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*Research article*

# Least squares estimation for non-ergodic weighted fractional Ornstein-Uhlenbeck process of general parameters

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**Abstract:** Let $B^{a,b} := \{B_t^{a,b}, t \geq 0\}$ be a weighted fractional Brownian motion of parameters $a > -1$, $|b| < 1$, $|b| < a + 1$. We consider a least square-type method to estimate the drift parameter $\theta > 0$ of the weighted fractional Ornstein-Uhlenbeck process $X := \{X_t, t \geq 0\}$ defined by $X_0 = 0$; $dX_t = \theta X_t dt + dB_t^{a,b}$. In this work, we provide least squares-type estimators for $\theta$ based continuous-time and discrete-time observations of $X$. The strong consistency and the asymptotic behavior in distribution of the estimators are studied for all $(a, b)$ such that $a > -1$, $|b| < 1$, $|b| < a + 1$. Here we extend the results of [1, 2] (resp. [3]), where the strong consistency and the asymptotic distribution of the estimators are proved for $-\frac{1}{2} < a < 0$, $-a < b < a + 1$ (resp. $-1 < a < 0$, $-a < b < a + 1$). Simulations are performed to illustrate the theoretical results.

**Keywords:** drift parameter estimation; weighted fractional Ornstein-Uhlenbeck process; strong consistency; asymptotic distribution
**Mathematics Subject Classification:** 62F12, 60F05, 60G15, 60H05

## 1. Introduction

Parameter estimation for non-ergodic type diffusion processes has been developed in several papers. For motivation and further references, we refer the reader to Basawa and Scott [4], Dietz and Kutoyants [5], Jacod [6] and Shimizu [7]. However, the statistical analysis for equations driven by fractional Brownian motion (fBm) is obviously more recent. The development of stochastic calculus with respect to the fBm allowed to study such models. In recent years, several researchers have been interested in studying statistical estimation problems for Gaussian Ornstein-Uhlenbeck processes. Estimation of the drift parameters in fractional-noise-driven Ornstein-Uhlenbeck processes is a problem that is both well-motivated by practical needs and theoretically challenging.

In this paper, we consider the weighted fractional Brownian motion (wfBm) $B^{a,b} := \{B_t^{a,b}, t \geq 0\}$ with parameters $(a, b)$ such that $a > -1$, $|b| < 1$ and $|b| < a + 1$, defined as a centered Gaussian process starting from zero with covariance

$$R^{a,b}(t,s) = E\left(B_t^{a,b} B_s^{a,b}\right) = \int_0^{s\wedge t} u^a \left[(t-u)^b + (s-u)^b\right] du, \quad s,t \geq 0. \tag{1.1}$$

For $a = 0$, $-1 < b < 1$, the wfBm is a fBm. The process $B^{a,b}$ was introduced by [8] as an extension of fBm. Moreover, it shares several properties with fBm, such as self-similarity, path continuity, behavior of increments, long-range dependence, non-semimartingale, and others. But, unlike fBm, the wfBm does not have stationary increments for $a \neq 0$. For more details about the subject, we refer the reader to [8].

In this work we consider the non-ergodic Ornstein-Uhlenbeck process $X := \{X_t, t \geq 0\}$ driven by a wfBm $B^{a,b}$, that is the unique solution of the following linear stochastic differential equation

$$X_0 = 0; \qquad dX_t = \theta X_t dt + dB_t^{a,b}, \tag{1.2}$$

where $\theta > 0$ is an unknown parameter.

An example of interesting problem related to (1.2) is the statistical estimation of $\theta$ when one observes $X$. In recent years, several researchers have been interested in studying statistical estimation problems for Gaussian Ornstein-Uhlenbeck processes. Let us mention some works in this direction in this case of Ornstein-Uhlenbeck process driven by a fractional Brownian motion $B^{0,b}$, that is, the solution of (1.2), where $a = 0$. Using the maximum likelihood approach (see [9]), the techniques used to construct maximum likelihood estimators for the drift parameter are based on Girsanov transforms for fractional Brownian motion and depend on the properties of the deterministic fractional operators (determined by the Hurst parameter) related to the fBm. In general, the MLE is not easily computable. On the other hand, using leat squares method, in the ergodic case corresponding to $\theta < 0$, the statistical estimation for the parameter $\theta$ has been studied by several papers, for instance [10–14] and the references therein. Further, in the non-ergodic case corresponding to $\theta > 0$, the estimation of $\theta$ has been considered by using least squares method, for example in [15–18] and the references therein.

Here our aim is to estimate the the drift parameter $\theta$ based on continuous-time and discrete-time observations of $X$, by using least squares-type estimators (LSEs) for $\theta$.
First we will consider the following LSE

$$\widetilde{\theta}_t = \frac{X_t^2}{2\int_0^t X_s^2 ds}, \quad t \geq 0, \tag{1.3}$$

as statistic to estimate $\theta$ based on the continuous-time observations $\{X_s,\ s \in [0,t]\}$ of (1.2), as $t \to \infty$. We will prove the strong consistency and the asymptotic behavior in distribution of the estimator $\widetilde{\theta}_t$ for all parameters $a > -1$, $|b| < 1$ and $|b| < a + 1$. Our results extend those proved in [1, 2], where $-\frac{1}{2} < a < 0$, $-a < b < a + 1$ only.

Further, from a practical point of view, in parametric inference, it is more realistic and interesting to consider asymptotic estimation for (1.2) based on discrete observations. So, we will assume that the process $X$ given in (1.2) is observed equidistantly in time with the step size $\Delta_n$: $t_i = i\Delta_n, i = 0, \ldots, n$, and $T_n = n\Delta_n$ denotes the length of the"observation window". Then we will consider the following estimators

$$\hat{\theta}_n = \frac{\sum_{i=1}^{n} X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})}{\Delta_n \sum_{i=1}^{n} X_{t_{i-1}}^2} \tag{1.4}$$

and

$$\check{\theta}_n = \frac{X_{T_n}^2}{2\Delta_n \sum_{i=1}^{n} X_{t_{i-1}}^2} \tag{1.5}$$

as statistics to estimate $\theta$ based on the sampling data $X_{t_i}, i = 0, \ldots, n$, as $\Delta_n \longrightarrow 0$ and $n \longrightarrow \infty$. We will study the asymptotic behavior and the rate consistency of the estimators $\hat{\theta}_n$ and $\check{\theta}_n$ for all parameters $a > -1$, $|b| < 1$ and $|b| < a + 1$. In this case, our results extend those proved in [3], where $-1 < a < 0$, $-a < b < a + 1$ only.

The rest of the paper is organized as follows. In Section 2, we present auxiliary results that are used in the calculations of the paper. In Section 3, we prove the consistency and the asymptotic distribution of the estimator $\widetilde{\theta}_t$ given in (1.3), based on the continuous-time observations of $X$. In Section 3, we study the asymptotic behavior and the rate consistency of the estimators $\hat{\theta}_n$ and $\check{\theta}_n$ defined in (1.4) and (1.5), respectively, based on the discrete-time observations of $X$. Our theoretical study is completed with simulations. We end the paper with a short review on some results from [15, 17] needed for the proofs of our results.

## 2. Auxiliary results

This section is devoted to prove some technical ingredients, which will be needed throughout this paper.
In the following lemma we provide a useful decomposition of the covariance function $R^{a,b}(t,s)$ of $B^{a,b}$.

**Lemma 2.1.** *Suppose that $a > -1$, $|b| < 1$ and $|b| < a+1$. Then we can rewrite the covariance $R^{a,b}(t,s)$ of $B^{a,b}$, given in (2.1) as*

$$R^{a,b}(t,s) = \beta(a+1, b+1)\left[t^{a+b+1} + s^{a+b+1}\right] - m(t,s), \tag{2.1}$$

*where $\beta(c,d) = \int_0^1 x^{c-1}(1-x)^{d-1}$ denotes the usual beta function, and the function $m(t,s)$ is defined by*

$$m(t,s) := \int_{s\wedge t}^{s\vee t} u^a (t \vee s - u)^b du. \tag{2.2}$$

*Proof.* We have for every $s, t \geq 0$,

$$\begin{aligned} & R^{a,b}(t,s) \\ = \; & E\left(B_t^{a,b} B_s^{a,b}\right) \end{aligned} \tag{2.3}$$

$$
\begin{aligned}
&= \int_0^{s\wedge t} u^a\left[(t-u)^b+(s-u)^b\right]du \\
&= \int_0^{s\wedge t} u^a\left[(t\vee s-u)^b+(t\wedge s-u)^b\right]du \\
&= \int_0^{s\wedge t} u^a(t\vee s-u)^b du+\int_0^{s\wedge t} u^a(t\wedge s-u)^b du \\
&= \int_0^{s\vee t} u^a(t\vee s-u)^b du-\int_{s\wedge t}^{s\vee t} u^a(t\vee s-u)^b du+\int_0^{s\wedge t} u^a(t\wedge s-u)^b du. \qquad (2.4)
\end{aligned}
$$

Further, making change of variables $x = u/t$, we have for every $t \geq 0$,

$$
\begin{aligned}
\int_0^t u^a(t-u)^b du &= t^b\int_0^t u^a\left(1-\frac{u}{t}\right)^b du \\
&= t^{a+b+1}\int_0^1 x^a(1-x)^b\,du \\
&= t^{a+b+1}\beta(a+1,b+1). \qquad (2.5)
\end{aligned}
$$

Therefore, combining (2.4) and (2.5), we deduce that

$$
\begin{aligned}
R^{a,b}(t,s) &= \beta(a+1,b+1)\left[(t\vee s)^{a+b+1}+(t\wedge s)^{a+b+1}\right]-\int_{s\wedge t}^{s\vee t} u^a(t\vee s-u)^b du \\
&= \beta(a+1,b+1)\left[t^{a+b+1}+s^{a+b+1}\right]-\int_{s\wedge t}^{s\vee t} u^a(t\vee s-u)^b du, \qquad (2.6)
\end{aligned}
$$

which proves (2.1). □

We will also need the following technical lemma.

**Lemma 2.2.** *We have as $t \longrightarrow \infty$,*

$$
I_t := t^{-a}e^{-\theta t}\int_0^t e^{\theta s}m(t,s)ds \longrightarrow \frac{\Gamma(b+1)}{\theta^{b+2}}, \qquad (2.7)
$$

$$
J_t := t^{-a}e^{-2\theta t}\int_0^t\int_0^t e^{\theta s}e^{\theta r}m(s,r)drds \longrightarrow \frac{\Gamma(b+1)}{\theta^{b+3}}, \qquad (2.8)
$$

*where $\Gamma(.)$ is the standard gamma function, whereas the function $m(t,s)$ is defined in (2.2).*

*Proof.* We first prove (2.7). We have,

$$
\begin{aligned}
t^{-a}e^{-\theta t}\int_0^t e^{\theta s}m(t,s)ds &= t^{-a}e^{-\theta t}\int_0^t e^{\theta s}\int_s^t u^a(t-u)^b duds \\
&= t^{-a}e^{-\theta t}\int_0^t duu^a(t-u)^b\int_0^u dse^{\theta s} \\
&= t^{-a}e^{-\theta t}\int_0^t duu^a(t-u)^b\frac{(e^{\theta u}-1)}{\theta}
\end{aligned}
$$

$$= \frac{t^{-a}e^{-\theta t}}{\theta}\int_0^t u^a(t-u)^b e^{\theta u}du - \frac{t^{-a}e^{-\theta t}}{\theta}\int_0^t u^a(t-u)^b du.$$

On the other hand, by the change of variables $x = t - u$, we get

$$\begin{aligned}
\frac{t^{-a}e^{-\theta t}}{\theta}\int_0^t u^a(t-u)^b e^{\theta u}du &= \frac{t^{-a}}{\theta}\int_0^t (t-x)^a x^b e^{-\theta x}dx \\
&= \frac{1}{\theta}\int_0^t \left(1-\frac{x}{t}\right)^a x^b e^{-\theta x}dx \\
&\longrightarrow \frac{1}{\theta}\int_0^\infty x^b e^{-\theta x}dx = \frac{\Gamma(b+1)}{\theta^{b+2}}
\end{aligned}$$

as $t \longrightarrow \infty$. Moreover, by the change of variables $x = u/t$,

$$\begin{aligned}
\frac{t^{-a}e^{-\theta t}}{\theta}\int_0^t u^a(t-u)^b du &= \frac{e^{-\theta t}}{\theta}t^b\int_0^t (u/t)^a(1-\frac{u}{t})^b dx \\
&= \frac{e^{-\theta t}}{\theta}t^{b+1}\int_0^t x^a(1-x)^b dx \\
&= \frac{e^{-\theta t}}{\theta}t^{b+1}\beta(a+1,b+1) \\
&\longrightarrow 0
\end{aligned}$$

as $t \longrightarrow \infty$. Thus the proof of the convergence (2.7) is done.
For (2.8), using L'Hôpital's rule, we obtain

$$\begin{aligned}
\lim_{t\to\infty} t^{-a}e^{-2\theta t}\int_0^t\int_0^t e^{\theta s}e^{\theta r}m(s,r)drds &= \lim_{t\to\infty}\frac{2\int_0^t\int_0^s e^{\theta s}e^{\theta r}m(s,r)drds}{t^a e^{2\theta t}} \\
&= \lim_{t\to\infty}\frac{2\int_0^t e^{\theta t}e^{\theta r}m(t,r)dr}{t^a e^{2\theta t}\left(2\theta+\frac{a}{t}\right)} \\
&= \lim_{t\to\infty}\frac{2}{\left(2\theta+\frac{a}{t}\right)}t^{-a}e^{-\theta t}\int_0^t e^{\theta r}m(t,r)dr \\
&= \frac{\Gamma(b+1)}{\theta^{b+3}},
\end{aligned}$$

where the latter equality comes from (2.7). Therefore the convergence (2.8) is proved. □

## 3. LSE based on continuous-time observation

In this section we will establish the consistency and the asymptotic distribution of the least square-type estimator $\widetilde{\theta}_t$ given in (1.3), based on the continuous-time observation $\{X_s,\ s \in [0,t]\}$ given by (1.2), as $t \to \infty$.

Recall that if $X \sim \mathcal{N}(m_1, \sigma_1)$ and $Y \sim \mathcal{N}(m_2, \sigma_2)$ are two independent random variables, then $X/Y$ follows a Cauchy-type distribution. For a motivation and further references, we refer the reader to [19], as well as [20]. Notice also that if $N \sim \mathcal{N}(0,1)$ is independent of $B^{a,b}$, then $N$ is independent of $Z_\infty$, since $Z_\infty := \int_0^\infty e^{-\theta s}B_s^{a,b}ds$ is a functional of $B^{a,b}$.

**Theorem 3.1.** *Assume that $a > -1$, $|b| < 1$, $|b| < a+1$, and let $\widetilde{\theta}_t$ be the estimator given in (1.3). Then, as $t \longrightarrow \infty$,*

$$\widetilde{\theta}_t \longrightarrow \theta \text{ almost surely.}$$

*Moreover, as $t \to \infty$,*

$$t^{-a/2} e^{\theta t} \left( \widetilde{\theta}_t - \theta \right) \xrightarrow{\texttt{law}} \frac{2\sigma_{B^{a,b}}}{\sqrt{E\left(Z_\infty^2\right)}} C(1),$$

*where $\sigma_{B^{a,b}} = \frac{\Gamma(b+1)}{\theta^{b+1}}$, $Z_\infty := \int_0^\infty e^{-\theta s} B_s^{a,b} ds$, whereas $C(1)$ is the standard Cauchy distribution with the probability density function $\frac{1}{\pi(1+x^2)}$; $x \in \mathbb{R}$.*

*Proof.* In order to prove this Theorem 3.1, using Theorem 6.1, it suffices to check that the assumptions $(\mathcal{H}1)$, $(\mathcal{H}2)$, $(\mathcal{H}3)$, $(\mathcal{H}4)$ hold.
Using (2.1) and the change of variables $x = (t-u)/(t-s)$, we get, for every $0 < s \leq t$,

$$\begin{aligned} E\left(B_t^{a,b} - B_s^{a,b}\right)^2 &= 2\int_s^t u^a (t-u)^b du \\ &= 2(t-s)^{b+1} \int_0^1 \left[t(1-x) + sx\right]^a x^b dx \\ &= 2t^a (t-s)^{b+1} \int_0^1 \left[(1-x) + \frac{s}{t} x\right]^a x^b dx \\ &=: I_{a,b}. \end{aligned}$$

Further, using the fact that $x \to (1-x) + (s/t)x$ is continuous and doesn't vanish on $[0,1]$, there exists constant $C_a$ depending only on $a$ such that

$$\begin{aligned} I_{a,b} &\leq 2C_a t^a (t-s)^{b+1} \int_0^1 x^b dx \\ &= 2C_a t^a (t-s)^{b+1} \frac{1}{b+1}. \end{aligned}$$

This implies

$$E\left(B_t^{a,b} - B_s^{a,b}\right)^2 \leq \frac{2C_a}{b+1} t^a (t-s)^{b+1}.$$

Furthermore, if $-1 < a < 0$, we have $t^a (t-s)^{b+1} \leq (t-s)^{a+b+1} = |t-s|^{(a+b+1)\wedge(b+1)}$, and if $a \geq 0$, we have $t^a (t-s)^{b+1} \leq T^a (t-s)^{b+1} = T^a |t-s|^{(a+b+1)\wedge(b+1)}$.
Consequently,for any fixed $T$, there exists a constant $C_{a,b}(T)$ depending only on $a, b, T$ such that, for every $0 < s \leq t \leq T$,

$$E\left(B_t^{a,b} - B_s^{a,b}\right)^2 \leq C_{a,b}(T) |t-s|^{(a+b+1)\wedge(b+1)},$$

Therefore, using the fact that $B^{a,b}$ is Gaussian, and Kolmogorov's continuity criterion, we deduce that $B^{a,b}$ has a version with $((a+b+1)\wedge(b+1) - \varepsilon)$-Hölder continuous paths for every $\varepsilon \in (0, (a+b+$

$1) \wedge (b+1))$. Thus $(\mathcal{H}1)$ holds for any $\delta$ in $(0, (a+b+1) \wedge (b+1))$.
On the other hand, according to (2.1) we have for every $t \geq 0$,

$$E\left(B_t^{a,b}\right)^2 = 2\beta(1+a, 1+b)t^{a+b+1},$$

which proves that $(\mathcal{H}2)$ holds for $\gamma = (a+b+1)/2$.
Now it remains to check that the assumptions $(\mathcal{H}3)$ and $(\mathcal{H}4)$ hold for $\nu = -a/2$ and $\sigma_{B^{a,b}} = \frac{\Gamma(b+1)}{\theta^{b+1}}$. Let us first compute the limiting variance of $t^{-a/2}e^{-\theta t}\int_0^t e^{\theta s}dB_s^{a,b}$ as $t \to \infty$. By (2.1) we obtain

$$\begin{aligned}
& E\left[\left(t^{-a/2}e^{-\theta t}\int_0^t e^{\theta s}dB_s^{a,b}\right)^2\right] = E\left[\left(t^{-a/2}e^{-\theta t}\left(e^{\theta t}B_t^{a,b} - \theta\int_0^t e^{\theta s}B_s^{a,b}ds\right)\right)^2\right] \\
= \;& t^{-a}\left(R^{a,b}(t,t) - 2\theta e^{-\theta t}\int_0^t e^{\theta s}R^{a,b}(t,s)ds + \theta^2 e^{-2\theta t}\int_0^t\int_0^t e^{\theta s}e^{\theta r}R^{a,b}(s,r)dsdr\right) \\
= \;& t^{-a}\Delta_{g_{B^{a,b}}}(t) + 2\theta I_t - \theta^2 J_t, \qquad (3.1)
\end{aligned}$$

where $I_t$, $J_t$ and $\Delta_{g_{B^{a,b}}}(t)$ are defined in (2.7), (2.8) and Lemma 6.1, respectively, whereas $g_{B^{a,b}}(s,r) = \beta(a+1, b+1)\left(s^{a+b+1} + r^{a+b+1}\right)$.
On the other hand, since $\frac{\partial g_{B^{a,b}}}{\partial s}(s,0) = \beta(a+1,b+1)(a+b+1)s^{a+b}$, and $\frac{\partial^2 g_{B^{a,b}}}{\partial s \partial r}(s,r) = 0$, it follows from (6.2) that

$$\begin{aligned}
t^{-a}\Delta_{g_{B^{a,b}}}(t) &= 2\beta(a+1,b+1)(a+b+1)t^{-a}e^{-2\theta t}\int_0^t s^{a+b}e^{\theta s}ds \\
&\leq 2\beta(a+1,b+1)e^{-\theta t}t^{a+b+1} \\
&\longrightarrow 0 \text{ as } t \to \infty. \qquad (3.2)
\end{aligned}$$

Combining (3.1), (3.2), (2.7) and (2.8), we get

$$E\left[\left(t^{-a/2}e^{-\theta t}\int_0^t e^{\theta s}dB_s^{a,b}\right)^2\right] \longrightarrow \frac{\Gamma(b+1)}{\theta^{b+1}} \quad \text{as } t \to \infty,$$

which implies that $(\mathcal{H}3)$ holds.
Hence, to finish the proof it remains to check that $(\mathcal{H}4)$ holds, that is, for all fixed $s \geq 0$

$$\lim_{t\to\infty} E\left(B_s^{a,b}t^{-a/2}e^{-\theta t}\int_0^t e^{\theta r}dB_r^{a,b}\right) = 0.$$

Let us consider $s < t$. According to (6.4), we can write

$$\begin{aligned}
& E\left(B_s^{a,b}t^{-a/2}e^{-\theta t}\int_0^t e^{\theta r}dB_r^{a,b}\right) \\
= \;& t^{-a/2}\left(R^{a,b}(s,t) - \theta e^{-\theta t}\int_0^t e^{\theta r}R^{a,b}(s,r)dr\right) \\
= \;& t^{-a/2}\left(R^{a,b}(s,t) - \theta e^{-\theta t}\int_s^t e^{\theta r}R^{a,b}(s,r)dr - \theta e^{-\theta t}\int_0^s e^{\theta r}R^{a,b}(s,r)dr\right)
\end{aligned}$$

$$= \quad t^{-a/2}\left(e^{-\theta(t-s)}R^{a,b}(s,s) + e^{-\theta t}\int_s^t e^{\theta r}\frac{\partial R^{a,b}}{\partial r}(s,r)dr - \theta e^{-\theta t}\int_0^s e^{\theta r}R^{a,b}(s,r)dr\right).$$

It is clear that $t^{-a/2}\left(e^{-\theta(t-s)}R^{a,b}(s,s) - \theta e^{-\theta t}\int_0^s e^{\theta r}R^{a,b}(s,r)dr\right) \longrightarrow 0$ as $t \to \infty$. Let us now prove that

$$t^{-a/2}e^{-\theta t}\int_s^t e^{\theta r}\frac{\partial R^{a,b}}{\partial r}(s,r)dr \longrightarrow 0$$

as $t \to \infty$. Using (1.1) we have for $s < r$

$$\frac{\partial R^{a,b}}{\partial r}(s,r) = b\int_0^s u^a(r-u)^{b-1}du$$

Applying L'Hôspital's rule we obtain

$$\begin{aligned}\lim_{t\to\infty} t^{-a/2}e^{-\theta t}\int_s^t e^{\theta r}\frac{\partial R^{a,b}}{\partial r}(s,r)dr &= \lim_{t\to\infty}\frac{bt^{-a/2}}{\theta+\frac{a}{2t}}\int_0^s u^a(t-u)^{b-1}du\\ &= \lim_{t\to\infty}\frac{bt^{b-1-\frac{a}{2}}}{\theta+\frac{a}{2t}}\int_0^s u^a(1-u/t)^{b-1}du\\ &\longrightarrow 0 \text{ as } t\to\infty,\end{aligned}$$

due to $b-1-\frac{a}{2} < 0$. In fact, if $-1 < a < 0$, we use $b < a+1$, then $b < a+1 < \frac{a}{2}+1$. Otherwise, if $a > 0$, we use $b < 1$, then $b-1-\frac{a}{2} < b-1- < 0$. Therefore the proof of Theorem 3.1 is complete.

□

## 4. LSEs based on discrete-time observations

In this section, our purpose is to study the asymptotic behavior and the rate consistency of the estimators $\hat{\theta}_n$ and $\check{\theta}_n$ based on the sampling data $X_{t_i}, i = 0,\ldots,n$ of (1.2), where $t_i = i\Delta_n, i = 0,\ldots,n$, and $T_n = n\Delta_n$ denotes the length of the "observation window".

**Definition 4.1.** *Let $\{Z_n\}$ be a sequence of random variables defined on a probability space $(\Omega,\mathcal{F},P)$. We say $\{Z_n\}$ is tight (or bounded in probability), if for every $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that,*

$$P(|Z_n| > M_\varepsilon) < \varepsilon, \quad \textit{for all } n.$$

### 4.1. The asymptotic behavior and the rate consistency of LSEs

**Theorem 4.1.** *Assume that $a > -1$, $|b| < 1$, $|b| < a+1$. Let $\hat{\theta}_n$ and $\check{\theta}_n$ be the estimators given in (1.4) and (1.5), respectively. Suppose that $\Delta_n \to 0$ and $n\Delta_n^{1+\alpha} \to \infty$ for some $\alpha > 0$. Then, as $n \to \infty$,*

$$\hat{\theta}_n \longrightarrow \theta, \quad \check{\theta}_n \longrightarrow \theta \quad \textit{almost surely},$$

*and for any $q \geq 0$,*

$$\Delta_n^q e^{\theta T_n}(\hat{\theta}_n - \theta) \textit{ and } \Delta_n^q e^{\theta T_n}(\check{\theta}_n - \theta) \textit{ are not tight.}$$

*In addition, if we assume that $n\Delta_n^3 \to 0$ as $n \to \infty$, the estimators $\hat{\theta}_n$ and $\check{\theta}_n$ are $\sqrt{T_n}$ – consistent in the sense that the sequences*

$$\sqrt{T_n}(\hat{\theta}_n - \theta) \textit{ and } \sqrt{T_n}(\check{\theta}_n - \theta) \textit{ are tight.}$$

*Proof.* In order to prove this Theorem 4.1, using Theorem 6.2, it suffices to check that the assumptions $(\mathcal{H}1)$, $(\mathcal{H}2)$, $(\mathcal{H}5)$ hold.

From the proof of Theorem 3.1, the assumptions $(\mathcal{H}1)$, $(\mathcal{H}2)$ hold. Now it remains to check that $(\mathcal{H}5)$ holds. In this case, the process $\zeta$ is defined as

$$\zeta_t := \int_0^t e^{-\theta s} dB_s^{a,b}, \qquad t \geq 0,$$

whereas the integral is interpreted in the Young sense (see Appendix).
Using the formula (6.4) and (6.3), we can write

$$\begin{aligned}
E\left[(\zeta_{t_i} - \zeta_{t_{i-1}})^2\right] &= E\left[\left(\int_{t_{i-1}}^{t_i} e^{-\theta s} dB_s^{a,b}\right)^2\right] \\
&= E\left[\left(e^{-\theta t_i} B_{t_i}^{a,b} - e^{-\theta t_{i-1}} B_{t_{i-1}}^{a,b} + \theta \int_{t_{i-1}}^{t_i} e^{-\theta s} B_s^{a,b} ds\right)^2\right] \\
&= \lambda_{g_{B^{a,b}}}(t_i, t_{i-1}) - \lambda_m(t_i, t_{i-1}) \\
&= \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} e^{-\theta(r+u)} \frac{\partial^2 g_{B^{a,b}}}{\partial r \partial u}(r,u) dr du - \lambda_m(t_i, t_{i-1}) \\
&= -\lambda_m(t_i, t_{i-1}),
\end{aligned}$$

where $\lambda_.(t_i, t_{i-1})$ is defined in Lemma 6.2, $g_{B^{a,b}}(s,r) = \beta(a+1, b+1)\left(s^{a+b+1} + r^{a+b+1}\right)$ and $\frac{\partial^2 g_{B^{a,b}}}{\partial s \partial r}(s,r) = 0$, whereas the term $\lambda_m(t_i, t_{i-1})$ is equal to

$$\begin{aligned}
\lambda_m(t_i, t_{i-1}) &= -2m(t_i, t_{i-1}) e^{-2\theta(t_{i-1}+t_i)} + 2\theta e^{-\theta t_i} \int_{t_{i-1}}^{t_i} m(r, t_i) e^{-\theta r} dr \\
&\quad -2\theta e^{-\theta t_{i-1}} \int_{t_{i-1}}^{t_i} m(r, t_{i-1}) e^{-\theta r} dr + \theta^2 \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} m(r,u) e^{-\theta(r+u)} dr du.
\end{aligned}$$

Combining this with the fact for every $t_{i-1} \leq u \leq r \leq t_i$, $i \geq 2$,

$$\begin{aligned}
|m(r,u)| &= \left|\int_u^r x^a (r-x)^b dx\right| \\
&\leq \begin{cases} \left|r^a \int_u^r (r-x)^b dx\right| & \text{if } -1 < a < 0 \\ \left|u^a \int_u^r (r-x)^b dx\right| & \text{if } a > 0 \end{cases} \\
&\leq \begin{cases} \frac{\Delta_n^{a+b+1}}{b+1} & \text{if } -1 < a \leq 0 \\ \frac{(n\Delta_n)^a \Delta_n^{b+1}}{b+1} & \text{if } a > 0 \end{cases}
\end{aligned}$$

together with $\Delta_n \longrightarrow 0$, we deduce that there is a positive constant $C$ such that

$$E\left[(\zeta_{t_i} - \zeta_{t_{i-1}})^2\right] \leq C \begin{cases} \frac{\Delta_n^{a+b+1}}{b+1} & \text{if } -1 < a \leq 0 \\ \frac{(n\Delta_n)^a \Delta_n^{b+1}}{b+1} & \text{if } a > 0, \end{cases}$$

which proves that the assumption $(\mathcal{H}5)$ holds. Therefore the desired result is obtained. □

### 4.2. Numerical results

For sample size $n = 2500$, we simulate 100 sample paths of the process $X$, given by (1.2), using software R. The Tables 1–8 below report the mean average values, the median values and the standard deviation values of the proposed estimators $\hat{\theta}_n$ and $\check{\theta}_n$ defined, respectively, by (1.4) and (1.5) of the true value of the parameter $\theta$. The results of the tables below show that the drift estimators $\hat{\theta}_n$ and $\check{\theta}_n$ perform well for different arbitrary values of $a$ and $b$ and they are strongly consistent, namely their values are close to the true values of the drift parameter $\theta$.

**Table 1.** The means, median and deviation values for $\widetilde{\theta}_n$, with $a = 0.5$ and $b = 0.9$.

| | $\theta = 0.5$ | $\theta = 0.9$ | $\theta = 2.5$ | $\theta = 7$ | $\theta = 10$ |
|---|---|---|---|---|---|
| Mean | 1.838178 | 2.039271 | 3.019508 | 7.027331 | 10.01776 |
| Median | 1.744118 | 1.906163 | 3.07125 | 7.02379 | 10.02108 |
| Std. dev. | 1.211776 | 1.007366 | 0.821718 | 0.1717033 | 0.0257764 |

**Table 2.** The means, median and deviation values for $\widetilde{\theta}_n$, with $a = 0.1$ and $b = 0.4$.

| | $\theta = 0.5$ | $\theta = 0.9$ | $\theta = 2.5$ | $\theta = 7$ | $\theta = 10$ |
|---|---|---|---|---|---|
| Mean | 1.259471 | 1.481005 | 2.39942 | 7.01006 | 10.02068 |
| Median | 1.170947 | 1.437582 | 2.552394 | 7.014911 | 10.01972 |
| Std. dev. | 0.9584086 | 0.8501618 | 1.004956 | 0.08241352 | 0.01018074 |

**Table 3.** The means, median and deviation values for $\hat{\theta}_n$, with $a = 0.5$ and $b = 0.9$.

| | $\theta = 0.5$ | $\theta = 0.9$ | $\theta = 2.5$ | $\theta = 7$ | $\theta = 10$ |
|---|---|---|---|---|---|
| Mean | 1.829429 | 2.03153 | 3.014143 | 7.01739 | 9.997761 |
| Median | 1.74294 | 1.896568 | 3.069177 | 7.013946 | 10.00107 |
| Std. dev. | 1.200806 | 1.008749 | 0.8246725 | 0.1711237 | 0.02569278 |

**Table 4.** The means, median and deviation values for $\hat{\theta}_n$, with $a = 0.1$ and $b = 0.4$.

| | $\theta = 0.5$ | $\theta = 0.9$ | $\theta = 2.5$ | $\theta = 7$ | $\theta = 10$ |
|---|---|---|---|---|---|
| Mean | 1.098053 | 1.388654 | 2.326678 | 6.998938 | 10.00066 |
| Median | 1.144919 | 1.402191 | 2.53695 | 7.005075 | 9.999712 |
| Std. dev. | 0.9961072 | 0.8888663 | 1.077326 | 0.08785867 | 0.01012159 |

**Table 5.** The means, median and deviation values for $\widetilde{\theta}_n$, with $a = 10$ and $b = 0.7$.

| | $\theta = 0.5$ | $\theta = 0.9$ | $\theta = 2.5$ | $\theta = 7$ | $\theta = 10$ |
|---|---|---|---|---|---|
| Mean | 6.284526 | 5.944562 | 7.676024 | 7.855939 | 9.586733 |
| Median | 5.620266 | 5.235174 | 6.698286 | 8.074982 | 9.919933 |
| Std. dev. | 4.851366 | 4.358735 | 6.545987 | 4.19334 | 2.664122 |

**Table 6.** The means, median and deviation values for $\widetilde{\theta}_n$, with $a = 5$ and $b = 0.9$.

| | $\theta = 0.5$ | $\theta = 0.9$ | $\theta = 2.5$ | $\theta = 7$ | $\theta = 10$ |
|---|---|---|---|---|---|
| Mean | 4.363369 | 3.81052 | 4.457953 | 6.959573 | 9.817441 |
| Median | 3.456968 | 3.650188 | 4.350376 | 7.172276 | 10.02188 |
| Std. dev. | 3.492142 | 2.865577 | 2.699623 | 1.759205 | 1.191188 |

**Table 7.** The means, median and deviation values for $\hat{\theta}_n$, with $a = 10$ and $b = 0.7$.

| | $\theta = 0.5$ | $\theta = 0.9$ | $\theta = 2.5$ | $\theta = 7$ | $\theta = 10$ |
|---|---|---|---|---|---|
| Mean | 5.882283 | 5.609962 | 7.379757 | 7.729786 | 9.526597 |
| Median | 5.337299 | 5.041629 | 6.502311 | 8.05469 | 9.900039 |
| Std. dev. | 4.882747 | 4.38566 | 6.319027 | 4.253017 | 2.698502 |

**Table 8.** The means, median and deviation values for $\hat{\theta}_n$, with $a = 5$ and $b = 0.9$.

| | $\theta = 0.5$ | $\theta = 0.9$ | $\theta = 2.5$ | $\theta = 7$ | $\theta = 10$ |
|---|---|---|---|---|---|
| Mean | 4.328267 | 3.782118 | 4.440072 | 6.946132 | 9.797283 |
| Median | 3.451499 | 3.590002 | 4.344927 | 7.161959 | 10.00186 |
| Std. dev. | 3.46757 | 2.86794 | 2.688184 | 1.758228 | 1.190694 |

## 5. Conclusions

To conclude, in this paper we provide least squares-type estimators for the drift parameter $\theta$ of the weighted fractional Ornstein-Uhlenbeck process $X$, given by (1.2), based continuous-time and discrete-time observations of $X$. The novelty of our approach is that it allows, comparing with the literature on statistical inference for $X$ discussed in [1–3], to consider the general case $a > -1$, $|b| < 1$ and $|b| < a+1$. More precisely,

- We estimate the drift parameter $\theta$ of (1.2) based on the continuous-time observations $\{X_s,\ s \in [0, t]\}$, as $t \to \infty$. We prove the strong consistency and the asymptotic behavior in distribution of the estimator $\widetilde{\theta}_t$ for all parameters $a > -1$, $|b| < 1$ and $|b| < a + 1$. Our results extend those proved in [1, 2], where $-\frac{1}{2} < a < 0$, $-a < b < a + 1$ only.
- Suppose that the process $X$ given in (1.2) is observed equidistantly in time with the step size $\Delta_n$: $t_i = i\Delta_n, i = 0, \dots, n$. We estimate the drift parameter $\theta$ of (1.2) on the sampling data $X_{t_i}, i = 0, \dots, n$, as $\Delta_n \longrightarrow 0$ and $n \longrightarrow \infty$. We study the asymptotic behavior and the rate consistency of the estimators $\hat{\theta}_n$ and $\check{\theta}_n$ for all parameters $a > -1$, $|b| < 1$ and $|b| < a + 1$. In this case, our results extend those proved in [3], where $-1 < a < 0$, $-a < b < a + 1$ only.

The proofs of the asymptotic behavior of the estimators are based on a new decomposition of the covariance function $R^{a,b}(t, s)$ of the wfBm $B^{a,b}$ (see Lemma 2.1), and slight extensions of results [15] and [17] (see Theorem 6.1 and Theorem 6.2 in Appendix).

## 6. Appendix

Here we present some ingredients needed in the paper.

Let $G = (G_t, t \geq 0)$ be a continuous centered Gaussian process defined on some probability space $(\Omega, \mathcal{F}, P)$ (Here, and throughout the text, we assume that $\mathcal{F}$ is the sigma-field generated by $G$). In this section we consider the non-ergodic case of Gaussian Ornstein-Uhlenbeck processes $X = \{X_t, t \geq 0\}$ given by the following linear stochastic differential equation

$$X_0 = 0; \quad dX_t = \theta X_t dt + dG_t, \quad t \geq 0, \tag{6.1}$$

where $\theta > 0$ is an unknown parameter. It is clear that the linear equation (6.1) has the following explicit solution

$$X_t = e^{\theta t}\zeta_t, \qquad t \geq 0,$$

where

$$\zeta_t := \int_0^t e^{-\theta s} dG_s, \qquad t \geq 0,$$

whereas this latter integral is interpreted in the Young sense.

Let us introduce the following required assumptions.

($\mathcal{H}1$) The process $G$ has Hölder continuous paths of some order $\delta \in (0, 1]$.

($\mathcal{H}2$) For every $t \geq 0$, $E\left(G_t^2\right) \leq ct^{2\gamma}$ for some positive constants $c$ and $\gamma$.

($\mathcal{H}3$) There is constant $\nu$ in $\mathbb{R}$ such that the limiting variance of $t^\nu e^{-\theta t}\int_0^t e^{\theta s} dG_s$ exists as $t \longrightarrow \infty$, that is, there exists a constant $\sigma_G > 0$ such that

$$\lim_{t\to\infty} E\left[\left(t^\nu e^{-\theta t}\int_0^t e^{\theta s} dG_s\right)^2\right] = \sigma_G^2.$$

($\mathcal{H}4$) For $\nu$ given in ($\mathcal{H}3$), we have all fixed $s \geq 0$

$$\lim_{t\to\infty} E\left(G_s t^\nu e^{-\theta t}\int_0^t e^{\theta r} dG_r\right) = 0.$$

($\mathcal{H}5$) There exist positive constants $\rho, C$ and a real constant $\mu$ such that

$$E\left[\left(\zeta_{t_i} - \zeta_{t_{i-1}}\right)^2\right] \leq C(n\Delta_n)^\mu \Delta_n^\rho e^{-2\theta t_i} \text{ for every } i = 1, \ldots, n, n \geq 1.$$

The following theorem is a slight extension of the main result in [15], and it can be established following the same arguments as in [15].

**Theorem 6.1.** *Assume that* ($\mathcal{H}1$) *and* ($\mathcal{H}2$) *hold and let* $\widetilde{\theta}_t$ *be the estimator of the form (1.3). Then, as* $t \longrightarrow \infty$,

$$\widetilde{\theta}_t \longrightarrow \theta \textit{ almost surely.}$$

*Moreover, if* $(\mathcal{H}1)$–$(\mathcal{H}4)$ *hold, then, as* $t \to \infty$,

$$t^{\nu} e^{\theta t}\left(\widetilde{\theta}_t - \theta\right) \xrightarrow{law} \frac{2\sigma_G}{\sqrt{E\left(Z_\infty^2\right)}} C(1),$$

*where* $Z_\infty := \int_0^\infty e^{-\theta s} G_s ds$, *whereas* $C(1)$ *is the standard Cauchy distribution with the probability density function* $\frac{1}{\pi(1+x^2)}$; $x \in \mathbb{R}$.

The following theorem is also a slight extension of the main result in [17], and it can be proved following line by line the proofs given in [17].

**Theorem 6.2.** *Assume that* $(\mathcal{H}1)$, $(\mathcal{H}2)$ *and* $(\mathcal{H}5)$ *hold. Let* $\hat{\theta}_n$ *and* $\check{\theta}_n$ *be the estimators of the forms (1.4) and (1.5), respectively. Suppose that* $\Delta_n \to 0$ *and* $n\Delta_n^{1+\alpha} \to \infty$ *for some* $\alpha > 0$. *Then, as* $n \to \infty$,

$$\hat{\theta}_n \longrightarrow \theta, \quad \check{\theta}_n \longrightarrow \theta \quad \textit{almost surely,}$$

*and for any* $q \geq 0$,

$$\Delta_n^q e^{\theta T_n}(\hat{\theta}_n - \theta) \textit{ and } \Delta_n^q e^{\theta T_n}(\check{\theta}_n - \theta) \textit{ are not tight.}$$

*In addition, if we assume that* $n\Delta_n^3 \to 0$ *as* $n \to \infty$, *the estimators* $\hat{\theta}_n$ *and* $\check{\theta}_n$ *are* $\sqrt{T_n}$ − *consistent in the sense that the sequences*

$$\sqrt{T_n}(\hat{\theta}_n - \theta) \textit{ and } \sqrt{T_n}(\check{\theta}_n - \theta) \textit{ are tight.}$$

**Lemma 6.1** ( [15]). *Let* $g : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ *be a symmetric function such that* $\frac{\partial g}{\partial s}(s,r)$ *and* $\frac{\partial^2 g}{\partial s \partial r}(s,r)$ *integrable on* $(0,\infty) \times [0,\infty)$. *Then, for every* $t \geq 0$,

$$\begin{aligned}\Delta_g(t) &:= g(t,t) - 2\theta e^{-\theta t}\int_0^t g(s,t)e^{\theta s}ds + \theta^2 e^{-2\theta t}\int_0^t\int_0^t g(s,r)e^{\theta(s+r)}drds \\ &= 2e^{-2\theta t}\int_0^t e^{\theta s}\frac{\partial g}{\partial s}(s,0)ds + 2e^{-2\theta t}\int_0^t dse^{\theta s}\int_0^s dr\frac{\partial^2 g}{\partial s\partial r}(s,r)e^{\theta r}. \end{aligned} \tag{6.2}$$

**Lemma 6.2** ( [17]). *Let* $g : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ *be a symmetric function such that* $\frac{\partial g}{\partial s}(s,r)$ *and* $\frac{\partial^2 g}{\partial s \partial r}(s,r)$ *integrable on* $(0,\infty) \times [0,\infty)$. *Then, for every* $t \geq s \geq 0$,

$$\begin{aligned}\lambda_g(t,s) &:= g(t,t)e^{-2\theta t} + g(s,s)e^{-2\theta s} - 2g(s,t)e^{-2\theta(s+t)} + 2\theta e^{-\theta t}\int_s^t g(r,t)e^{-\theta r}dr \\ &\quad -2\theta e^{-\theta s}\int_s^t g(r,s)e^{-\theta r}dr + \theta^2\int_s^t\int_s^t g(r,u)e^{-\theta(r+u)}drdu \\ &= \int_s^t\int_s^t e^{-\theta(r+u)}\frac{\partial^2 g}{\partial r\partial u}(r,u)drdu. \end{aligned} \tag{6.3}$$

Let us now recall the Young integral introduced in [21]. For any $\alpha \in (0,1]$, we denote by $\mathcal{H}^\alpha([0,T])$ the set of $\alpha$-Hölder continuous functions, that is, the set of functions $f : [0,T] \to \mathbb{R}$ such that

$$|f|_\alpha := \sup_{0\leq s<t\leq T}\frac{|f(t)-f(s)|}{(t-s)^\alpha} < \infty.$$

We also set $|f|_\infty = \sup_{t\in[0,T]} |f(t)|$, and we equip $\mathcal{H}^\alpha([0,T])$ with the norm $\|f\|_\alpha := |f|_\alpha + |f|_\infty$. Let $f \in \mathcal{H}^\alpha([0,T])$, and consider the operator $T_f : C^1([0,T]) \to C^0([0,T])$ defined as

$$T_f(g)(t) = \int_0^t f(u)g'(u)du, \quad t \in [0,T].$$

It can be shown (see, e.g., [22, Section 3.1]) that, for any $\beta \in (1-\alpha, 1)$, there exists a constant $C_{\alpha,\beta,T} > 0$ depending only on $\alpha, \beta$ and $T$ such that, for any $g \in \mathcal{H}^\beta([0,T])$,

$$\left\| \int_0^\cdot f(u)g'(u)du \right\|_\beta \leq C_{\alpha,\beta,T} \|f\|_\alpha \|g\|_\beta.$$

We deduce that, for any $\alpha \in (0,1)$, any $f \in \mathcal{H}^\alpha([0,T])$ and any $\beta \in (1-\alpha, 1)$, the linear operator $T_f : C^1([0,T]) \subset \mathcal{H}^\beta([0,T]) \to \mathcal{H}^\beta([0,T])$, defined as $T_f(g) = \int_0^\cdot f(u)g'(u)du$, is continuous with respect to the norm $\|\cdot\|_\beta$. By density, it extends (in an unique way) to an operator defined on $\mathcal{H}^\beta$. As consequence, if $f \in \mathcal{H}^\alpha([0,T])$, if $g \in \mathcal{H}^\beta([0,T])$ and if $\alpha + \beta > 1$, then the (so-called) Young integral $\int_0^\cdot f(u)dg(u)$ is well-defined as being $T_f(g)$ (see [21]).

The Young integral obeys the following formula. Let $f \in \mathcal{H}^\alpha([0,T])$ with $\alpha \in (0,1)$ and $g \in \mathcal{H}^\beta([0,T])$ with $\beta \in (0,1)$ such that $\alpha + \beta > 1$. Then $\int_0^\cdot g_u df_u$ and $\int_0^\cdot f_u dg_u$ are well-defined as the Young integrals. Moreover, for all $t \in [0,T]$,

$$f_t g_t = f_0 g_0 + \int_0^t g_u df_u + \int_0^t f_u dg_u. \tag{6.4}$$

## Conflict of interest

All authors declare that there is no conflict of interest in this paper.

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