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Research article

Automorphism group of the commuting graph of 2×2 matrix ring over $\mathbb{Z}_{p^s}$

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Abstract: Let R be a ring with identity. The commuting graph of R is the graph associated to R whose vertices are non-central elements in R, and distinct vertices A and B are adjacent if and only if $AB = BA$. In this paper, we completely determine the automorphism group of the commuting graph of 2×2 matrix ring over $\mathbb{Z}_{p^s}$, where $\mathbb{Z}_{p^s}$ is the ring of integers modulo p^s, p is a prime and s is a positive integer.

Keywords: commuting graph; automorphism group; matrix ring
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1. Introduction

Let R be a ring with identity, and let $C(R)$ be the center of R. The commuting graph $\Gamma(R)$ of R is the graph associated to R whose vertices are the elements of $R \setminus C(R)$ such that distinct vertices A and B are adjacent if and only if $AB = BA$. For the purpose of investigating the structures of a group or a ring, there are many associated graphs that have been studied extensively. Let $M_n(F)$ denote the ring of $n \times n$ matrices over F, where F is a field and $n \geq 2$ an arbitrary integer. In [1], if F is a finite field and $\Gamma(R) \cong \Gamma(M_n(F))$, then $|R| = |M_n(F)|$. Furthermore, if F is a prime field and $n = 2$, then $R \cong M_2(F)$. In [2], this result still holds if it is just assumed that F is a finite field. There are also some graph-theoretic properties of the commuting graphs that have been investigated, such as connectivity and domination number. In [3], Akbari et al. showed that $\Gamma(M_n(F))$ is a connected graph if and only if every field extension of F of degree n contains a proper intermediate field. Also it is shown that for two fields F and E and integers $n, m \geq 2$, if $\Gamma(M_n(F)) \cong \Gamma(M_m(E))$, then $n = m$ and $|F| = |E|$.

The commuting graph of a finite group $\Delta(G)$ is the graph whose vertex set is G with $x, y \in G$, $x \neq y$, joined by an edge whenever $xy = yx$, where G is a finite group. The graph $\Delta(G)$ has been studied in [4–7]. The set of all automorphisms of a graph forms a group known as the graph's automorphism group. The automorphism group of a graph describes its symmetries. In [6], it is proved that the automorphism group of $\Delta(G)$ is abelian if and only if $|G| \leq 2$. With the wreath product, Mirzargar et al. [7] determined the automorphism group of $\Delta(G)$, where G is an AC-group. In [8], it is proved that the automorphism group of $\Gamma(M_2(F))$ is a direct product of symmetric groups, where F is a finite field. In this paper, motivated by these works, we extend the finite field to the ring of integers modulo p^s, and we completely determine the automorphism group of $\Gamma(M_2(\mathbb{Z}_{p^s}))$, where $\mathbb{Z}_{p^s}$ is the ring of integers modulo p^s, p is a prime and s is a positive integer. This paper is organized as follows. In section 2, we give some preliminaries, notation, lemmas and definition of the wreath product. In section 3, we show that the automorphism group of $\Gamma(M_2(\mathbb{Z}_{p^s}))$ is a subgroup of a direct product of some wreath products, and we completely characterize it in Theorem 3.8.

2. Preliminaries and notation

In this paper, let $M_2(\mathbb{Z}_{p^s})$ denote the 2×2 matrix ring over $\mathbb{Z}_{p^s}$, we write it R for short. Let E_{ij} denote the matrix in R having 1 in its (i, j) entry and zeros elsewhere, and let E denote the identity matrix. It is well known that $C(R) = \{aE \mid a \in \mathbb{Z}_{p^s}\}$. For $A \in R$, $C_R(A) = \{B \in R \mid AB = BA\}$ is called the centralizer of A in R. For the ring R, let us denote by $U(R)$ and $D(R)$ the unit group and the zero divisor set of R respectively. The commuting graph of R is the graph with vertices $R \setminus C(R)$, and distinct vertices A and B are adjacent if and only if $AB = BA$. In a graph G, if x is adjacent to y (denoted by $[x, y]$), then we say that x is a neighbor of y or that y is a neighbor of x. Let $N(x)$ denote the neighbors of x in G. A graph automorphism of a graph G is a bijection on vertex set (denoted by $V(G)$) which preserves adjacency. For $a \in \mathbb{Z}_{p^s}$, let $\langle a\rangle$ be the ideal of $\mathbb{Z}_{p^s}$ generated by a, we will denote by $\mathrm{Ann}(a)$ the set $\{b \in \mathbb{Z}_{p^s} \mid ab = 0\}$, and by $\mathrm{Ass}(a)$ the set $\{ua \mid u \in U(\mathbb{Z}_{p^s})\}$. Write $T = \{0, 1, \cdots, p-1\} \subseteq \mathbb{Z}_{p^s}$. The subset of T consisting of all non-zero elements is denoted by T^*. Let us denote by S_n the symmetric group of degree n. For a set D, we will denote by $|D|$ the size of D, and by S_D the symmetric group on D.

Lemma 2.1. [9, p. 328] *Every non-zero element in $\mathbb{Z}_{p^s}$ can be written uniquely as*

$$t_0 + t_1 p + \cdots + t_{s-1}p^{s-1},$$

where $t_i \in T$, $i \in \{0, 1, \cdots, s-1\}$. Furthermore, $|\langle p^i\rangle| = p^{s-i}$, $|\mathrm{Ass}(p^i)| = p^{s-i} - p^{s-i-1}$, and $\mathrm{Ann}(p^i) = \langle p^{s-i}\rangle$.

Definition 2.2. [10, p. 172] *Let D and Q be groups, let Ω be a finite Q-set, and let $K = \prod_{\omega\in\Omega} D_\omega$, where $D_\omega \cong D$ for all $\omega \in \Omega$. Then the wreath product of D by Q, denoted by $D \wr_\Omega Q$, is the semidirect product of K by Q, where Q acts on K by $q \cdot (d_\omega) = (d_{q\omega})$ for $q \in Q$ and $(d_\omega) \in \prod_{\omega\in\Omega} D_\omega$.*

Lemma 2.3. ([10, p. 178] or [11, Theorem 2.1.6]) *Let $X = B_1 \cup \cdots \cup B_m$ be a partition of a set X in which each B_i has k elements. If $G = \{g \in S_X | \text{for each } i, \text{ there is } j \text{ with } g(B_i) = B_j\}$, then $G \cong S_k \wr_{\Omega_m} S_m$, where $\Omega_m = \{1, 2, \cdots, m\}$.*

Let $X = \bigcup_{i_1=1}^{m_1} B_{i_1}$ be a partition of a set X in which each B_{i_1} has same size. Let $B_{i_1} = \bigcup_{i_2=1}^{m_2} B_{i_1,i_2}$ be a partition of a set B_{i_1} in which each B_{i_1,i_2} has same size, where $i_1 = 1, 2, \cdots, m_1$. Continuing in this way we obtain partitions

$$X = \bigcup_{i_j=1}^{m_j} \cdots \bigcup_{i_1=1}^{m_1} B_{i_1,\cdots,i_j}$$

of X in which each $B_{i_1,\cdots,i_j}$ has same size for $j = 1, \cdots, k$. With this notation, by Lemma 2.3, we have the following:

Corollary 2.4. ([12, p. 93] or [11, Theorem 2.1.15]) *Let G be the largest subgroup of S_X preserving above partitions and $|B_{i_1,\cdots,i_k}| = m_{k+1}$. Then $G = \{g \in S_X | \text{for each } i_j, \text{ there is } i'_j \text{ with } g(B_{i_1,\cdots,i_j}) = B_{i'_1,\cdots,i'_j},\ j = 1, \cdots, k\}$. Moreover, $G \cong (\cdots(S_{m_{k+1}} \wr_{\Omega_{m_k}} S_{m_k}) \wr \cdots \wr_{\Omega_{m_2}} S_{m_2}) \wr_{\Omega_{m_1}} S_{m_1}$, where $\Omega_{m_i} = \{1, 2, \cdots, m_i\}$ for $i = 1, 2, \cdots, k + 1$.*

With the associativity of the wreath product (see [10, Theorem 7.26]), we will simply write $(\cdots(S_{m_{k+1}} \wr_{\Omega_{m_k}} S_{m_k}) \wr \cdots \wr_{\Omega_{m_2}} S_{m_2}) \wr_{\Omega_{m_1}} S_{m_1}$ as $S_{m_{k+1}} \wr S_{m_k} \wr \cdots \wr S_{m_2} \wr S_{m_1}$. In [11, p. 68], the iterated wreath product $S_{m_{k+1}} \wr S_{m_k} \wr \cdots \wr S_{m_2} \wr S_{m_1}$ consists of all $f_{k+1} \wr f_k \wr \cdots \wr f_2 \wr f_1$, where $f_1 \in S_{m_1}$ and

$$f_j = \prod_{i_{j-1}=1}^{m_{j-1}} \prod_{i_{j-2}=1}^{m_{j-2}} \cdots \prod_{i_1=1}^{m_1} g_{j,i_1,\cdots,i_{j-2},i_{j-1}} \in \prod^{m_{j-1}m_{j-2}\cdots m_1} S_{m_j}, \tag{2.1}$$

$j = 2, 3, \cdots, k + 1$, with the action on $\prod_{j=1}^{k+1} \Omega_{m_j}$ defined by

$$(f_{k+1} \wr f_k \wr \cdots \wr f_2 \wr f_1)(x_1, x_2, \cdots, x_{k+1}) = (y_1, y_2, \cdots, y_{k+1}), \tag{2.2}$$

where $y_1 = f_1(x_1)$ and $y_j = g_{j,y_1,y_2,\cdots,y_{j-1}}(x_j)$, $j = 2, 3, \cdots, k + 1$ for all $(x_1, x_2, \cdots, x_{k+1}) \in \prod_{j=1}^{k+1} \Omega_{m_j}$ and $f_{k+1} \wr f_k \wr \cdots \wr f_2 \wr f_1 \in S_{m_{k+1}} \wr S_{m_k} \wr \cdots \wr S_{m_2} \wr S_{m_1}$.

3. Automorphisms of $\Gamma(R)$

Let R_0 denote the set $\{aE_{11} + bE_{12} + cE_{21} + dE_{22} \mid a-d \in U(\mathbb{Z}_{p^s}) \text{ or } b \in U(\mathbb{Z}_{p^s}) \text{ or } c \in U(\mathbb{Z}_{p^s})\}$. Then $R \setminus R_0 = \{aE_{11} + bE_{12} + cE_{21} + dE_{22} \mid a-d \in D(\mathbb{Z}_{p^s}),\ b \in D(\mathbb{Z}_{p^s}) \text{ and } c \in D(\mathbb{Z}_{p^s})\}$. Since $|D(\mathbb{Z}_{p^s})| = p^{s-1}$, an easy computation shows that $|R \setminus R_0| = p^{4s-3}$. Therefore $|R_0| = p^{4s} - p^{4s-3}$. For $A, B \in R$, we write $A \sim B$ if there exist $a \in U(\mathbb{Z}_{p^s})$ and $b \in \mathbb{Z}_{p^s}$ such that $A = aB + bE$. A trivial verification shows that $\sim$ is an equivalence relation on R. Set $[A] = \{B \in R \mid B \sim A\}$. It follows immediately that $[A]$ is the equivalence class of A on R under the equivalence relation of $\sim$.

Lemma 3.1. *Every equivalence class in R_0 has size $p^{2s} - p^{2s-1}$. Moreover, there are $p^{2s} + p^{2s-1} + p^{2s-2}$ distinct equivalence classes in R_0.*

Proof. Assume that $A = aE_{11} + bE_{12} + cE_{21} + dE_{22} \in R_0$, where $a - d \in U(\mathbb{Z}_{p^s})$ or $b \in U(\mathbb{Z}_{p^s})$ or $c \in U(\mathbb{Z}_{p^s})$. Let $A_1 = a_1A + b_1E$ and $A_2 = a_2A + b_2E \in [A]$, where $a_1, a_2 \in U(\mathbb{Z}_{p^s})$ and $b_1, b_2 \in \mathbb{Z}_{p^s}$. We claim that if $a_1 \neq a_2$ or $b_1 \neq b_2$, then $A_1 \neq A_2$. If $a_1 = a_2$ and $b_1 \neq b_2$, then $A_1 - A_2 = (b_1 - b_2)E$. It is clear that $A_1 \neq A_2$. If $a_1 \neq a_2$ and $b_1 = b_2$, then $A_1 - A_2 = ((a_1 - a_2)(a - d) + (a_1 - a_2)d)E_{11} + (a_1 - a_2)bE_{12} + (a_1 - a_2)cE_{21} + (a_1 - a_2)dE_{22}$. If $(a_1 - a_2)d = 0$, then $(a_1 - a_2)(a - d) \neq 0$ or $(a_1 - a_2)b \neq 0$ or $(a_1 - a_2)c \neq 0$ (i.e., $A_1 \neq A_2$), since $a_1 - a_2 \neq 0$, $a - d \in U(\mathbb{Z}_{p^s})$ or $b \in U(\mathbb{Z}_{p^s})$ or $c \in U(\mathbb{Z}_{p^s})$. If $(a_1 - a_2)d \neq 0$, then it is obvious that $A_1 \neq A_2$. If $a_1 \neq a_2$ and $b_1 \neq b_2$, then $A_1 - A_2 = ((a_1 - a_2)(a - d) + (a_1 - a_2)d + b_1 - b_2)E_{11} + (a_1 - a_2)bE_{12} + (a_1 - a_2)cE_{21} + ((a_1 - a_2)d + b_1 - b_2)E_{22}$. Similarly, we have $A_1 \neq A_2$. It is well known that $|U(\mathbb{Z}_{p^s})| = p^s - p^{s-1}$. So $|[A]| = p^{2s} - p^{2s-1}$.

It is easily seen that if $A \in R_0$, then $[A] \subseteq R_0$. This fact makes it obvious that R_0 is the disjoint union of some equivalence classes. Since $|R_0| = p^{4s} - p^{4s-3}$, there are exactly $p^{2s} + p^{2s-1} + p^{2s-2}$ equivalence classes in R_0. □

In fact, a trivial verification shows that the set of equivalence class representatives in R_0 is

$$\{E_{11}+aE_{12}+bE_{21},\ aE_{11}+E_{12}+bE_{21},\ aE_{11}+bE_{12}+E_{21},\\ E_{11}+cE_{12}+bE_{21},\ E_{11}+bE_{12}+cE_{21},\ bE_{11}+E_{12}+cE_{21},\\ E_{11}+cE_{12}+dE_{21}\mid a,b\in\langle p\rangle,\ c,d\in U(\mathbb{Z}_{p^s})\}.$$

We denote this set by P_0. By Lemma 3.1, we can write

$$P_0=\{A_{0,1},A_{0,2},\cdots,A_{0,p^{2s}+p^{2s-1}+p^{2s-2}}\}.$$

It is immediate that $R_0=\bigcup_{i_0=1}^{|P_0|}[A_{0,i_0}]$.

Let $j\in\{1,2,\cdots,s-1\}$. Set $P_j=p^jP_0$. Since $\mathbb{Z}_{p^s}$ is a principal ideal ring,

$$P_j=\{p^jE_{11}+aE_{12}+bE_{21},\ aE_{11}+p^jE_{12}+bE_{21},\ aE_{11}+bE_{12}+p^jE_{21},\\ p^jE_{11}+cE_{12}+bE_{21},\ p^jE_{11}+bE_{12}+cE_{21},\ bE_{11}+p^jE_{12}+cE_{21},\\ p^jE_{11}+cE_{12}+dE_{21}\mid a,b\in\langle p^{j+1}\rangle,\ c,d\in \mathrm{Ass}(p^j)\}.$$

From Lemma 3.1, $|P_j|=p^{2s-2j}+p^{2s-2j-1}+p^{2s-2j-2}$. Write $P_j=\{A_{j,1},A_{j,2},\cdots,A_{j,|P_j|}\}$. Set

$$R_j=\bigcup_{i_j=1}^{|P_j|}[A_{j,i_j}]. \tag{3.1}$$

Accordingly, there are seven forms in $\bigcup_{j=0}^{s-1}P_j$. For example, let $j,k\in\{0,1,\cdots,s-1\}$, if $A_{j,i_j}=p^jE_{11}+a_1E_{12}+b_1E_{21}$, $A_{k,i_k}=a_2E_{11}+p^kE_{12}+b_2E_{21}$, where $a_1,b_1\in\langle p^{j+1}\rangle$, $a_2,b_2\in\langle p^{k+1}\rangle$, then we say that A_{j,i_j} and A_{k,i_k} have different forms.

Lemma 3.2. *Let $R_j=\bigcup_{i_j=1}^{|P_j|}[A_{j,i_j}]$, where $j=0,1,\ldots,s-1$. Then*

$$R=\bigcup_{j=0}^{s-1}R_j\bigcup C(R)=\bigcup_{j=0}^{s-1}(\cup_{i_j=1}^{|P_j|}[A_{j,i_j}])\bigcup C(R)$$

is a partition of R.

Proof. By the definition of $C(R)$, we have $C(R)\cap R_j=\varnothing$ for all $j\in\{0,1,\cdots,s-1\}$. By construction, $C(R)\not\subseteq R_0$ and hence $C(R)\not\subseteq R_j$ for $j\in\{1,2,\cdots,s-1\}$. Let $A_{j,i_j}\in P_j$. Then $A_{j,i_j}=p^jA_{0,i_0}$ for a certain $A_{0,i_0}\in P_0$. Consequently, $[A_{j,i_j}]=[p^jA_{0,i_0}]=\{ap^jA_{0,i_0}+bE\mid a\in U(\mathbb{Z}_{p^s})$ and $b\in\mathbb{Z}_{p^s}\}=\{aA_{0,i_0}+bE\mid a\in\mathrm{Ass}(p^j)$ and $b\in\mathbb{Z}_{p^s}\}$. By Lemma 2.1, in much the same way as Lemma 3.1, the size of an equivalence class in R_j is $p^{2s-j}-p^{2s-j-1}$. It follows that $|R_j|=p^{4s-3j}-p^{4s-3j-3}$. Then

$$\sum_{j=0}^{s-1}|R_j|+|C(R)|=\sum_{j=0}^{s-1}(p^{4s-3j}-p^{4s-3j-3})+p^s=p^{4s}=|R|.$$

It remains to prove that $R_{j_1}\cap R_{j_2}=\varnothing$ for all $j_1\neq j_2\in\{0,1,\cdots,s-1\}$. Assume that $A\in R_{j_1}\cap R_{j_2}\neq\varnothing$. Then there exist $a_1,a_2\in U(\mathbb{Z}_{p^s})$, $b_1,b_2\in\mathbb{Z}_{p^s}$, $A_{j_1,i_{j_1}}\in P_{j_1}$ and $A_{j_2,i_{j_2}}\in P_{j_2}$ such that $A=a_1A_{j_1,i_{j_1}}+b_1E=a_2A_{j_2,i_{j_2}}+b_2E$. It implies that $A_{j_1,i_{j_1}}=a_1^{-1}a_2A_{j_2,i_{j_2}}+a_1^{-1}(b_2-b_1)E$. Since the (2, 2) entries of $A_{j_1,i_{j_1}}$ and $A_{j_2,i_{j_2}}$ are equal to 0, $a_1^{-1}(b_2-b_1)=0$. Thus, $A_{j_1,i_{j_1}}=a_1^{-1}a_2A_{j_2,i_{j_2}}$. Suppose that $A_{j_1,i_{j_1}}=p^{j_1}E_{11}+\star p^{j_1+1}E_{12}+\star E_{21}$ and $A_{j_2,i_{j_2}}=p^{j_2}E_{11}+\star p^{j_2+1}E_{12}+\star E_{21}$. We thus get $j_1=j_2$. This contradicts our assumption $j_1\neq j_2$. Similarly, we obtain contradictions in the other cases of $A_{j_1,i_{j_1}}$ and $A_{j_2,i_{j_2}}$. This completes the proof. □

Lemma 3.3. *Let $A \in [A_{j,i_j}]$, $B \in [A_{k,i_k}]$, where $j, k \in \{0, 1, \cdots, s-1\}$, $A_{j,i_j} \in P_j$ and $A_{k,i_k} \in P_k$.*

(i) *Let $j + k \leq s - 1$. Then $AB = BA$ if and only if $p^k A_{j,i_j} = p^j A_{k,i_k}$.*

(ii) *Let $j + k > s - 1$. Then $AB = BA$.*

Proof. It is easily seen that $AB = BA$ if and only if $A_{j,i_j} A_{k,i_k} = A_{k,i_k} A_{j,i_j}$.

(i) Suppose that $A_{j,i_j} = p^j E_{11} + a_1 E_{12} + b_1 E_{21}$, $A_{k,i_k} = a_2 E_{11} + p^k E_{12} + b_2 E_{21}$, where $a_1, b_1 \in \langle p^{j+1} \rangle$, $a_2, b_2 \in \langle p^{k+1} \rangle$. Then $A_{j,i_j} A_{k,i_k} = \star E_{11} + p^{j+k} E_{12} + \star E_{21}$, $A_{k,i_k} A_{j,i_j} = \star E_{11} + \star p^{j+k+2} E_{12} + \star E_{21}$. Obviously, $A_{j,i_j} A_{k,i_k} \neq A_{k,i_k} A_{j,i_j}$. By similar arguments, it is easy to check that $A_{j,i_j} A_{k,i_k} \neq A_{k,i_k} A_{j,i_j}$ when A_{j,i_j} and A_{k,i_k} have different forms.

Without loss of generality we assume that $j \geq k$. Now suppose that $A_{j,i_j} A_{k,i_k} = A_{k,i_k} A_{j,i_j}$, where $A_{j,i_j} = p^j E_{11} + a_1 E_{12} + b_1 E_{21}$, $A_{k,i_k} = p^j E_{11} + a_2 E_{12} + b_2 E_{21}$, $a_1, b_1 \in \langle p^{j+1} \rangle$, $a_2, b_2 \in \langle p^{k+1} \rangle$. By Lemma 2.1, we can assume that $a_1 = \sum_{i=j+1}^{s-1} r_i p^i$, $b_1 = \sum_{i=j+1}^{s-1} t_i p^i$, $a_2 = \sum_{i=k+1}^{s-1} u_i p^i$ and $b_2 = \sum_{i=k+1}^{s-1} v_i p^i$, where $r_i, t_i, u_i, v_i \in T$. Since $A_{j,i_j} A_{k,i_k} = A_{k,i_k} A_{j,i_j}$, it is obvious that $r_{j+1} = u_{k+1}$, $r_{j+2} = u_{k+2}$, $\cdots$, $r_{s-k-1} = u_{s-j-1}$, and $t_{j+1} = v_{k+1}$, $t_{j+2} = v_{k+2}$, $\cdots$, $t_{s-k-1} = v_{s-j-1}$. It is immediately that $p^k A_{j,i_j} = p^j A_{k,i_k}$. In other cases we conclude similarly that $p^k A_{j,i_j} = p^j A_{k,i_k}$.

Conversely, suppose that $p^k A_{j,i_j} = p^j A_{k,i_k}$. An easy computation shows that it occurs only when A_{j,i_j} and A_{k,i_k} have same form. Assume that $A_{j,i_j} = p^j E_{11} + a_1 E_{12} + b_1 E_{21}$, $A_{k,i_k} = p^j E_{11} + a_2 E_{12} + b_2 E_{21}$ with $a_1 = \sum_{i=j+1}^{s-1} r_i p^i$, $b_1 = \sum_{i=j+1}^{s-1} t_i p^i$, $a_2 = \sum_{i=k+1}^{s-1} u_i p^i$ and $b_2 = \sum_{i=k+1}^{s-1} v_i p^i$, where $r_i, t_i, u_i, v_i \in T$. Since $p^k A_{j,i_j} = p^j A_{k,i_k}$, it is easy to check that $r_{j+1} = u_{k+1}$, $r_{j+2} = u_{k+2}$, $\cdots$, $r_{s-k-1} = u_{s-j-1}$, and $t_{j+1} = v_{k+1}$, $t_{j+2} = v_{k+2}$, $\cdots$, $t_{s-k-1} = v_{s-j-1}$. It is clear that $A_{j,i_j} A_{k,i_k} = A_{k,i_k} A_{j,i_j}$. The proof for other cases is similar.

(ii) If $j + k > s - 1$, then $A_{j,i_j} A_{k,i_k} = 0 = A_{k,i_k} A_{j,i_j}$. Therefore, $AB = BA$. □

For fixed $j, k \in \{0, 1, \cdots, s-1\}$ and $i_k \in \{1, 2, \cdots, |P_k|\}$, set

$$R_j^{k,i_k} = \{[A_{j,i_j}] \subseteq R_j \mid p^k A_{j,i_j} = p^j A_{k,i_k}\}.$$

By Lemma 3.3, we have the following proposition.

Proposition 3.4. *Let $A \in [A_{k,i_k}]$, where $k \in \{0, 1, \cdots, s-1\}$ and $A_{k,i_k} \in P_k$.*

(i) $C_R(A) = \bigcup_{j=0}^{s-1} [p^j A_{0,i_0}] \bigcup C(R)$.

(ii) *Let $0 < k \leq s - 1$. Then* $C_R(A) = \bigcup_{j=0}^{s-k-1} R_j^{k,i_k} \bigcup_{j=s-k}^{s-1} R_j \bigcup C(R)$.

For fixed $k, j \in \{0, 1, \cdots, s-1\}$, $k \geq j$, $i_k \in \{1, 2, \cdots, |P_k|\}$, $i_{k+1} \in \{1, 2, \cdots, |P_{k+1}|\}$, $\cdots$, $i_{s-1} \in \{1, 2, \cdots, |P_{s-1}|\}$, if $p^{s-1-k} A_{k,i_k} = p^{s-1-(k+1)} A_{k+1,i_{k+1}} = \cdots = p^0 A_{s-1,i_{s-1}}$, then set

$$R_{j,i_{s-1},\cdots,i_{k+1},i_k} = \{[A_{j,i_j}] \subseteq R_j \mid p^{k-j} A_{j,i_j} = A_{k,i_k}\},$$

$$N_{k-1}^{i_k} = \{i_{k-1} \in \{1, \cdots, |P_{k-1}|\} \mid p A_{k-1,i_{k-1}} = A_{k,i_k}\}.$$

Since $p^{s-1-j} P_j = p^{s-1-(j+1)} P_{j+1} = \cdots = P_{s-1}$,

$$R_j = \bigcup_{i_{s-1}=1}^{|P_{s-1}|} R_{j,i_{s-1}} = \cdots = \bigcup_{i_j \in N_j^{i_{j+1}}} \bigcup_{i_{j+1} \in N_{j+1}^{i_{j+2}}} \cdots \bigcup_{i_{s-1}=1}^{|P_{s-1}|} R_{j,i_{s-1},\cdots,i_{j+1},i_j}.$$

Lemma 3.5. *Let* $0 \le j \le k \le s-1$, $A_{k,i_k} \in P_k$, $A_{k+1,i_{k+1}} \in P_{k+1}, \cdots, A_{s-1,i_{s-1}} \in P_s$ *and* $p^{s-1-k}A_{k,i_k} = p^{s-1-(k+1)}A_{k+1,i_{k+1}} = \cdots = p^0 A_{s-1,i_{s-1}}$. *Then the number of equivalence classes in* $R_{j,i_{s-1},\cdots,i_{k+1},i_k}$ *is* $p^{2(k-j)}$.

Proof. From the construction of P_j and P_k, we know that $p^{k-j}P_j = p^k P_0 = P_k$. Define two maps $f : \langle p^{j+1}\rangle \to \langle p^{k+1}\rangle$ by $\sum_{i=j+1}^{s-1} t_i p^i \mapsto \sum_{i=j+1}^{s-k+j-1} t_i p^{i+k-j}$ and $g : \mathrm{Ass}(p^j) \to \mathrm{Ass}(p^k)$ by $\sum_{i=j}^{s-1} t_i p^i \mapsto \sum_{i=j}^{s-k+j-1} t_i p^{i+k-j}$, where $t_j \in T^*$, $t_i \in T$, $i = j+1, j+2, \cdots, s-1$. Clearly, f, g are surjective, and we have $\ker(f) = \{\sum_{i=s-k+j}^{s-1} t_i p^i \mid t_i \in T,\ i = s-k+j, s-k+j+1, \cdots, s-1\} = \langle p^{s-k+j}\rangle$ and $\ker(g) = \{p^j + \sum_{i=s-k+j}^{s-1} t_i p^i \mid t_i \in T,\ i = s-k+j, s-k+j+1, \cdots, s-1\}$. By Lemma 2.1 and $|T| = p$, $|\ker(f)| = |\ker(g)| = p^{k-j}$. Then the size of the inverse image of each element in $\langle p^{k+1}\rangle$ and $\mathrm{Ass}(p^k)$ under f and g is p^{k-j} respectively. Moreover, it is evident that the number of solutions of $p^{k-j}X = A_{k,i_k}$ in P_j is $p^{2(k-j)}$. In fact, the number of equivalence classes in $R_{j,i_{s-1},\cdots,i_{k+1},i_k}$ is equal to the number of solutions of $p^{k-j}X = A_{k,i_k}$ in P_j, which completes the proof. □

From Lemma 3.5, $|N_{k-1}^{i_k}| = p^2$ for all $k \in \{1, 2, \cdots, s-1\}$ and $i_k \in \{1, 2, \cdots |P_k|\}$. Recall that $\Omega_{p^2} = \{1, 2, \cdots, p^2\}$. It is easily seen that there exists a unique map $\varphi_{i_k} : N_{k-1}^{i_k} \to \Omega_{p^2}$ such that for $i, j \in N_{k-1}^{i_k}$, if $i < j$, then $\varphi_{i_k}(i) < \varphi_{i_k}(j)$. Let $i'_k \in \{1, 2, \cdots |P_k|\}$. Define a map

$$\varphi_{k-1}^{i'_k} : N_{k-1}^{i_k} \to N_{k-1}^{i'_k} \tag{3.2}$$

by $i \mapsto j$ if $\varphi_{i_k}(i) = \varphi_{i'_k}(j)$.

Corollary 3.6. *Let* $R = M_2(\mathbb{Z}_{p^s})$*, with p prime and s positive integer. Let* $A, B \in R$*. Then* $C_R(A) = C_R(B)$ *if and only if* $[A] = [B]$.

Proof. If $A, B \in C(R)$, it is obviously that $C_R(A) = R = C_R(B)$ if and only if $[A] = C(R) = [B]$. If $A \in C(R)$ and $B \notin C(R)$, it is clear that $C_R(A) = R \ne C_R(B)$. Similarly, if $A \notin C(R)$ and $B \in C(R)$, then $C_R(A) \ne C_R(B)$.

Now let $A, B \in R \setminus C(R)$. Suppose that $C_R(A) = C_R(B)$, where $A \in [A_{j,i_j}]$, $B \in [A_{k,i_k}]$, $j, k \in \{0, 1, \cdots, s-1\}$. We claim that $j = k$ and $i_j = i_k$. If $j = 0$ and $k \ne 0$, by Proposition 3.4, we know that $C_R(A) \ne C_R(B)$, a contradiction. Similarly, if $j \ne 0$ and $k = 0$, then $C_R(A) \ne C_R(B)$, a contradiction. If $0 < j \ne k \le s-1$, then $\bigcup_{l=s-j}^{s-1} R_l \ne \bigcup_{l=s-k}^{s-1} R_l$. By Proposition 3.4 (ii), $C_R(A) = \bigcup_{l=0}^{s-j-1} R_l^{j,i_j} \bigcup_{l=s-j}^{s-1} R_l \ne \bigcup_{l=0}^{s-k-1} R_k^{k,i_k} \bigcup_{l=s-k}^{s-1} R_l = C_R(B)$, a contradiction. If $j = k = 0$ and $i_j \ne i_k$, then $[A_{0,i_j}] \ne [A_{0,i_k}]$. By Proposition 3.4 (i), $C_R(A) = [A_{0,i_j}] \bigcup_{l=1}^{s-1} [p^l A_{0,i_j}] \ne [A_{0,i_k}] \bigcup_{l=1}^{s-1} [p^l A_{0,i_k}] = C_R(B)$, a contradiction. If $0 < j = k \le s-1$ and $i_j \ne i_k$, then $A_{j,i_j} \ne A_{j,i_k}$. Thus, by the proof of Lemma 3.5, $R_0^{j,i_j} = R_{0,i_{s-1},\cdots,i_j} \ne R_{0,i_{s-1},\cdots,i_k} = R_0^{j,i_k}$. Furthermore, $C_R(A) = R_0^{j,i_j} \bigcup_{l=1}^{s-j-1} R_l^{j,i_j} \bigcup_{l=s-j}^{s-1} R_l \ne R_0^{j,i_k} \bigcup_{l=1}^{s-j-1} R_l^{j,i_k} \bigcup_{l=s-j}^{s-1} R_l = C_R(B)$ by Proposition 3.4 (ii), a contradiction. Therefore $j = k$ and $i_j = i_k$ as claimed. This means that $A_{j,i_j} = A_{k,i_k}$ (i.e. $[A] = [B]$). The converse is straightforward. □

Corollary 3.7. *Let* $R = M_2(\mathbb{Z}_{p^s})$*, with p prime and s positive integer. If* $f \in Aut(\Gamma(R))$*, then* $f(R_j) = R_j$ *for* $j \in \{0, 1, \cdots, s-1\}$*, where* R_j *is as defined in* (3.1).

Proof. For $j = 0, 1, \cdots, s-1$, if $A \in R_j$, then $|C_R(A) \setminus C(R)| = p^{2s+2j} - p^s$ by Proposition 3.4 and the proof of Lemma 3.5. This means that if $A \in R_j$, $B \in R_k$ and $j \ne k$, then $|N(A)| \ne |N(B)|$, where $j, k \in \{0, 1, \cdots, s-1\}$. Since automorphisms of a graph must preserve the number of neighbors of vertices, $f(R_j) = R_j$, where $j \in \{0, 1, \cdots, s-1\}$. □

Recall that a graph automorphism of a graph G is a bijection on vertex set which preserves adjacency. If $|V(G)| = n$, then in the obvious way $\mathrm{Aut}(G)$ is isomorphic to a subgroup of S_n. Specifically, $\mathrm{Aut}(G) = \{f \in S_n \mid for\ all\ x, y \in V(G),\ [x, y] \Leftrightarrow [f(x), f(y)]\}$. It is easy to show that $\mathrm{Aut}(G) = \{f \in S_n \mid for\ all\ x \in V(G),\ f(N(x)) = N(f(x))\}$. For $\Gamma(R)$, $N(A) = C_R(A) \setminus \{C(R) \cup A\}$. This means that $\mathrm{Aut}(\Gamma(R)) = \{f \in S_{\sum_{j=0}^{s-1}|R_j|} \mid for\ all\ A \in V(\Gamma(R)),\ f(N(A)) = N(f(A))\}$.

We now prove our main result about the automorphism group of the commuting graph of $M_2(\mathbb{Z}_{p^s})$. To state it, we need to define a group. For each $j \in \{0, 1, \cdots, s-1\}$ denote

$$G_{s-1-j} = S_{p^{2s-j}-p^{2s-j-1}} \wr \underbrace{S_{p^2} \wr \cdots \wr S_{p^2}}_{s-1-j} \wr S_{p^2+p+1}.$$

Let G be a subset of $\prod_{j=0}^{s-1} G_{s-1-j}$ and define:

$$\begin{aligned} G = \{(&h_0 \wr g_0 \wr \cdots \wr g_{s-2} \wr g_{s-1}, h_1 \wr g_1 \wr \cdots \wr g_{s-2} \wr g_{s-1}, \cdots, h_{s-1} \wr g_{s-1}) \\ &\mid h_j \wr g_j \wr \cdots \wr g_{s-2} \wr g_{s-1} \in G_{s-1-j},\ j = 0, 1, \ldots, s-1\}. \end{aligned} \tag{3.3}$$

The multiplication law of the iterated wreath product is defined in [11, p. 68], the proof that G is a subgroup of $\prod_{j=0}^{s-1} G_{s-1-j}$ is routine.

Theorem 3.8. *Let $R = M_2(\mathbb{Z}_{p^s})$, with p prime and s positive integer. Then* $\mathrm{Aut}(\Gamma(R)) \cong G$*, where G is a group defined in* (3.3).

Proof. By Lemma 3.2 and Corollary 3.7, $\mathrm{Aut}(\Gamma(R))$ is isomorphic to a subgroup of $\prod_{j=0}^{s-1} S_{R_j}$. So $f \in \mathrm{Aut}(\Gamma(R))$ can be written as a product $\prod_{j=0}^{s-1} f_j$, where $f_j \in S_{R_j}$. We claim that

$$\{f_j \in S_{R_j} \mid (\cdots, f_j, \cdots) = f \in \mathrm{Aut}(\Gamma(R))\} \cong G_{s-1-j},$$

where $j = 0, 1, \cdots, s-1$.

Let $j \in \{1, \cdots, s-1\}$ and $(\cdots, f_j, \cdots) = f \in \mathrm{Aut}(\Gamma(R))$. Assume that $A \in [A_{j,i_j}]$, $B \in [A_{j,i'_j}]$ with $f_j(A) = B$. By Proposition 3.4 (ii) and $f(N(A)) = N(f(A))$,

$$f(R_0^{j,i_j} \bigcup_{k=1}^{s-j-1} R_k^{j,i_j} \bigcup_{k=s-j}^{s-1} R_k) = R_0^{j,i'_j} \bigcup_{k=1}^{s-j-1} R_k^{j,i'_j} \bigcup_{k=s-j}^{s-1} R_k.$$

Then $f(R_0^{j,i_j}) = R_0^{j,i'_j}$ by Corollary 3.7. It is immediate that $f([A_{s-1,i_{s-1}}]) = [A_{s-1,i'_{s-1}}]$ by Proposition 3.4 (i), where $[A_{s-1,i_{s-1}}] = p^{s-1}R_0^{j,i_j}$, $[A_{s-1,i'_{s-1}}] = p^{s-1}R_0^{j,i'_j}$. Since Proposition 3.4 (ii) and $f(N([A_{s-1,i_{s-1}}])) = N(f([A_{s-1,i_{s-1}}]))$,

$$f(R_0^{s-1,i_{s-1}} \bigcup_{k=1}^{s-1} R_k) = R_0^{s-1,i'_{s-1}} \bigcup_{k=1}^{s-1} R_k.$$

Thus $f(R_0^{s-1,i_{s-1}}) = R_0^{s-1,i'_{s-1}}$. It is evident that $f(p^j R_0^{s-1,i_{s-1}}) = p^j R_0^{s-1,i'_{s-1}}$ by Proposition 3.4 (i), i.e.,

$$f_j(R_{j,i_{s-1}}) = R_{j,i'_{s-1}}.$$

Similarly, we have

$$f_j(R_{j,i_{s-1},i_{s-2}}) = R_{j,i'_{s-1},i'_{s-2}},$$

$$\cdots$$

$$f_j(R_{j,i_{s-1},\cdots,i_{j+2},i_{j+1}}) = R_{j,i'_{s-1},\cdots,i'_{j+2},i'_{j+1}},$$

where $i_{s-2}, i'_{s-2} \in \{1, \cdots, |P_{s-2}|\}, \cdots, i_{j+2}, i'_{j+2} \in \{1, \cdots, |P_{j+2}|\}$, $i_{j+1}, i'_{j+1} \in \{1, \cdots, |P_{j+1}|\}$ with

$$A_{s-2,i_{s-2}} = p^{s-2-j} A_{j,i_j},\ A_{s-2,i'_{s-2}} = p^{s-2-j} A_{j,i'_j},$$

$$\cdots$$

$$A_{j+2,i_{j+2}} = p^2 A_{j,i_j},\ A_{j+2,i'_{j+2}} = p^2 A_{j,i'_j},$$

$$A_{j+1,i_{j+1}} = pA_{j,i_j},\ A_{j+1,i'_{j+1}} = pA_{j,i'_j}.$$

Obviously,

$$f_j(R_{j,i_{s-1},\cdots,i_{j+1},i_j}) = f_j([A_{j,i_j}]) = [A_{j,i'_j}] = R_{j,i'_{s-1},\cdots,i'_{j+1},i'_j}.$$

Hence, for $i_{s-1} \in \{1, 2, \cdots, |P_{s-1}|\}$, $i_{s-2} \in N_{s-2}^{i_{s-1}}$, $\cdots$, $i_{j+1} \in N_{j+1}^{i_{j+2}}$, $i_j \in N_j^{i_{j+1}}$, there are $i'_{s-1} \in \{1, 2, \cdots, |P_{s-1}|\}$, $i'_{s-2} \in N_{s-2}^{i'_{s-1}}, \cdots, i'_{j+1} \in N_{j+1}^{i'_{j+2}}$, $i'_j \in N_j^{i'_{j+1}}$ such that

$$f_j(R_{j,i_{s-1}}) = R_{j,i'_{s-1}},$$

$$f_j(R_{j,i_{s-1},i_{s-2}}) = R_{j,i'_{s-1},i'_{s-2}},$$

$$\cdots$$

$$f_j(R_{j,i_{s-1},\cdots,i_{j+1},i_j}) = R_{j,i'_{s-1},\cdots,i'_{j+1},i'_j}.$$

By Lemma 3.5, $|N_k^{i_{k+1}}| = p^2$, $k = j, j+1, \cdots, s-2$. In the proof of Lemma 3.2, we know that $|[A]| = p^{2s-j} - p^{2s-j-1}$ for $A \in R_j$. Therefore $\{f_j \in S_{R_j} \mid (\cdots, f_j, \cdots) = f \in \mathrm{Aut}(\Gamma(R))\} \cong G_{s-1-j}$ by Corollaries 2.4 and 3.6. The proof for $j = 0$ is similar.

From the above proof, it follows that $\mathrm{Aut}(\Gamma(R))$ is a subgroup of $\prod_{j=0}^{s-1} G_{s-1-j}$. Let $j \in \{0, 1, \cdots, s-2\}$. Let ϕ_j be an isomorphism between $\{f_j \in S_{R_j} \mid (\cdots, f_j, \cdots) = f \in \mathrm{Aut}(\Gamma(R))\}$ and G_{s-1-j}. Suppose that $(\cdots, f_j, f_{j+1}, \cdots) = f \in \mathrm{Aut}(\Gamma(R))$, where $\phi_j(f_j) = h_j \wr g_j \wr \cdots \wr g_{s-2} \wr g_{s-1} \in G_{s-1-j}$. As defined in (2.1), $g_{s-1} \in S_{p^2+p+1}$,

$$g_k = \prod_{i_{k+1} \in N_{k+1}^{i_{k+2}}} \prod_{i_{k+2} \in N_{k+2}^{i_{k+3}}} \cdots \prod_{i_{s-1}=1}^{p^2+p+1} g_{k,i_{s-1},\cdots,i_{k+2},i_{k+1}} \in \prod_{i_{k+1} \in N_{k+1}^{i_{k+2}}} \prod_{i_{k+2} \in N_{k+2}^{i_{k+3}}} \cdots \prod_{i_{s-1}=1}^{p^2+p+1} S_{N_k^{i_{k+1}}},$$

$k = s-2, s-3, \cdots, j$, and

$$h_j = \prod_{i_j \in N_j^{i_{j+1}}} \prod_{i_{j+1} \in N_{j+1}^{i_{j+2}}} \cdots \prod_{i_{s-1}=1}^{p^2+p+1} h_{j,i_{s-1},\cdots,i_{j+1},i_j} \in \prod_{i_j \in N_j^{i_{j+1}}} \prod_{i_{j+1} \in N_{j+1}^{i_{j+2}}} \cdots \prod_{i_{s-1}=1}^{p^2+p+1} S_{[A_{j,i_j}]}.$$

As the action defined in (2.2), we define $f_j(R_{j,i_{s-1}}) = R_{j,y_{s-1}}$,

$$f_j(R_{j,i_{s-1},\cdots,i_{k+1},i_k}) = R_{j,y_{s-1},\cdots,y_{k+1},y_k},$$

where $y_{s-1} = g_{s-1}(i_{s-1})$, $y_k = g_{k,y_{s-1},\cdots,y_{k+2},y_{k+1}}(\varphi_k^{y_{k+1}}(i_k))$, $\varphi_k^{y_{k+1}}$ is defined in (3.2), $k = s-2, s-3, \cdots, j$ and $f_j(aA_{j,i_j} + bE) = h_{j,y_{s-1},\cdots,y_{j+1},y_j}(aA_{j,y_j} + bE)$ for all $a \in \mathrm{Ass}(p^j)$, $b \in \mathbb{Z}_{p^s}$. Suppose that $\phi_{j+1}(f_{j+1}) = h_{j+1} \wr g'_{j+1} \wr \cdots \wr g'_{s-2} \wr g'_{s-1} \in G_{s-1-(j+1)}$. We next claim that $g_{j+1} = g'_{j+1}, g_{j+2} = g'_{j+2}, \cdots, g_{s-1} = g'_{s-1}$. If there exists $k \in \{j+1, j+2, \cdots, s-1\}$ such that $g_{j+1} = g'_{j+1}, \cdots, g_{k-1} = g'_{k-1}, g_k \neq g'_k, g_{k+1} = g'_{k+1}, \cdots, g_{s-1} = g'_{s-1}$, then there exist $i_{s-1} \in \{1, 2, \cdots, p^2+p+1\}, \cdots, i_{k+1} \in N_{k+1}^{i_{k+2}}, i_k \in N_k^{i_{k+1}}$ such that $y_k \neq y'_k$, where y_k, y'_k are defined above. Assume that $f_j(R_{j,i_{s-1},\cdots,i_{k+1},i_k}) = R_{j,y_{s-1},\cdots,y_{k+1},y_k}$ and $f_{j+1}(R_{j+1,i_{s-1},\cdots,i_{k+1},i_k}) = R_{j+1,y_{s-1},\cdots,y_{k+1},y'_k}$. By Proposition 3.4 (i) and $f(N(A)) = N(f(A))$ for all $A \in R \setminus C(R)$, $f_0(R_{0,i_{s-1},\cdots,i_k}) = R_{0,y_{s-1},\cdots,y_{k+1},y_k}$ and $f_0(R_{0,i_{s-1},\cdots,i_k}) = R_{0,y_{s-1},\cdots,y_{k+1},y'_k}$. Since $y_k \neq y'_k$, $R_{0,y_{s-1},\cdots,y_{k+1},y_k} \neq R_{0,y_{s-1},\cdots,y_{k+1},y'_k}$, i.e., $f_0(R_{0,i_{s-1},\cdots,i_k}) \neq f_0(R_{0,i_{s-1},\cdots,i_k})$, which is impossible. By this claim, we know that $f \in \mathrm{Aut}(\Gamma(R))$ can be written as $(h_0 \wr g_0 \wr \cdots \wr g_{s-2} \wr g_{s-1}, h_1 \wr g_1 \wr \cdots \wr g_{s-2} \wr g_{s-1}, \cdots, h_{s-1} \wr g_{s-1})$, where $h_j \wr g_j \wr \cdots \wr g_{s-2} \wr g_{s-1} \in G_{s-1-j}$, $j = 0, 1, \ldots, s-1$. Therefore $\mathrm{Aut}(\Gamma(R)) \cong G$. □

4. Conclusions

In this paper, we show that the automorphism group of $\Gamma(M_2(\mathbb{Z}_{p^s}))$ is a subgroup of a direct product of some wreath products, and we completely characterize it in Theorem 3.8.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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