AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(11): 12698–12712.
DOI:10.3934/math.2021732
Received: 16 July 2021
Accepted: 03 September 2021
Published: 06 September 2021

Research article

Weighted boundedness of multilinear pseudo-differential operators

Dazhao Chen*

School of Science, Shaoyang University, Shaoyang 422000, Hunan, China

* **Correspondence:** Email: chendazhao27@163.com.

Abstract: In this paper, the weighted boundedness for some multilinear operators generated by the pseudo-differential operators and the weighted Lipschitz functions are obtained.

Keywords: multilinear operators; pseudo-differential operator; weighted Lipschitz space
Mathematics Subject Classification: 42B20, 42B25

1. Introduction

As the development of singular integral operators, their commutators and multilinear operators have been well studied (see [4–10]). In [4–10][20][21], the authors proved that the commutators and multilinear operators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$; Chanillo (see [2]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [4][19], the boundedness for the commutators and multilinear operators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)(1 < p < \infty)$ spaces are obtained. In [1][14], the weighted boundedness for the commutators generated by the singular integral operators and BMO and Lipschitz functions on $L^p(R^n)(1 < p < \infty)$ spaces are obtained. The purpose of this paper is to prove the weighted boundedness on Lebesgue spaces for some multilinear operators associated to the pseudo-differential operators and the weighted Lipschitz functions. To do this, we first prove a sharp function estimate for the multilinear operators. Our results are new, even in the unweighted cases.

2. Preliminaries and Theorems

Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a locally integrable function f, let $f(Q) = \int_Q f(x)dx$, $f_Q = |Q|^{-1}\int_Q f(x)dx$ and

$$f^{\#}(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y) - f_Q|dy.$$

It is well-known that (see [13,23])

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

For $1 \leq p < \infty$ and $0 \leq \eta < n$, let

$$M_{\eta,p}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-p\eta/n}} \int_Q |f(y)|^p dy \right)^{1/p},$$

which is the Hardy-Littlewood maximal function when $p = 1$ and $\eta = 0$.

We write $f \in C_0^\infty(R^n)$ if f has any order derivative and compact support set on R^n.

The A_1 weight is defined by (see [13]) $A_1 = \{w > 0 : M(w)(x) \leq Cw(x), a.e.\}$.

The $A(p, q)$ weight is defined by (see [16]), for $1 < p, q < \infty$,

$$A(p, q) = \left\{ w > 0 : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)^q dx \right)^{1/q} \left(\frac{1}{|Q|} \int_Q w(x)^{-p/(p-1)} dx \right)^{(p-1)/p} < \infty \right\}.$$

For $1 < p < \infty$, given a weight function w, the weighted Lebesgue space $L^p(w)$ is the space of functions f such that

$$\|f\|_{L^p(w)} = \left(\int_{R^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty.$$

For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}(w)$ be the weighted homogeneous Triebel-Lizorkin space. For $0 < \beta < 1$, the weighted Lipschitz space $Lip_\beta(w)$ is the space of functions f such that

$$\|f\|_{Lip_\beta(w)} = \sup_Q \frac{1}{w(Q)^{1+\beta/n}} \int_Q |f(y) - f_Q| dy < \infty.$$

We say a symbol $\sigma(x, \xi)$ is in the class $S^m_{\rho,\delta}$ or $\sigma \in S^m_{\rho,\delta}(-\infty < m < +\infty, 0 < \rho, \delta < +\infty)$, if for $x, \xi \in R^n$,

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial \xi^\beta} \sigma(x, \xi) \right| \leq C_{\alpha,\beta}(1 + |\xi|)^{m-\rho|\beta|+\delta|\alpha|}.$$

A pseudo-differential operator with symbol $\sigma(x, \xi) \in S^m_{\rho,\delta}$ is defined by

$$T(f)(x) = \int_{R^n} e^{2\pi i x \cdot \xi} \sigma(x, \xi) \hat{f}(\xi) d\xi,$$

where f is a Schwartz function and $\hat{f}$ denotes the Fourier transform of f. We know there exists a kernel $K(x, y)$ such that

$$T(f)(x) = \int_{R^n} K(x, x - y) f(y) dy,$$

where, formally,

$$K(x, y) = \int_{R^n} e^{2\pi i (x-y) \cdot \xi} \sigma(x, \xi) d\xi.$$

In [11], the boundedness of the pseudo-differential operators with symbol $\sigma \in S^{-\beta}_{1-\theta,\delta}(\beta < n\theta/2, 0 \leq \delta < 1 - \theta, 0 < \theta < 1)$ is obtained. In [18], the boundedness of the pseudo-differential operators with symbol of order 0 and $-\infty$ is obtained. In [4], the sharp function estimate of the pseudo-differential operators with symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0 < \theta < 1, 0 \leq \delta < 1-\theta)$ is obtained. In [22], the boundedness of the pseudo-differential operators and their commutators with symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0 < \theta < 1, 0 \leq \delta < 1-\theta)$ is obtained. In [15–17], the boundedness of the multilinear and Toeplitz type operators associated to pseudo-differential operators with symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0 < \theta < 1, 0 \leq \delta < 1-\theta)$ is obtained. Our works are motivated by these papers.

Suppose T is a pseudo-differential operator with symbol $\sigma(x,\xi) \in S^m_{\rho,\delta}$. Let m_j be the positive integers($j = 1, \cdots, l$), $m_1 + \cdots + m_l = L$ and A_j be the functions on $R^n(j = 1, \cdots, l)$. Set, for $1 \leq j \leq L$,

$$R_{m_j+1}(A_j; x, y) = b_j(x) - \sum_{|\alpha| \leq m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x-y)^\alpha.$$

The multilinear operator associated to T is defined by

$$T^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^l R_{m_j+1}(A_j; x, y)}{|x-y|^m} K(x, x-y) f(y) dy.$$

Note that when $m = 0$, T^A is just the multilinear commutators of T and A (see [20,21]). While when $m > 0$, it is non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [4–8,10]). The purpose of this paper is to study the weighted boundedness properties for the multilinear operator. We shall prove the following theorems in Section 3.

Theorem 1. *Suppose T is the pseudo-differential operator with symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0 < \theta < 1, 0 \leq \delta < 1-\theta)$. Let $0 < \beta < 1$, $w \in A_1$ and $D^\alpha A_j \in Lip_\beta(w)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then there exists a constant $C > 0$ such that for every $f \in C_0^\infty(R^n)$, $2 \leq s < \infty$ and $\tilde{x} \in R^n$,*

$$(T^A(f))^\#(\tilde{x}) \leq C \prod_{j=1}^l \left(\sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{Lip_\beta(w)} \right) w(\tilde{x})^{l + l\beta/n} M_{l\beta,s}(f)(\tilde{x}).$$

Theorem 2. *Suppose T is the pseudo-differential operator with symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0 < \theta < 1, 0 \leq \delta < 1-\theta)$. Let $0 < \beta < 1$, $w \in A_1$ and $D^\alpha A_j \in Lip_\beta(w)$ for all α with $|\alpha| = m_j$ and $j = 1, \cdots, l$. Then T^A is bounded from $L^p(w)$ to $L^q(w^{1-lq})$ for $2 \leq p < n/l\beta$ and $1/p - 1/q = l\beta/n$, that is*

$$||T^A(f)||_{L^q(w^{1-lq})} \leq C \prod_{j=1}^l \left(\sum_{|\alpha_j| = m_j} ||D^{\alpha_j} A_j||_{Lip_\beta(w)} \right) ||f||_{L^p(w)}.$$

3. Proofs of Theorems

To prove the theorems, we need the following lemmas.

Lemma 1. *(see [5]) Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = L$ and some $q > n$. Then, for $x \neq y$,*

$$|R_L(A; x, y)| \leq C|x-y|^L \sum_{|\alpha| = L} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Lemma 2. *(see [2,19]) Suppose that $0 < \eta < n$, $1 \le s < p < n/\eta$, $1/q = 1/p - \eta/n$ and $w \in A(p,q)$. Then*

$$\|M_{\eta,s}(f)\|_{L^q(w^q)} \le C\|f\|_{L^p(w^p)}.$$

Lemma 3. *(see [12,14]) For any cube Q, $b \in Lip_\beta(w)$, $0 < \beta < 1$ and $w \in A_1$, we have*

$$\sup_{x\in Q} |b(x) - b_Q| \le C\|b\|_{Lip_\beta(w)} w(Q)^{1+\beta/n}|Q|^{-1}.$$

Lemma 4. *(see [3]) Let T be the pseudo-differential operator with symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0 < \theta < 1, 0 \le \delta < 1-\theta)$. Then, for every $f \in L^p(R^n)$, $1 < p < \infty$,*

$$\|T(f)\|_{L^p} \le C\|f\|_{L^p}.$$

Lemma 5. *(see [3]) Let $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}(0 < \theta < 1, 0 \le \delta < 1-\theta)$ and K be the kernel of the pseudo-differential operator T with symbol $\sigma \in S^{-n\theta/2}_{1-\theta,\delta}$. Then, for $|x_0 - x| \le d < 1$ and $k \ge 1$,*

$$\left(\int_{(2^k d)^{1-\theta} \le |y-x_0| < (2^{k+1}d)^{1-\theta}} |K(x, x-y) - K(x_0, x_0 - y)|^2 dy\right)^{1/2} \le C\frac{|x_0 - x|^{(1-\theta)(m-n/2)}}{(2^k d)^{m(1-\theta)}},$$

provided m is an integer such that $n/2 < m < n/2 + 1/(1-\theta)$.

Lemma 6. *(see [3]) Let $\sigma \in S^0_{\rho,\delta}(0 < \rho, \delta < 1)$ and*

$$K(x, w) = \int_{R^n} e^{2\pi i w\cdot\xi}\sigma(x,\xi)d\xi.$$

Then, for $|w| \ge 1/4$ and any integer $N \ge 1$,

$$|K(x, w)| \le C_N|w|^{-2N}.$$

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds for any cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$:

$$\frac{1}{|Q|}\int_Q |T^A(f)(x) - C_0|dx \le C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right) w(\tilde{x})^{l+l\beta/n} M_{l\beta,s}(f)(\tilde{x}).$$

Without loss of generality, we may assume $l = 2$. Fixed a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. We consider the following two cases:

Case 1. $d \le 1$. In this case, let Q^* be the cube concentric with Q of side length $d^{1-\theta}$. Set $\tilde{Q} = 5\sqrt{n}Q^*$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!}(D^\alpha A_j)_{\tilde{Q}}x^\alpha$, then $R_{m_j}(A_j; x, y) = R_{m_j}(\tilde{A}_j; x, y)$ and $D^\alpha\tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We split $f = g + h$ with $g = f\chi_{\tilde{Q}}$ and $h = f\chi_{R^n\setminus\tilde{Q}}$. Write

$$\begin{aligned} T^A(f)(x) &= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x-y|^L} K(x, x-y)h(y)dy \\ &+ \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^L} K(x, x-y)g(y)dy \end{aligned}$$

$$
\begin{aligned}
&- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^L} D^{\alpha_1}\tilde{A}_1(y) K(x, x-y) g(y) dy \\
&- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^L} D^{\alpha_2}\tilde{A}_2(y) K(x, x-y) g(y) dy \\
&+ \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{A}_1(y) D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^L} K(x, x-y) g(y) dy,
\end{aligned}
$$

then

$$
\begin{aligned}
& \frac{1}{|Q|}\int_Q \left| T^A(f)(x) - T^{\tilde{A}}(h)(x_0) \right| dx \\
\leq\ & \frac{1}{|Q|}\int_Q \left| \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^L} K(x, x-y) g(y) dy \right| dx \\
& + \frac{C}{|Q|}\int_Q \left| \sum_{|\alpha_1|=m_1} \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^L} D^{\alpha_1}\tilde{A}_1(y) K(x, x-y) g(y) dy \right| dx \\
& + \frac{C}{|Q|}\int_Q \left| \sum_{|\alpha_2|=m_2} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^L} D^{\alpha_2}\tilde{A}_2(y) K(x, x-y) g(y) dy \right| dx \\
& + \frac{C}{|Q|}\int_Q \left| \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{A}_1(y) D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^L} K(x, x-y) g(y) dy \right| dx \\
& + \frac{1}{|Q|}\int_Q \left| T^{\tilde{A}}(h)(x) - T^{\tilde{A}}(h)(x_0) \right| dx \\
:=\ & I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

Now, let us estimate I_1, I_2, I_3, I_4 and I_5, respectively. First, by Lemmas 1 and 3, we get

$$
\begin{aligned}
|R_L(\tilde{A}_j; x, y)| &\leq C|x-y|^L \sum_{|\alpha|=L} \sup_{x\in\tilde{Q}} |D^\alpha A_j(x) - (D^\alpha A_j)_{\tilde{Q}}| \\
&\leq C|x-y|^L \frac{w(\tilde{Q})^{1+\beta/n}}{|\tilde{Q}|} \sum_{|\alpha|=L} ||D^\alpha A_j||_{Lip_\beta(w)}.
\end{aligned}
$$

Now, let $\sigma(x,\xi) = \sigma(x,\xi)|\xi|^{n\theta/2}|\xi|^{-n\theta/2} = q(x,\xi)|\xi|^{-n\theta/2}$. We have $q(x,\xi) \in S^0_{1-\theta,\delta}$. Let S be the pseudo-differential operator with symbol $q(x,\xi)$, by the Hardy-Littlewood-Sobolev fractional integration theorem and the L^2-boundedness of S (see [3,23]). We obtain, for $1/p = 1/2 - \theta/2$, $1 < p < \infty$ and $2 \leq s < \infty$,

$$
\begin{aligned}
I_1 &\leq C\prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Lip_\beta(w)} \right) \left(\frac{w(\tilde{Q})^{2+2\beta/n}}{|\tilde{Q}|^2} \right) \frac{1}{|Q|} \int_Q |T(g)(x)| dx \\
&\leq C\prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} ||D^{\alpha_j} A_j||_{Lip_\beta(w)} \right) \left(\frac{w(\tilde{Q})^{2+2\beta/n}}{|\tilde{Q}|^2} \right) \left(\frac{1}{|Q|} \int_{R^n} |T(g)(x)|^p dx \right)^{1/p}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)\left(\frac{w(\tilde{Q})^{2+2\beta/n}}{|\tilde{Q}|^2}\right)|Q|^{-1/p}\left(\int_{R^n}|S(g)(x)|^2dx\right)^{1/2}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)\left(\frac{w(\tilde{Q})^{2+2\beta/n}}{|\tilde{Q}|^2}\right)|Q|^{-1/p}\left(\int_{R^n}|g(x)|^2dx\right)^{1/2}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)\left(\frac{w(\tilde{Q})^{2+2\beta/n}}{|\tilde{Q}|^2}\right)\frac{|\tilde{Q}|^{1/2}}{|Q|^{1/p}}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^2dx\right)^{1/2}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)\left(\frac{w(\tilde{Q})^{2+2\beta/n}}{|\tilde{Q}|^2}\right)\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^sdx\right)^{1/s}\\
&= C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)\left(\frac{w(\tilde{Q})}{|\tilde{Q}|}\right)^{2+2\beta/n}\left(\frac{1}{|\tilde{Q}|^{1-2\beta s/n}}\int_{\tilde{Q}}|f(x)|^sdx\right)^{1/s}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)w(\tilde{x})^{2+2\beta/n}M_{2\beta,s}(f)(\tilde{x}).
\end{aligned}
$$

For I_2, by Lemma 1 and Hölder's inequality, we get, for $1 < p < \infty$,

$$
\begin{aligned}
I_2 &\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{Lip_\beta(w)}\frac{w(\tilde{Q})^{1+\beta/n}}{|\tilde{Q}|}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|}\int_Q|T(D^{\alpha_1}\tilde{A}_1g)(x)|dx\\
&\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{Lip_\beta(w)}\frac{w(\tilde{Q})^{1+\beta/n}}{|\tilde{Q}|}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1g)(x)|^pdx\right)^{1/p}\\
&\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{Lip_\beta(w)}\frac{w(\tilde{Q})^{1+\beta/n}}{|\tilde{Q}|}\sum_{|\alpha_1|=m_1}|Q|^{-1/p}\left(\int_{R^n}|S(D^{\alpha_1}\tilde{A}_1g)(x)|^2dx\right)^{1/2}\\
&\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{Lip_\beta(w)}\frac{w(\tilde{Q})^{1+\beta/n}}{|\tilde{Q}|}\sum_{|\alpha_1|=m_1}|Q|^{-1/p}\left(\int_{R^n}|D^{\alpha_1}\tilde{A}_1(x)g(x)|^2dx\right)^{1/2}\\
&\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{Lip_\beta(w)}\frac{w(\tilde{Q})^{1+\beta/n}}{|\tilde{Q}|}\frac{|\tilde{Q}|^{1/2}}{|Q|^{1/p}}\\
&\quad\times\sum_{|\alpha_1|=m}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|D^{\alpha_1}A_1(x)-(D^{\alpha}A_j)_{\tilde{Q}}|^2|f(x)|^2dx\right)^{1/2}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)\left(\frac{w(\tilde{Q})^{2+2\beta/n}}{|\tilde{Q}|^2}\right)\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(y)|^sdy\right)^{1/s}\\
&= C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)\left(\frac{w(\tilde{Q})}{|\tilde{Q}|}\right)^{2+2\beta/n}\left(\frac{1}{|\tilde{Q}|^{1-2\beta s/n}}\int_{\tilde{Q}}|f(x)|^sdx\right)^{1/s}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)w(\tilde{x})^{2+2\beta/n}M_{2\beta,s}(f)(\tilde{x}).
\end{aligned}
$$

For I_3, similarly to the proof of I_2, we get, for $1 < p < \infty$,

$$I_3 \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{Lip_\beta(w)} \right) w(\tilde{x})^{2+2\beta/n} M_{2\beta,s}(f)(\tilde{x}).$$

Similarly, for I_4, we obtain

$$\begin{aligned}
I_4 &\leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \frac{1}{|Q|} \int_Q |T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 g)(x)| dx \\
&\leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{R^n} |T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 g)(x)|^p dx \right)^{1/p} \\
&\leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} |Q|^{-1/p} \left(\int_{R^n} |S(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 g)(x)|^2 dx \right)^{1/2} \\
&\leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} |Q|^{-1/p} \left(\int_{R^n} |D^{\alpha_1}\tilde{A}_1(x) D^{\alpha_2}\tilde{A}_2(x)|^2 |g(x)|^2 dx \right)^{1/2} \\
&\leq C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \frac{|\tilde{Q}|^{1/2}}{|Q|^{1/p}} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |D^{\alpha_1}\tilde{A}_1(x) D^{\alpha_2}\tilde{A}_2(x)|^2 |f(x)|^2 dx \right)^{1/2} \\
&\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{Lip_\beta(w)} \right) \left(\frac{w(\tilde{Q})^{2+2\beta/n}}{|\tilde{Q}|^2} \right) \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(y)|^s dy \right)^{1/s} \\
&= C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{Lip_\beta(w)} \right) \left(\frac{w(\tilde{Q})}{|\tilde{Q}|} \right)^{2+2\beta/n} \left(\frac{1}{|\tilde{Q}|^{1-2\beta s/n}} \int_{\tilde{Q}} |f(x)|^s dx \right)^{1/s} \\
&\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{Lip_\beta(w)} \right) w(\tilde{x})^{2+2\beta/n} M_{2\beta,s}(f)(\tilde{x}).
\end{aligned}$$

For I_5, we write

$$\begin{aligned}
&T^{\tilde{A}}(h)(x) - T^{\tilde{A}}(h)(x_0) \\
&= \int_{R^n} \left(\frac{K(x, x-y)}{|x-y|^L} - \frac{K(x_0, x_0-y)}{|x_0-y|^L} \right) \prod_{j=1}^{2} R_{m_j}(\tilde{A}_j; x, y) h(y) dy \\
&+ \int_{R^n} \left(R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; x_0, y) \right) \frac{R_{m_2}(\tilde{A}_2; x, y)}{|x_0-y|^L} K(x_0, x_0-y) h(y) dy \\
&+ \int_{R^n} \left(R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y) \right) \frac{R_{m_1}(\tilde{A}_1; x_0, y)}{|x_0-y|^L} K(x_0, x_0-y) h(y) dy \\
&- \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} D^{\alpha_1}\tilde{A}_1(y) h(y) \\
&\times \left[\frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^L} K(x, x-y) - \frac{R_{m_2}(\tilde{A}_2; x_0, y)(x_0-y)^{\alpha_1}}{|x_0-y|^L} K(x_0, x_0-y) \right] dy
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}D^{\alpha_2}\tilde{A}_2(y)h(y)\\
&\times\left[\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^L}K(x,x-y)-\frac{R_{m_1}(\tilde{A}_1;x_0,y)(x_0-y)^{\alpha_2}}{|x_0-y|^L}K(x_0,x_0-y)\right]dy\\
&+\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^L}K(x,x-y)-\frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^L}K(x_0,x_0-y)\right]\\
&\times D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)h(y)dy\\
=\ &I_5^{(1)}+I_5^{(2)}+I_5^{(3)}+I_5^{(4)}+I_5^{(5)}+I_5^{(6)}.
\end{aligned}
$$

Note that $|x-y|\sim|x_0-y|$ for $x\in\tilde{Q}$ and $y\in R^n\setminus\tilde{Q}=\bigcup_{k=0}^{\infty}(Q(x_0,(2^{k+1}d)^{1-\theta})\setminus Q(x_0,(2^kd)^{1-\theta}))$, and by Lemmas 1 and 5, we get

$$
|R_{m_j}(\tilde{A}_j;x,y)|\le\sum_{|\alpha|=m_j}||D^\alpha A_j||_{Lip_\beta(w)}\left(\frac{w(Q(x_0,(2^{k+1}d)^{1-\theta}))^{1+\beta/n}}{|Q(x_0,(2^{k+1}d)^{1-\theta})|}\right)|x-y|^{m_j},
$$

then

$$
\begin{aligned}
|I_5^{(1)}|\ \le\ &\sum_{k=0}^{\infty}\int_{(2^kd)^{1-\theta}\le|y-x_0|<(2^{k+1}d)^{1-\theta}}|K(x,x-y)-K(x_0,x_0-y)|\\
&\times\frac{1}{|x-y|^L}\prod_{j=1}^{2}|R_{m_j}(\tilde{A}_j;x,y)||f(y)|dy\\
&+\sum_{k=0}^{\infty}\int_{(2^kd)^{1-\theta}\le|y-x_0|<(2^{k+1}d)^{1-\theta}}\left|\frac{1}{|x-y|^L}-\frac{1}{|x_0-y|^L}\right|\\
&\times|K(x_0,x_0-y)|\prod_{j=1}^{2}|R_{m_j}(\tilde{A}_j;x,y)||f(y)|dy\\
\le\ &C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{Lip_\beta(w)}\right)\\
&\times\sum_{k=0}^{\infty}\frac{w(Q(x_0,(2^{k+1}d)^{1-\theta}))^{2+2\beta/n}}{|Q(x_0,(2^{k+1}d)^{1-\theta})|^2}\left(\int_{|y-x_0|<(2^{k+1}d)^{1-\theta}}|f(y)|^2dy\right)^{1/2}\\
&\times\left(\int_{(2^kd)^{1-\theta}\le|y-x_0|<(2^{k+1}d)^{1-\theta}}|K(x,x-y)-K(x_0,x_0-y)|^2dy\right)^{1/2}\\
&+C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{Lip_\beta(w)}\right)\\
&\times\sum_{k=0}^{\infty}\frac{w(Q(x_0,(2^{k+1}d)^{1-\theta}))^{2+2\beta/n}}{|Q(x_0,(2^{k+1}d)^{1-\theta})|^2}\left(\int_{|y-x_0|<(2^{k+1}d)^{1-\theta}}|f(y)|^2dy\right)^{1/2}\\
&\times\left(\int_{(2^kd)^{1-\theta}\le|y-x_0|<(2^{k+1}d)^{1-\theta}}\frac{|x_0-x|^2}{|x_0-y|^2}|K(x_0,x_0-y)|^2dy\right)^{1/2},
\end{aligned}
$$

for the second term above, similarly to the proof of Lemma 2.1 in [3], we have

$$\left(\int_{(2^k d)^{1-\theta}\le|y-x_0|<(2^{k+1}d)^{1-\theta}} \frac{|x_0-x|^2}{|x_0-y|^2}|K(x_0,x_0-y)|^2dy\right)^{1/2} \le C\frac{|x_0-x|^{(1-\theta)(m-n/2)}}{(2^kd)^{m(1-\theta)}},$$

thus, by Lemma 5 and recall that $n/2<m$,

$$\begin{aligned}
|I_5^{(1)}| &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)\sum_{k=0}^{\infty}\frac{w(Q(x_0,(2^{k+1}d)^{1-\theta}))^{2+2\beta/n}}{|Q(x_0,(2^{k+1}d)^{1-\theta})|^2}\\
&\times \frac{d^{(1-\theta)(m-n/2)}}{(2^kd)^{m(1-\theta)}}\left(\int_{|y-x_0|<(2^{k+1}d)^{1-\theta}}|f(y)|^2dy\right)^{1/2}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)\sum_{k=1}^{\infty}2^{k(1-\theta)(n/2-m)}\\
&\times \left(\frac{w(Q(x_0,(2^kd)^{1-\theta}))}{|Q(x_0,(2^kd)^{1-\theta})|}\right)^{2+2\beta/n}\left(\frac{1}{|Q(x_0,(2^kd)^{1-\theta})|^{1-2\beta s/n}}\int_{Q(x_0,(2^kd)^{1-\theta})}|f(y)|^sdy\right)^{1/s}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)w(\tilde{x})^{2+2\beta/n}M_{2\beta,s}(f)(\tilde{x})\sum_{k=1}^{\infty}2^{k(1-\theta)(n/2-m)}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)w(\tilde{x})^{2+2\beta/n}M_{2\beta,s}(f)(\tilde{x}).
\end{aligned}$$

For $I_5^{(2)}$, by the formula (see [5]):

$$R_{m_j}(\tilde{A}_j;x,y)-R_{m_j}(\tilde{A}_j;x_0,y)=\sum_{|\eta|<m_j}\frac{1}{\eta!}R_{m_j-|\eta|}(D^\eta\tilde{A}_j;x,x_0)(x-y)^\eta$$

and Lemma 5, we get

$$\begin{aligned}
|I_5^{(2)}| &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}A_j\|_{Lip_\beta(w)}\right)\sum_{k=0}^{\infty}\frac{w(Q(x_0,(2^{k+1}d)^{1-\theta}))^{2+2\beta/n}}{|Q(x_0,(2^{k+1}d)^{1-\theta})|^2}\\
&\times \int_{(2^kd)^{1-\theta}\le|y-x_0|<(2^{k+1}d)^{1-\theta}}\frac{|x-x_0|}{|x_0-y|}|K(x_0,x_0-y)||f(y)|dy\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)\sum_{k=1}^{\infty}2^{k(1-\theta)(n/2-m)}\\
&\times \left(\frac{w(Q(x_0,(2^kd)^{1-\theta}))}{|Q(x_0,(2^kd)^{1-\theta})|}\right)^{2+2\beta/n}\left(\frac{1}{|Q(x_0,(2^kd)^{1-\theta})|^{1-2\beta s/n}}\int_{Q(x_0,(2^kd)^{1-\theta})}|f(y)|^sdy\right)^{1/s}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}A_j\|_{Lip_\beta(w)}\right)w(\tilde{x})^{2+2\beta/n}M_{2\beta,s}(f)(\tilde{x}).
\end{aligned}$$

Similarly,

$$|I_5^{(3)}| \le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{Lip_\beta(w)}\right) w(\tilde{x})^{2+2\beta/n} M_{2\beta,s}(f)(\tilde{x}).$$

For $I_5^{(4)}$, notice that for $b \in Lip_\beta(w)$, $w \in A_1$ and $x \in Q$, we have

$$|b_Q - b_{2^kQ}| \le Ckw(x)w(2^kQ)^{\beta/n}\|b\|_{Lip_\beta(w)},$$

thus, similarly to the estimates of $I_5^{(1)}$ and $I_5^{(2)}$, we obtain

$$\begin{aligned}
|I_5^{(4)}| &\le C \sum_{|\alpha_1|=m_1} \int_{R^n\setminus\tilde{Q}} \left|\frac{(x-y)^{\alpha_1}K(x,x-y)}{|x-y|^L} - \frac{(x_0-y)^{\alpha_1}K(x_0,x_0-y)}{|x_0-y|^L}\right| |R_{m_2}(\tilde{A}_2;x,y)| \\
&\times (|D^{\alpha_1}A_1(y) - (D^{\alpha_1}A_1)_{Q(x_0,(2^{k+1}d)^{1-\theta})}| + |(D^{\alpha_1}A_1)_{Q(x_0,(2^{k+1}d)^{1-\theta})} - (D^{\alpha_1}A_1)_{\tilde{Q}}|)|f(y)|dy \\
&+ C \sum_{|\alpha_1|=m_1} \int_{R^n\setminus\tilde{Q}} |R_{m_2}(\tilde{A}_2;x,y) - R_{m_2}(\tilde{A}_2;x_0,y)| \frac{|(x_0-y)^{\alpha_1}K(x_0,x_0-y)|}{|x_0-y|^L} \\
&\times (|D^{\alpha_1}A_1(y) - (D^{\alpha_1}A_1)_{Q(x_0,(2^{k+1}d)^{1-\theta})}| + |(D^{\alpha_1}A_1)_{Q(x_0,(2^{k+1}d)^{1-\theta})} - (D^{\alpha_1}A_1)_{\tilde{Q}}|)|f(y)|dy \\
&\le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{Lip_\beta(w)}\right) \sum_{k=1}^{\infty} k \frac{w(Q(x_0,(2^kd)^{1-\theta}))^{1+\beta/n}}{|Q(x_0,(2^kd)^{1-\theta})|} \cdot \frac{d^{(1-\theta)(m-n/2)}}{(2^kd)^{m(1-\theta)}} \\
&\times \left[\frac{w(Q(x_0,(2^kd)^{1-\theta}))^{1+\beta/n}}{|Q(x_0,(2^kd)^{1-\theta})|} + w(\tilde{x})w(Q(x_0,(2^kd)^{1-\theta}))^{\beta/n}\right] \left(\int_{|y-x_0|<(2^kd)^{1-\theta}} |f(y)|^2 dy\right)^{1/2} \\
&\le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{Lip_\beta(w)}\right) w(\tilde{x}) \sum_{k=1}^{\infty} k 2^{k(1-\theta)(n/2-m)} \left(\frac{w(Q(x_0,(2^{k+1}d)^{1-\theta}))}{|Q(x_0,(2^{k+1}d)^{1-\theta})|}\right)^{1+2\beta/n} \\
&\times \left(\frac{1}{|Q(x_0,(2^kd)^{1-\theta})|^{1-2\beta s/n}} \int_{Q(x_0,(2^kd)^{1-\theta})} |f(y)|^s dy\right)^{1/s} \\
&\le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{Lip_\beta(w)}\right) w(\tilde{x})^{2+2\beta/n} M_{2\beta,s}(f)(\tilde{x}).
\end{aligned}$$

Similarly,

$$|I_5^{(5)}| \le C \prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{Lip_\beta(w)}\right) w(\tilde{x})^{2+2\beta/n} M_{2\beta,s}(f)(\tilde{x}).$$

For $I_5^{(6)}$, we get

$$\begin{aligned}
|I_5^{(6)}| &\le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \int_{R^n\setminus\tilde{Q}} \left|\frac{(x-y)^{\alpha_1+\alpha_2}K(x,x-y)}{|x-y|^L} - \frac{(x_0-y)^{\alpha_1+\alpha_2}K(x_0,x_0-y)}{|x_0-y|^L}\right| \\
&\times (|D^{\alpha_1}A_1(y) - (D^{\alpha_1}A_1)_{Q(x_0,(2^{k+1}d)^{1-\theta})}| + |(D^{\alpha_1}A_1)_{Q(x_0,(2^{k+1}d)^{1-\theta})} - (D^{\alpha_1}A_1)_{\tilde{Q}}|) \\
&\times (|D^{\alpha_1}A_1(y) - (D^{\alpha_2}A_2)_{Q(x_0,(2^{k+1}d)^{1-\theta})}| + |(D^{\alpha_1}A_2)_{Q(x_0,(2^{k+1}d)^{1-\theta})} - (D^{\alpha_2}A_2)_{\tilde{Q}}|)|f(y)|dy \\
&\le C \prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{Lip_\beta(w)}\right) \sum_{k=1}^{\infty} k^2 \frac{d^{(1-\theta)(m-n/2)}}{(2^kd)^{m(1-\theta)}}
\end{aligned}$$

$$
\begin{aligned}
&\times \left[\frac{w(Q(x_0,(2^k d)^{1-\theta}))^{1+\beta/n}}{|Q(x_0,(2^k d)^{1-\theta})|} + w(\tilde{x})w(Q(x_0,(2^k d)^{1-\theta}))^{\beta/n}\right]^2 \left(\int_{|y-x_0|<(2^k d)^{1-\theta}} |f(y)|^2 dy\right)^{1/2} \\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)\sum_{k=1}^{\infty} k^2 2^{k(1-\theta)(n/2-m)} \\
&\times \left[w(\tilde{x})\left(\frac{w(Q(x_0,(2^k d)^{1-\theta}))}{|Q(x_0,(2^k d)^{1-\theta})|}\right)^{\beta/n}\right]^2 \left(\frac{1}{|Q(x_0,(2^k d)^{1-\theta})|^{1-2\beta s/n}}\int_{Q(x_0,(2^k d)^{1-\theta})} |f(y)|^s dy\right)^{1/s} \\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right) w(\tilde{x})^{2+2\beta/n} M_{2\beta,s}(f)(\tilde{x}).
\end{aligned}
$$

Thus

$$
|T^{\tilde{A}}(f)(x) - T^{\tilde{A}}(f)(x_0)| \le C\sum_{|\alpha|=m} \|D^{\alpha}A\|_{Lip_\beta(w)} w(\tilde{x})^{2+2\beta/n} M_{2\beta,s}(f)(\tilde{x})
$$

and

$$
I_5 \le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right) w(\tilde{x})^{2+2\beta/n} M_{2\beta,s}(f)(\tilde{x}).
$$

Case 2. $d > 1$. In this case, let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!}(D^{\alpha}A_j)_{\tilde{Q}} x^{\alpha}$, then $R_{m_j}(A_j; x, y) = R_{m_j}(\tilde{A}_j; x, y)$ and $D^{\alpha}\tilde{A}_j = D^{\alpha}A_j - (D^{\alpha}A_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. Write, for $f = f\chi_{\tilde{Q}} + f\chi_{R^n\setminus\tilde{Q}} = f_1 + f_2$,

$$
\begin{aligned}
& \frac{1}{|Q|}\int_Q |T^A(f)(x)|dx \\
\le\ & \frac{1}{|Q|}\int_Q \left|\int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^L} K(x, x-y) f_1(y) dy\right| dx \\
& + \frac{C}{|Q|}\int_Q \left|\sum_{|\alpha_1|=m_1}\int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^L} D^{\alpha_1}\tilde{A}_1(y) K(x, x-y) f_1(y) dy\right| dx \\
& + \frac{C}{|Q|}\int_Q \left|\sum_{|\alpha_2|=m_2}\int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^L} D^{\alpha_2}\tilde{A}_2(y) K(x, x-y) f_1(y) dy\right| dx \\
& + \frac{C}{|Q|}\int_Q \left|\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1}\tilde{A}_1(y) D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^L} K(x, x-y) f_1(y) dy\right| dx \\
& + \frac{1}{|Q|}\int_Q |T^{\tilde{A}}(f_2)(x)|dx \\
:=\ & J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}
$$

Similarly to the proof of I_1, I_2, I_3 and I_4, by the L^s-boundedness of T (see Lemma 4), we get,

$$
J_1 \le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)\left(\frac{w(\tilde{Q})^{2+2\beta/n}}{|\tilde{Q}|^2}\right)\left(\frac{1}{|Q|}\int_{R^n} |T(f_1)(x)|^s dx\right)^{1/s}
$$

$$
\begin{aligned}
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)\left(\frac{w(\tilde{Q})^{2+2\beta/n}}{|\tilde{Q}|^2}\right)\left(\frac{1}{|Q|}\int_{R^n}|f_1(x)|^s dx\right)^{1/s}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)\left(\frac{w(\tilde{Q})}{|\tilde{Q}|}\right)^{2+2\beta/n}\left(\frac{1}{|\tilde{Q}|^{1-2\beta s/n}}\int_{\tilde{Q}}|f(x)|^s dx\right)^{1/s}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)w(\tilde{x})^{2+2\beta/n}M_{2\beta,s}(f)(\tilde{x});\\
J_2 &\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{Lip_\beta(w)}\sum_{|\alpha_1|=m_1}\left(\frac{w(\tilde{Q})^{1+\beta/n}}{|\tilde{Q}|}\right)\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1f_1)(x)|^s dx\right)^{1/s}\\
&\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{Lip_\beta(w)}\sum_{|\alpha_1|=m_1}\left(\frac{w(\tilde{Q})^{1+\beta/n}}{|\tilde{Q}|}\right)\left(\frac{1}{|Q|}\int_{R^n}|D^{\alpha_1}\tilde{A}_1(x)f_1(x)|^s dx\right)^{1/s}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)\left(\frac{w(\tilde{Q})}{|\tilde{Q}|}\right)^{2+2\beta/n}\left(\frac{1}{|\tilde{Q}|^{1-2\beta s/n}}\int_{\tilde{Q}}|f(x)|^s dx\right)^{1/s}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}A_j\|_{Lip_\beta(w)}\right)w(\tilde{x})^{2+2\beta/n}M_{2\beta,s}(f)(\tilde{x});\\
J_3 &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}A_j\|_{Lip_\beta(w)}\right)w(\tilde{x})^{2+2\beta/n}M_{2\beta,s}(f)(\tilde{x});\\
J_4 &\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1D^{\alpha_2}\tilde{A}_2f_1)(x)|^s dx\right)^{1/s}\\
&\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\left(\frac{1}{|Q|}\int_{R^n}|D^{\alpha_1}\tilde{A}_1(x)D^{\alpha_2}\tilde{A}_2(x)f_1(x)|^s dx\right)^{1/s}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{Lip_\beta(w)}\right)\left(\frac{w(\tilde{Q})}{|\tilde{Q}|}\right)^{2+2\beta/n}\left(\frac{1}{|\tilde{Q}|^{1-2\beta s/n}}\int_{\tilde{Q}}|f(x)|^s dx\right)^{1/s}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}\|D^{\alpha}A_j\|_{Lip_\beta(w)}\right)w(\tilde{x})^{2+2\beta/n}M_{2\beta,s}(f)(\tilde{x}).
\end{aligned}
$$

For J_5, we write

$$
\begin{aligned}
&T^{\tilde{A}}(f_2)(x) = \int_{R^n}\frac{\prod_{j=1}^{2}R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^L}K(x,x-y)f_2(y)dy\\
&-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^L}K(x,x-y)D^{\alpha_1}\tilde{A}_1(y)f_2(y)dy\\
&-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^L}K(x,x-y)D^{\alpha_2}\tilde{A}_2(y)f_2(y)dy
\end{aligned}
$$

$$+\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^L}K(x,x-y)D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)f_2(y)dy.$$

Similarly to the proof of I_5 and by using Lemma 6, we get

$$\begin{aligned}
&|T^{\tilde{A}}(f_2)(x)|\\
\le\ & C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{Lip_\beta(w)}\right)\sum_{k=0}^{\infty}\left(\frac{w(2^{k+1}\tilde{Q})^{2+2\beta/n}}{|2^{k+1}\tilde{Q}|^2}\right)\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|x-y|^{-2n}|f(y)|dy\\
&+C\sum_{|\alpha|=m_2}||D^\alpha A_2||_{Lip_\beta(w)}\sum_{|\alpha_1|=m_1}\sum_{k=0}^{\infty}\left(\frac{w(2^{k+1}\tilde{Q})^{1+\beta/n}}{|2^{k+1}\tilde{Q}|}\right)\\
&\times\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|x-y|^{-2n}|D^{\alpha_1}\tilde{A}_1(y)||f(y)|dy\\
&+C\sum_{|\alpha|=m_1}||D^\alpha A_1||_{Lip_\beta(w)}\sum_{|\alpha_2|=m_2}\sum_{k=0}^{\infty}\left(\frac{w(2^{k+1}\tilde{Q})^{1+\beta/n}}{|2^{k+1}\tilde{Q}|}\right)\\
&\times\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|x-y|^{-2n}|D^{\alpha_2}\tilde{A}_2(y)||f(y)|dy\\
&+C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}|x-y|^{-2n}|D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||f(y)|dy\\
\le\ & C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{Lip_\beta(w)}\right)\\
&\times d^{-n}\sum_{k=1}^{\infty}\left(\frac{w(2^{k+1}\tilde{Q})}{|2^{k+1}\tilde{Q}|}\right)^{2+2\beta/n}2^{-kn}\left(\frac{1}{|2^k\tilde{Q}|^{1-2\beta s/n}}\int_{2^k\tilde{Q}}|f(y)|^s dy\right)^{1/s}\\
\le\ & C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{Lip_\beta(w)}\right)w(\tilde{x})^{2+2\beta/n}M_{2\beta,s}(f)(\tilde{x}),
\end{aligned}$$

thus

$$|J_5|\le C\prod_{j=1}^{2}\left(\sum_{|\alpha|=m_j}||D^\alpha A_j||_{Lip_\beta(w)}\right)w(\tilde{x})^{2+2\beta/n}M_{2\beta,s}(f)(\tilde{x}).$$

This completes the proof of the Theorem 1.

Proof Theorem 2. We choose s with $2\le s<p$ in Theorem 1, notice that $w^{1-lq}\in A_\infty$ and $w^{1/p}\in A(p,q)$. By using Lemma 2, we get

$$\begin{aligned}
&||T^A(f)||_{L^q(w^{1-ql})}\le||M(T^A(f))||_{L^q(w^{1-lq})}\\
\le\ & C||(T^A(f))^{\#}||_{L^q(w^{1-ql})}\\
\le\ & C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Lip_\beta(w)}\right)||M_{l\beta,s}(f)w^{l+l\beta/n}||_{L^q(w^{1-ql})}\\
=\ & C\prod_{j=1}^{l}\left(\sum_{|\alpha_j|=m_j}||D^{\alpha_j}A_j||_{Lip_\beta(w)}\right)||M_{l\beta,s}(f)||_{L^q(w^{q/p})}
\end{aligned}$$

$$\leq \quad C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{Lip_\beta(w)} \right) \|f\|_{L^p(w)}.$$

This completes the proof of the theorem.

Acknowledgments

The author would like to express his deep gratitude to the referee for his/her valuable comments and suggestions.

This research was supported by the National Natural Science Foundation of China (Grant No. 11901126) and the Natural Science Foundation of Hunan Province(No.2021JJ30630) and the Scientific Research Funds of Hunan Provincial Education Department (Grant No. 19B509).

Conflict of interest

The author declares no conflicts of interest in this paper.

References

1. S. Bloom, A commutator theorem and weighted *BMO*, *Trans. Amer. Math. Soc.*, **292** (1985), 103–122.
2. S. Chanillo, A note on commutators, *Indiana Univ. Math. J.*, **31** (1982), 7–16.
3. S. Chanillo, A. Torchinsky, Sharp function and weighted L^p estimates for a class of pseudo-differential operators, *Ark. Math.*, **24** (1986), 1–25.
4. W. G. Chen, Besov estimates for a class of multilinear singular integrals, *Acta Math. Sinica*, **16** (2000), 613–626.
5. J. Cohen, A sharp estimate for a multilinear singular integral on R^n, *Indiana Univ. Math. J.*, **30** (1981), 693–702.
6. J. Cohen, J. Gosselin, On multilinear singular integral operators on R^n, *Studia Math.*, **72** (1982), 199–223.
7. J. Cohen, J. Gosselin, A BMO estimate for multilinear singular integral operators, *Illinois J. Math.*, **30** (1986), 445–465.
8. R. Coifman, Y. Meyer, *Wavelets, Calderón-Zygmund and multilinear operators*, Cambridge Studies in Advanced Math., Vol. 48, Cambridge University Press, Cambridge, 1997.
9. R. R. Coifman, R. Rochberg, G. Weiss, Factorization theorems for Hardy spaces in several variables, *Ann. Math.*, **103** (1976), 611–635.
10. Y. Ding, S. Z. Lu, Weighted boundedness for a class rough multilinear operators, *Acta Math. Sinica*, **17** (2001), 517–526.
11. C. Fefferman, L^p bounds for pseudo-differential operators, *Israel J. Math.*, **14** (1973), 413–417.
12. J. Garcia-Cuerva, *Weighted H^p spaces*, Warszawa: Instytut Matematyczny Polskiej Akademi Nauk, 1979.

13. J. Garcia-Cuerva, J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Math., Vol. 116, Amsterdam, 1985.
14. B. Hu, J. Gu, Necessary and sufficient conditions for boundedness of some commutators with weighted Lipschitz spaces, *J. Math. Anal. Appl.*, **340** (2008), 598–605.
15. L. Z. Liu, Sharp and weighted boundedness for multilinear operators associated with pseudo-differential operators on Morrey space, *J. Contemp. Math. Anal.*, **45** (2010), 136–150.
16. L. Z. Liu, Boundedness for multilinear operators of pseudo-differential operators for the extreme cases, *J. Math. Inequal.*, **4** (2010), 217–232.
17. L. Z. Liu, Sharp maximal function inequalities and boundedness for Toeplitz type operator associated to pseudo-differential operator, *J. Pseudo-Differ. Oper.*, **4** (2013), 91–112.
18. N. Miller, Weighted Sobolev spaces and pseudo-differential operators with smooth symbols, *Trans. Amer. Math. Soc.*, **269** (1982), 91–109.
19. M. Paluszynski, Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss, *Indiana Univ. Math. J.*, **44** (1995), 1–17.
20. C. Pérez, R. Trujillo-Gonzalez, Sharp weighted estimates for vector-valued singular integral operators and commutators, *Tohoku Math. J.*, **55** (2003), 109–129.
21. C. Pérez, R. Trujillo-Gonzalez, Sharp weighted estimates for multilinear commutators, *J. London Math. Soc.*, **65** (2002), 672–692.
22. M. Saidani, A. Lahmar-Benbernou, S. Gala, Pseudo-differential operators and commutators in multiplier spaces, *African Diaspora J. of Math.*, **6** (2008), 31–53.
23. E. M. Stein, *Harmonic analysis: real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton NJ, 1993.