



Research article

# On global well-posedness and decay of 3D Ericksen-Leslie system

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**Abstract:** In this paper, the small initial data global well-posedness and time decay estimates of strong solutions to the Cauchy problem of 3D incompressible liquid crystal system with general Leslie stress tensor are studied. First, assuming that  $\|u_0\|_{\dot{H}^{\frac{1}{2}+\varepsilon}} + \|d_0 - d_*\|_{\dot{H}^{\frac{3}{2}+\varepsilon}}$  ( $\varepsilon > 0$ ) is sufficiently small, we obtain the global well-posedness of strong solutions. Moreover, the  $L^p-L^2$  ( $\frac{3}{2} \leq p \leq 2$ ) type optimal decay rates of the higher-order spatial derivatives of solutions are also obtained. The  $\dot{H}^{-s}$  ( $0 \leq s < \frac{1}{2}$ ) negative Sobolev norms are shown to be preserved along time evolution and enhance the decay rates.

**Keywords:** Ericksen-Leslie system; strong solutions; global well-posedness; decay

**Mathematics Subject Classification:** 35Q35, 76D03, 35B40

## 1. Introduction

Consider the following incompressible Ericksen-Leslie system for the nematic liquid crystals [9, 21]:

$$\begin{cases} u_t + u \cdot \nabla u - \frac{\gamma}{Re} \Delta u + \nabla p = \frac{1-\gamma}{Re} \operatorname{div} \sigma, \\ (\gamma_1 N + \gamma_2 A \cdot d - h) \times d = 0, \\ \nabla \cdot u = 0, \quad |d| = 1, \\ (u, d)(\cdot, 0) = (u_0, d_0)(\cdot), \end{cases} \quad (1.1)$$

where the vector field  $u \in \mathbb{R}^3$  is the velocity field, the scalar function  $p \in \mathbb{R}$  represents the pressure and the unit vector field  $d$  stands for the director field,  $Re$ ,  $\gamma$ ,  $\gamma_1$  and  $\gamma_2$  are constants, with  $\gamma \in (0, 1)$  and  $Re > 0$  is the Reynolds number. The notations  $\sigma$ ,  $N$  and  $h$  in system (1.1) means the Cauchy stress tensor, the co-rotational derivative and the molecular field respectively, given by

$$\sigma = \sigma^E + \sigma^L, \quad N = \partial_t d + u \cdot \nabla d - \Omega \cdot d,$$

and

$$h = \nabla \cdot \left( \frac{\partial W}{\partial(\nabla d)} \right) - \frac{\partial W}{\partial d} = \partial_\alpha W_{p^\alpha}(d, \nabla d) - W_d(d, \nabla d),$$

with

$$A = \frac{1}{2}(\nabla u + \nabla u^T), \quad \Omega = \frac{1}{2}(\nabla u - \nabla u^T),$$

where  $\sigma^E$  denotes the Ericksen stress tensor,  $\sigma^L$  stands for the Leslie stress tensor and  $W = W(d, \nabla d)$  represents the free energy, respectively.

The main purpose of this paper is to consider the problems of the isotropic case, hence we take the free energy  $W(d, \nabla d) = \frac{1}{2}|\nabla d|^2$  and consequently the Ericksen stress tensor  $\sigma^E$  satisfies

$$\sigma^E = -(\nabla d)^T \frac{\partial W}{\partial d} = -\nabla d \odot \nabla d.$$

Moreover, the Leslie stress tensor satisfies the following general expression:

$$\sigma^L = \alpha_1(d \cdot A \cdot d)d \otimes d + \alpha_2 N \otimes d + \alpha_3 d \otimes N + \alpha_4 A + \alpha_5(A \cdot d) \otimes d + \alpha_6 d \otimes (A \cdot d),$$

where  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$  and  $\alpha_6$  are Leslie coefficients.  $\alpha_i$  ( $1 \leq i \leq 6$ ),  $\gamma_1 > 0$  and  $\gamma_2$  satisfy

$$\gamma_1 = \alpha_3 - \alpha_2, \quad \gamma_2 = \alpha_6 - \alpha_5, \quad \alpha_2 + \alpha_3 = \gamma_2. \quad (1.2)$$

The Ericksen-Leslie system can be used to describes the evolutionary behavior of nematic liquid crystal flows (we refer to the monographs [1, 5] for a detailed presentation of the physical foundations of continuum theories of liquid crystals). As the generalized Ericksen-Leslie system is so complicated, many earlier works treated the simplified (or approximated) system of (1.1). Motivated by work on the harmonic heat flow, Lin and Liu [24] considered the mathematical analysis for the dynamical system of the Ginzburg-Landau approximate system (which involves a penalty term  $\frac{1-|d|^2}{\epsilon^2}d$  to relax the constraint  $|d| = 1$ ) of a simplified Ericksen-Leslie system

$$\begin{cases} u_t + u \cdot \nabla u - \frac{\gamma}{Re} \Delta u + \nabla p = \frac{1-\gamma}{Re} \operatorname{div} \sigma, \\ d_t + u \cdot \nabla d + \Omega \cdot d - \mu_1 A \cdot d - \frac{1}{\epsilon^2}(|d|^2 - 1)d = 0, \\ \nabla \cdot u = 0. \end{cases} \quad (1.3)$$

The authors proved the global existence of weak solutions and the local existence and uniqueness of strong solutions. Moreover, Liu and Shen [26], Sun and Liu [33] and Cavaterra et al. [3] also considered the nematic liquid crystals with Ginzburg-Landau approximate system. However, because the corresponding a priori estimates of (1.3) established in the above papers depend crucially on the parameter  $\epsilon$ , the results established there cannot be applied to obtain the existence of solutions to the original liquid crystal systems, by letting the parameter  $\epsilon$  go to 0. If the Leslie stress  $\sigma^L$  is neglected in (1.1), the simplest system preserving the basic energy law is obtained in the following:

$$\begin{cases} u_t + u \cdot \nabla u - \Delta u + \nabla p = -\nabla \cdot (\nabla d \odot \nabla d), \\ d_t + u \cdot \nabla d - \Delta d = |\nabla d|^2 d, \\ \nabla \cdot u = 0. \end{cases} \quad (1.4)$$

In [34], Wang studied the well-posedness result with rough data. Huang and Wang [19] considered the BKM type blow-up criterion for system (1.4). There are also many classical results for system (1.4), see for instance, [10, 11, 25, 27, 38] and the reference cited therein. In the case when  $|\nabla d|^2 d$  in (1.4) is replaced by  $\frac{1}{\varepsilon}(|d|^2 - 1)d$ , we can refer to [8, 23, 41] and the reference cited therein. On the other hand, some papers also focus on the mathematical analysis of density dependent incompressible liquid crystal system and the compressible liquid crystal system (cf. [6, 13, 15, 18, 39] and the reference therein).

Taking the vector cross-product to Eq (1.1)<sub>2</sub> with  $d$ , using the constraint  $|d| = 1$ , we can rewrite (1.1)<sub>2</sub> as

$$\gamma_1 N + \gamma_2(A \cdot d - (d \cdot A \cdot d)d) = \Delta d + |\nabla d|^2 d. \quad (1.5)$$

Consider the relation (1.2) and Eq (1.5), rewrite the Leslie stress tensor  $\sigma^L$  as

$$\begin{aligned} \sigma^L = & \alpha'_1(d \cdot A \cdot d)d \otimes d + \alpha_4 A + \alpha'_5(A \cdot d) \otimes d + \alpha'_6 d \otimes (A \cdot d) \\ & + \frac{1}{\gamma_1}(\alpha \Delta d \otimes d + \alpha_3 d \otimes \Delta d) + \frac{\gamma_2}{\gamma_1} |\nabla d|^2 d \otimes d, \end{aligned} \quad (1.6)$$

where

$$\alpha'_1 = \alpha_1 + \frac{\gamma_2}{\gamma_1}(\alpha_2 + \alpha_3), \quad \alpha'_5 = \alpha_5 - \frac{\gamma_2}{\gamma_1}\alpha_2, \quad \alpha'_6 = \alpha_6 - \frac{\gamma_2}{\gamma_1}\alpha_3. \quad (1.7)$$

Combining (1.2) and (1.13), we easily obtain

$$\alpha'_5 = \alpha'_6, \quad \alpha_3 - \alpha_2 = \gamma_1, \quad \alpha_2 + \alpha_3 = \gamma_2. \quad (1.8)$$

Wang et al. [35] and Gong et al. [15] pointed out that the energy of Ericksen-Leslie system (1.1) is dissipated if and only if

$$\begin{cases} \alpha_4 \geq 0, \\ 2\alpha_4 + \alpha_5 + \alpha_6 - \frac{\gamma_2^2}{\gamma_1} = 2\alpha_4 + \alpha'_5 + \alpha'_6 \geq 0, \\ \frac{3}{2}\alpha_4 + \alpha_5 + \alpha_6 + \alpha_1 = \frac{3}{2}\alpha_4 + \alpha'_5 + \alpha'_6 + \alpha'_1 \geq 0. \end{cases} \quad (1.9)$$

In this paper, we use the new expression (1.6) of the Leslie stress tensor, consider the following Cauchy problem:

$$\begin{cases} u_t + u \cdot \nabla u - \frac{\gamma}{Re} \Delta u + \nabla p = \frac{1-\gamma}{Re} \operatorname{div} \left[ -\nabla d \odot \nabla d + \alpha'_1(d \cdot A \cdot d)d \otimes d + \alpha_4 A \right. \\ \quad \left. + \alpha'_5(A \cdot d) \otimes d + \alpha'_6 d \otimes (A \cdot d) + \frac{1}{\gamma_1}(\alpha \Delta d \otimes d + \alpha_3 d \otimes \Delta d) + \frac{\gamma_2}{\gamma_1} |\nabla d|^2 d \otimes d \right], \\ \gamma_1(d_t + u \cdot \nabla d - \Omega \cdot d) + \gamma_2(A \cdot d - (d \cdot A \cdot d)d) = \Delta d + |\nabla d|^2 d, \\ \nabla \cdot u = 0, \quad |d| = 1, \\ (u, d)(\cdot, 0) = (u_0, d_0)(\cdot), \end{cases} \quad (1.10)$$

For system (1.10) without the key property  $|d| = 1$ , Wang et al. [35] studied the existence of unique local strong solution provided that the initial data  $(u_0, \nabla d_0) \in H^4(\mathbb{R}^3)$ . Moreover, Gong et al. [15] assumed that  $(u_0, d_0 - d_*) \in H^2(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$  with  $\nabla \cdot u_0 = 0$  and  $|d_0| = 1$ , proved that there exists a local strong solution for system (1.10). In this paper, we list Gong et al.'s result in the following:

**Lemma 1.1** ([15]). *Let  $d_*$  be a constant unit vector. Suppose that  $(u_0, d_0 - d_*) \in H^2(\mathbb{R}^3) \times H^3(\mathbb{R}^3)$  with  $\nabla \cdot u_0 = 0$  and  $|d_0| = 1$ . Then, there exists a small time  $\tilde{T} > 0$  and a unique strong solution  $(u, d)$  to system (1.1) satisfying*

$$\begin{cases} u \in C([0, \tilde{T}]; H^1) \cap L^\infty(0, \tilde{T}; H^2) \cap L^2(0, \tilde{T}; W^{2,6}), \\ d - d_* \in C([0, \tilde{T}]; H^2) \cap L^\infty(0, \tilde{T}; H^3) \cap L^2(0, \tilde{T}; H^4), \\ u_t \in L^2(0, \tilde{T}; H^1), \quad d_t \in L^\infty(0, \tilde{T}; H^1) \cap L^2(0, \tilde{T}; H^2). \end{cases}$$

The main purpose of this paper is to study the global well-posedness and decay estimates for system (1.10). Next, we give a notation of this paper.

**Notation 1.2.** In this paper, we use  $H^k(\mathbb{R}^3)$  ( $k \in \mathbb{R}$ ) to denote the usual Sobolev spaces with norm  $\|\cdot\|_{H^s}$  and  $L^p(\mathbb{R}^3)$ ,  $1 \leq p \leq \infty$  to denote the usual  $L^p$  spaces with norm  $\|\cdot\|_{L^p}$ . We also introduce the homogeneous negative index Sobolev space  $\dot{H}^{-s}(\mathbb{R}^3)$ :

$$\dot{H}^{-s}(\mathbb{R}^3) := \{f \in L^2(\mathbb{R}^3) : \|\|\xi\|^{-s} \hat{f}(\xi)\|_{L^2} < \infty\}$$

endowed with the norm  $\|f\|_{\dot{H}^{-s}} := \|\|\xi\|^{-s} \hat{f}(\xi)\|_{L^2}$ . The symbol  $\nabla^l$  with an integer  $l \geq 0$  stands for the usual spatial derivatives of order  $l$ . For instance, we define

$$\nabla^l z = \{\partial_x^\alpha z_i \mid |\alpha| = l, i = 1, 2, 3\}, \quad z = (z_1, z_2, z_3).$$

If  $l < 0$  or  $l$  is not a positive integer,  $\nabla^l$  stands for  $\Lambda^l$  defined by

$$\Lambda^s f(x) = \int_{\mathbb{R}^3} |\xi|^s \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi,$$

where  $\hat{f}$  is the Fourier transform of  $f$ .

The first purpose of this paper is to establish the global well-posedness for system (1.10) provided that the initial data is sufficiently small. We prove the following theorem:

**Theorem 1.3.** *Suppose that  $(u_0, d_0 - d_*) \in H^N(\mathbb{R}^3) \times H^{N+1}(\mathbb{R}^3)$  with  $N \geq 2$ ,  $|d_0| = 1$  and  $\nabla \cdot u_0 = 0$ . There exists a sufficiently small constant  $K > 0$  and any  $\varepsilon > 0$  such that if*

$$\|u_0\|_{\dot{H}^{\frac{1}{2}+\varepsilon}} + \|d_0 - d_*\|_{\dot{H}^{\frac{3}{2}+\varepsilon}} < K, \quad (1.11)$$

holds, there exists a unique global solution  $(u, d)$  satisfying

$$\begin{aligned} & \|u\|_{H^N}^2 + \|d - d_*\|_{H^N}^2 + \|\nabla d\|_{H^N}^2 + \int_0^t (\|\nabla u\|_{H^N}^2 + \|\nabla d\|_{H^N}^2 + \|\Delta d\|_{H^N}^2) ds \\ & \leq \|u_0\|_{H^N}^2 + \|d_0 - d_*\|_{H^N}^2 + \|\nabla d_0\|_{H^N}^2, \quad \forall t \geq 0. \end{aligned}$$

**Remark 1.4.** It is worth pointing out that the constant  $\varepsilon$  in Theorem 1.3 can not be reduced to 0 because of the Sobolev's embedding

$$\|d\|_{L^\infty} \leq \|d\|_{L^6}^{\frac{2\varepsilon}{1+2\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}^{\frac{1}{1+2\varepsilon}} \leq \|\nabla d\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}^{\frac{1}{1+2\varepsilon}} \leq \|\nabla d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}. \quad (1.12)$$

**Remark 1.5.** Wang et al. [35] also obtained the global well-posedness of strong solutions for system (1.1) provided that  $\|\nabla d_0\|_{H^{2s}} + \|u_0\|_{H^{2s}}$  ( $s \geq 2$ ) is sufficiently small. Compare with [35], our results only need  $\|u_0\|_{\dot{H}^{\frac{1}{2}+\varepsilon}} + \|d_0 - d_*\|_{\dot{H}^{\frac{3}{2}+\varepsilon}}$  ( $\varepsilon > 0$ ) is sufficiently small, in a sense, it can be seen as an improvement of the global well-posedness result of [35].

**Remark 1.6.** In the study of small initial data global well-posedness, the main difficulties are caused by the Leslie stress tensor and the co-rotational derivative. Since the structure of Leslie stress tensor and the co-rotational derivative are so complicated, it is difficult to obtain the energy estimates for higher order derivatives of the solution. In order to overcome those difficulties, one assume that  $\|\Lambda^{\frac{1}{2}+\varepsilon}u\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon}(d - d^*)\|_{L^2}$  is sufficiently small, use the assumption  $|d| = 1$  and the basic energy law, obtain suitable estimates. Moreover, one of the main step to overcome the difficult caused by the co-rotational derivative is the estimate  $\|d\|_{L^\infty}$ . We adopt the estimate (1.12), by interpolation inequality, obtain the result of Theorem 1.3. However, since the estimate (1.12) does not holds for the case  $\varepsilon = 0$ , the result of Theorem 1.3 is not a perfect result, maybe it can be improved in the future. We leave it as an open problem to be considered latter.

The second purpose of this paper is to show the time decay rate of solutions for system (1.1), i. e., we prove the following theorem:

**Theorem 1.7.** *Assume that all the assumptions of Theorem 1.3 holds. Let  $N \in \mathbb{N}^+$  and  $N \geq 3$ . Then, if  $(u_0, d_0 - d_*, \nabla d_0) \in L^p(\mathbb{R}^3) \times H^N(\mathbb{R}^3)$  with  $(\frac{3}{2} \leq p \leq 2)$  and  $l = 0, 1, \dots, N - 1$ ,*

$$\|\nabla^l u(t)\|_{L^2} + \|\nabla^l (d - d_*)\|_{L^2} + \|\nabla^{l+1} d\|_{L^2} \leq C(1+t)^{-[\frac{3}{2}(\frac{1}{p}-\frac{1}{2})+\frac{l}{2}]}, \quad \forall t \geq 0. \quad (1.13)$$

**Remark 1.8.** To study the decay estimates of the dissipative equations, one of the main tools is the Fourier splitting method, which was introduced by Schonbek in the 1980s (see [30, 31]), then it becomes a standard way to establish the decay rate of solutions. Many classical decay results on incompressible hydrodynamics equations have been obtained by using this Fourier splitting method, see for example [2, 4, 22, 28, 40, 42] and the reference therein. It is worth pointing out that Guo and Wang [17] also introduced a powerful method–pure energy method to study the decay estimates for compressible Navier-Stokes equations. By using their method, the decay estimates for some dissipative equations has been obtained (see for instance [7, 14, 29, 37, 43] and the reference cited therein). In this paper, since the structure of the equations is so complex and it is difficult to obtain the decay estimates by using Fourier splitting method. Hence, we adopt Guo and Wang’s method (see [17, 36]) to obtain the optimal decay rate. The negative Sobolev norm estimates for the solution are shown to be preserved along time evolution and enhance the decay rates.

This paper is organized as follows. In Section 2, we give the proof of Theorem 1.3 on the small initial data global well-posedness of system (1.10); Theorem 1.7 on the decay rate of solutions are shown in the last section.

## 2. Proof of Theorem 1.3

In the proof of lemmas and theorems, we frequently employ the Gagliardo-Nirenberg inequality:

**Lemma 2.1** ([12]). Let  $u \in L^q(\mathbb{R}^n)$ ,  $\nabla^m u \in L^r(\mathbb{R}^n)$ ,  $1 \leq q, r \leq \infty$ . Then, there exists a positive constant  $C = C(n, m, j, a, q, r)$ , such that

$$\|\nabla^j u\|_{L^p} \leq C \|\nabla^m u\|_{L^r}^a \|u\|_{L^q}^{1-a},$$

where

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}, \quad 1 \leq p \leq \infty, \quad 0 \leq j \leq m, \quad \frac{j}{m} \leq a \leq 1.$$

The Kato-Ponce inequality is of great importance in the proof of Theorem 1.3.

**Lemma 2.2** ([20]). Let  $1 < p < \infty$ ,  $s > 0$ . There exists a positive constant  $C$  such that

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{q_1}} \|g\|_{L^{q_2}}),$$

and

$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}} \|\Lambda^s g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{q_1}} \|g\|_{L^{q_2}}),$$

where  $p_1, q_1, p_2, q_2 \in (1, \infty)$  satisfying  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$ .

For convenience, we set  $\mu = \frac{Re}{1-\gamma}$ ,  $\nu = \frac{\gamma}{1-\gamma}$ ,  $S(d, M, M) = \Gamma(d, M) : M$  and

$$\Gamma(d, A) = \alpha'_1(d \cdot A \cdot d)d \otimes d + \alpha_4 A + \alpha'_5(A \cdot d) \otimes d + \alpha'_6 d \otimes (A \cdot d). \quad (2.1)$$

Simple calculation shows that [35]:

$$S(d, A, A) \geq 0.$$

In the following, we give the proof of Theorem 1.3.

*Proof of Theorem 1.3.* Multiplying (1.10)<sub>1</sub> by  $u$ , integrating over  $\mathbb{R}^3$ , using the divergence free condition, we find that

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^3} \frac{1}{2} \mu |u|^2 dx + \nu \int_{\mathbb{R}^3} |\nabla u|^2 dx \\ &= - \int_{\mathbb{R}^3} [-\nabla d \odot \nabla d + \Gamma(d, A)] |\nabla u| dx \\ & \quad - \int_{\mathbb{R}^3} \left[ \frac{1}{\gamma_1} (\alpha_2 \Delta d \otimes d + \alpha_3 d \otimes \Delta d) + \frac{\gamma_2}{\gamma_1} |\nabla d|^2 d \otimes d \right] : \nabla u dx \\ &= \int_{\mathbb{R}^3} \left[ \nabla d \odot \nabla d : \nabla u - S(d, A, D) - \frac{\gamma_2}{\gamma_1} d \otimes (\Delta d + |\nabla d|^2 d) : A - d \otimes \Delta d : \Omega \right] dx. \end{aligned} \quad (2.2)$$

Note that  $|d| = 1$ , we have  $\|d\|_{L^\infty} \leq C$  and  $d \cdot d = 1$ . Hence,  $\Delta(d \cdot d) = 0$ , which implies that

$$\nabla \cdot (d \nabla d) = |\nabla d|^2 + d \Delta d = 0.$$

Therefore, we have

$$d \cdot \Delta d = -|\nabla d|^2, \quad \operatorname{div}(\nabla d \odot \nabla d) = \nabla \cdot \left( \frac{|\nabla d|^2}{2} \right) + \nabla d \cdot \Delta d.$$

On the basis of the above equalities, multiplying (1.10)<sub>2</sub> by  $-\Delta d - |\nabla d|^2 d$ , integrating over  $\mathbb{R}^3$ , applying integration by parts and using the divergence free condition, we deduce that

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla d|^2 dx + \frac{1}{\gamma_1} \int_{\mathbb{R}^3} |\Delta d + |\nabla d|^2 d|^2 dx \\
 &= \int_{\mathbb{R}^3} \left[ \frac{\gamma_2}{\gamma_1} A : d \otimes (\Delta d + |\nabla d|^2 d) + \Omega : d \otimes \Delta d + (u \cdot \nabla) d \cdot \Delta d \right] dx \\
 &= \int_{\mathbb{R}^3} \left\{ \frac{\gamma_2}{\gamma_1} A : d \otimes (\Delta d + |\nabla d|^2 d) + \Omega : d \otimes \Delta d \right. \\
 &\quad \left. + u \cdot \left[ \operatorname{div}(\nabla d \odot \nabla d) - \nabla \left( \frac{|\nabla d|^2}{2} \right) \right] \right\} dx \\
 &= \int_{\mathbb{R}^3} \left[ \frac{\gamma_2}{\gamma_1} A : d \otimes (\Delta d + |\nabla d|^2 d) + \Omega : d \otimes \Delta d - \nabla u : \nabla d \odot \nabla d \right] dx.
 \end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3) together gives the basic energy identity of system (1.10):

$$\frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) + \nu \|\nabla u\|_{L^2}^2 + \frac{1}{\gamma_1} \|\Delta d + |\nabla d|^2 d\|_{L^2}^2 + S(d, A, A) = 0. \tag{2.4}$$

which implies that

$$\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \int_0^t (\|\nabla u\|_{L^2}^2 + \|\Delta d + |\nabla d|^2 d\|_{L^2}^2) ds \leq \|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2. \tag{2.5}$$

Taking  $\Lambda^{\frac{1}{2}+\varepsilon}$  to (1.1)<sub>1</sub>, multiplying by  $\Lambda^{\frac{1}{2}+\varepsilon} u$ , integrating over  $\mathbb{R}^3$ , we deduce that

$$\begin{aligned}
 & \frac{\mu}{2} \frac{d}{dt} (\|\Lambda^{\frac{1}{2}+\varepsilon} u\|_{L^2}^2 + (\nu + \alpha_4) \|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2}^2) \\
 &= - \int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}+\varepsilon} (u \cdot \nabla u) \cdot \Lambda^{\frac{1}{2}+\varepsilon} u dx - \int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}+\varepsilon} [\nabla \cdot (\nabla d \odot \nabla d)] \cdot \Lambda^{\frac{1}{2}+\varepsilon} u dx \\
 &\quad - \alpha'_1 \int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}+\varepsilon} [\nabla \cdot ((d \cdot A \cdot d) d \otimes d)] \cdot \Lambda^{\frac{1}{2}+\varepsilon} u dx \\
 &\quad + \alpha'_5 \int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}+\varepsilon} [\nabla \cdot [(A \cdot d) \otimes d]] \cdot \Lambda^{\frac{1}{2}+\varepsilon} u dx + \alpha'_6 \int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}+\varepsilon} [\nabla \cdot [d \otimes (A \cdot d)]] \cdot \Lambda^{\frac{1}{2}+\varepsilon} u dx \\
 &\quad + \frac{1}{\gamma_1} \int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}+\varepsilon} [\nabla \cdot [\alpha_2 \Delta d \otimes d + \alpha_3 d \otimes \Delta d]] \cdot \Lambda^{\frac{1}{2}+\varepsilon} u dx \\
 &\quad + \frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}+\varepsilon} [\nabla \cdot [|\nabla d|^2 d \otimes d]] \cdot \Lambda^{\frac{1}{2}+\varepsilon} u dx \\
 &= J_1 + \dots + J_7.
 \end{aligned} \tag{2.6}$$

By using Lemmas 2.1 and 2.2, the condition  $|d| = 1$ , (1.12) and (2.5), we estimate  $J_1 - J_7$  term by term:

$$\begin{aligned}
 |J_1| &\leq C \|\Lambda^{\frac{1}{2}+\varepsilon} u\|_{L^6} \|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2} \|u\|_{L^3} \leq C \|\Lambda^{\frac{1}{2}} u\|_{L^2} \|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2}^2 \\
 &\leq C \|u\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}} \|\Lambda^{\frac{1}{2}+\varepsilon} u\|_{L^2}^{\frac{1}{1+2\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2}^2 \\
 &\leq C (\|u\|_{L^2} + \|\Lambda^{\frac{1}{2}+\varepsilon} u\|_{L^2}) \|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2}^2,
 \end{aligned} \tag{2.7}$$

$$\begin{aligned}
|J_2| &\leq C\|\Lambda^{\frac{1}{2}+\varepsilon}u\|_{L^6}\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}\|\nabla d\|_{L^3} \\
&\leq C\|\Lambda^{\frac{3}{2}}d\|_{L^2}(\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}^2) \\
&\leq C\|\nabla d\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}}\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}^{\frac{1}{1+2\varepsilon}}(\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}^2) \\
&\leq C(\|\nabla d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2})(\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}^2),
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
|J_3| &\leq C\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}\|\Lambda^{\frac{1}{2}+\varepsilon}[(d \cdot A \cdot d)d \otimes d]\|_{L^2} \\
&\leq C\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}(\|\Lambda^{\frac{1}{2}+\varepsilon}d\|_{L^6}\|\nabla u\|_{L^3}\|d\|_{L^\infty}^3 + \|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}\|d\|_{L^\infty}^4) \\
&\leq C\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}(\|\nabla u\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2})\|d\|_{L^\infty}^3 \\
&\quad + C\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2\|d\|_{L^\infty}^3(\|\nabla d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}) \\
&\leq C\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}(\|\nabla u\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}) \\
&\quad + C\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2(\|\nabla d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}) \\
&\leq C(\|\nabla d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2})(\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2),
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
|J_4| + |J_5| &\leq C\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}(\|\Lambda^{\frac{1}{2}+\varepsilon}[(A \cdot d) \otimes d]\|_{L^2} + \|\Lambda^{\frac{1}{2}+\varepsilon}[d \otimes (A \cdot d)]\|_{L^2}) \\
&\leq C\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}(\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}\|d\|_{L^\infty}^2 + \|\nabla u\|_{L^3}\|\Lambda^{\frac{1}{2}+\varepsilon}d\|_{L^6}\|d\|_{L^\infty}) \\
&\leq C\|d\|_{L^\infty}(\|\nabla d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2})\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2 \\
&\quad + C\|d\|_{L^\infty}\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}(\|\nabla u\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}) \\
&\leq C(\|\nabla d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2})\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2 \\
&\quad + C\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}(\|\nabla u\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}) \\
&\leq C(\|\nabla d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2})(\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2),
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
|J_6| &\leq C\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}\|\Lambda^{\frac{1}{2}+\varepsilon}[\alpha_2\Delta d \otimes d + \alpha_3d \otimes \Delta d]\|_{L^2} \\
&\leq C\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}(\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}\|d\|_{L^\infty} + \|\Delta d\|_{L^{\frac{3}{1-\varepsilon}}}\|\Lambda^{\frac{1}{2}+\varepsilon}d\|_{L^{\frac{6}{1+2\varepsilon}}}) \\
&\leq C\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}(\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}\|\nabla d\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}}\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}^{\frac{1}{1+2\varepsilon}} \\
&\quad + \|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}\|\nabla d\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}}\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}^{\frac{1}{1+2\varepsilon}}) \\
&\leq C(\|\nabla d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2})(\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}^2),
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
|J_7| &\leq C\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}\|\Lambda^{\frac{1}{2}+\varepsilon}[(\nabla d)^2d \otimes d]\|_{L^2} \\
&= C\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}\|\Lambda^{\frac{1}{2}+\varepsilon}[(d \cdot \Delta d)d \otimes d]\|_{L^2} \\
&\leq C\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}(\|\Lambda^{\frac{1}{2}+\varepsilon}d\|_{L^{\frac{6}{1+2\varepsilon}}}\|\Delta d\|_{L^{\frac{3}{1-\varepsilon}}}\|d\|_{L^\infty}^2 + \|\Lambda^{\frac{1}{2}+\varepsilon}\Delta d\|_{L^2}\|d\|_{L^\infty}^3) \\
&\leq C\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}(\|\Lambda^{\frac{1}{2}+\varepsilon}d\|_{L^{\frac{6}{1+2\varepsilon}}}\|\Delta d\|_{L^{\frac{3}{1-\varepsilon}}} + \|\Lambda^{\frac{1}{2}+\varepsilon}\Delta d\|_{L^2}\|d\|_{L^\infty}) \\
&\leq C\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}(\|\Lambda^{\frac{3}{2}}d\|_{L^2}\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2} + \|\Lambda^{\frac{1}{2}+\varepsilon}\Delta d\|_{L^2}\|\nabla d\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}}\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}^{\frac{1}{1+2\varepsilon}}) \\
&\leq C(\|\nabla d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2})(\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}^2).
\end{aligned} \tag{2.12}$$



It then follows from (2.6)–(2.12) that

$$\begin{aligned} & \frac{d}{dt} \|\Lambda^{\frac{1}{2}+\varepsilon} u\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2}^2 \\ & \leq C(\|u\|_{L^2} + \|\Lambda^{\frac{1}{2}+\varepsilon} u\|_{L^2} + \|\nabla d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}) \\ & \quad \times (\|\nabla u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2}^2). \end{aligned} \quad (2.13)$$

Taking  $\Lambda^{\frac{3}{2}+\varepsilon}$  to (1.1)<sub>2</sub>, multiplying by  $\Lambda^{\frac{3}{2}+\varepsilon} d$ , integrating over  $\mathbb{R}^3$ , we deduce that

$$\begin{aligned} & \frac{\gamma_1}{2} \frac{d}{dt} \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2}^2 \\ & = -\gamma_1 \int_{\mathbb{R}^3} \Lambda^{\frac{3}{2}+\varepsilon} (u \cdot \nabla d) \cdot \Lambda^{\frac{3}{2}+\varepsilon} d \, dx + \gamma_1 \int_{\mathbb{R}^3} \Lambda^{\frac{3}{2}+\varepsilon} (\Omega \cdot d) \cdot \Lambda^{\frac{3}{2}+\varepsilon} d \, dx \\ & \quad - \gamma_2 \int_{\mathbb{R}^3} \Lambda^{\frac{3}{2}+\varepsilon} (A \cdot d) \cdot \Lambda^{\frac{3}{2}+\varepsilon} d \, dx + \int_{\mathbb{R}^3} \Lambda^{\frac{3}{2}+\varepsilon} ((d \cdot A \cdot d) d) \cdot \Lambda^{\frac{3}{2}+\varepsilon} d \, dx \\ & \quad + \int_{\mathbb{R}^3} \Lambda^{\frac{3}{2}+\varepsilon} (|\nabla d|^2 d) \cdot \Lambda^{\frac{3}{2}+\varepsilon} d \, dx \\ & = I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \quad (2.14)$$

By using Lemmas 2.1 and 2.2, the condition  $|d| = 1$ , (1.12) and (2.5), we arrive at

$$\begin{aligned} |I_1| & \leq C \|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2} \left( \|\Lambda^{\frac{1}{2}+\varepsilon} u\|_{L^6} \|\nabla d\|_{L^3} + \|u\|_{L^3} \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^6} \right) \\ & \leq C \|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2} \left( \|\Lambda^{\frac{1}{2}+\varepsilon} u\|_{L^6} \|\nabla d\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}^{\frac{1}{1+2\varepsilon}} \right. \\ & \quad \left. + \|u\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}} \|\Lambda^{\frac{1}{2}+\varepsilon} u\|_{L^2}^{\frac{1}{1+2\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^6} \right) \\ & \leq C(\|u\|_{L^2} + \|\nabla d\|_{L^2} + \|\Lambda^{\frac{1}{2}+\varepsilon} u\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}) (\|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2}^2), \end{aligned} \quad (2.15)$$

$$\begin{aligned} |I_2| + |I_3| & \leq C \|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2} \left( \|\Lambda^{\frac{1}{2}+\varepsilon} d\|_{L^{\frac{6}{2\varepsilon+1}}} \|\nabla u\|_{L^{\frac{3}{1-\varepsilon}}} + \|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2} \|d\|_{L^\infty} \right) \\ & \leq C \|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2} \left( \|\nabla d\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}^{\frac{1}{1+2\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2} \right. \\ & \quad \left. + \|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2} \|\nabla d\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}^{\frac{1}{1+2\varepsilon}} \right) \\ & \leq C(\|\nabla d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}) (\|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2}^2), \end{aligned} \quad (2.16)$$

$$\begin{aligned} |I_4| & \leq C \|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2} \|\Lambda^{\frac{1}{2}+\varepsilon} [(d \cdot A \cdot d) d]\|_{L^2} \\ & \leq C \|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2} \left( \|\Lambda^{\frac{1}{2}+\varepsilon} d\|_{L^{\frac{6}{1+2\varepsilon}}} \|\nabla u\|_{L^{\frac{3}{1-\varepsilon}}} \|d\|_{L^\infty}^2 + \|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2} \|d\|_{L^\infty}^3 \right) \\ & \leq C \|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2} \left( \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2} \|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2} \|d\|_{L^\infty} \right) \\ & \leq C \|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2} \left( \|\nabla d\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}^{\frac{1}{1+2\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2} \right. \\ & \quad \left. + \|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2} \|\nabla d\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}^{\frac{1}{1+2\varepsilon}} \right) \\ & \leq C(\|\nabla d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}) (\|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2}^2), \end{aligned} \quad (2.17)$$

and

$$\begin{aligned}
|I_5| &\leq C\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}\|\Lambda^{\frac{1}{2}+\varepsilon}(|\nabla d|^2d)\|_{L^2} \\
&= C\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}\|\Lambda^{\frac{1}{2}+\varepsilon}[(d\cdot\Delta d)d]\|_{L^2} \\
&\leq C\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}\left(\|\Lambda^{\frac{1}{2}+\varepsilon}d\|_{L^{\frac{6}{1+2\varepsilon}}}\|\Delta d\|_{L^{\frac{3}{1-\varepsilon}}}\|d\|_{L^\infty}+\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}\|d\|_{L^\infty}^2\right) \\
&\leq C\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}\left(\|\Lambda^{\frac{3}{2}}d\|_{L^2}\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}+\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}\|d\|_{L^\infty}\right) \\
&\leq C\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}\left(\|\nabla d\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}}\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}^{\frac{1}{1+2\varepsilon}}\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}\right. \\
&\quad \left.+\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}\|\nabla d\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}}\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}^{\frac{1}{1+2\varepsilon}}\right) \\
&\leq C(\|\nabla d\|_{L^2}+\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2})\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}^2.
\end{aligned} \tag{2.18}$$

Summing up (2.15)–(2.18), we derive that

$$\begin{aligned}
&\frac{\gamma_1}{2}\frac{d}{dt}\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}^2+\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}^2 \\
&\leq C(\|u\|_{L^2}+\|\nabla d\|_{L^2}+\|\Lambda^{\frac{1}{2}+\varepsilon}u\|_{L^2}+\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2})(\|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2+\|\Lambda^{\frac{5}{2}+\varepsilon}d\|_{L^2}^2).
\end{aligned} \tag{2.19}$$

Now, we study the  $L^2$  estimate for  $d - d_*$ . Multiplying (1.1)<sub>2</sub> by  $d - d_*$ , integrating over  $\mathbb{R}^3$ , we arrive at

$$\begin{aligned}
&\frac{\gamma_1}{2}\frac{d}{dt}\|d - d_*\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 \\
&= -\gamma_1 \int_{\mathbb{R}^3} u \cdot \nabla d \cdot (d - d_*) dx + \gamma_1 \int_{\mathbb{R}^3} \Omega \cdot d \cdot (d - d_*) dx - \gamma_2 \int_{\mathbb{R}^3} A \cdot d \cdot (d - d_*) dx \\
&\quad + \gamma_2 \int_{\mathbb{R}^3} (d \cdot A \cdot d) d \cdot (d - d_*) dx + \int_{\mathbb{R}^3} |\nabla d|^2 d \cdot (d - d_*) dx \\
&= L_1 + L_2 + L_3 + L_4 + L_5.
\end{aligned} \tag{2.20}$$

On the basis of Lemmas 2.1 and 2.2, we can bound  $L_1$ – $L_5$  as

$$\begin{aligned}
|L_1| &\leq C\|u\|_{L^3}\|\nabla d\|_{L^2}\|d - d_*\|_{L^6} \\
&\leq C\|u\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}}\|\Lambda^{\frac{1}{2}+\varepsilon}u\|_{L^2}^{\frac{1}{1+2\varepsilon}}\|\nabla d\|_{L^2}^2 \\
&\leq C(\|u\|_{L^2}+\|\Lambda^{\frac{1}{2}+\varepsilon}u\|_{L^2})\|\nabla d\|_{L^2}^2,
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
|L_2| + |L_3| &\leq C\|\nabla u\|_{L^3}\|d\|_{L^6}\|d - d_*\|_{L^2} \\
&\leq C\|d - d_*\|_{L^2}(\|\nabla d\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}}u\|_{L^2}^2) \\
&\leq C\|d - d_*\|_{L^2}(\|\nabla d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}+\varepsilon}u\|_{L^2}^2),
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
|L_4| &\leq C\|d\|_{L^6}\|\nabla u\|_{L^2}\|d\|_{L^6}\|d - d_*\|_{L^6}\|d\|_{L^\infty} \\
&\leq C\|\nabla u\|_{L^2}\|\nabla d\|_{L^2}^3\|\nabla d\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}}\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}^{\frac{1}{1+2\varepsilon}} \\
&\leq C\|\nabla d\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}}\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2}^{\frac{1}{1+2\varepsilon}}\|\nabla d\|_{L^2}^2 \\
&\leq C(\|\nabla d\|_{L^2}+\|\Lambda^{\frac{3}{2}+\varepsilon}d\|_{L^2})\|\nabla d\|_{L^2}^2,
\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}
 |L_5| &\leq C \|\nabla d\|_{L^6} \|\nabla d\|_{L^2} \|d\|_{L^6} \|d - d_*\|_{L^\infty} \\
 &\leq C \|\Delta d\|_{L^2} \|\nabla d\|_{L^2}^2 \|\nabla d\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}^{\frac{1}{1+2\varepsilon}} \\
 &\leq C \|\nabla d\|_{L^2}^2 \|\nabla d\|_{L^2}^{\frac{2\varepsilon}{1+2\varepsilon}} \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}^{\frac{1}{1+2\varepsilon}} \\
 &\leq C (\|\nabla d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}) \|\nabla d\|_{L^2}^2.
 \end{aligned} \tag{2.24}$$

Combining (2.20)–(2.24) together gives

$$\begin{aligned}
 &\frac{d}{dt} \|d - d_*\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 \\
 &\leq C (\|u\|_{L^2} + \|\Lambda^{\frac{1}{2}+\varepsilon} u\|_{L^2} + \|d - d_*\|_{L^2} + \|\nabla d\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}) \\
 &\quad \times (\|\nabla d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2}^2).
 \end{aligned} \tag{2.25}$$

Adding (2.4), (2.13), (2.19) and (2.25) together, it yields that

$$\begin{aligned}
 &\frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \|d - d_*\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}+\varepsilon} u\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}^2) \\
 &\quad + \|\nabla u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|\Delta d + |\nabla d|^2 d\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2}^2 \\
 &\leq C (\|u\|_{L^2} + \|d - d_*\|_{L^2} + \|\nabla d\|_{L^2} + \|\Lambda^{\frac{1}{2}+\varepsilon} u\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}) \\
 &\quad \times (\|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|\Delta d + |\nabla d|^2 d\|_{L^2}^2).
 \end{aligned} \tag{2.26}$$

Taking  $K$  small enough such that

$$\|u_0\|_{L^2} + \|d_0 - d_*\|_{L^2} + \|\nabla d_0\|_{L^2} + \|\Lambda^{\frac{1}{2}+\varepsilon} u_0\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon} d_0\|_{L^2} < K < \frac{1}{C},$$

then,  $\|u\|_{L^2} + \|d - d_*\|_{L^2} + \|\nabla d\|_{L^2} + \|\Lambda^{\frac{1}{2}+\varepsilon} u\|_{L^2} + \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}$  is decreasing. So, for any  $0 < T < \infty$ , we have

$$\begin{aligned}
 &\frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \|d - d_*\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|\Lambda^{\frac{1}{2}+\varepsilon} u\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}+\varepsilon} d\|_{L^2}^2) \\
 &\quad + \|\nabla u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|\Delta d + |\nabla d|^2 d\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}+\varepsilon} u\|_{L^2}^2 + \|\Lambda^{\frac{5}{2}+\varepsilon} d\|_{L^2}^2 \leq 0.
 \end{aligned}$$

which implies that

$$u \in L^\infty(0, T; H^{\frac{1}{2}+\varepsilon}) \cap L^2(0, T; H^{\frac{3}{2}+\varepsilon}), \quad d - d_* \in L^\infty(0, T; H^{\frac{3}{2}+\varepsilon}) \cap L^2(0, T; H^{\frac{5}{2}+\varepsilon}). \tag{2.27}$$

Moreover, under the condition (1.11), we can easily obtain the higher order norm estimates

$$\begin{aligned}
 &\frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \|d - d_*\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|\nabla^N u\|_{L^2}^2 + \|\nabla^{N+1} u\|_{L^2}^2) \\
 &\quad + \|\nabla u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|\Delta d\|_{L^2}^2 + \|\nabla^{N+1} u\|_{L^2}^2 + \|\nabla^{N+2} u\|_{L^2}^2 \leq 0,
 \end{aligned} \tag{2.28}$$

which is equivalent to

$$\frac{1}{2} \frac{d}{dt} (\|u\|_{H^N}^2 + \|d - d_*\|_{H^N}^2 + \|\nabla d\|_{H^N}^2) + \|\nabla u\|_{H^N}^2 + \|\nabla d\|_{H^N}^2 + \|\Delta d\|_{H^N}^2 \leq 0. \tag{2.29}$$

Hence, for all  $t \geq 0$ ,

$$\begin{aligned} & \|u\|_{H^N}^2 + \|d - d_*\|_{H^N}^2 + \|\nabla d\|_{H^N}^2 + \int_0^t (\|\nabla u\|_{H^N}^2 + \|\nabla d\|_{H^N}^2 + \|\Delta d\|_{H^N}^2) ds \\ & \leq \|u_0\|_{H^N}^2 + \|d_0 - d_*\|_{H^N}^2 + \|\nabla d_0\|_{H^N}^2, \end{aligned} \quad (2.30)$$

this complete the proof.  $\square$

### 3. Proof of Theorem 1.7

The following special Sobolev interpolation lemma will be used in the proof of Theorem 1.7.

**Lemma 3.1** ([17, 36]). *Let  $s, l \geq 0$ , then*

$$\|\nabla^l f\|_{L^2(\mathbb{R}^3)} \leq \|\nabla^{l+1} f\|_{L^2(\mathbb{R}^3)}^{1-\theta} \|f\|_{\dot{H}^{-s}(\mathbb{R}^3)}^\theta, \quad \text{with } \theta = \frac{1}{l+1+s}.$$

We also introduce the Hardy-Littlewood-Sobolev theorem, which implies the following  $L^p$  type inequality.

**Lemma 3.2** ([16, 32]). *Let  $0 \leq s < \frac{3}{2}$ ,  $1 < p \leq 2$  and  $\frac{1}{2} + \frac{s}{3} = \frac{1}{p}$ , then*

$$\|f\|_{\dot{H}^{-s}(\mathbb{R}^3)} \lesssim \|f\|_{L^p(\mathbb{R}^3)}. \quad (3.1)$$

Next, we will derive the evolution of the negative Sobolev norm of the solution. First of all, we establish the following lemma.

**Lemma 3.3.** *For  $s \in (0, \frac{1}{2}]$ , we have*

$$\begin{aligned} & \frac{d}{dt} (\|\Lambda^{-s} u\|_{L^2}^2 + \|\Lambda^{-s}(d - d_*)\|_{L^2}^2 + \|\Lambda^{-s} \nabla d\|_{L^2}^2) \\ & \leq C(\|\nabla u\|_{H^2}^2 + \|\nabla d\|_{H^3}^2) (\|\Lambda^{-s} u\|_{L^2} + \|\Lambda^{-s}(d - d_*)\|_{L^2}). \end{aligned} \quad (3.2)$$

**PROOF.** Taking  $\Lambda^{-s}$  to (1.1)<sub>1</sub>, multiplying by  $\Lambda^s u$ , integrating over  $\mathbb{R}^3$ , we deduce that

$$\begin{aligned} & \frac{\mu}{2} \frac{d}{dt} \|\Lambda^{-s} u\|_{L^2}^2 + (\nu + \alpha_4) \|\Lambda^{-s} \nabla u\|_{L^2}^2 \\ & = - \int_{\mathbb{R}^3} \Lambda^{-s} (u \cdot \nabla u) \cdot \Lambda^{-s} u dx - \int_{\mathbb{R}^3} \Lambda^{-s} [\nabla \cdot (\nabla d \odot \nabla d)] \cdot \Lambda^{-s} u dx \\ & \quad - \alpha'_1 \int_{\mathbb{R}^3} \Lambda^{-s} [\nabla \cdot ((d \cdot A \cdot d) d \otimes d)] \cdot \Lambda^{-s} u dx \\ & \quad + \alpha'_5 \int_{\mathbb{R}^3} \Lambda^{-s} [\nabla \cdot [(A \cdot d) \otimes d]] \cdot \Lambda^{-s} u dx + \alpha'_6 \int_{\mathbb{R}^3} \Lambda^{-s} [\nabla \cdot [d \otimes (A \cdot d)]] \cdot \Lambda^{-s} u dx \\ & \quad + \frac{1}{\gamma_1} \int_{\mathbb{R}^3} \Lambda^{-s} [\nabla \cdot [\alpha_2 \Delta d \otimes d + \alpha_3 d \otimes \Delta d]] \cdot \Lambda^{-s} u dx \\ & \quad + \frac{\gamma_2}{\gamma_1} \int_{\mathbb{R}^3} \Lambda^{-s} [\nabla \cdot [|\nabla d|^2 d \otimes d]] \cdot \Lambda^{-s} u dx \\ & = K_1 + \dots + K_7. \end{aligned} \quad (3.3)$$

If  $s \in [0, \frac{1}{2}]$ , then  $\frac{1}{2} + \frac{s}{3} < 1$  and  $\frac{3}{s} \geq 6$ . By using the estimate (3.1) of Riesz potential in Lemma 3.2 and the estimate (2.29), we find that

$$\begin{aligned}
 |K_1| &\leq \|\Lambda^{-s}(u \cdot \nabla u)\|_{L^2} \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim \|u \cdot \nabla u\|_{L^{\frac{1}{\frac{1}{2} + \frac{s}{3}}}} \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim \|u\|_{L^{\frac{3}{s}}} \|\nabla u\|_{L^2} \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim \|\nabla u\|_{L^2}^{\frac{1}{2} + s} \|\nabla^2 u\|_{L^2}^{\frac{1}{2} - s} \|\nabla u\|_{L^2} \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim \|\Lambda^{-s}u\|_{L^2} (\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2),
 \end{aligned} \tag{3.4}$$

$$\begin{aligned}
 |K_2| &\leq \|\Lambda^{-s} \nabla \cdot (\nabla d \odot \nabla d)\|_{L^2} \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim \|\nabla \cdot (\nabla d \odot \nabla d)\|_{L^{\frac{1}{\frac{1}{2} + \frac{s}{3}}}} \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim \|\nabla d\|_{L^{\frac{3}{s}}} \|\nabla^2 d\|_{L^2} \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim \|\nabla^2 d\|_{L^2}^{\frac{1}{2} + s} \|\nabla^3 d\|_{L^2}^{\frac{1}{2} - s} \|\nabla^2 d\|_{L^2} \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim \|\Lambda^{-s}u\|_{L^2} (\|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2),
 \end{aligned} \tag{3.5}$$

$$\begin{aligned}
 |K_3| &\leq \|\Lambda^{-s} \nabla((d \cdot A \cdot d)d \otimes d)\|_{L^2} \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim \|\nabla((d \cdot A \cdot d)d \otimes d)\|_{L^{\frac{1}{\frac{1}{2} + \frac{s}{3}}}} \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim (\|d\|_{L^\infty}^3 \|\nabla d\|_{L^{\frac{3}{s}}} \|\nabla u\|_{L^2} + \|d\|_{L^\infty}^3 \|d\|_{L^{\frac{3}{s}}} \|\Delta u\|_{L^2}) \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim (\|\nabla d\|_{L^{\frac{3}{s}}} \|\nabla u\|_{L^2} + \|d\|_{L^{\frac{3}{s}}} \|\Delta u\|_{L^2}) \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim (\|\nabla^2 d\|_{L^2}^{\frac{1}{2} + s} \|\nabla^3 d\|_{L^2}^{\frac{1}{2} - s} \|\nabla u\|_{L^2} + \|\nabla d\|_{L^2}^{\frac{1}{2} + s} \|\nabla^2 d\|_{L^2}^{\frac{1}{2} - s} \|\Delta u\|_{L^2}) \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim \|\Lambda^{-s}u\|_{L^2} (\|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2),
 \end{aligned} \tag{3.6}$$

$$\begin{aligned}
 |K_4| + |K_5| &\leq (\|\Lambda^{-s} [\nabla \cdot [(A \cdot d) \otimes d]]\|_{L^2} + \|\Lambda^{-s} [\nabla \cdot [d \otimes (A \cdot d)]]\|_{L^2}) \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim (\|\nabla \cdot [(A \cdot d) \otimes d]\|_{L^{\frac{1}{\frac{1}{2} + \frac{s}{3}}}} + \|\nabla \cdot [d \otimes (A \cdot d)]\|_{L^{\frac{1}{\frac{1}{2} + \frac{s}{3}}}}) \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim (\|d\|_{L^\infty} \|\nabla d\|_{L^{\frac{3}{s}}} \|\nabla u\|_{L^2} + \|d\|_{L^\infty} \|d\|_{L^{\frac{3}{s}}} \|\Delta u\|_{L^2}) \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim (\|\nabla d\|_{L^{\frac{3}{s}}} \|\nabla u\|_{L^2} + \|d\|_{L^{\frac{3}{s}}} \|\Delta u\|_{L^2}) \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim (\|\nabla^2 d\|_{L^2}^{\frac{1}{2} + s} \|\nabla^3 d\|_{L^2}^{\frac{1}{2} - s} \|\nabla u\|_{L^2} + \|\nabla d\|_{L^2}^{\frac{1}{2} + s} \|\nabla^2 d\|_{L^2}^{\frac{1}{2} - s} \|\Delta u\|_{L^2}) \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim \|\Lambda^{-s}u\|_{L^2} (\|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2),
 \end{aligned} \tag{3.7}$$

$$\begin{aligned}
 |K_6| &\lesssim \|\Lambda^{-s} [\nabla \cdot [\alpha_2 \Delta d \otimes d + \alpha_3 d \otimes \Delta d]]\|_{L^2} \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim \left( \|\nabla \cdot [\Delta d \otimes d]\|_{L^{\frac{1}{\frac{1}{2} + \frac{s}{3}}}} + \|\nabla \cdot [d \otimes \Delta d]\|_{L^{\frac{1}{\frac{1}{2} + \frac{s}{3}}}} \right) \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim \|\nabla \Delta d\|_{L^2} \|d\|_{L^{\frac{3}{s}}} + \|\Delta d\|_{L^2} \|\nabla d\|_{L^{\frac{3}{s}}} \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim \left( \|\nabla \Delta d\|_{L^2} \|\nabla d\|_{L^2}^{\frac{1}{2} + s} \|\nabla^2 d\|_{L^2}^{\frac{1}{2} - s} + \|\Delta d\|_{L^2} \|\nabla^2 d\|_{L^2}^{\frac{1}{2} + s} \|\nabla^3 d\|_{L^2}^{\frac{1}{2} - s} \right) \|\Lambda^{-s}u\|_{L^2} \\
 &\lesssim \|\Lambda^{-s}u\|_{L^2} (\|\nabla d\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2),
 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
|K_7| &\lesssim \|\Lambda^{-s} \nabla \cdot [|\nabla d|^2 d \otimes d]\|_{L^2} \|\Lambda^{-s} u\|_{L^2} \\
&\lesssim \|\nabla \cdot [|\nabla d|^2 d \otimes d]\|_{L^{\frac{1}{\frac{1}{2} + \frac{s}{3}}}} \|\Lambda^{-s} u\|_{L^2} \\
&\lesssim (\|d\|_{L^\infty}^2 \|\nabla d\|_{L^{\frac{3}{s}}} \|\Delta d\|_{L^2} + \|d\|_{L^{\frac{3}{s}}} \|\nabla d\|_{L^6}^3) \|\Lambda^{-s} u\|_{L^2} \\
&\lesssim (\|\nabla d\|_{L^{\frac{3}{s}}} \|\Delta d\|_{L^2} + \|d\|_{L^{\frac{3}{s}}} \|\Delta d\|_{L^2}) \|\Lambda^{-s} u\|_{L^2} \\
&\lesssim (\|\Delta d\|_{L^2}^{\frac{1}{2} + s} \|\nabla^3 d\|_{L^2}^{\frac{1}{2} - s} \|\nabla^2 d\|_{L^2} + \|\nabla d\|_{L^2}^{\frac{1}{2} + s} \|\nabla^2 d\|_{L^2}^{\frac{1}{2} - s} \|\Delta d\|_{L^2}) \|\Lambda^{-s} u\|_{L^2} \\
&\lesssim \|\Lambda^{-s} u\|_{L^2} (\|\nabla d\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2).
\end{aligned} \tag{3.9}$$

Combining (3.3)–(3.9) together gives

$$\frac{\mu}{2} \frac{d}{dt} \|\Lambda^{-s} u\|_{L^2}^2 + (\nu + \alpha_4) \|\Lambda^{-s} \nabla u\|_{L^2}^2 \lesssim \|\Lambda^{-s} u\|_{L^2} (\|\nabla u\|_{H^1}^2 + \|\nabla d\|_{H^2}^2). \tag{3.10}$$

Taking  $\Lambda^{-s}$  to (1.1)<sub>2</sub>, multiplying by  $\Lambda^{-s}(d - d_*)$ , integrating over  $\mathbb{R}^3$ , we deduce that

$$\begin{aligned}
&\frac{\gamma_1}{2} \frac{d}{dt} \|\Lambda^{-s}(d - d_*)\|_{L^2}^2 + \|\Lambda^{-s} \nabla d\|_{L^2}^2 \\
&= -\gamma_1 \int_{\mathbb{R}^3} \Lambda^{-s}(u \cdot \nabla d) \cdot \Lambda^{-s}(d - d_*) dx + \gamma_1 \int_{\mathbb{R}^3} \Lambda^{-s}(\Omega \cdot d) \cdot \Lambda^{-s}(d - d_*) dx \\
&\quad - \gamma_2 \int_{\mathbb{R}^3} \Lambda^{-s}(A \cdot d) \cdot \Lambda^{-s}(d - d_*) dx + \int_{\mathbb{R}^3} \Lambda^{-s}((d \cdot A \cdot d)d) \cdot \Lambda^{-s}(d - d_*) dx \\
&\quad + \int_{\mathbb{R}^3} \Lambda^{-s}(|\nabla d|^2 d) \cdot \Lambda^{-s}(d - d_*) dx \\
&= L_1 + L_2 + L_3 + L_4 + L_5.
\end{aligned} \tag{3.11}$$

If  $s \in [0, \frac{1}{2}]$ , then  $\frac{1}{2} + \frac{s}{3} < 1$  and  $\frac{3}{s} \geq 6$ . By using the estimate (3.1) of Riesz potential in Lemma 3.2 and the estimate (2.29), we find that

$$\begin{aligned}
|L_1| &\lesssim \|\Lambda^{-s}(u \cdot \nabla d)\|_{L^2} \|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim \|u \cdot \nabla d\|_{L^{\frac{1}{\frac{1}{2} + \frac{s}{3}}}} \|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim \|u\|_{L^{\frac{3}{s}}} \|\nabla d\|_{L^2} \|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim \|\nabla u\|_{L^2}^{\frac{1}{2} + s} \|\nabla^2 u\|_{L^2}^{\frac{1}{2} - s} \|\nabla d\|_{L^2} \|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim \|\Lambda^{-s}(d - d_*)\|_{L^2} (\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2),
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
|L_2| + |L_3| &\leq (\|\Lambda^{-s}(\Omega \cdot d)\|_{L^2} + \|\Lambda^{-s}(A \cdot d)\|_{L^2}) \|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim (\|\Omega \cdot d\|_{L^{\frac{1}{\frac{1}{2} + \frac{s}{3}}}} + \|A \cdot d\|_{L^{\frac{1}{\frac{1}{2} + \frac{s}{3}}}}) \|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim \|d\|_{L^{\frac{3}{s}}} \|\nabla u\|_{L^2} \|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim \|\nabla d\|_{L^2}^{\frac{1}{2} + s} \|\nabla^2 d\|_{L^2}^{\frac{1}{2} - s} \|\nabla u\|_{L^2} \|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim \|\Lambda^{-s}(d - d_*)\|_{L^2} (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2),
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
|L_4| &\leq \|\Lambda^{-s} \nabla((d \cdot A \cdot d)d)\|_{L^2} \|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim \|\nabla((d \cdot A \cdot d)d)\|_{L^{\frac{1}{\frac{1}{2} + \frac{s}{3}}}} \|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim (\|d\|_{L^\infty}^2 \|\nabla d\|_{L^{\frac{3}{s}}} \|\nabla u\|_{L^2} + \|d\|_{L^\infty}^2 \|d\|_{L^{\frac{3}{s}}} \|\Delta u\|_{L^2}) \|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim (\|\nabla d\|_{L^{\frac{3}{s}}} \|\nabla u\|_{L^2} + \|d\|_{L^{\frac{3}{s}}} \|\Delta u\|_{L^2}) \|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim (\|\nabla^2 d\|_{L^2}^{\frac{1}{2} + s} \|\nabla^3 d\|_{L^2}^{\frac{1}{2} - s} \|\nabla u\|_{L^2} + \|\nabla d\|_{L^2}^{\frac{1}{2} + s} \|\nabla^2 d\|_{L^2}^{\frac{1}{2} - s} \|\Delta u\|_{L^2}) \|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim \|\Lambda^{-s}(d - d_*)\|_{L^2} (\|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2),
\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}
|L_5| &\leq \|\Lambda^{-s} \nabla \cdot [|\nabla d|^2 d]\|_{L^2} \|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim \|\nabla \cdot [|\nabla d|^2 d]\|_{L^{\frac{1}{\frac{1}{2} + \frac{s}{3}}}} \|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim (\|d\|_{L^\infty} \|\nabla d\|_{L^{\frac{3}{s}}} \|\Delta d\|_{L^2} + \|\nabla d\|_{L^{\frac{3}{s}}} \|\nabla d\|_{L^2} \|\nabla d\|_{L^\infty}) \|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim (\|\nabla d\|_{L^{\frac{3}{s}}} \|\Delta d\|_{L^2} + \|\nabla d\|_{L^{\frac{3}{s}}} \|\nabla d\|_{L^2}) \|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim (\|\Delta d\|_{L^2}^{\frac{1}{2} + s} \|\nabla^3 d\|_{L^2}^{\frac{1}{2} - s} \|\nabla^2 d\|_{L^2} + \|\Delta d\|_{L^2}^{\frac{1}{2} + s} \|\nabla^3 d\|_{L^2}^{\frac{1}{2} - s} \|\nabla d\|_{L^2}) \|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim \|\Lambda^{-s}(d - d_*)\|_{L^2} (\|\nabla d\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2).
\end{aligned} \tag{3.15}$$

Combining (3.11)–(3.15) together, we obtain

$$\frac{\gamma_1}{2} \frac{d}{dt} \|\Lambda^{-s}(d - d_*)\|_{L^2}^2 + \|\Lambda^{-s} \nabla d\|_{L^2}^2 \lesssim \|\Lambda^{-s}(d - d_*)\|_{L^2} (\|\nabla u\|_{H^1}^2 + \|\nabla d\|_{H^2}^2). \tag{3.16}$$

Taking  $\Lambda^{-s} \nabla$  to (1.1)<sub>2</sub>, multiplying by  $\Lambda^{-s} \nabla d$ , integrating over  $\mathbb{R}^3$ , we deduce that

$$\begin{aligned}
&\frac{\gamma_1}{2} \frac{d}{dt} \|\Lambda^{-s} \nabla d\|_{L^2}^2 + \|\Lambda^{-s} \nabla^2 d\|_{L^2}^2 \\
&= -\gamma_1 \int_{\mathbb{R}^3} \Lambda^{-s} \nabla(u \cdot \nabla d) \cdot \Lambda^{-s} \nabla d dx + \gamma_1 \int_{\mathbb{R}^3} \Lambda^{-s} \nabla(\Omega \cdot d) \cdot \Lambda^{-s} \nabla d dx \\
&\quad - \gamma_2 \int_{\mathbb{R}^3} \Lambda^{-s} \nabla(A \cdot d) \cdot \Lambda^{-s} \nabla d dx + \int_{\mathbb{R}^3} \Lambda^{-s} \nabla((d \cdot A \cdot d)d) \cdot \Lambda^{-s} \nabla d dx \\
&\quad + \int_{\mathbb{R}^3} \Lambda^{-s} \nabla(|\nabla d|^2 d) \cdot \Lambda^{-s} \nabla d dx \\
&= W_1 + W_2 + W_3 + W_4 + W_5.
\end{aligned} \tag{3.17}$$

By using the estimate (3.1) of Riesz potential in Lemma 3.2 and the estimate (2.29), we find that

$$\begin{aligned}
|W_1| &\leq \|\Lambda^{-s} \nabla(u \cdot \nabla d)\|_{L^2} \|\Lambda^{-s} \nabla d\|_{L^2} \\
&\lesssim \|\nabla(u \cdot \nabla d)\|_{L^{\frac{1}{\frac{1}{2} + \frac{s}{3}}}} \|\Lambda^{-s} \nabla d\|_{L^2} \\
&\lesssim (\|u\|_{L^{\frac{3}{s}}} \|\Delta d\|_{L^2} + \|\nabla d\|_{L^{\frac{3}{s}}} \|\nabla u\|_{L^2}) \|\Lambda^{-s} \nabla d\|_{L^2} \\
&\lesssim (\|\nabla u\|_{L^2}^{\frac{1}{2} + s} \|\nabla^2 u\|_{L^2}^{\frac{1}{2} - s} \|\nabla^2 d\|_{L^2} + \|\nabla^2 d\|_{L^2}^{\frac{1}{2} + s} \|\nabla^3 d\|_{L^2}^{\frac{1}{2} - s} \|\nabla u\|_{L^2}) \|\Lambda^{-s} \nabla d\|_{L^2} \\
&\lesssim \|\Lambda^{-s} \nabla d\|_{L^2} (\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2),
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
|W_2| + |W_3| &\leq (\|\Lambda^{-s}\nabla(\Omega \cdot d)\|_{L^2} + \|\Lambda^{-s}\nabla(A \cdot d)\|_{L^2})\|\Lambda^{-s}\nabla d\|_{L^2} \\
&\lesssim (\|\nabla(\Omega \cdot d)\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}} + \|\nabla(A \cdot d)\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}})\|\Lambda^{-s}\nabla d\|_{L^2} \\
&\lesssim (\|d\|_{L^{\frac{3}{s}}} \|\Delta u\|_{L^2} + \|\nabla d\|_{L^{\frac{3}{s}}} \|\nabla u\|_{L^2})\|\Lambda^{-s}(d - d_*)\|_{L^2} \\
&\lesssim (\|\nabla d\|_{L^2}^{\frac{1}{2}+s} \|\nabla^2 d\|_{L^2}^{\frac{1}{2}-s} \|\Delta u\|_{L^2} + \|\Delta d\|_{L^2}^{\frac{1}{2}+s} \|\nabla^3 d\|_{L^2}^{\frac{1}{2}-s} \|\nabla u\|_{L^2})\|\Lambda^{-s}\nabla d\|_{L^2} \\
&\lesssim \|\Lambda^{-s}\nabla d\|_{L^2} (\|\nabla u\|_{L^2}^2 + \|\nabla^2 d\|_{L^2}^2 + \|\nabla d\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2),
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
|W_4| &\leq \|\Lambda^{-s}\nabla^2((d \cdot A \cdot d)d)\|_{L^2} \|\Lambda^{-s}\nabla d\|_{L^2} \\
&\lesssim \|\nabla^2((d \cdot A \cdot d)d)\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}} \|\Lambda^{-s}\nabla d\|_{L^2} \\
&\lesssim (\|d\|_{L^\infty}^2 \|\nabla^2 d\|_{L^{\frac{3}{s}}} \|\nabla u\|_{L^2} + \|d\|_{L^\infty}^2 \|d\|_{L^{\frac{3}{s}}} \|\nabla^3 u\|_{L^2})\|\Lambda^{-s}\nabla d\|_{L^2} \\
&\lesssim (\|\nabla^2 d\|_{L^{\frac{3}{s}}} \|\nabla u\|_{L^2} + \|d\|_{L^{\frac{3}{s}}} \|\nabla \Delta u\|_{L^2})\|\Lambda^{-s}\nabla d\|_{L^2} \\
&\lesssim (\|\nabla^3 d\|_{L^2}^{\frac{1}{2}+s} \|\nabla^4 d\|_{L^2}^{\frac{1}{2}-s} \|\nabla u\|_{L^2} + \|\nabla d\|_{L^2}^{\frac{1}{2}+s} \|\nabla^2 d\|_{L^2}^{\frac{1}{2}-s} \|\nabla \Delta u\|_{L^2})\|\Lambda^{-s}\nabla d\|_{L^2} \\
&\lesssim \|\Lambda^{-s}\nabla d\|_{L^2} (\|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2 + \|\nabla^4 d\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2),
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
|W_5| &\lesssim \|\Lambda^{-s}\nabla^2[|\nabla d|^2 d]\|_{L^2} \|\Lambda^{-s}\nabla d\|_{L^2} \\
&\lesssim \|\nabla^2[|\nabla d|^2 d]\|_{L^{\frac{1}{\frac{1}{2}+\frac{s}{3}}}} \|\Lambda^{-s}\nabla d\|_{L^2} \\
&\lesssim (\|d\|_{L^\infty} \|\nabla d\|_{L^{\frac{3}{s}}} \|\nabla \Delta d\|_{L^2} + \|\nabla d\|_{L^{\frac{3}{s}}} \|\nabla^2 d\|_{L^2} \|\nabla d\|_{L^\infty})\|\Lambda^{-s}\nabla d\|_{L^2} \\
&\lesssim (\|\nabla d\|_{L^{\frac{3}{s}}} \|\nabla \Delta d\|_{L^2} + \|\nabla d\|_{L^{\frac{3}{s}}} \|\nabla^2 d\|_{L^2})\|\Lambda^{-s}\nabla d\|_{L^2} \\
&\lesssim (\|\Delta d\|_{L^2}^{\frac{1}{2}+s} \|\nabla^3 d\|_{L^2}^{\frac{1}{2}-s} \|\nabla^3 d\|_{L^2} + \|\Delta d\|_{L^2}^{\frac{1}{2}+s} \|\nabla^3 d\|_{L^2}^{\frac{1}{2}-s} \|\nabla^2 d\|_{L^2})\|\Lambda^{-s}\nabla d\|_{L^2} \\
&\lesssim \|\Lambda^{-s}\nabla d\|_{L^2} (\|\nabla^2 d\|_{L^2}^2 + \|\nabla^3 d\|_{L^2}^2).
\end{aligned} \tag{3.21}$$

Combining (3.17)–(3.21) together, we obtain

$$\frac{\gamma_1}{2} \frac{d}{dt} \|\Lambda^{-s}\nabla d\|_{L^2}^2 + \|\Lambda^{-s}\nabla^2 d\|_{L^2}^2 \lesssim \|\Lambda^{-s}\nabla d\|_{L^2} (\|\nabla u\|_{H^2}^2 + \|\nabla d\|_{H^3}^2). \tag{3.22}$$

Adding (3.10), (3.16) and (3.22) together, we obtain (3.2). Hence, the proof is complete.  $\square$

The proof of Theorem 1.7 will be given in the following. We only consider the case  $N \in \mathbb{N}^+$ , the case  $N \in \mathbb{R}^+/\mathbb{N}^+$  can be obtained by interpolation.

*Proof of Theorem 1.7.* Note that  $N \in \mathbb{N}^+$ . We prove (1.13) holds for  $s \in [0, \frac{1}{2}]$ . Define

$$\mathcal{E}_{-s}(t) := \|\Lambda^{-s}u(t)\|_{L^2}^2 + \|\Lambda^{-s}(d - d_*)\|_{L^2}^2 + \|\Lambda^{-s}\nabla d\|_{L^2}^2.$$

For inequality (3.2), integrating in time, by the bounds (2.30), we have

$$\begin{aligned}
\mathcal{E}_{-s}(t) &\leq \mathcal{E}_{-s}(0) + C \int_0^t (\|\nabla u\|_{H^2}^2 + \|\nabla d\|_{H^3}^2) \sqrt{\mathcal{E}_{-s}(\tau)} d\tau \\
&\leq C_0 \left( 1 + \sup_{0 \leq \tau \leq t} \sqrt{\mathcal{E}_{-s}(\tau)} \right),
\end{aligned}$$



which means that

$$\|\Lambda^{-s}u\|_{L^2}^2 + \|\Lambda^{-s}(d - d_*)\|_{L^2}^2 + \|\Lambda^{-s}\nabla d\|_{L^2}^2 \leq C_0. \quad (3.23)$$

Moreover, if  $l = 1, 2, \dots, N - 1$ , we may use Lemma 3.1 to have

$$\|\nabla^{l+1}f\|_{L^2} \geq C\|\Lambda^{-s}f\|_{L^2}^{-\frac{1}{l+s}}\|\nabla^l f\|_{L^2}^{1+\frac{1}{l+s}}.$$

Then, by this facts and (3.23), we get

$$\|\nabla^l(\nabla u, \nabla d, \nabla^2 d)\|_{L^2}^2 \geq C_0(\|\nabla^l(u, d - d_*, \nabla d)\|_{L^2}^2)^{1+\frac{1}{l+s}}.$$

Hence, for  $l = 1, 2, \dots, N - 1$ ,

$$\|\nabla^l(\nabla u, \nabla d, \nabla^2 d)\|_{H^{N-l-1}}^2 \geq C_0(\|\nabla^l(u, d - d_*, \nabla d)\|_{H^{N-l-1}}^2)^{1+\frac{1}{l+s}}.$$

Thus, we deduce from (2.29) the following inequality

$$\begin{aligned} & \frac{d}{dt}(\|\nabla^l u\|_{H^{N-l}}^2 + \|\nabla^l(d - d_*)\|_{H^{N-l}}^2 + \|\nabla^{l+1}d\|_{H^{N-l}}^2) \\ & + C_0(\|u\|_{H^{N-l}}^2 + \|d - d_*\|_{H^{N-l}}^2 + \|\nabla d\|_{H^{N-l}}^2)^{1+\frac{1}{l+s}} \leq 0, \text{ for } l = 1, \dots, N - 1. \end{aligned}$$

Solving this inequality directly gives

$$\|\nabla^l u\|_{H^{N-l}}^2 + \|\nabla^l(d - d_*)\|_{H^{N-l}}^2 + \|\nabla^{l+1}d\|_{H^{N-l}}^2 \leq C_0(1+t)^{-l-s}, \quad \text{for } l = 1, 2, \dots, N - 1. \quad (3.24)$$

Note that the Hardy-Littlewood-Sobolev theorem (Lemma 3.2) implies that for  $p \in [\frac{3}{2}, 2]$ ,  $L^p(\mathbb{R}^3) \subset \dot{H}^{-s}(\mathbb{R}^3)$  with  $s = 3(\frac{1}{p} - \frac{1}{2}) \in [0, \frac{1}{2})$ . Therefore, based on (3.24), for  $l = 0, 1, \dots, N - 1$ , we obtain

$$\|\nabla^l u\|_{H^{N-l}}^2 + \|\nabla^l(d - d_*)\|_{H^{N-l}}^2 + \|\nabla^{l+1}d\|_{H^{N-l}}^2 \leq C(1+t)^{-[\frac{3}{2}(\frac{1}{p}-\frac{1}{2})+\frac{1}{2}]}.$$

Then, the inequality (1.13) holds and we complete the proof of Theorem 1.7. □

## 4. Conclusions

In this paper, for the Cauchy problem of 3D incompressible liquid crystal system with general Leslie stress tensor, we consider the small initial data global well-posedness and time decay estimates of strong solutions. The main tools to study the main results are energy estimates and negative Sobolev norm estimates. We first establish the global well-posedness result provide that  $\|u_0\|_{\dot{H}^{\frac{1}{2}+\varepsilon}} + \|d_0 - d_*\|_{\dot{H}^{\frac{3}{2}+\varepsilon}}$  ( $\varepsilon > 0$ ) is sufficiently small, which can be seen as an improvement of [35]. Moreover, we obtain the  $\dot{H}^{-s}$  ( $0 \leq s < \frac{1}{2}$ ) negative Sobolev norm estimates. By using those estimates, the  $L^p-L^2$  ( $\frac{3}{2} \leq p \leq 2$ ) type optimal decay rates of the higher-order spatial derivatives of solutions are shown. We believe our results on global well-posedness and decay estimates will attract the attentions of the related readers.

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## Conflict of interest

The authors declare that they have no competing interests.

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