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Research article

Spatial decay estimates for the Fochheimer equations interfacing with a Darcy equations

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Abstract: Spatial decay estimates for the Fochheimer fluid interfacing with a Darcy flow in a semi-infinite pipe was studied. The exponential decay result can be obtained by integrating a first-order differential inequality. The result can be seen as the usage of Saint-Venant's principle for the interfacing fluids.

Keywords: Saint-Venant's principle; decay estimates; Forchheimer fluid; Darcy fluid
Mathematics Subject Classification: 35B30, 35K55, 35Q35

1. Introduction

The model equations (Brinkman, Forchheimer and Darcy equations) have been widely studied by many authors. If the net flow into the infinite end of the cylinder is zero, then the velocity can decay to zero as the distance tends to infinity. The result can be explained by Saint-Venant's Principle. Early results on Saint-Venant's principle mainly focused on the elliptic equations. Boley [1] in 1856 firstly proved that Saint-Venant's principle was valid for the heat equations. Then, many efforts were to the parabolic equations. References [2–4] gave a review of recent development on Saint-Venant's principle.

There has been substantial interest in spatial decay estimates for the model equations in porous medium. In fact, the Brinkman-Forchheimer type equations have been studied by many papers in the literature. Following the paper [5] which studied the model equations (Brinkman, Darcy, Forchheimer and Brinkman-Forchheimer equations) describing flow in a porous medium, several papers have appeared (see [6–12]). Ames and Payne [13], and Franchi and Straughan [14] analyzed certain structural stability questions and Payne and Straughan [9] studied the question of continuous dependence of solutions of both systems on the initial-time geometry for bounded spatial domains both forward and backward in time. Other questions for these systems have been treated by Ames and Payne [15], Franchi [16], Morro and Straughan [17], Qin and Kaloni [18], and Richardson and Straughan [19]. For more recent work, one may refer to [20]. In that paper, Payne and Song examined the time-dependent double diffusive convection in Brinkman flow in a semi-infinite cylinder. Under appropriate initial and boundary conditions the authors established the exponential decay of solutions in energy norm with distance from the finite end of the cylinder. Other results for models of Brinkman, Forchheimer and Darcy equations were found in [21–28]. Some new results about properties of solutions for fluids in porous medium may be found in [29–36].

In [37], the authors studied the spatial decay for the stokes flow interfacing with a Darcy flow in a cylinder. Under homogeneous initial lateral surface boundary conditions and some other interface conditions, they established the exponential decay estimates for the energy expression. For a review of other porous interface problems, one could see [38–40]. Some new results about the structural stability of the interfacing problems may be found in [41–45]. In [46], Payne and Song obtained the spatial decay result for flows in a porous medium. For other Saint-Venant's principal results on penetrative convention, one could see [5]. Most of these papers studied only one fluid in a domain. In reality, there usually exist two or more fluids interfacing with each other in a domain. People want to know the behavior of their solutions. Inspired by paper [37], we continue to study these interfacing problems. We replace the Brinkman term Δu_i by a nonlinear item $b|u|u_i$. The nonlinear term is difficult to tackle. We cannot follow the method used in [37]. A new method should be developed to deal with this nonlinear term. We want to establish exponential decay results for the interfacing problems. We have never seen such results for the interfacing fluids in literature except [37].

We assume that one part is filled with the Forchheimer flow, while the other part is filled with the Darcy fluid. Some new results for the Forchheimer-Darcy equations may be found in [47–53].

Let

$$\Omega = \Omega_1 \cup \Omega_2 \tag{1.1}$$

be the interior of a semi-infinite cylinder. The generators of the cylinder are paralleled to the x_3-axis. Ω_1 is a portion lying above the x_1x_3-plane. While Ω_2 is a portion lying below the x_1x_3-plane. L denotes the common plane boundary of Ω_1 and Ω_2. The plane $x_3 = 0$, L, and a lateral surface Γ_1 can bound Ω_1. The plane $x_3 = 0$, L, and a lateral surface Γ_2 can bound Ω_2. We further define (see Figure 1)

$$\Omega_1 = \{(x_1, x_2) \in D_1, x_3 > 0\}, \tag{1.2}$$

$$\Omega_2 = \{(x_1, x_2) \in D_2, x_3 > 0\}. \tag{1.3}$$

D_1 is the cross-section of Ω_1, and D_2 is the cross-section of Ω_2. We can easily get the results $x_2 > 0$ ($x \in \Omega_1$) and $x_2 < 0$ ($x \in \Omega_2$). We assume that the fluid satisfies the Forchheimer equations in Ω_1 and the Darcy equations in Ω_2.

The Forchheimer model is believed accurate when the flow velocity is too large for Darcy's law to be valid and additionally the porosity is not too small. They usually use the Boussinesq approximation to get the equation.

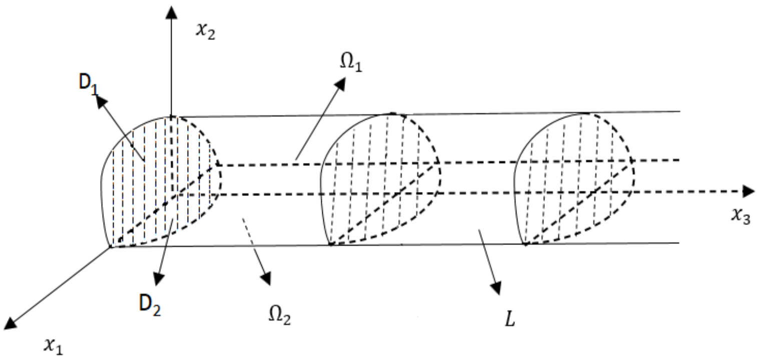

Figure 1. The domain of Ω_1 and Ω_2.

The Forchheimer equations are the governing equations in Ω_1 (see [54]).

$$
\begin{aligned}
&b|u|u_i + (1+\gamma T)u_i = -p_{,i} + g_i T,\\
&\frac{\partial u_i}{\partial x_i} = 0,\\
&\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \kappa \Delta T.
\end{aligned} \tag{1.4}
$$

Here u_i denotes the velocity, T denotes the temperature and p denotes the pressure. b and γ are positive constants. The gravity field is represented by the vector g_i. We assume

$$
|g_i g_i| \leq \zeta^2. \tag{1.5}
$$

κ is the thermal diffusivity. In Ω_2, the governing equations are the Darcy equations

$$
\begin{aligned}
&\frac{b}{k} v_i = -\frac{\partial q}{\partial x_i} + g_i S,\\
&\frac{\partial v_i}{\partial x_i} = 0,\\
&\frac{\partial S}{\partial t} + v_i \frac{\partial S}{\partial x_i} = \kappa_s \Delta S,
\end{aligned} \tag{1.6}
$$

where v_i denotes the velocity, S denotes the temperature and q denotes the pressure. k denotes the permeability and κ_s denotes the thermal diffusivity of the porous medium. We impose the following boundary and initial conditions:

$$
\begin{aligned}
&u_i n_i = 0, T = 0 \quad on \quad \Gamma_1 \times \{t > 0\},\\
&v_i n_i = 0, S = 0 \quad on \quad \Gamma_2 \times \{t > 0\},\\
&T = 0 \quad in \quad \Omega_1 \times \{t = 0\},\\
&S = 0 \quad in \quad \Omega_2 \times \{t = 0\},\\
&u_3 = f_3, T = g \geq 0 \quad on \quad D_1 \times \{x_3 = 0\} \times \{t > 0\},\\
&v_3 = h_3, S = \tau \quad on \quad D_2 \times \{x_3 = 0\} \times \{t > 0\}.
\end{aligned} \tag{1.7}
$$

We assume when $x_3 \to \infty$, the following conditions are satisfied

$$|u|, |v|, |T|, |S| = O(1),$$
$$|u_3|, |v_3|, |\nabla T|, |\nabla S|, |p|, |q| = o(x_3^{-1}), \tag{1.8}$$

uniformly in x_1, x_2.

At last, we impose the same conditions at the interface as [37]:

$$u_2 = v_2, \quad T = S, \quad \kappa\frac{\partial T}{\partial x_2} = \kappa_s\frac{\partial S}{\partial x_2}, \quad q = p, \tag{1.9}$$

on $L \times \{t > 0\}$.

We will use the following notations for convenience (see Figure 2).

$$\begin{aligned}
&\Omega_i(z) = \Omega_i \cap \{x_3 > z\}, \quad i = 1, 2,\\
&D_i(z) = \Omega_i \cap \{x_3 = z\}, \quad i = 1, 2,\\
&L(z) = L \cap \{x_3 > z\},\\
&\Gamma_i(z) = \Gamma_i \cap \{x_3 > z\}, \quad i = 1, 2,\\
&\partial D_i(z) = Boundary \quad of \quad D_i(z), \quad i = 1, 2,\\
&D(z) = D_1(z) \cup D_2(z), \quad i = 1, 2.
\end{aligned} \tag{1.10}$$

We want to formulate a first-order differential inequality for a weighted energy expression. An inequality which will imply exponential decay.

In the present paper, the partial differentiation with respect to the direction x_k is defined by $, k$. Thus, $u_{,i}$ denotes $\frac{\partial u}{\partial x_i}$. The usual summation convection is used in this paper. The repeated Latin subscripts is used to sum from 1 to 3. While repeated Greek subscripts is used to sum from 1 to 2. Hence we have $u_{i,i} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}$, $u_{\alpha,\alpha} = \sum_{\alpha=1}^{2} \frac{\partial u_i}{\partial x_i}$.

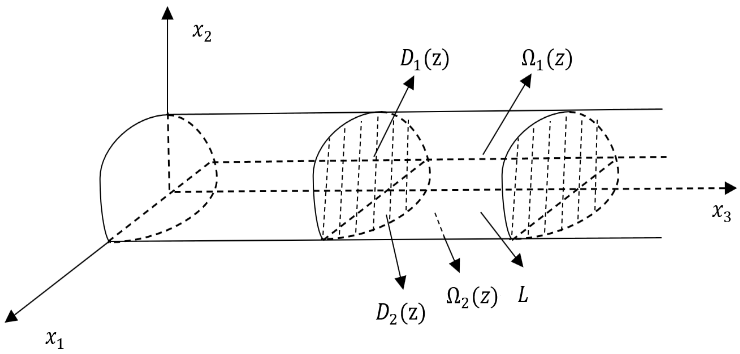

Figure 2. The domain of $\Omega_1(z)$ and $\Omega_2(z)$.

2. Weighted energy $\psi(z, t)$

In this section, we want to derive a weighted energy $\psi(z, t)$. If we define a function

$$f(z, t) = b\int_0^t \int_{\Omega_1(z)} (\xi - z)(u_i u_i)^{\frac{3}{2}} dx d\eta + \int_0^t \int_{\Omega_1(z)} (\xi - z)(1 + \gamma T)u_i u_i dx d\eta, \tag{2.1}$$

using the Eq (1.4) and integrating by parts, we have

$$f(z,t)=\int_0^t\int_{\Omega_1(z)} pu_3\,dxd\eta+\int_0^t\int_{\Omega_1(z)} g_i(\xi-z)Tu_i\,dxd\eta-\int_0^t\int_{L(z)}(\xi-z)u_2p\,dAd\eta. \tag{2.2}$$

Using the interface condition (1.9), we have

$$\begin{aligned}-\int_0^t\int_{L(z)}(\xi-z)u_2p\,dAd\eta&=-\int_0^t\int_{L(z)}(\xi-z)v_2q\,dAd\eta\\&=\int_0^t\int_{\Omega_2(z)}[(\xi-z)qv_i]_{,i}\,dxd\eta\\&=\int_0^t\int_{\Omega_2(z)}(\xi-z)v_iq_{,i}\,dxd\eta+\int_0^t\int_{\Omega_2(z)}qv_3\,dxd\eta.\end{aligned} \tag{2.3}$$

Using the Eq (1.6) and integrating by parts, we obtain

$$\begin{aligned}\int_0^t\int_{\Omega_2(z)}(\xi-z)v_iq_{,i}\,dxd\eta&=\int_0^t\int_{\Omega_2(z)}(\xi-z)v_i\left(-\frac{b}{k}v_i+g_iS\right)dxd\eta\\&=-\frac{b}{k}\int_0^t\int_{\Omega_2(z)}(\xi-z)v_iv_i\,dxd\eta+\int_0^t\int_{\Omega_2(z)}(\xi-z)g_iSv_i\,dxd\eta.\end{aligned} \tag{2.4}$$

If we define

$$\begin{aligned}E_1(z,t)=b&\int_0^t\int_{\Omega_1(z)}(\xi-z)(u_iu_i)^{\frac{3}{2}}\,dxd\eta+\int_0^t\int_{\Omega_1(z)}(\xi-z)(1+\gamma T)u_iu_i\,dxd\eta\\&+\frac{b}{k}\int_0^t\int_{\Omega_2(z)}(\xi-z)v_iv_i\,dxd\eta,\end{aligned} \tag{2.5}$$

Combining (2.2)–(2.4), we have

$$\begin{aligned}E_1(z,t)=&\int_0^t\int_{\Omega_1(z)}(\xi-z)g_iTu_i\,dxd\eta+\int_0^t\int_{\Omega_2(z)}g_i(\xi-z)v_iS\,dxd\eta\\&+\int_0^t\int_{\Omega_1(z)}pu_3\,dxd\eta+\int_0^t\int_{\Omega_2(z)}qv_3\,dxd\eta.\end{aligned} \tag{2.6}$$

We now begin to deal with items involving T and S. We know

$$\begin{aligned}\int_0^t\int_{\Omega_1(z)}(\xi-z)T_{,i}T_{,i}dxd\eta =& -\int_0^t\int_{\Omega_1(z)}(\xi-z)TT_{,ii}dxd\eta-\int_0^t\int_{\Omega_1(z)}TT_{,3}dxd\eta\\ &+\int_0^t\int_{L(z)}(\xi-z)TT_{,2}dAd\eta\\ &=-\frac{1}{\kappa}\int_0^t\int_{\Omega_1(z)}(\xi-z)T\left(T_{,t}+u_iT_{,i}\right)dxd\eta\\ &-\int_0^t\int_{\Omega_1(z)}TT_{,3}dxd\eta+\int_0^t\int_{L(z)}(\xi-z)TT_{,2}dAd\eta\\ &=-\frac{1}{2\kappa}\int_{\Omega_1(z)}(\xi-z)T^2dx|_{\eta=t}+\frac{1}{2\kappa}\int_0^t\int_{\Omega_1(z)}T^2u_3dxd\eta\\ &-\frac{1}{2\kappa}\int_0^t\int_{L(z)}(\xi-z)T^2u_2dAd\eta-\int_0^t\int_{\Omega_1(z)}TT_{,3}dxd\eta\\ &+\int_0^t\int_{L(z)}(\xi-z)TT_{,2}dAd\eta.\end{aligned}\tag{2.7}$$

Following the same procedure, we obtain the following results for items contain S

$$\begin{aligned}\int_0^t\int_{\Omega_2(z)}(\xi-z)S_{,i}S_{,i}dxd\eta =& -\frac{1}{2\kappa_s}\int_{\Omega_2(z)}(\xi-z)S^2dx|_{\eta=t}+\frac{1}{2\kappa_s}\int_0^t\int_{\Omega_2(z)}S^2v_3dxd\eta\\ &+\frac{1}{2\kappa_s}\int_0^t\int_{L(z)}(\xi-z)v_2S^2dAd\eta-\int_0^t\int_{\Omega_2(z)}SS_{,3}dxd\eta\\ &-\int_0^t\int_{L(z)}(\xi-z)SS_{,2}dxd\eta.\end{aligned}\tag{2.8}$$

We define

$$\begin{aligned}E_2(z,t)=&\ \kappa\int_0^t\int_{\Omega_1(z)}(\xi-z)T_{,i}T_{,i}dxd\eta+\kappa_s\int_0^t\int_{\Omega_2(z)}(\xi-z)S_{,i}S_{,i}dxd\eta\\ &+\frac{1}{2}\int_{\Omega_1(z)}(\xi-z)T^2dx+\frac{1}{2}\int_{\Omega_2(z)}(\xi-z)S^2dx,\end{aligned}\tag{2.9}$$

We now define

$$\begin{aligned}\psi(z,t)=&\ E_1(z,t)+AE_2(z,t)\\ =&\ b\int_0^t\int_{\Omega_1(z)}(\xi-z)(u_iu_i)^{\frac{3}{2}}dxd\eta+\int_0^t\int_{\Omega_1(z)}(\xi-z)(1+\gamma T)u_iu_idxd\eta\\ &+\frac{b}{k}\int_0^t\int_{\Omega_2(z)}(\xi-z)v_iv_idxd\eta+A\kappa\int_0^t\int_{\Omega_1(z)}(\xi-z)T_{,i}T_{,i}dxd\eta\\ &+A\kappa_s\int_0^t\int_{\Omega_2(z)}(\xi-z)S_{,i}S_{,i}dxd\eta+\frac{A}{2}\int_{\Omega_1(z)}(\xi-z)T^2dx\\ &+\frac{A}{2}\int_{\Omega_2(z)}(\xi-z)S^2dx,\end{aligned}\tag{2.10}$$

where A is a positive constant to be determined later.

Combining (2.6)–(2.10), we can also get

$$\begin{aligned}\psi(z,t) = &\int_0^t\int_{\Omega_1(z)}(\xi - z)g_iTu_idxd\eta + \int_0^t\int_{\Omega_2(z)}g_i(\xi - z)v_iS\,dxd\eta\\&+\int_0^t\int_{\Omega_1(z)}pu_3dxd\eta + \int_0^t\int_{\Omega_2(z)}qv_3dxd\eta + \frac{A}{2}\int_0^t\int_{\Omega_1(z)}T^2u_3dxd\eta\\&- A\kappa\int_0^t\int_{\Omega_1(z)}TT_{,3}dxd\eta + \frac{A}{2}\int_0^t\int_{\Omega_2(z)}v_3S^2dxd\eta - \kappa_sA\int_0^t\int_{\Omega_2(z)}SS_{,3}dxd\eta\\&=\sum_{i=1}^{8}K_i.\end{aligned} \tag{2.11}$$

In this paper, we want to obtain a first-order differential inequality for $\psi(z,t)$.

3. Decay estimates

In the proof of our main result, we will use the following Lemmas:

Lemma 1. *(see [37]) We suppose $\bar{\Omega}$ is a bounded region in R^3 which has Lipschitz boundary. χ is a bounded function in $\bar{\Omega}$ satisfies $\int_{\bar{\Omega}}\chi dx = 0$. There exists a vector function ω_i satifies*

$$\omega_{i,i} = \chi \quad on \quad \bar{\Omega}, \quad \omega_i = 0 \quad on \quad \partial\bar{\Omega}, \tag{3.1}$$

and

$$\int_{\bar{\Omega}}\omega_{i,j}\omega_{i,j}dx \le C\int_{\bar{\Omega}}\chi^2dx, \tag{3.2}$$

with C is a constant which is dependent on the shape of $\bar{\Omega}$.

Lemma 2. *(See [37]) The temperatures T and S satisfy the following maximum estimates:*

$$\max\{|T|, |S|\} \le T_M, \tag{3.3}$$

with $T_M = \max\{\sup_{D_1(0)\times[0,\infty]}|g|, \sup_{D_2(0)\times[0,\infty]}|\tau|\}$. For $g \ge 0$, using the maximum principle, we can easily get $T \ge 0$ in $\Omega_1 \times \{t \ge 0\}$.

Lemma 3. *For K_n defined in* (2.11)*, we have the following estimates:*

$$K_1 + K_2 + K_5 + K_6 + K_7 + K_8 \le \frac{1}{2}\psi(z,t) + n_1\left[-\frac{\partial\psi(z,t)}{\partial z}\right], \tag{3.4}$$

with n_1 is a positive constant to be defined later.

Proof. We give a bound for $K_1 + K_2$.

$$
\begin{aligned}
K_1 + K_2 &= \int_0^t \int_{\Omega_1(z)} (\xi - z) g_i T u_i dx d\eta + \int_0^t \int_{\Omega_2(z)} g_i(\xi - z) v_i S\, dx d\eta \\
&\le \frac{\zeta^2}{2} \int_0^t \int_{\Omega_1(z)} (\xi - z) T^2 dx d\eta + \frac{1}{2} \int_0^t \int_{\Omega_1(z)} (\xi - z)(1 + \gamma T) u_i u_i dx d\eta \\
&+ \frac{\zeta^2}{2} \int_0^t \int_{\Omega_2(z)} (\xi - z) S^2 dx d\eta + \frac{1}{2} \int_0^t \int_{\Omega_2(z)} (\xi - z) v_i v_i dx d\eta \\
&\le \frac{\zeta^2}{2\lambda} \int_0^t \int_{\Omega_1(z)} (\xi - z) T_{,\alpha} T_{,\alpha} dx d\eta + \frac{1}{2} \int_0^t \int_{\Omega_1(z)} (\xi - z)(1 + \gamma T) u_i u_i dx d\eta \\
&+ \frac{\zeta^2}{2\upsilon} \int_0^t \int_{\Omega_2(z)} (\xi - z) S_{,\alpha} S_{,\alpha} dx d\eta + \frac{1}{2} \int_0^t \int_{\Omega_2(z)} (\xi - z) v_i v_i dx d\eta,
\end{aligned} \tag{3.5}
$$

where λ is the lowest eigenvalue of

$$
\begin{aligned}
&U_{,\alpha\alpha} + \lambda U = 0 \quad in \quad D_1, \\
&U = 0 \quad on \quad \partial D_1 \cap \Gamma_1, \\
&U_{,\alpha} = 0 \quad on \quad \partial D_1 \cap L,
\end{aligned} \tag{3.6}
$$

and υ is the lowest eigenvalue of

$$
\begin{aligned}
&V_{,\alpha\alpha} + \upsilon V = 0 \quad in \quad D_2, \\
&V = 0 \quad on \quad \partial D_2 \cap \Gamma_2, \\
&V_{,\alpha} = 0 \quad on \quad \partial D_2 \cap L.
\end{aligned} \tag{3.7}
$$

For $K_5 + K_6$, we have

$$
\begin{aligned}
K_5 + K_6 &= \frac{A}{2} \int_0^t \int_{\Omega_1(z)} T^2 u_3 dx d\eta - A\kappa \int_0^t \int_{\Omega_1(z)} T T_{,3} dx d\eta \\
&\le \frac{AT_M}{4} \int_0^t \int_{\Omega_1(z)} T^2 dx d\eta + \frac{AT_M}{4} \int_0^t \int_{\Omega_1(z)} (1 + \gamma T) u_i u_i dx d\eta \\
&+ \frac{A\kappa}{2} \int_0^t \int_{\Omega_1(z)} T^2 dx d\eta + \frac{A\kappa}{2} \int_0^t \int_{\Omega_1(z)} T_{,3}^2 dx d\eta \\
&\le \left[\frac{AT_M}{4\lambda} + \frac{A\kappa}{2\lambda}\right] \int_0^t \int_{\Omega_1(z)} T_{,i} T_{,i} dx d\eta \\
&+ \frac{AT_M}{4} \int_0^t \int_{\Omega_1(z)} (1 + \gamma T) u_i u_i dx d\eta + \frac{A\kappa}{2} \int_0^t \int_{\Omega_1(z)} T_{,3}^2 dx d\eta \\
&\le \left[\frac{AT_M}{4\lambda} + \frac{A\kappa}{2\lambda} + \frac{A\kappa}{2}\right] \int_0^t \int_{\Omega_1(z)} T_{,i} T_{,i} dx d\eta \\
&+ \frac{AT_M}{4} \int_0^t \int_{\Omega_1(z)} (1 + \gamma T) u_i u_i dx d\eta.
\end{aligned} \tag{3.8}
$$

Similarly, we can get

$$
\begin{aligned}
K_7+K_8 &= \frac{A}{2}\int_0^t\int_{\Omega_2(z)} v_3S^2dxd\eta-\kappa_sA\int_0^t\int_{\Omega_2(z)} SS_{,3}dxd\eta\\
&\le \frac{AT_M}{4}\int_0^t\int_{\Omega_2(z)} v_3^2dxd\eta+\frac{AT_M}{4\upsilon}\int_0^t\int_{\Omega_2(z)} S_{,i}S_{,i}dxd\eta\\
&+\left[\frac{\kappa_sA}{2\upsilon}+\frac{\kappa_sA}{2}\right]\int_0^t\int_{\Omega_2(z)} S_{,i}S_{,i}dxd\eta\\
&\le \frac{AT_M}{4}\int_0^t\int_{\Omega_2(z)} v_3^2dxd\eta+\left[\frac{\kappa_sA}{2\upsilon}+\frac{\kappa_sA}{2}+\frac{AT_M}{4\upsilon}\right]\int_0^t\int_{\Omega_2(z)} S_{,i}S_{,i}dxd\eta.
\end{aligned}
\tag{3.9}
$$

Combining (3.5), (3.8), (3.9) and (2.10), and choosing $A>\max\left\{\frac{\zeta^2}{\kappa\lambda},\frac{\zeta^2}{\kappa_s\upsilon}\right\}$, we have

$$
K_1+K_2+K_5+K_6+K_7+K_8\le\frac{1}{2}\psi(z,t)+n_1\left[-\frac{\partial\psi(z,t)}{\partial z}\right], \tag{3.10}
$$

with $n_1=\max\left\{\frac{AT_M}{4\lambda\kappa}+\frac{1}{2\lambda}+\frac{1}{2},\frac{AT_M}{4},\frac{\kappa AT_M}{4b},\frac{1}{2\upsilon}+\frac{1}{2}+\frac{T_M}{4\upsilon\kappa_s}\right\}$. □

Lemma 4. *For K_n defined in* (2.11)*, we have the following estimates:*

$$
K_3+K_4\le\tilde{k}_5\left[-\frac{\partial\psi(z,t)}{\partial z}\right]+\frac{b}{2}\int_0^t\left[\int_{\Omega_1(z)}(u_ju_j)^{\frac{3}{2}}dx\right]^{\frac{4}{3}}d\eta, \tag{3.11}
$$

where $\tilde{k}_5$ is a computable constant.

Proof. We define

$$
J=\int_{\Omega_1(z)} pu_3dx|_{s=\eta}+\int_{\Omega_2(z)} qv_3dx|_{s=\eta}.
$$

We then rewritten J as

$$
J=\int_{\Omega_1(z)\cup\Omega_2(z)}\sigma\omega_3dx|_{s=\eta}=\int_{\Omega_1(z)} pu_3dx|_{s=\eta}+\int_{\Omega_2(z)} qv_3dx|_{s=\eta}, \tag{3.12}
$$

with

$$
\sigma=\begin{cases} p & in\quad \Omega_1\times\{t>0\},\\ q & in\quad \Omega_2\times\{t>0\},\end{cases} \tag{3.13}
$$

and

$$
\omega_3=\begin{cases} u_3 & in\quad \Omega_1\times\{t>0\},\\ v_3 & in\quad \Omega_2\times\{t>0\}.\end{cases} \tag{3.14}
$$

Therefore, we find

$$
J=\sum_{n=0}^{\infty}\int_{z+na}^{z+(n+1)a}\int_{D_1(z)} pu_3dx|_{s=\eta}+\sum_{n=0}^{\infty}\int_{z+na}^{z+(n+1)a}\int_{D_2(z)} qv_3dx|_{s=\eta}, \tag{3.15}
$$

with a is an arbitrary positive constant. We note that

$$
\int_{\Omega_1(z)} u_{i,i}dx|_{s=\eta}+\int_{\Omega_2(z)} v_{i,i}dx|_{s=\eta}=0. \tag{3.16}
$$

Using integration by parts, we have

$$-\int_{D_1(z)} u_3 dA|_{s=\eta} - \int_{L(z)\cap D(z)} u_2 dA|_{s=\eta} - \int_{D_2(z)} v_3 dA|_{s=\eta} + \int_{L(z)\cap D(z)} v_2 dA|_{s=\eta} = 0. \tag{3.17}$$

We have

$$\int_{D_1(z)} u_3 dA + \int_{D_2(z)} v_3 dA = 0.$$

Thus, we can obtain

$$\int_D w_3 dA|_{s=\eta} = 0. \tag{3.18}$$

Using the result of Lemma 1, we have

$$\begin{aligned}
\int_z^{z+a}\int_D \sigma w_3 dx|_{s=\eta} = &-\int_z^{z+a}\int_D \omega_j \sigma_{,i} dx|_{s=\eta}\\
&\le b\left[\int_z^{z+a}\int_{D_1} (\omega_j\omega_j)^{\frac{3}{2}} dx|_{s=\eta}\right]^{\frac{1}{3}} \left[\int_z^{z+a}\int_{D_1} (u_j u_j)^{\frac{3}{2}} dx|_{s=\eta}\right]^{\frac{2}{3}}\\
&+\left[\int_z^{z+a}\int_{D_1} (1+\gamma T)\omega_j\omega_j dx\right]^{\frac{1}{2}} \left[\int_z^{z+a}\int_{D_1} (1+\gamma T) u_i u_i dx|_{s=\eta}\right]^{\frac{1}{2}}\\
&+\left[\int_z^{z+a}\int_{D_1} \omega_j\omega_j dx|_{s=\eta}\right]^{\frac{1}{2}} \left[\int_z^{z+a}\int_{D_1} T^2 dx|_{s=\eta}\right]^{\frac{1}{2}}\\
&+\frac{b}{k}\int_z^{z+a}\int_{D_2} (\omega_j v_j) dx - \int_z^{z+a}\int_{D_2} \omega_j g_j S\, dx.
\end{aligned} \tag{3.19}$$

We have

$$\begin{aligned}
&\int_z^{z+a}\int_{D_1} (\omega_j\omega_j)^{\frac{3}{2}} dx|_{s=\eta}\\
&\le \left(\int_z^{z+a}\int_{D_1} (\omega_j\omega_j)^2 dx|_{s=\eta}\right)^{\frac{1}{2}} \left(\int_z^{z+a}\int_{D_1} \omega_j\omega_j dx|_{s=\eta}\right)^{\frac{1}{2}}\\
&\le \tilde{k}^{\frac{1}{2}} \left(\int_z^{z+a}\int_{D_1} \omega_{j,i}\omega_{j,i} dx|_{s=\eta}\right)^{\frac{1}{4}} \left(\int_z^{z+a}\int_{D_1} \omega_j\omega_j dx|_{s=\eta}\right)^{\frac{5}{4}}\\
&\le \frac{C^{\frac{1}{4}}\tilde{k}^{\frac{1}{2}}}{\hat{\Lambda}^{\frac{5}{4}}} \left(\int_z^{z+a}\int_{D_1} {\omega_3}^2 dx|_{s=\eta}\right)^{\frac{1}{4}} \left(\int_z^{z+a}\int_{D_1} \omega_{j,i}\omega_{j,i} dx|_{s=\eta}\right)^{\frac{5}{4}}\\
&\le \frac{C^{\frac{3}{2}}\tilde{k}^{\frac{1}{2}}}{\hat{\Lambda}^{\frac{5}{4}}} \left(\int_z^{z+a}\int_{D_1} {\omega_3}^2 dx|_{s=\eta}\right)^{\frac{3}{2}}\\
&\le \frac{C^{\frac{3}{2}}\tilde{k}^{\frac{1}{2}}}{\hat{\Lambda}^{\frac{5}{4}}} \left(\int_z^{z+a}\int_{D_1} {u_3}^2 dx|_{s=\eta} + \int_z^{z+a}\int_{D_2} {v_3}^2 dx|_{s=\eta}\right)^{\frac{3}{2}}\\
&\le \frac{2C^{\frac{3}{2}}\tilde{k}^{\frac{1}{2}}}{\hat{\Lambda}^{\frac{5}{4}}} \left(\int_z^{z+a}\int_{D_1} {u_3}^2 dx|_{s=\eta}\right)^{\frac{3}{2}} + \frac{2C^{\frac{3}{2}}\tilde{k}^{\frac{1}{2}}}{\hat{\Lambda}^{\frac{5}{4}}} \left(\int_z^{z+a}\int_{D_2} {v_3}^2 dx|_{s=\eta}\right)^{\frac{3}{2}}\\
&\le \frac{2C^{\frac{3}{2}}\tilde{k}^{\frac{1}{2}}a^{\frac{1}{3}}|D|^{\frac{1}{3}}}{\hat{\Lambda}^{\frac{5}{4}}} \int_z^{z+a}\int_{D_1} (u_i u_i)^{\frac{3}{2}} dx|_{s=\eta} + \frac{2C^{\frac{3}{2}}\tilde{k}^{\frac{1}{2}}}{\hat{\Lambda}^{\frac{5}{4}}} \left(\int_z^{z+a}\int_{D_2} {v_3}^2 dx|_{s=\eta}\right)^{\frac{3}{2}},
\end{aligned} \tag{3.20}$$

with $\hat{\wedge}$ is the first eigenvalue of the following problem.

$$\begin{aligned}
&\Delta\phi + \hat{\wedge}\phi = 0 \quad in \quad D \cap \{z < x_3 < z + a\}, \\
&\phi = 0 \quad on \quad (\Gamma_1 \cup \Gamma_2) \cap \{z < x_3 < z + a\}, \\
&\phi = 0 \quad on \quad D \quad for \quad x_3 = z, x_3 = z + a,
\end{aligned} \tag{3.21}$$

and $\tilde{k}$ is the constant satisfies the following poincar$\acute{e}$ inequality

$$\int_\Omega (\omega_j\omega_j)^2 dx \leq \tilde{k}\left(\int_\Omega \omega_{j,i}\omega_{j,i} dx\right)^{\frac{1}{2}} \left(\int_\Omega \omega_j\omega_j dx\right)^{\frac{3}{2}}. \tag{3.22}$$

Using (3.20), we have

$$\begin{aligned}
&\left[\int_z^{z+a}\int_{D_1} (\omega_j\omega_j)^{\frac{3}{2}} dx|_{s=\eta}\right]^{\frac{1}{3}} \times \left[\int_z^{z+a}\int_{D_1} (u_ju_j)^{\frac{3}{2}} dx|_{s=\eta}\right]^{\frac{2}{3}} \\
&\leq \left[\left(\frac{2C^{\frac{3}{2}}\tilde{k}^{\frac{1}{2}}a^{\frac{1}{3}}|D|^{\frac{1}{3}}}{\hat{\wedge}^{\frac{5}{4}}}\right)^{\frac{1}{3}} \left(\int_z^{z+a}\int_{D_1} (u_iu_i)^{\frac{3}{2}} dx|_{s=\eta}\right)^{\frac{1}{3}} + \left(\frac{2C^{\frac{3}{2}}\tilde{k}^{\frac{1}{2}}}{\hat{\wedge}^{\frac{5}{4}}}\right)^{\frac{1}{3}} \left(\int_z^{z+a}\int_{D_2} {v_3}^2 dx|_{s=\eta}\right)^{\frac{1}{2}}\right] \\
&\times \left[\int_z^{z+a}\int_{D_1} (u_ju_j)^{\frac{3}{2}} dx|_{s=\eta}\right]^{\frac{2}{3}} \\
&\leq [\left(\frac{2C^{\frac{3}{2}}\tilde{k}^{\frac{1}{2}}a^{\frac{1}{3}}|D|^{\frac{1}{3}}}{\hat{\wedge}^{\frac{5}{4}}}\right)^{\frac{1}{3}} \int_z^{z+a}\int_{D_1} (u_ju_j)^{\frac{3}{2}} dx|_{s=\eta} + \frac{1}{2}\left(\frac{2C^{\frac{3}{2}}\tilde{k}^{\frac{1}{2}}}{\hat{\wedge}^{\frac{5}{4}}}\right)^{\frac{2}{3}} \int_z^{z+a}\int_{D_2} v_3^2 dx|_{s=\eta} \\
&+ \frac{1}{2}\left[\int_z^{z+a}\int_{D_1} (u_ju_j)^{\frac{3}{2}} dx|_{s=\eta}\right]^{\frac{4}{3}}.
\end{aligned} \tag{3.23}$$

Using the Schwarz and poincar$\acute{e}$ inequalities, we can get

$$\begin{aligned}
&\left[\int_z^{z+a}\int_{D_1} (1+\gamma T)\omega_j\omega_j dx|_{s=\eta}\right]^{\frac{1}{2}} \left[\int_z^{z+a}\int_{D_1} (1+\gamma T)u_iu_i dx|_{s=\eta}\right]^{\frac{1}{2}} \\
&\leq \left[\frac{(1+\gamma T_M)}{\hat{\wedge}}\int_z^{z+a}\int_D \omega_{j,i}\omega_{j,i} dx|_{s=\eta}\right]^{\frac{1}{2}} \left[\int_z^{z+a}\int_{D_1} (1+\gamma T)u_iu_i dx|_{s=\eta}\right]^{\frac{1}{2}} \\
&\leq \left[\frac{C(1+\gamma T_M)}{\hat{\wedge}}\int_z^{z+a}\int_D w_3w_3 dx|_{s=\eta}\right]^{\frac{1}{2}} \left[\int_z^{z+a}\int_{D_1} (1+\gamma T)u_iu_i dx|_{s=\eta}\right]^{\frac{1}{2}} \\
&\leq \left[\frac{C(1+\gamma T_M)}{\hat{\wedge}}\right]^{\frac{1}{2}} \int_z^{z+a}\int_{D_1} (1+\gamma T)u_ju_j dx|_{s=\eta} \\
&+ \left[\frac{C(1+\gamma T_M)}{\hat{\wedge}}\int_z^{z+a}\int_{D_2} v_iv_i dx|_{s=\eta}\right]^{\frac{1}{2}} \left[\int_z^{z+a}\int_{D_1} (1+\gamma T)u_iu_i dx|_{s=\eta}\right]^{\frac{1}{2}} \\
&\leq \left\{\left[\frac{C(1+\gamma T_M)}{\hat{\wedge}}\right]^{\frac{1}{2}} + \frac{C(1+\gamma T_M)}{2\hat{\wedge}}\right\} \int_z^{z+a}\int_{D_1} (1+\gamma T)u_ju_j dx|_{s=\eta} \\
&+ \frac{1}{2}\int_z^{z+a}\int_{D_2} v_iv_i dx|_{s=\eta}.
\end{aligned} \tag{3.24}$$

Following the same procedure as (3.24), we have

$$\begin{aligned}&\left[\int_z^{z+a}\int_{D_1}\omega_j\omega_j dx|_{s=\eta}\right]^{\frac{1}{2}}\times\left[\int_z^{z+a}\int_{D_1}T^2dx|_{s=\eta}\right]^{\frac{1}{2}}\\&\leq\frac{C}{2\hat{\wedge}}\int_z^{z+a}\int_{D_1}(1+\gamma T)u_ju_jdx|_{s=\eta}+\frac{C}{2\hat{\wedge}}\int_z^{z+a}\int_{D_2}v_jv_jdx|_{s=\eta}\\&+\frac{1}{2}\int_z^{z+a}\int_{D_1}T^2dx|_{s=\eta}.\end{aligned}\tag{3.25}$$

We can also get

$$\begin{aligned}&\frac{b}{k}\int_z^{z+a}\int_{D_2}\omega_jv_jdx|_{s=\eta}-\int_z^{z+a}\int_{D_2}\omega_jg_jS\,dx|_{s=\eta}\\&\leq\frac{C}{\hat{\wedge}}\int_z^{z+a}\int_{D_1}(1+\gamma T)u_ju_jdx|_{s=\eta}+\left(\frac{b^2}{2k^2}+\frac{C}{\hat{\wedge}}\right)\int_z^{z+a}\int_{D_2}v_jv_jdx|_{s=\eta}\\&+\frac{1}{2}\int_z^{z+a}\int_{D_2}S^2dx|_{s=\eta}.\end{aligned}\tag{3.26}$$

Inserting (3.23)–(3.26) into (3.19), we obtain

$$\begin{aligned}\int_z^{z+a}\int_D\sigma w_3dx|_{s=\eta}\leq{}&b\left(\frac{2C^{\frac{3}{2}}\tilde{k}^{\frac{1}{2}}a^{\frac{1}{3}}|D|^{\frac{1}{3}}}{\hat{\wedge}^{\frac{5}{4}}}\right)^{\frac{1}{3}}\left[\int_z^{z+a}\int_{D_1}(u_ju_j)^{\frac{3}{2}}dx|_{s=\eta}\right]\\&+\frac{b}{2}\left[\int_z^{z+a}\int_{D_1}(u_ju_j)^{\frac{3}{2}}dx\right]^{\frac{4}{3}}+\tilde{k}_1\int_z^{z+a}\int_{D_1}(1+\gamma T)u_ju_jdx|_{s=\eta}\\&+\tilde{k}_2\int_z^{z+a}\int_{D_2}v_iv_idx|_{s=\eta}+\tilde{k}_3\int_z^{z+a}\int_{D_1}T^2dx|_{s=\eta}\\&+\tilde{k}_4\int_z^{z+a}\int_{D_2}S^2dx|_{s=\eta},\end{aligned}\tag{3.27}$$

where $\tilde{k}_1$, $\tilde{k}_2$, $\tilde{k}_3$ and $\tilde{k}_4$ are computable positive constance.

Thus

$$\begin{aligned}\int_{\Omega(z)}\sigma w_3dx|_{s=\eta}\leq{}&b\left(\frac{2C^{\frac{3}{2}}\tilde{k}^{\frac{1}{2}}a^{\frac{1}{3}}|D|^{\frac{1}{3}}}{\hat{\wedge}^{\frac{5}{4}}}\right)^{\frac{1}{3}}\left[\int_{\Omega_1(z)}(u_ju_j)^{\frac{3}{2}}dx|_{s=\eta}\right]\\&+\frac{b}{2}\left[\int_{\Omega_1(z)}(u_ju_j)^{\frac{3}{2}}dx\right]^{\frac{4}{3}}+\tilde{k}_1\int_{\Omega_1(z)}(1+\gamma T)u_ju_jdx|_{s=\eta}\\&+\tilde{k}_2\int_{\Omega_2(z)}v_iv_idx|_{s=\eta}+\tilde{k}_3\int_{\Omega_1(z)}T^2dx|_{s=\eta}\\&+\tilde{k}_4\int_{\Omega_2(z)}S^2dx|_{s=\eta}.\end{aligned}\tag{3.28}$$

We can easily get

$$K_3+K_4\leq\tilde{k}_5\left[-\frac{\partial\psi(z,t)}{\partial z}\right]+\frac{b}{2}\int_0^t\left[\int_{\Omega_1(z)}(u_ju_j)^{\frac{3}{2}}dx\right]^{\frac{4}{3}}d\eta,\tag{3.29}$$

where $\tilde{k}_5$ is a computable constant. □

Lemma 5. *The velocity u_i satisfies*

$$\max_{\eta} \int_{\Omega_1(z)} (u_j u_j)^{\frac{3}{2}} dx|_{s=\eta} \leq \frac{2}{\delta} \int_0^t \int_{\Omega_1(0)} (u_j u_j)^{\frac{3}{2}} dx d\eta, \tag{3.30}$$

with δ is a positive constant.

Proof. From the definition of $\Omega_1(z)$, we have

$$\max_{\eta} \int_{\Omega_1(z)} (u_j u_j)^{\frac{3}{2}} dx|_{s=\eta} \leq \max_{\eta} \int_{\Omega_1(0)} (u_j u_j)^{\frac{3}{2}} dx|_{s=\eta}. \tag{3.31}$$

We now define a function

$$F(s) = \int_{\Omega_1(0)} (u_j u_j)^{\frac{3}{2}} dx. \tag{3.32}$$

The following method was used in [23] in deriving (3.8). Since $F(s)$ is continuous on [0,t], there exists a $\tilde{t} \in [0, t]$ such that

$$\max_{s} F(s) = F(\tilde{t}).$$

The following discussions will be divided into three cases. If $\tilde{t} = 0$, there exists a $0 < \delta_1 < t$, when $s \in (0, \delta_1)$, we have

$$F(s) \geq \frac{1}{2} F(\tilde{t}).$$

We can get

$$\int_0^{\delta_1} F(s) ds \geq \frac{1}{2} \delta_1 F(\tilde{t}).$$

We can easily get

$$F(\tilde{t}) \leq \frac{2}{\delta_1} \int_0^t F(s) ds. \tag{3.33}$$

If $\tilde{t} = t$, there exists a $0 < \delta_2 < t$, when $s \in (t - \delta_2, t)$, we also get

$$F(\tilde{t}) \leq \frac{2}{\delta_2} \int_0^t F(s) ds. \tag{3.34}$$

If $\tilde{t} \in (0, t)$, there exists a $0 < \delta_3 < t - \tilde{t}$, when $s \in (\tilde{t}, \tilde{t} + \delta_3)$, we can get

$$F(\tilde{t}) \leq \frac{2}{\delta_3} \int_0^t F(s) ds. \tag{3.35}$$

A combination of (3.33)–(3.35) gives

$$F(\tilde{t}) \leq \frac{2}{\delta} \int_0^t F(s)ds, \tag{3.36}$$

with $\delta = \min\{\delta_1, \delta_2, \delta_3\}$.

Inserting (3.36) and (3.32) into (3.37), we obtain

$$\max_{\eta} \int_{\Omega_1(z)} (u_j u_j)^{\frac{3}{2}} dx|_{s=\eta} \leq \frac{2}{\delta} \int_0^t \int_{\Omega_1(0)} (u_j u_j)^{\frac{3}{2}} dx d\eta. \tag{3.37}$$

□

Lemma 6. *From the definition of* $\psi(z,t)$ *in* (2.11)*, we can get*

$$\psi(z,t) \leq \left(\tilde{k}_6 + \tilde{k}_7 \left[\psi(0,t)\right]^{\frac{1}{3}}\right)\left[-\frac{\partial \psi(z,t)}{\partial z}\right], \tag{3.38}$$

where $\tilde{k}_6$ *and* $\tilde{k}_7$ *are computable positive constants.*

Proof. We know

$$\int_0^t \left[\int_{\Omega_1(z)} (u_j u_j)^{\frac{3}{2}} dx\right]^{\frac{4}{3}} d\eta \leq \max_{\eta} \left[\int_{\Omega_1(z)} (u_j u_j)^{\frac{3}{2}} dx|_{s=\eta}\right]^{\frac{1}{3}} \int_0^t \int_{\Omega_1(z)} (u_j u_j)^{\frac{3}{2}} dx d\eta. \tag{3.39}$$

Inserting (3.30) and (3.39) into (3.11), we have

$$K_3 + K_4 \leq \tilde{k}_5 \left[-\frac{\partial \psi(z,t)}{\partial z}\right] + \frac{b}{2^{\frac{2}{3}} \delta^{\frac{1}{3}}} \left[\int_0^t \int_{\Omega_1(0)} (u_j u_j)^{\frac{3}{2}} dx d\eta\right]^{\frac{1}{3}} \int_0^t \int_{\Omega_1(z)} (u_j u_j)^{\frac{3}{2}} dx d\eta. \tag{3.40}$$

From the definition of $\psi(z,t)$ in (2.10), we have

$$K_3 + K_4 \leq \tilde{k}_5 \left[-\frac{\partial \psi(z,t)}{\partial z}\right] + \frac{b}{2^{\frac{2}{3}} \delta^{\frac{1}{3}}} \left[\psi(0,t)\right]^{\frac{1}{3}} \left[-\frac{\partial \psi(z,t)}{\partial z}\right]. \tag{3.41}$$

A combination of (3.4) and (3.41) gives

$$\psi(z,t) \leq \left(\tilde{k}_6 + \tilde{k}_7 \left[\psi(0,t)\right]^{\frac{1}{3}}\right)\left[-\frac{\partial \psi(z,t)}{\partial z}\right], \tag{3.42}$$

where $\tilde{k}_6$ and $\tilde{k}_7$ are computable positive constants.

In this part, we will get the following main result. □

Theorem 1. *The energy* $\psi(z,t)$ *defined in* (2.10) *satisfies the following decay estimates:*

$$\psi(z,t) \leq \psi(0,t) e^{-\left(\tilde{k}_6 + \tilde{k}_7 \left[\psi(0,t)\right]^{\frac{1}{3}}\right)^{-1} z}. \tag{3.43}$$

Proof. We rewrite (3.38) as

$$\psi(z,t) + \left(\tilde{k}_6 + \tilde{k}_7\left[\psi(0,t)\right]^{\frac{1}{3}}\right)\frac{\partial\psi(z,t)}{\partial z} \leq 0,$$

We can easily get

$$e^{\left(\tilde{k}_6+\tilde{k}_7\left[\psi(0,t)\right]^{\frac{1}{3}}\right)^{-1}z}\left(\left(\tilde{k}_6 + \tilde{k}_7\left[\psi(0,t)\right]^{\frac{1}{3}}\right)^{-1}\psi(z,t) + \frac{\partial\psi(z,t)}{\partial z}\right) \leq 0, \tag{3.44}$$

An integration of (3.44) from 0 to z, we get

$$\psi(z,t) \leq \psi(0,t)e^{-\left(\tilde{k}_6+\tilde{k}_7\left[\psi(0,t)\right]^{\frac{1}{3}}\right)^{-1}z}. \tag{3.45}$$

Inequality (3.45) shows the desired decay estimates. □

4. Conclusions

In this paper, we study the spatial decay estimates for the Forchheimer-Darcy interfacing problems in a semi-infinite pipe. We get the Saint-Venant type result for the interfacing fluids. We can extend the result to the equations which the velocity eqaution contains a nonlinear term. We have never seen similar results in literature. Our method is useful to study other interfacing problems in porous medium. If the velocity equation contains the derivative of time, the problem will become very complex. We can not get similar result by using the method proposed in this paper. For the pressure term is difficult to tackle. We must seek new method to overcome this difficulty. We will discuss this problem in the following paper. We can further study the structural stability for these equations based on the result obtained in this paper. We think it would be interesting.

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Conflict of interest

The authors declare that they have no competing interests.

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