits_{k=1}^{\infty} p_k z^k \in \mathcal{P}$, then $|p_k| \leq 2$ for all $k \geq 1$ and the inequality is sharp for $p_\lambda(z) = \dfrac{1+\lambda z}{1-\lambda z}$, $|\lambda| \leq 1$.*

Lemma 3. *[12] If $p(z) = 1 + \sum\limits_{k=1}^{\infty} p_k z^k \in \mathcal{P}$ and v is complex number, then*

$$\left| p_2 - v p_1^2 \right| \leq 2 \max \{1; |2v - 1|\}$$

and the result is sharp for the functions

$$p_1(z) = \frac{1+z}{1-z} \quad \textit{and} \quad p_2(z) = \frac{1+z^2}{1-z^2}.$$

3. Main results

In general, the functions in $\mathcal{K}_\lambda^\beta(\alpha;\ s, t;\ \psi;\ h;\ C, D)$ and $\mathcal{L}_\lambda^\beta(\alpha;\ s, t;\ \psi;\ h;\ C, D)$ are not univalent. Hence the inverse function of f defined in the unit disc is not guaranteed. However, it is always possible to find the inverse of a function in a smaller disk. The following theorem is based on the assumption that if g is the inverse of f.

Throughout this paper, we let once for all

$$\vartheta_n = \sum_{k=1}^{n} t^{k-1} s^{n-k}. \tag{3.1}$$

Theorem 4. *If $f \in \mathcal{K}_\lambda^\beta(\alpha;\ s, t;\ \psi;\ h;\ C, D)$, is given by* (1.1) *, then for the coefficients of $g = f^{-1}$ the following estimates hold:*

$$|b_2| \leq \frac{2|L_1|(C - D)}{|2\lambda(1+\alpha) + (\beta - 1)\{(1-\alpha)\vartheta_2 + 2\alpha\vartheta_3\}|\, |\Theta_2|} \tag{3.2}$$

and

$$|b_3| \leq \frac{2|L_1|(C-D)}{|3\lambda(1+2\alpha)+(\beta-1)\{(1-\alpha)\vartheta_3+3\alpha\vartheta_4\}|\,|\Theta_3|}\max\{1;\ |2\nu-1|\} \tag{3.3}$$

with

$$\nu := \frac{1}{4}\left[(D+1)L_1+2\left(1-\frac{L_2}{L_1}\right)+\mathcal{M}+\mathcal{N}\right], \tag{3.4}$$

where

$$\begin{aligned}\mathcal{M} := &\ \frac{L_1(C-D)\left[4\lambda(\lambda-1)(1+\alpha)^2+4\lambda(1+\alpha)(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}\right]}{2\left[2\lambda(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}\right]^2}\\ &+\frac{L_1(C-D)(\beta-1)(\beta-2)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}^2}{2\left[2\lambda(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}\right]^2}\end{aligned} \tag{3.5}$$

and

$$\mathcal{N} := \frac{8L_1(C-D)\left[3\lambda(1+2\alpha)+(\beta-1)\{(1-\alpha)\vartheta_3+3\alpha\vartheta_4\}\right]\Theta_3}{\left[2\lambda(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}\right]^2\Theta_2^2}. \tag{3.6}$$

Proof. If $f \in \mathcal{K}_\lambda^\beta(\alpha;\ s,t;\ \psi;\ h;\ C,D)$, then by the definition of subordination, there exists a function w analytic in $\mathbb{U}$, with $w(0)=0$ and $|w(z)|<1$, $z \in \mathbb{U}$, such that

$$\begin{aligned}&\frac{[(s-t)z]^{1-\beta}\left[\mathcal{H}'(z)+\alpha z\mathcal{H}''(z)\right]^\lambda}{\{(1-\alpha)\left[\mathcal{H}(sz)-\mathcal{H}(tz)\right]+\alpha z\left[s\mathcal{H}'(sz)-t\mathcal{H}'(tz)\right]\}^{1-\beta}}\\ =\ &\frac{(C+1)\psi(w(z))-(C-1)}{(D+1)\psi(w(z))-(D-1)},\ z\in\mathbb{U}.\end{aligned}$$

Thus, let $\ell \in \mathcal{P}$ be of the form $\ell(z) = 1+\sum\limits_{k=1}^{\infty} p_k z^k$ and defined by

$$\ell(z) := \frac{1+w(z)}{1-w(z)},\ z\in\mathbb{U}.$$

A simple computation shows that

$$\begin{aligned}w(z) &= \frac{\ell(z)-1}{\ell(z)+1} = \frac{p_1z+p_2z^2+p_3z^3+\ldots}{2+p_1z+p_2z^2+p_3z^3+\ldots}\\ &= \frac{1}{2}p_1z+\frac{1}{2}\left(p_2-\frac{1}{2}p_1^2\right)z^2+\frac{1}{2}\left(p_3-p_1p_2+\frac{1}{4}p_1^3\right)z^3+\cdots,\ z\in\mathbb{U}\end{aligned}$$

and considering

$$\begin{aligned}&\frac{(C+1)\psi(w(z))-(C-1)}{(D+1)\psi(w(z))-(D-1)}\\ =\ &1+\frac{L_1p_1(C-D)}{4}z+\frac{(C-D)L_1}{4}\left[p_2-p_1^2\left(\frac{(D+1)L_1+2\left(1-\frac{L_2}{L_1}\right)}{4}\right)\right]z^2+\cdots,\end{aligned}$$

we have

$$\frac{[(s-t)z]^{1-\beta}\left[\mathcal{H}'(z)+\alpha z\mathcal{H}''(z)\right]^{\lambda}}{\left[(1-\alpha)\left[\mathcal{H}(sz)-\mathcal{H}(tz)\right]+\alpha z\left[s\mathcal{H}'(sz)-t\mathcal{H}'(tz)\right]\right]^{1-\beta}}=1+\frac{L_1p_1(C-D)}{4}z$$
$$+\frac{(C-D)L_1}{4}\left[p_2-p_1^2\left(\frac{(D+1)L_1+2\left(1-\frac{L_2}{L_1}\right)}{4}\right)\right]z^2+\cdots,\ z\in\mathbb{U}. \tag{3.7}$$

The left hand side of (3.7) will be of the form

$$\begin{aligned}
&\frac{[(s-t)z]^{1-\beta}\left[\mathcal{H}'(z)+\alpha z\mathcal{H}''(z)\right]^{\lambda}}{\left[(1-\alpha)\left[\mathcal{H}(sz)-\mathcal{H}(tz)\right]+\alpha z\left[s\mathcal{H}'(sz)-t\mathcal{H}'(tz)\right]\right]^{1-\beta}}\\
&=1+\left[2\lambda(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}\right]\Theta_2a_2z\\
&\quad+\Big\{\left[3\lambda(1+2\alpha)+(\beta-1)\left\{(1-\alpha)\vartheta_3+3\alpha\vartheta_4\right\}\right]\Theta_3a_3\\
&\quad+\Big[2\lambda(\lambda-1)(1+\alpha)^2+2\lambda(1+\alpha)(\beta-1)\left\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\right\}\\
&\quad+\frac{(\beta-1)(\beta-2)}{2}\left\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\right\}^2\Big]\Theta_2^2a_2^2\Big\}z^2+\cdots,\ z\in\mathbb{U},
\end{aligned} \tag{3.8}$$

where Θ_k are the corresponding coefficients from the power series expansion of h, which may be real or complex.

From (3.7) and (3.8) we obtain

$$a_2=\frac{L_1p_1(C-D)}{\left[2\lambda(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}\right]\Theta_2} \tag{3.9}$$

and

$$\begin{aligned}
a_3=&\frac{L_1(C-D)}{\left[3\lambda(1+2\alpha)+(\beta-1)\left\{(1-\alpha)\vartheta_3+3\alpha\vartheta_4\right\}\right]\Theta_3}\Bigg[p_2-\frac{1}{4}\Bigg((D+1)L_1+2\left(1-\frac{L_2}{L_1}\right)\\
&+\frac{L_1(C-D)\left[4\lambda(\lambda-1)(1+\alpha)^2+4\lambda(1+\alpha)(\beta-1)\left\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\right\}\right]}{2\left[2\lambda(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}\right]^2}\\
&+\frac{L_1(C-D)(\beta-1)(\beta-2)\left\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\right\}^2}{2\left[2\lambda(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}\right]^2}\Bigg)p_1^2\Bigg].
\end{aligned} \tag{3.10}$$

From (2.2) we see that $b_2=-a_2$, and applying Lemma 2 for (3.9) we obtain the inequality (3.2).

Also, from (2.2) we have

$$\begin{aligned}
b_3&=\frac{(-1)^4}{3!}\begin{vmatrix}3a_2 & 1\\ 6a_3 & 4a_2\end{vmatrix}\\
&=2a_2^2-a_3\\
&=\frac{2L_1^2p_1^2(C-D)^2}{\left[2\lambda(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}\right]^2\Theta_2^2}\\
&-\frac{L_1(C-D)}{\left[3\lambda(1+2\alpha)+(\beta-1)\left\{(1-\alpha)\vartheta_3+3\alpha\vartheta_4\right\}\right]\Theta_3}\Bigg[p_2-\frac{1}{4}\Bigg((D+1)L_1+2\left(1-\frac{L_2}{L_1}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{L_1(C-D)\left[4\lambda(\lambda-1)(1+\alpha)^2+4\lambda(1+\alpha)(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}\right]}{2\left[2\lambda(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}\right]^2} \\
&+ \quad \left.\frac{L_1(C-D)(\beta-1)(\beta-2)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}^2}{2\left[2\lambda(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}\right]^2}\right)p_1^2\Bigg] \\
&= \quad \frac{-L_1(C-D)}{\left[3\lambda(1+2\alpha)+(\beta-1)\{(1-\alpha)\vartheta_3+3\alpha\vartheta_4\}\right]\Theta_3} \\
&\times \quad \left[p_2\frac{1}{4}\left\{(D+1)L_1+2\left(1-\frac{L_2}{L_1}\right)+\mathcal{M}+\mathcal{N}\right\}p_1^2\right],
\end{aligned}
$$

where $\mathcal{M}$ and $\mathcal{N}$ are given by (3.5) and (3.6) respectively. Now using Lemma 2 we get (3.3), with ν given by (3.4). □

Theorem 5. *If $f \in \mathcal{L}_\lambda^\beta(\alpha;\ s,t;\ \psi;\ h;\ C,D)$ is given by* (1.1) *then for the coefficients of $g = f^{-1}$ the following estimates hold:*

$$
|b_2| \le \frac{2|L_1|(C-D)}{|\lambda(1+q)(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+\alpha(1+q)\vartheta_3\}|\,|\Theta_2|} \tag{3.11}
$$

and

$$
|b_3| \le \frac{2|L_1|(C-D)}{\left|\lambda[3]_q(1+\alpha+q\alpha)+(\beta-1)\left\{(1-\alpha)\vartheta_3+\alpha[3]_q\vartheta_4\right\}\right|\,|\Theta_3|}\max\{1;\ |2\tau-1|\} \tag{3.12}
$$

with

$$
\tau := \frac{1}{4}\left[(D+1)L_1+2\left(1-\frac{L_2}{L_1}\right)+\mathcal{M}_q+\mathcal{N}_q\right], \tag{3.13}
$$

where

$$
\begin{aligned}
\mathcal{M}_q := &\frac{L_1(C-D)\left[\lambda(\lambda-1)(1+q)^2(1+2\alpha)^2+2\lambda(1+\alpha)(\beta-1)\{(1-\alpha)\vartheta_2+\alpha(1+q)\vartheta_3\}\right]}{2\left[\lambda(1+q)(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+\alpha(1+q)\vartheta_3\}\right]^2} \\
&+\frac{L_1(C-D)(\beta-1)(\beta-2)\{(1-\alpha)\vartheta_2+\alpha(1+q)\vartheta_3\}^2}{2\left[\lambda(1+q)(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+\alpha(1+q)\vartheta_3\}\right]^2}
\end{aligned} \tag{3.14}
$$

and

$$
\mathcal{N}_q := \frac{2L_1(C-D)\left[\lambda[3]_q(1+\alpha+q\alpha)+(\beta-1)\left\{(1-\alpha)\vartheta_3+\alpha[3]_q\vartheta_4\right\}\right]\Theta_3}{\left[\lambda(1+q)(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+\alpha(1+q)\vartheta_3\}\right]^2\Theta_2^2}. \tag{3.15}
$$

Proof. Let $f \in \mathcal{L}_\lambda^\beta(\alpha;\ s,t;\ \psi;\ h;\ C,D)$, then from (1.11) we have

$$
\begin{aligned}
&1+\left[2\lambda(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}\right]\Theta_2 a_2 z \\
&+\Big\{\left[3\lambda(1+2\alpha)+(\beta-1)\{(1-\alpha)\vartheta_3+3\alpha\vartheta_4\}\right]\Theta_3 a_3 \\
&+\Big[2\lambda(\lambda-1)(1+\alpha)^2+2\lambda(1+\alpha)(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}+
\end{aligned}
$$

$$\frac{(\beta-1)(\beta-2)}{2}\left\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\right\}^2\Big]\Theta_2^2a_2^2\Big\}z^2+\cdots=1+\frac{L_1p_1(C-D)}{4}z+$$

$$\frac{(C-D)L_1}{4}\left[p_2-p_1^2\left(\frac{(D+1)L_1+2\left(1-\frac{L_2}{L_1}\right)}{4}\right)\right]z^2+\cdots. \tag{3.16}$$

From (3.16) we can prove the assertion of Theorem 5 by the following the steps as in Theorem 4. □

The impact of the well-known Janowski function on

$$\mathcal{J}(z):=1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2,\ z\in\mathbb{U}, \tag{3.17}$$

was recently studied by Malik et al. [13]. Following the same steps as in Theorem 1 of [10] we get

$$\mathcal{J}(z)=1+\frac{8}{\pi^2}z+\frac{16}{3\pi^2}z^2+\ldots,\ z\in\mathbb{U}. \tag{3.18}$$

Replacing the values of L_1, L_2 and L_3 of Theorem 4 with the corresponding coefficients of the power series (3.18) we obtain the next result:

Theorem 6. *If $f\in\mathcal{K}_\lambda^\beta(\alpha;\ s,t;\ \mathcal{J};\ h;\ C,D)$ is given by* (1.1) *with $\mathcal{J}$ defined as in* (3.17) *and for the coefficients of $g=f^{-1}$ the following estimates hold:*

$$|b_2|\le\frac{16(C-D)}{\pi^2\left|2\lambda(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}\right||\Theta_2|}$$

and

$$\begin{aligned}|b_3|\le{}&\frac{16(C-D)}{\pi^2\left|3\lambda(1+2\alpha)+(\beta-1)\{(1-\alpha)\vartheta_3+3\alpha\vartheta_4\}\right||\Theta_3|}\max\left\{1;\ \frac{4}{\pi^2}\left|\left(D+1-\frac{\pi^2}{6}\right)\right.\right.\\&+\frac{8(C-D)\left[3\lambda(1+2\alpha)+(\beta-1)\{(1-\alpha)\vartheta_3+3\alpha\vartheta_4\}\right]\Theta_3}{\left[2\lambda(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}\right]^2\Theta_2^2}\\&\frac{(C-D)\left[4\lambda(\lambda-1)(1+\alpha)^2+4\lambda(1+\alpha)(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}\right]}{2\left[2\lambda(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}\right]^2}\\&\left.\left.+\frac{(C-D)(\beta-1)(\beta-2)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}^2}{2\left[2\lambda(1+\alpha)+(\beta-1)\{(1-\alpha)\vartheta_2+2\alpha\vartheta_3\}\right]^2}\right|\right\}.\end{aligned}$$

If we let $\alpha=\beta=t=0$, $s=\lambda=1$ and $h(z)=z+\sum_{k=2}^{\infty}\frac{(2)_{k-1}}{(1)_{k-1}}z^k$, $z\in\mathbb{U}$, in Theorem 6, we obtain the following result:

Corollary 7. *[13, Theorem 4] If $f\in\mathcal{K}_1^0(0;\ 1,0;\ \mathcal{J};\ h;\ C,D)$ with $\mathcal{J}$ defined as in* (3.17)*, is given by* (1.1)*, then for the coefficients of $g=f^{-1}$ the following estimate hold:*

$$|b_k|\le\frac{4(C-D)}{k(k-1)\pi^2},\ k=2,\ 3.$$

We let $L_c^b f$ to denote the well-known Carlson-Shaffer operator [5] which can be obtained by replacing $h(z) = z + \sum_{k=2}^{\infty} \frac{(b)_{k-1}}{(c)_{k-1}} z^k$ in (1.7).

Corollary 8. *If $L_c^b f \in \mathcal{A}$ satisfies the condition*

$$\frac{[(s-t)z]\,[(L_c^b f(z))' + \alpha z(L_c^b f(z))'']}{[(1-\alpha)\,[L_c^b f(sz) - L_c^b f(tz)] + \alpha z\,[s(L_c^b f(sz))' - t(L_c^b f(tz))']]} \prec z + \sqrt{1+z^2},$$

then for $\kappa \in \mathbb{C}$,

$$|b_2| \le \left|\frac{(c)_2}{(b)_2}\right| \frac{4}{|2(1+\alpha) - \{(1-\alpha)\vartheta_2 + 2\alpha\vartheta_3\}|}$$

and

$$|b_3| \le \left|\frac{(c)_3}{(b)_3}\right| \frac{4}{|3(1+2\alpha) - \{(1-\alpha)\vartheta_3 + 3\alpha\vartheta_4\}|} \max\{1;\ |2\kappa - 1|\}$$

with

$$\kappa := \frac{1}{4}\left[1 - \frac{2(1+2\alpha)\,\{(1-\alpha)\vartheta_2 + 2\alpha\vartheta_3\}}{[2(1+\alpha) - \{(1-\alpha)\vartheta_2 + 2\alpha\vartheta_3\}]^2} + \frac{16(b)_3[(c)_2]^2\,[2(1+2\alpha) - \{(1-\alpha)\vartheta_3 + 2\alpha\vartheta_4\}]}{(c)_3[(b)_2]^2\,[2(1+\alpha) - \{(1-\alpha)\vartheta_2 + 2\alpha\vartheta_3\}]^2}\right].$$

4. Fekete-Szegö inequality for the functions of $\mathcal{K}_\lambda^\beta(\alpha;\ s,t;\ \psi;\ h;\ C,D)$ and $\mathcal{L}_\lambda^\beta(\alpha;\ s,t;\ \psi;\ h;\ C,D)$

We will give the solution of the Fekete-Szegö problem for the functions that belong to the classes we defined in the first section.

Theorem 9. *If $f \in \mathcal{K}_\lambda^\beta(\alpha;\ s,t;\ \psi;\ h;\ C,D)$ is given by* (1.1)*, then for all $\mu \in \mathbb{C}$ we have*

$$\left|a_3 - \mu a_2^2\right| \le \frac{2|L_1|(C-D)}{|\,[3\lambda(1+2\alpha) + (\beta-1)\,\{(1-\alpha)\vartheta_3 + 3\alpha\vartheta_4\}]\,||\Theta_3|} \max\{1;\ |2\rho - 1|\}$$

with $\mathcal{M}$ and $\mathcal{N}$ is defined as in (3.5) *and* (3.6) *respectively, ρ is given by $\rho := \frac{1}{4}\Big[(D+1)L_1 + 2\left(1 - \frac{L_2}{L_1}\right) + \frac{\mu\mathcal{N}}{2} + \mathcal{M}\Big]$. The inequality is sharp for each $\mu \in \mathbb{C}$.*

Proof. If $f \in \mathcal{K}_\lambda^\beta(\alpha;\ s,t;\ \psi;\ h;\ C,D)$, in view of the relations (3.9) and (3.10), for $\mu \in \mathbb{C}$ we have

$$\begin{aligned}\left|a_3 - \mu a_2^2\right| &= \left|\frac{L_1(C-D)}{[3\lambda(1+2\alpha) + (\beta-1)\,\{(1-\alpha)\vartheta_3 + 3\alpha\vartheta_4\}]\,\Theta_3}\left[p_2 - \frac{1}{4}\left((D+1)L_1\right.\right.\right. \\ &\quad \left.\left.+2\left(1 - \frac{L_2}{L_1}\right) + \mathcal{M}\right)p_1^2\right] - \left.\frac{\mu L_1^2 p_1^2 (C-D)^2}{[2\lambda(1+\alpha) + (\beta-1)\{(1-\alpha)\vartheta_2 + 2\alpha\vartheta_3\}]^2\,\Theta_2^2}\right| \\ &= \left|\frac{L_1(C-D)}{[3\lambda(1+2\alpha) + (\beta-1)\,\{(1-\alpha)\vartheta_3 + 3\alpha\vartheta_4\}]\,\Theta_3}\left[p_2 - \frac{1}{4}\left((D+1)L_1\right.\right.\right. \\ &\quad \left.\left.\left.+2\left(1 - \frac{L_2}{L_1}\right) + \mathcal{M} + \frac{\mu\mathcal{N}}{2}\right)p_1^2\right]\right| \\ &\le \frac{|L_1|(C-D)}{|\,[3\lambda(1+2\alpha) + (\beta-1)\,\{(1-\alpha)\vartheta_3 + 3\alpha\vartheta_4\}]\,||\Theta_3|}\end{aligned}$$

$$\left[2+\frac{1}{4}|p_1|^2\left(\left|\frac{L_2}{L_1}-(D+1)L_1-\mathcal{M}-\frac{\mu\mathcal{N}}{2}\right|-2\right)\right]. \tag{4.1}$$

Now if $\left|\frac{L_2}{L_1}-(D+1)L_1-\mathcal{M}-\frac{\mu\mathcal{N}}{2}\right|\le 2$, from (4.1) we obtain

$$\left|a_3-\mu a_2^2\right|\le\frac{2|L_1|(C-D)}{|3\lambda(1+2\alpha)+(\beta-1)\{(1-\alpha)\vartheta_3+3\alpha\vartheta_4\}|\,|\Theta_3|}. \tag{4.2}$$

Further, if $\left|\frac{L_2}{L_1}-(D+1)L_1-\mathcal{M}-\frac{\mu\mathcal{N}}{2}\right|\ge 2$, from (4.1) we deduce

$$\left|a_3-\mu a_2^2\right|\le\frac{2|L_1|(C-D)}{|3\lambda(1+2\alpha)+(\beta-1)\{(1-\alpha)\vartheta_3+3\alpha\vartheta_4\}|\,|\Theta_3|}\left(\left|\frac{L_2}{L_1}-(D+1)L_1-\mathcal{M}-\frac{\mu\mathcal{N}}{2}\right|\right). \tag{4.3}$$

An examination of the proof shows that the equality for (4.2) holds if $p_1=0$, $p_2=2$. Equivalently, by Lemma 3 we have $p(z^2)=p_2(z)=\dfrac{1+z^2}{1-z^2}$. Therefore, the extremal function of the class $\mathcal{K}_\lambda^\beta(\alpha;\ s,t;\ \psi;\ h;\ C,D)$ is given by

$$\frac{[(s-t)z]^{1-\beta}\left[\mathcal{H}'(z)+\alpha z\mathcal{H}''(z)\right]^\lambda}{\left[(1-\alpha)\left[\mathcal{H}(sz)-\mathcal{H}(tz)\right]+\alpha z\left[s\mathcal{H}'(sz)-t\mathcal{H}'(tz)\right]\right]^{1-\beta}}=\frac{(C+1)p(z^2)-(C-1)}{(D+1)p(z^2)-(D-1)}.$$

Similarly, the equality for (4.3) holds if $p_2=2$. Equivalently, by Lemma 3 we have $p(z)=p_1(z)=\dfrac{1+z}{1-z}$. Therefore, the extremal function in $\mathcal{K}_\lambda^\beta(\alpha;\ s,t;\ \psi;\ h;\ C,D)$ is given by

$$\frac{[(s-t)z]^{1-\beta}\left[\mathcal{H}'(z)+\alpha z\mathcal{H}''(z)\right]^\lambda}{\left[(1-\alpha)\left[\mathcal{H}(sz)-\mathcal{H}(tz)\right]+\alpha z\left[s\mathcal{H}'(sz)-t\mathcal{H}'(tz)\right]\right]^{1-\beta}}=\frac{(C+1)p_1(z)-(C-1)}{(D+1)p_1(z)-(D-1)}$$

and the proof of the theorem is complete. □

Using Theorem 5, we can obtain the following result.

Theorem 10. *If $f\in\mathcal{L}_\lambda^\beta(\alpha;\ s,t;\ \psi;\ h;\ C,D)$ is given by* (1.1)*, then for all $\mu\in\mathbb{C}$ we have*

$$\left|a_3-\mu a_2^2\right|\le\frac{2|L_1|(C-D)}{\left[\lambda[3]_q(1+\alpha+q\alpha)+(\beta-1)\left\{(1-\alpha)\vartheta_3+\alpha[3]_q\vartheta_4\right\}\right]\Theta_3}\max\{1;\ |2\varrho-1|\}$$

with $\mathcal{M}_q$ and $\mathcal{N}_q$ is defined as in (3.14) *and* (3.15) *respectively, ϱ is given by*

$$\varrho:=\frac{1}{4}\left[(D+1)L_1+2\left(1-\frac{L_2}{L_1}\right)+\frac{\mu\mathcal{N}_q}{2}+\mathcal{M}_q\right].$$

The inequality is sharp for each $\mu\in\mathbb{C}$.

If we replace $h(z)=z+\sum_{n=2}^{\infty}z^n$, $C=s=\lambda=1$, $\alpha=\beta=0$, $D=t=-1$ and $\psi(z)=\hat{p}_{k,0}(z)$ in Theorem 10, we get the following result.

Corollary 11. *[17, Theorem 2.1] Let* $\hat{p}_{\eta,\sigma}(z) = 1 + \delta_1 z + \delta_2 z^2 + \cdots$ $(\delta_j = p_j(\eta, \sigma),\ j = 1, 2, 3, \cdots)$ *be defined as* (1.3)*. If* $f \in \mathcal{S}_s^*(p_\eta)$ *(see Remark 2 (i)), then for* $\mu \in \mathbb{C}$ *we have*

$$| a_3 - \mu a_2^2 | \leq \frac{\delta_1}{[3]_q - 1} \max\left\{1, \left|\frac{\delta_2}{\delta_1} - \frac{\mu\delta_1([3]_q - 1)}{[2]_q^2}\right|\right\},$$

where δ_1 *is given by* (1.5).

If we replace $h(z) = z + \sum_{n=2}^{\infty} z^n$, $C = s = \lambda = 1, \beta = 0$, $D = t = -1$ and $\psi(z) = \hat{p}_{\eta,0}(z)$ in Theorem 9, we have

Corollary 12. *[9, Theorem 2.1] Let* $\hat{p}_{\eta,\sigma}(z) = 1 + \delta_1 z + \delta_2 z^2 + \cdots$ $(\delta_j = p_j(\eta, \sigma),\ j = 1, 2, 3, \cdots)$ *be defined as* (1.3) *and* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in \mathcal{P}$*. If* $f \in \mathcal{M}_s(p_\eta)$ *(see Remark 2 (ii)), then we have*

$$a_2 = \frac{\delta_1 p_1}{4(1+\alpha)}, \quad a_3 = \frac{\delta_1}{4(1+2\alpha)}\left[p_2 - \frac{p_1^2}{2}\left(1 - \frac{\delta_2}{\delta_1}\right)\right]$$

and for any complex number μ,

$$| a_3 - \mu a_2^2 | \leq \frac{\delta_1}{2(1+2\alpha)} \max\left\{1, \left|\frac{\delta_2}{\delta_1} - \frac{\mu\delta_1(1+2\alpha)}{2(1+\alpha)^2}\right|\right\}.$$

If $h(z) = z + \sum_{n=2}^{\infty} z^n$, $\psi(z) = z + \sqrt{1+z^2}$, $t = \alpha = \beta = 0$, $\lambda = s = C = 1$ and $D = -1$ in Theorem 9, we get the following result.

Corollary 13. *[19] If* $f \in \mathcal{A}$ *satisfies the following condition*

$$\frac{zf'(z)}{f(z)} \prec z + \sqrt{1+z^2},$$

then $|a_2| \leq 1$, $|a_3| \leq \frac{3}{4}$ *and* $|a_3 - \mu a_2^2| \leq \max\left\{\frac{1}{2}, |\mu - \frac{3}{4}|\right\}$.

5. Conclusions

We unify and extend various classes of analytic function by defining λ-pseudo starlike function using subordination and Hadamard product. New extensions were discussed in details. Further, by replacing the ordinary differentiation with quantum differentiation, we have attempted at the discretization of some of the well-known results. For other several results which are closely related to the results presented here, refer to [3, 13, 22] and references provided therein.

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Conflict of interest

The authors declare that they have no competing interests.

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