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Research article

Maximal and minimal iterative positive solutions for p-Laplacian Hadamard fractional differential equations with the derivative term contained in the nonlinear term

Limin Guo[1,*], **Lishan Liu**[2] **and Ying Wang**[3,*]

[1] School of Science, Changzhou Institute of Technology, Changzhou 213002, China
[2] School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China
[3] School of Mathematicacs and Statistics, Linyi University, Linyi 276000, China

* **Correspondence:** Email: guolimin811113@163.com, lywy1981@163.com.

Abstract: In this paper, the maximal and minimal iterative positive solutions are investigated for a singular Hadamard fractional differential equation boundary value problem with a boundary condition involving values at infinite number of points. Green's function is deduced and some properties of Green's function are given. Based upon these properties, iterative schemes are established for approximating the maximal and minimal positive solutions.

Keywords: Hadamard fractional differential equation; iterative positive solution; positive solution; infinite-point
Mathematics Subject Classification: 34B16, 34B18

1. Introduction

We consider the following Hadamard fractional differential equation

$$\phi_p({}^H D^{\alpha}_{1+}u(t)) + f(t, u(t), {}^H D^{\mu}_{1+}u(t)) = 0,\ 1 < t < e, \tag{1.1}$$

with nonlocal boundary conditions

$$u^{(i)}(1) = 0, i = 0, 1, 2, \ldots, n-2, {}^H D^{p_1}_{1+}u(e) = \sum_{j=1}^{\infty} \eta_j {}^H D^{p_2}_{1+}u(\xi_j), \tag{1.2}$$

where $\alpha, \mu \in \mathbb{R}^+(\mathbb{R}^+ = [0, +\infty))$, $n-1 < \alpha \le n, n \ge 3$, $0 \le \mu \le n-2$, $\eta_j \ge 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1$ $(j = 1, 2\cdots)$, $\phi_p(s) = |s|^{p-2}s, p > 1$, $\phi_p^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $p_1 \in [1, n-2]$, $0 \le p_2 \le p_1$,

$f(t, x, y)$ may be singular at $t = 1$ and ${}^H D^{\alpha}_{1^+}u$, ${}^H D^{p_i}_{1^+}u(i = 1, 2)$ are the standard Hadamard derivatives. The existence of maximal and minimal positive solutions are obtained by iterative sequence for the boundary value problem (1.1) and (1.2) under certain conditions.

Compared with classical integer order differential equations, fractional order differential model have the advantages of simple modeling, accurate description and clear physical meaning of parameters for complex problems, and is one of the important tools for mathematical modeling of complex mechanics, physics, medicine and other processes. It has been noticed that most of the work on the topic is based on Riemann-Liuville and Caputo derivatives, for more details readers can refer to [2, 3, 5–12, 14–18] and the references therein, there is another kind of fractional derivatives in the literature due to Hadamard [13], which is named as Hadamard derivative and differs from the preceding ones in the sense that its definition involves logarithmic function of arbitrary exponent. Although many researchers are paying more and more attention to Hadamard fractional differential equation, but the solutions of Hadamard fractional differential equations are still very few, the study of the topic is still in its primary stage. About the details and recent developments on Hadamard fractional differential equations, we refer the reader to [1, 2, 4, 19, 21, 22]. In [4], Ahmad et al. considered fractional integro-differential inclusions of Hadamard and Riemann-Liouville type:

$${}^H D^{\alpha}\left(x(t) - \Sigma_{i=1}^{m} IA^{\beta_i} h_i(t, x(t))\right) \in F(t, x(t)), 1 \le t \le e,$$

with a initial value u(1)=0, where ${}^H D^{\alpha}$ denotes Hadamard fractional derivative of order α, $0 < \alpha \le 1$, and ${}^H I^{\gamma}$ denotes Hadamard fractional integral of order $\gamma > 0$, $\gamma \in \{\beta_1, \beta_2, \ldots, \beta_m\}$. In [19], Thiramanus et al. considered the following Hadamard fractional differential equation:

$${}^H D^{\beta} \phi_p({}^H D^{u}(t)) = a(t) f(u(t)), t \in (1, T), T > 1,$$

with boundary conditions

$$u(T) = \lambda^H I^{\sigma} u(\eta),^H D^{\alpha} u(1) = 0, u(1) = 0,$$

where ${}^H D^{\alpha}$ and ${}^H I^{\sigma}$ denote Hadamard fractional derivative of order α and the Hadamard fractional integral of order σ, respectively. $\phi_p(s)$ is a $p-$ Laplacian operator, i.e., $\phi_p(s) = |s|^{p-2}s$ for $p > 1$, $(\phi_p)^{-1}(s) = \phi_q(s)$, where $\frac{1}{p} + \frac{1}{q} = 1$. In [22], Yukunthorn et al. considered the following fractional differential equation:

$$\begin{aligned} {}^c D^{\alpha} u(t) &= f(t, u(t), v(t)), t \in (1, e), 1 < \alpha \le 2, \\ {}^c D^{\beta} v(t) &= g(t, u(t), v(t)), t \in (1, e), 1 < \beta \le 2, \end{aligned}$$

subject to integral boundary condition

$$u(1) = 0, u(e) = I^{\gamma} u(\sigma_1), \quad v(1) = 0, v(e) = I^{\gamma} u(\sigma_2),$$

where $\gamma > 0$, $1 < \sigma_1 < e, 1 < \sigma_2 < e,^H D^{\kappa}$ and ${}^H I^{\gamma}$ denote Hadamard fractional derivative of order κ and Hadamard fractional integral of order γ, and $f, g : [1, e] \times \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ are continuous functions. In [23], the author considered the following fractional differential equation:

$$D^{\alpha}_{0+} u(t) + g(t) f(t, u(t)) = 0,\ 0 < t < 1,$$

with boundary conditions

$$u(0) = u'(0) = \ldots = u^{(n-2)}(0) = 0, u^{(i)}(1) = \sum_{i=1}^{\infty} \alpha_j u(\xi_j),$$

where $\alpha \in \mathbb{R}^+$, $n-1 < \alpha \leq n, n > 3$, $i \in [1, n-2]$ is a fixed integer, $\alpha_j \geq 0$, $0 < \xi_1 < \xi_2 < \ldots < \xi_{j-1} < \xi_j < \ldots < 1$ $(j = 1, 2, \cdots)$, f is allowed to have singularities with respect to both time and space variables. Various theorems were established for the existence and multiplicity of positive solutions. The existence of positive solutions are established under some sufficient conditions by u_0-positive linear operator and the fixed point theorem.

Motivated by the excellent results above, in this paper, we investigate the existence of maximal and minimal positive solutions for singular Hadamard fractional differential equation with infinite-point boundary value conditions (1.2). Compared with [20, 23], the fractional derivative is involved in the nonlinear term in this paper, and the result is more precise this is because that the positive solutions we obtained are iterative solutions. Compared with [23], the derivatives in our paper are Hadamard fractional derivatives.

Now we list a condition below to be used later in the paper.

(H_0): $f : (1, e] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, and there exists a constant $0 < \epsilon < 1$ such that $(\ln t)^\epsilon \phi_q(f(t, x_0, x_1))$ is continuous on $[1, e] \times \mathbb{R}^+ \times \mathbb{R}^+$.

2. Preliminaries and lemmas

For the convenience of the reader, we first present some basic definitions and lemmas which are useful for the following research are given, and which can be found in the recent literature such as [13].

Definition 2.1. ([13]). The Hadamard fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}^1_+$ is given by

$$^{H}I^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_1^t (\ln \frac{t}{s})^{\alpha - 1} \frac{y(s)}{s} ds.$$

Definition 2.2. ([13]). The Hadamard fractional derivative of order $\alpha > 0$ of a continuous function $y : (0, \infty) \to \mathbb{R}^1_+$ is given by

$$^{H}D^{\alpha} y(t) = \frac{1}{\Gamma(n - \alpha)} \left(t \frac{d}{dt} \right)^n \int_1^t \frac{y(s)}{s \left(\ln \frac{t}{s} \right)^{\alpha - n + 1}} ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1. ([13]). If $\alpha, \gamma, \beta > 0$, then

$$^{H}I^{\alpha}_a \left(\ln(\frac{t}{a})^{\beta - 1} \right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha)} \left(\ln \frac{x}{a} \right)^{\beta + \alpha - 1},$$

$$^{H}D^{\alpha}_a \left(\ln(\frac{t}{a})^{\beta - 1} \right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} \left(\ln \frac{x}{a} \right)^{\beta - \alpha - 1}.$$

Lemma 2.2. ([23]). Suppose that $\alpha > 0$ and $u \in C[1,\infty) \cap L^1[1,\infty)$, then the solution of Hadamard fractional differential equation ${}^H D^{\alpha}_{1^+} u(t) = 0$ is

$$u(t) = c_1(\ln t)^{\alpha-1} + c_2(\ln t)^{\alpha-2} + \cdots + c_n(\ln t)^{\alpha-n}, c_i \in R, i = 0, 1, \cdots, n, n = [\alpha] + 1.$$

Lemma 2.3. ([23]). Suppose that $\alpha > 0, \alpha$ is not natural number. If $u \in C[1,\infty) \cap L^1[1,\infty)$, then

$$u(t) = {}^H I^{\alpha}_{1^+} {}^H D^{\alpha}_{1^+} u(t) + \sum_{k=1}^{n} c_k(\ln t)^{\alpha-k},$$

for $t \in (1, e]$, where $c_k \in R$ $(k = 1, 2, \cdots, n)$, and $n = [\alpha] + 1$.

We consider the linear fractional differential equation

$$\phi_p({}^H D^{\alpha}_{1^+} u(t)) + g(t) = 0,\ 1 < t < e, \tag{2.1}$$

with boundary condition (1.2).

Lemma 2.4. Given $g \in L^1(1, e) \cap C(1, e)$, then the Eq (2.1) with boundary condition (1.2) can be expressed by

$$u(t) = \int_1^e G(t, s)\frac{\phi_q(g(s))}{s}ds,\ t \in [1, e], \tag{2.2}$$

where

$$G(t, s) = G_1(t, s) + \frac{(\ln t)^{\alpha-1}}{\Delta}\sum_{j=1}^{\infty} \eta_j G_2(\xi_j, s),$$

in which

$$G_1(t, s) = \frac{1}{\Gamma(\alpha)}\begin{cases} (\ln t)^{\alpha-1}(\ln \frac{e}{s})^{\alpha-p_1-1} - (\ln \frac{t}{s})^{\alpha-1}, & 1 \le s \le t \le e, \\ (\ln t)^{\alpha-1}(\ln \frac{e}{s})^{\alpha-p_1-1}, & 1 \le t \le s \le e, \end{cases} \tag{2.3}$$

$$G_2(t, s) = \frac{1}{\Gamma(\alpha - p_2)}\begin{cases} (\ln t)^{\alpha-p_2-1}(\ln \frac{e}{s})^{\alpha-p_1-1} - (\ln \frac{t}{s})^{\alpha-p_2-1}, & 1 \le s \le t \le e, \\ (\ln t)^{\alpha-p_2-1}(\ln \frac{e}{s})^{\alpha-p_1-1}, & 1 \le t \le s \le e, \end{cases} \tag{2.4}$$

$$\Delta = \frac{\Gamma(\alpha)}{\Gamma(\alpha - p_1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha - p_2)}\sum_{i=1}^{\infty} \eta_j(\ln \xi_j)^{\alpha-p_2-1}.$$

Proof. By means of the Lemma 2.3, we can reduce (2.1) to an equivalent integral equation

$$u(t) = -{}^H I^{\alpha}_{0^+}\phi_q(g(t)) + C_1(\ln t)^{\alpha-1} + C_2(\ln t)^{\alpha-2} + \cdots + C_n(\ln t)^{\alpha-n},$$

for some $C_1, C_2, \cdots, C_n \in \mathbb{R}^1$. From $u(1) = u'(1) = \cdots = u^{(n-2)}(1) = 0$ of (1.2), we have $C_2 = C_3 = \cdots = C_n = 0$, then

$$u(t) = -\int_1^t \frac{(\ln \frac{t}{s})^{\alpha-1}}{\Gamma(\alpha)}\frac{\phi_q(g(s))}{s}dt + C_1(\ln t)^{\alpha-1}, \tag{2.5}$$

by simple calculation, we have

$${}^H D^{p_i}_{1^+} u(t) = -{}^H I^{\alpha-p_i}_{1^+}\phi_q(g(t)) + C_1\frac{\Gamma(\alpha)}{\Gamma(\alpha - p_i)}(\ln t)^{\alpha-p_i-1}, i = 1, 2. \tag{2.6}$$

Substituting (2.6) into ${}^{H}D_{1^+}^{p_1}u(e) = \sum_{j=1}^{\infty}\eta_j^{H}D_{1^+}^{p_2}u(\xi_j)$, we have

$$C_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-p_1)} -{}^{H}I_{1^+}^{\alpha-p_1}\phi_q(g(e)) = \sum_{i=1}^{\infty}\eta_j\left(C_1\frac{\Gamma(\alpha)}{\Gamma(\alpha-p_2)}(\ln\xi_j)^{\alpha-p_2-1} -{}^{H}I_{1^+}^{\alpha-p_2}\phi_q(g(\xi_j))\right),$$

then

$$C_1 = \frac{\frac{1}{\Gamma(\alpha-p_1)}\int_1^e(\ln\frac{e}{s})^{\alpha-p_1-1}\frac{\phi_q(g(s))}{s}ds - \sum_{i=1}^{\infty}\eta_j\frac{1}{\Gamma(\alpha-p_2)}\int_1^{\xi_j}(\ln\frac{\xi_j}{s})^{\alpha-p_2-1}\frac{\phi_q(g(s))}{s}ds}{\frac{\Gamma(\alpha)}{\Gamma(\alpha-p_1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_2)}\sum_{i=1}^{\infty}\eta_j(\ln\xi_j)^{\alpha-p_2-1}}. \tag{2.7}$$

Substituting (2.7) into (2.5), we have

$$\begin{aligned}
u(t) =& -\frac{1}{\Gamma(\alpha)}\int_1^t(\ln\frac{t}{s})^{\alpha-1}\frac{\phi_q(g(s))}{s}ds \\
&+ (\ln t)^{\alpha-1}\frac{\frac{1}{\Gamma(\alpha-p_1)}\int_1^e(\ln\frac{e}{s})^{\alpha-p_0-1}\frac{\phi_q(g(s))}{s}ds - \sum_{i=1}^{\infty}\eta_j\frac{1}{\Gamma(\alpha-p_2)}\int_1^{\xi_j}\ln(\frac{\xi_j}{s})^{\alpha-p_2-1}\frac{\phi_q(y(s))}{s}ds}{\frac{\Gamma(\alpha)}{\Gamma(\alpha-p_1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_2)}\sum_{i=1}^{\infty}\eta_j(\ln\xi_j)^{\alpha-p_2-1}} \\
=& -\frac{1}{\Gamma(\alpha)}\int_1^t(\ln\frac{t}{s})^{\alpha-1}\frac{\phi_q(g(s))}{s}ds \\
&+ \frac{1}{\Gamma(\alpha)}\frac{\frac{\Gamma(\alpha)}{\Gamma(\alpha-p_1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_2)}\sum_{i=1}^{\infty}\eta_j(\ln\xi_j)^{\alpha-p_2-1} + \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_2)}\sum_{i=1}^{\infty}\eta_j(\ln\xi_j)^{\alpha-p_2-1}}{\frac{\Gamma(\alpha)}{\Gamma(\alpha-p_1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_2)}\sum_{i=1}^{\infty}\eta_j(\ln\xi_j)^{\alpha-p_2-1}} \\
&\times \int_1^e(\ln t)^{\alpha-1}(\ln\frac{e}{s})^{\alpha-p_1-1}\frac{\phi_q(g(s))}{s}ds - (\ln t)^{\alpha-1}\frac{\sum_{i=1}^{\infty}\eta_j\frac{1}{\Gamma(\alpha-p_2)}\int_1^{\xi_j}(\ln\frac{\xi_j}{s})^{\alpha-p_2-1}\frac{\phi_q(g(s))}{s}ds}{\frac{\Gamma(\alpha)}{\Gamma(\alpha-p_1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_2)}\sum_{i=1}^{\infty}\eta_j(\ln\xi_j)^{\alpha-p_2-1}} \\
=& -\frac{1}{\Gamma(\alpha)}\int_1^t(\ln\frac{t}{s})^{\alpha-1}\frac{\phi_q(g(s))}{s}ds + \frac{1}{\Gamma(\alpha)}\int_1^e(\ln t)^{\alpha-1}(\ln\frac{e}{s})^{\alpha-p_1-1}\frac{\phi_q(g(s))}{s}ds \\
&+ \frac{\sum_{i=1}^{\infty}\eta_j(\ln\xi_j)^{\alpha-p_2-1}}{\frac{\Gamma(\alpha)}{\Gamma(\alpha-p_1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_2)}\sum_{i=1}^{\infty}\eta_j(\ln\xi_j)^{\alpha-p_2-1}}\int_1^e(\ln t)^{\alpha-1}(\ln\frac{e}{s})^{\alpha-p_1-1}\frac{\phi_q(g(s))}{s}ds \\
&- (\ln t)^{\alpha-1}\frac{\sum_{i=1}^{\infty}\eta_j\frac{1}{\Gamma(\alpha-p_2)}\int_1^{\xi_j}(\ln\frac{\xi_j}{s})^{\alpha-p_2-1}\frac{\phi_q(g(s))}{s}ds}{\frac{\Gamma(\alpha)}{\Gamma(\alpha-p_1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-p_2)}\sum_{j=1}^{\infty}\eta_j(\ln\xi_j)^{\alpha-p_2-1}}
\end{aligned}$$

$$= \int_1^e G_1(t,s)\frac{\phi_q(g(s))}{s}ds + \frac{(\ln t)^{\alpha-1}}{\Delta}\left(\frac{1}{\Gamma(\alpha-p_2)}\sum_{j=1}^{\infty}\eta_j(\ln\xi_j)^{\alpha-p_2-1}\int_1^e\left(\ln\frac{e}{s}\right)^{\alpha-p_1-1}\frac{\phi_q(g(s))}{s}ds\right.$$
$$\left. - \sum_{j=1}^{\infty}\eta_j\frac{1}{\Gamma(\alpha-p_2)}\int_1^{\xi_j}\ln(\frac{\xi_j}{s})^{\alpha-p_2-1}\frac{\phi_q(g(s))}{s}ds\right)$$
$$= \int_1^e G_1(t,s)\frac{\phi_q(g(s))}{s}ds + \frac{(\ln t)^{\alpha-1}}{\Delta}\sum_{j=1}^{\infty}\eta_j\frac{1}{\Gamma(\alpha-p_2)}\left[\int_1^e(\ln\xi_j)^{\alpha-p_2-1}\left(\ln\frac{e}{s}\right)^{\alpha-p_1-1}\frac{\phi_q(g(s))}{s}ds\right.$$
$$\left. - \int_1^{\xi_j}\ln(\frac{\xi_j}{s})^{\alpha-p_2-1}\frac{\phi_q(g(s))}{s}ds\right]$$
$$= \int_1^e\left(G_1(t,s) + \frac{(\ln t)^{\alpha-1}}{\Delta}\sum_{i=1}^{\infty}\eta_j G_2(\xi_j,s)\right)\frac{\phi_q(g(s))}{s}ds$$
$$= \int_1^e G(t,s)\frac{\phi_q(g(s))}{s}ds,$$

where Δ is as (2.2). Moreover, by simple calculation, we have

$$ {}^HD^{\mu}_{1^+}G(t,s) = {}^H D^{\mu}_{1^+}G_1(t,s) + \frac{\Gamma(\alpha)}{\Delta\Gamma(\alpha-\mu)}(\ln t)^{\alpha-1-\mu}\sum_{j=1}^{\infty}\eta_j G_2(\xi_j,s), \tag{2.8}$$

and

$$ {}^HD^{\mu}_{1^+}G_1(t,s) = \frac{1}{\Gamma(\alpha-\mu)}\begin{cases}(\ln t)^{\alpha-1-\mu}(\ln\frac{e}{s})^{\alpha-p_1-1} - (\ln\frac{t}{s})^{\alpha-1-\mu}, & 1\le s\le t\le e,\\ (\ln t)^{\alpha-1-\mu}(\ln\frac{e}{s})^{\alpha-p_1-1}, & 1\le t\le s\le e.\end{cases}$$

It is easy to check that $G(t,s)$ and ${}^HD^{\mu}_{0^+}G(t,s)$ are uniformly continuous on $[1,e]\times[1,e]$. □

Lemma 2.5. The functions G_1 and G_2 given by (2.2) have the following properties:

(1) $G_1(t,s)\ge\frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}\ln s(\ln\frac{e}{s})^{\alpha-p_1-1}$, $\forall t,s\in[1,e]$;
(2) $G_1(t,s)\le\frac{1}{\Gamma(\alpha-1)}(\ln s)(\ln\frac{e}{s})^{\alpha-p_1-1}$,$\forall t,s\in[1,e]$;
(3) $G_1(t,s)\le\frac{1}{\Gamma(\alpha-1)}(\ln t)^{\alpha-1}(\ln\frac{e}{s})^{\alpha-p_1-1}$;
(4) $G(t,s)\le J(s)$, $J(s)=\frac{1}{\Gamma(\alpha-1)}(\ln s)(\ln\frac{e}{s})^{\alpha-p_1-1}+\frac{1}{\Delta}\sum_{i=1}^{\infty}\eta_j G_2(\xi_j,s)$;
$G(t,s)\le\overline{J}(s)$, $\overline{J}(s)=\frac{1}{\Gamma(\alpha-\mu-1)}(\ln s)(\ln\frac{e}{s})^{\alpha-p_1-1}+\frac{1}{\Delta}\sum_{i=1}^{\infty}\eta_j G_2(\xi_j,s)$ for all $t,s\in[1,e]$;
(5) $\frac{1}{\alpha-1}(\ln t)^{\alpha-1}J(s)\le G(t,s)\le\sigma(\ln t)^{\alpha-1}$, $\frac{1}{\alpha-\mu-1}(\ln t)^{\alpha-\mu-1}\overline{J}(s)\le D^{\mu}G(t,s)\le\overline{\sigma}(\ln t)^{\alpha-\mu-1}$, where

$$\sigma = \frac{1}{\Gamma(\alpha)}(\ln\frac{e}{s})^{\alpha-p_1-1}\left((\alpha-1)+\frac{1}{\Delta}\sum_{j=1}^{\infty}\eta_j\frac{1}{\Gamma(\alpha-p_2)}(\ln\xi_j)^{\alpha-p_2-1}\right),$$

$$\overline{\sigma} = \frac{1}{\Gamma(\alpha-\mu)}(\ln\frac{e}{s})^{\alpha-p_1-1}\left((\alpha-\mu-1)+\frac{1}{\Delta}\sum_{j=1}^{\infty}\eta_j\frac{1}{\Gamma(\alpha-p_2)}(\ln\xi_j)^{\alpha-p_2-1}\right),$$

for $\forall t,s\in[1,e]$.

Proof. (1) For $1 \le s \le t \le e$, notice that $p_1 \ge 1$, we get $(1-s)^{p_1} \le (1-s)$. Hence,

$$\begin{aligned}
G_1(t,s) =& \frac{1}{\Gamma(\alpha)}\left((\ln t)^{\alpha-1}(\ln \frac{e}{s})^{\alpha-p_1-1} - (\ln \frac{t}{s})^{\alpha-1}\right)\\
=& \frac{1}{\Gamma(\alpha)}\left[(\ln t)^{\alpha-1}(\ln e - \ln s)^{\alpha-p_1-1} - (\ln t - \ln s)^{\alpha-1}\right]\\
=& \frac{1}{\Gamma(\alpha)}\left[(\ln t)^{\alpha-1}(\ln e - \ln s)^{\alpha-p_1-1} - (\ln t)^{\alpha-1}(1-\frac{\ln s}{\ln t})^{\alpha-1}\right]\\
\ge& \frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}\left[(\ln e - \ln s)^{\alpha-p_1-1} - (\ln e - \ln s)^{\alpha-p_1-1+p_1}\right]\\
\ge& \frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}(\ln e - \ln s)^{\alpha-p_1-1}\left[1 - (\ln \frac{e}{s})^{p_1}\right]\\
\ge& \frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}(\ln e - \ln s)^{\alpha-p_1-1}[1-(1-\ln s)]\\
=& \frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}\ln s(\ln e - \ln s)^{\alpha-p_1-1}.
\end{aligned}$$

For $1 \le t \le s \le e$, we get

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}(\ln \frac{e}{s})^{\alpha-p_1-1} \ge \frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}\ln s(\ln \frac{e}{s})^{\alpha-p_1-1},$$

hence, (1) holds.

(2) For $1 \le s \le t \le e$, notice that $\alpha - p_1 - 1 > 0$, we get

$$\begin{aligned}
G_1(t,s) =& \frac{1}{\Gamma(\alpha)}\left((\ln t)^{\alpha-1}(\ln \frac{e}{s})^{\alpha-p_1-1} - (\ln \frac{t}{s})^{\alpha-1}\right)\\
=& \frac{1}{\Gamma(\alpha)}(\ln \frac{e}{s})^{-p_1}\left[(\ln t \ln \frac{e}{s})^{\alpha-1} - (\ln \frac{e}{s})^{p_1}(\ln \frac{t}{s})^{\alpha-1}\right]\\
\le& \frac{1}{\Gamma(\alpha)}(\ln \frac{e}{s})^{-p_1}\left[(\ln t \ln \frac{e}{s})^{\alpha-1} - (\ln \frac{e}{s})^{\alpha-1}(\ln \frac{t}{s})^{\alpha-1}\right]\\
=& \frac{1}{\Gamma(\alpha)}(\ln \frac{e}{s})^{-p_1}(\alpha-1)\int_{\ln \frac{e}{s}\ln \frac{t}{s}}^{\ln t \ln \frac{e}{s}} x^{\alpha-2}dx\\
\le& \frac{1}{\Gamma(\alpha)}(\ln \frac{e}{s})^{-p_1}(\alpha-1)(\ln t)^{\alpha-2}(\ln \frac{e}{s})^{\alpha-2}\left[\ln t \ln \frac{e}{s} - \ln \frac{e}{s}\ln \frac{t}{s}\right]\\
=& \frac{1}{\Gamma(\alpha-1)}(\ln \frac{e}{s})^{-p_1}(\ln t)^{\alpha-2}(\ln \frac{e}{s})^{\alpha-2}\ln s \ln \frac{e}{s}\\
\le& \frac{1}{\Gamma(\alpha-1)}\ln s(\ln \frac{e}{s})^{\alpha-p_1-1}.
\end{aligned}$$

For $1 \le t \le s \le e$, and $\alpha > 2$, we get

$$\begin{aligned}
G_1(t,s) =& \frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}(\ln \frac{e}{s})^{\alpha-p_1-1}\\
\le& \frac{1}{\Gamma(\alpha-1)}\ln s(\ln \frac{e}{s})^{\alpha-p_1-1}.
\end{aligned}$$

(3) By (2), for $t, s \in (1, e)$, we have

$$G_1(t,s) \le \frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}(\ln \frac{e}{s})^{\alpha-p_1-1} \le \frac{1}{\Gamma(\alpha-1)}(\ln t)^{\alpha-1}(\ln \frac{e}{s})^{\alpha-p_1-1},$$

hence, (3) holds.

(4) By (1) and (2), we get

$$\begin{aligned} G(t,s) =& G_1(t,s) + \frac{(\ln t)^{\alpha-1}}{\Delta}\sum_{j=1}^{\infty}\eta_j G_2(\xi_j, s) \\ \le& \frac{1}{\Gamma(\alpha-1)}(\ln s)(\ln \frac{e}{s})^{\alpha-p_1-1} + \frac{1}{\Delta}\sum_{j=1}^{\infty}\eta_j G_2(\xi_j, s) = J(s). \end{aligned}$$

(5) By (1) and (3), we get

$$\begin{aligned} G(t,s) =& G_1(t,s) + \frac{(\ln t)^{\alpha-1}}{\Delta}\sum_{j=1}^{\infty}\eta_j G_2(\xi_j, s) \\ \ge& \frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}\ln s(\ln \frac{e}{s})^{\alpha-p_1-1} + \frac{(\ln t)^{\alpha-1}}{\Delta}\sum_{j=1}^{\infty}\eta_j G_2(\xi_j, s)) \\ \ge& \frac{1}{\alpha-1}(\ln t)^{\alpha-1}\left(\frac{1}{\Gamma(\alpha-1)}(\ln s)(\ln \frac{e}{s})^{\alpha-p_1-1} + \frac{1}{\Delta}\sum_{i=1}^{\infty}\eta_j G_2(\xi_j, s))\right) \\ =& \frac{1}{\alpha-1}(\ln t)^{\alpha-1}J(s). \end{aligned}$$

By (3), we have

$$\begin{aligned} G(t,s) =& G_1(t,s) + \frac{(\ln t)^{\alpha-1}}{\Delta}\sum_{j=1}^{\infty}\eta_j G_2(\xi_j, s) \\ \le& \frac{1}{\Gamma(\alpha-1)}(\ln t)^{\alpha-1}(\ln \frac{e}{s})^{\alpha-p_1-1} + \frac{(\ln t)^{\alpha-1}}{\Delta}\sum_{j=1}^{\infty}\eta_j G_2(\xi_j, s) \\ \le& \frac{1}{\Gamma(\alpha-1)}(\ln t)^{\alpha-1}(\ln \frac{e}{s})^{\alpha-p_1-1} + \frac{(\ln t)^{\alpha-1}}{\Delta}\sum_{i=1}^{\infty}\eta_j \frac{1}{\Gamma(\alpha-p_2)}(\ln \xi_j)^{\alpha-p_2-1}(\ln \frac{e}{s})^{\alpha-p_1-1} \\ \le& \frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}(\ln \frac{e}{s})^{\alpha-p_1-1}\left((\alpha-1) + \frac{1}{\Delta}\sum_{j=1}^{\infty}\eta_j \frac{1}{\Gamma(\alpha-p_2)}(\ln \xi_j)^{\alpha-p_2-1}\right) \\ \le& \sigma(\ln t)^{\alpha-1}, \end{aligned}$$

hence, (5) holds. Similarly, we have

$$G(t,s) \le \overline{J}(s), \overline{J}(s) = \frac{1}{\Gamma(\alpha-\mu-1)}(\ln s)(\ln \frac{e}{s})^{\alpha-p_1-1} + \frac{1}{\Delta}\sum_{i=1}^{\infty}\eta_j G_2(\xi_j, s),$$

$$\frac{1}{\alpha-\mu-1}(\ln t)^{\alpha-\mu-1}\overline{J}(s) \le D_{1^+}^{\mu}G(t,s) \le \overline{\sigma}(\ln t)^{\alpha-\mu-1}.$$

□

Let $E = \{u(t)|u(t) \in C[1,e],^H D^{\mu}_{1^+}u(t) \in C[1,e]\}$ be a Banach space with the norm

$$\|u(t)\| = \max\left\{\max_{t\in[1,e]}|u(t)|, \max_{t\in[1,e]}\ ^H D^{\mu}_{1^+}|u(t)|\right\},$$

and E is endowed with an order relation $u \leq v$ if $u(t) \leq v(t),^H D^{\mu}_{1^+}u(t) \leq^H D^{\mu}_{1^+}v(t)$. Moreover, we define a normal cone of E by

$$K = \left\{u \in E : u(t) \geq 0,^H D^{\mu}_{1^+}u(t) \geq 0,\ t \in [1,e]\right\},$$

clearly, K is a normal cone, and define an operator

$$Tu(t) = \int_1^e G(t,s)\frac{\phi_q(f(s,u(s),^H D^{\mu}_{1^+}u(s)))}{s}ds,\ u \in K.(\star)$$

Problems (1.1) and (1.2) have a positive solution if and only if u is a fixed point of T in K.

Lemma 2.6. *The operator $T : K \to E$ is continuous.*

Proof. First, for $u \in P$, by the continuity of $G(t,s)$, $(\ln s)^{\epsilon}\phi_q(f(s,u(s),^H D^{\mu}_{1^+}u(s)))$, and the integrability of $(\ln s)^{-\epsilon}$,

$$Tu(t) = \int_1^e G(t,s)\frac{\phi_q(f(s,u(s),^H D^{\mu}_{1^+}u(s)))}{s}ds,\ u \in K$$

is well defined on K. It thus follows from the uniform continuity of $G(t,s)$ in $[1,e]\times[1,e]$ and

$$|Tu(t_2) - Tu(t_1)| \leq \int_1^e |G(t_2,s) - G(t_1,s)|(\ln s)^{-\epsilon}(\ln s)^{\epsilon}\frac{\phi_q(f(s,u(s),^H D^{\mu}_{1^+}u(s)))}{s}ds$$

that $Tu \in C[1,e],\ u \in K$. Furthermore, by the uniform continuity of ${}^H D^{\mu}_{1^+}G(t,s)$ for $t,s \in [1,e]$, we get

$${}^H D^{\mu}_{1^+}(Tu)(t) = \int_1^e {}^H D^{\mu}_{1^+}G(t,s)\frac{\phi_q(f(s,u(s),^H D^{\mu}_{1^+}u(s)))}{s}ds \in C[1,e].$$

Let $u_n,\ u \in K,\ u_n \to u$ in $C^1[1,e]$. Since $G(t,s),^H D^{\mu}_{1^+}G(t,s)$ are uniformly continuous, there exists $M > 0$ such that

$$\max\left\{G(t,s),^H D^{\mu}_{1^+}G(t,s)\right\} \leq M,\ t,\ s \in [1,e].$$

On the other hand, since $u_n \to u$ in $C^1[1,e]$, there exists $\Upsilon > 0$ such that $\|u_n\| \leq \Upsilon$ $(n = 1,2,\cdots)$, and then $\|u\| \leq \Upsilon$. Furthermore, $(\ln s)^{\epsilon}\phi_q(f(s,x_0,x_1))$ is continuous on $[1,e]\times\mathbb{R}^+\times\mathbb{R}^+$, so $(\ln s)^{\epsilon}\phi_q(f(s,x_0,x_1))$ is uniformly continuous on $[1,e]\times[0,\Upsilon]\times[0,\Upsilon]$. Hence, for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $s_1,\ s_2 \in [1,e],\ x_0^1,\ x_0^2\ x_1^1,\ x_1^2 \in [0,\Upsilon]$, $|s_1 - s_2| < \delta, |x_0^1 - x_0^2| < \delta, |x_1^1 - x_1^2| < \delta$, we have

$$|(\ln s_1)^{\epsilon}\phi_q(f(s_1,x_0^1,x_1^1)) - (\ln s_2)^{\epsilon}\phi_q(f(s_2,x_0^2,x_1^2))| < \varepsilon. \tag{2.9}$$

By $\|u_n - u\| \to 0$, for the above $\delta > 0$, there exists N_0 such that, $n > N_0$, we have

$$|u_n(t) - u(t)|,\ |^H D^{\mu}_{1^+}u_n(s) -^H D^{\mu}_{1^+}u(s)| \leq \|u_n - u\| < \delta,\ \text{for any } t \in [1,e].$$

Hence, for any $t \in [1, e]$, $n > N_0$, by (2.9), we have

$$\left|(\ln t)^{\epsilon}\phi_q(f(t, u_n(t), {}^H D^{\mu}_{1^+}u_n(t)) - (\ln t)^{\epsilon}\phi_q(f(t, u(t), {}^H D^{\mu}_{1^+}u(t)))\right| < \varepsilon. \tag{2.10}$$

Thus, for $n > N,\ t \in [1, e]$, by (2.10), we have

$$\begin{aligned}
&|(Tu_n)(t) - (Tu)(t)| \\
&= \left|\int_1^e G(t,s)\frac{\phi_q(f\left(s, u_n(s), {}^H D^{\mu}_{1^+}u_n(s))\right)}{s}ds - \int_1^e G(t,s)\frac{\phi_q(f\left(s, u(s), {}^H D^{\mu}_{1^+}u(s))\right)}{s}ds\right| \\
&= \left|\int_1^e G(t,s)(\ln s)^{-\epsilon}\left((\ln s)^{\epsilon}\phi_q(f\left(s, u_n(s), {}^H D^{\mu}_{1^+}u_n(s))\right) - (\ln s)^{\epsilon}\phi_q(f\left(s, u(s), {}^H D^{\mu}_{1^+}u(s))\right)\right)\frac{ds}{s}\right| \\
&\leq M\int_1^e (\ln s)^{-\epsilon}\left((\ln s)^{\epsilon}\phi_q(f(s, u_n(s), {}^H D^{\mu}_{1^+}u_n(s))) - (\ln s)^{\epsilon}\phi_q(f(s, u(s), {}^H D^{\mu}_{1^+}u(s)))\right)\frac{ds}{s} \\
&\leq M\varepsilon\int_1^e (\ln s)^{-\epsilon}\frac{ds}{s},
\end{aligned}$$

and

$$\begin{aligned}
&|{}^H D^{\mu}_{1^+}(Tu_n)(t) - {}^H D^{\mu}_{1^+}(Tu)(t)| \\
&= \left|\int_1^e {}^H D^{\mu}_{1^+}G(t,s)\frac{\phi_q(f\left(s, u(s), {}^H D^{\mu}_{1^+}u(s))\right)}{s}ds - \int_1^e D^{\mu}_{1^+}G(t,s)\frac{\phi_q(f\left(s, u(s), D^{\mu}_{1^+}u(s))\right)}{s}ds\right| \\
&= \left|\int_1^e D^{\mu}_{1^+}G(t,s)(\ln s)^{-\epsilon}\left((\ln s)^{\epsilon}\phi_q(f(s, u_n(s), {}^H D^{\mu}_{1^+}u_n(s))) - (\ln s)^{\epsilon}\phi_q(f(s, u(s), {}^H D^{\mu}_{1^+}u(s)))\right)\frac{ds}{s}\right| \\
&\leq M\int_1^e (\ln s)^{-\epsilon}\left((\ln s)^{\epsilon}\phi_q(f(s, u_n(s), {}^H D^{\mu}_{1^+}u_n(s))) - (\ln s)^{\epsilon}\phi_q(f(s, u(s), {}^H D^{\mu}_{1^+}u(s)))\right)\frac{ds}{s} \\
&\leq M\varepsilon\int_1^e (\ln s)^{-\epsilon}\frac{ds}{s},
\end{aligned}$$

and hence, we get $\|Tu_n - Tu\|_0 \to 0$, $\|{}^H D^{\mu}_{1^+}(Tu_n) - {}^H D^{\mu}_{1^+}(Tu)\|_0 \to 0$ $(n \to \infty)$. That is $\|Tu_n - Tu\| \to 0 (n \to \infty)$, namely T is continuous in the space E. □

Lemma 2.7. *$T : K \to K$ is completely continuous.*

Proof. From Lemma 2.5, we have $(Tu)(t) \geq 0, {}^H D^{\mu}_{1^+}(Tu)(t) \geq 0$, $t \in [1, e]$, then we have $T(K) \subset K$. Now we will prove that TV is relatively compact for bounded $V \subset K$. Since V is bounded, there exists $D > 0$ such that for any $u \in V$, $\|u\| \leq D$, and by the continuity of $(\ln t)^{\epsilon}\phi_q(f(t, x_0, x_1))$ on $[1, e]\times[0, D]\times[0, D]$, there exists $C > 0$ such that $|(\ln s)^{\epsilon}\phi_q(f(s, u(s), {}^H D^{\mu}_{1^+}u(s)))| \leq C$ for $s \in [1, e],\ u \in V$. Hence, for $t \in [1, e],\ u \in V$, we have

$$\begin{aligned}
|Tu(t)| &= \int_1^e G(t,s)\frac{\phi_q(f\left(s, u(s), {}^H D^{\mu}_{1^+}u(s))\right)}{s}ds \\
&= \int_1^e G(t,s)(\ln s)^{-\epsilon}(\ln s)^{\epsilon}\frac{\phi_q(f\left(s, u(s), {}^H D^{\mu}_{1^+}u(s))\right)}{s}ds \\
&\leq C\int_1^e J(s)(\ln s)^{-\epsilon}\frac{ds}{s} \\
&= CB_1,
\end{aligned}$$

where $B_1 = \int_1^e J(s)(\ln s)^{-\epsilon}\frac{ds}{s}$. Similarly, we can derive

$$|{}^H D^{\mu}_{1^+}(Tu)(t)| \le C\overline{B}_1,\ t \in [1,e],\ u \in V,$$

where $\overline{B}_1 = \int_1^e \overline{J}(s)(\ln s)^{-\epsilon}\frac{ds}{s}$, which shows that TV is bounded. Next we will verify that ${}^H D^{\mu}_{1^+}(TV)$ is equicontinuous. Let $t_1, t_2 \in [1,e], t_1 < t_2, u \in V$, we get

$$\begin{aligned}
&|{}^H D^{\mu}_{1^+}(Tu)(t_2) - {}^H D^{\mu}_{1^+}(Tu)(t_1)| \\
=&\Bigg|(\ln t_2)^{\alpha-1-\mu}\int_1^e \frac{(\ln\frac{e}{s})^{\alpha-p_1-1}}{\Gamma(\alpha-\mu)}\frac{\phi_q(f(s,u(s),{}^H D^{\mu}_{1^+}u(s)))}{s}ds \\
&-\int_1^{t_2}\frac{(\ln\frac{t_2}{s})^{\alpha-1-\mu}}{\Gamma(\alpha-\mu)}\frac{\phi_q(f(s,u(s),{}^H D^{\mu}_{1^+}u(s)))}{s}ds \\
&-(\ln t_1)^{\alpha-1-\mu}\int_1^e \frac{(\ln\frac{e}{s})^{\alpha-p_1-1}}{\Gamma(\alpha-\mu)}\frac{\phi_q(f(s,u(s),{}^H D^{\mu}_{1^+}u(s)))}{s}ds \\
&+\int_1^{t_1}\frac{(\ln\frac{t_1}{s})^{\alpha-1-\mu}}{\Gamma(\alpha-\mu)}\frac{\phi_q(f(s,u(s),{}^H D^{\mu}_{1^+}u(s)))}{s}ds\Bigg| \\
\le&|((\ln t_2)^{\alpha-1-\mu} - (\ln t_1)^{\alpha-1-\mu})|\int_1^e \frac{(\ln\frac{e}{s})^{\alpha-p_1-1}}{\Gamma(\alpha-\mu)}\frac{\phi_q(f(s,u(s),{}^H D^{\mu}_{1^+}u(s)))}{s}ds \\
&+\Bigg|\frac{1}{\Gamma(\alpha-\mu)}\int_1^{t_2}(\ln\frac{t_2}{s})^{\alpha-1-\mu}(\ln s)^{-\epsilon}(\ln s)^{\epsilon}\frac{\phi_q(f(s,u(s),{}^H D^{\mu}_{1^+}u(s)))}{s}ds \\
&-\frac{1}{\Gamma(\alpha-\mu)}\int_1^{t_1}(\ln\frac{t_1}{s})^{\alpha-1-\mu}(\ln s)^{-\epsilon}(\ln s)^{\epsilon}\frac{\phi_q(f(s,u(s),{}^H D^{\mu}_{1^+}u(s)))}{s}ds\Bigg| \\
\le&\frac{C}{\Gamma(\alpha-\mu)}((\ln t_2)^{\alpha-\mu-1} - (\ln t_1)^{\alpha-\mu-1}) \\
&+\frac{C}{\Gamma(\alpha-\mu)}\left[\int_1^{t_2}(\ln\frac{t_2}{s})^{\alpha-1-\mu}(\ln s)^{-\epsilon}\frac{ds}{s} - \int_1^{t_1}(\ln\frac{t_1}{s})^{\alpha-1-\mu}(\ln s)^{-\epsilon}\frac{ds}{s}\right].
\end{aligned}$$

Furthermore,

$$\int_1^t(\ln\frac{t}{s})^{\alpha-1-\mu}(\ln s)^{-\epsilon}\frac{ds}{s} = (\ln t)^{\alpha-\mu-\epsilon}\int_1^e(\ln\frac{e}{s})^{\alpha-1-\mu}(\ln s)^{-\epsilon}\frac{ds}{s}.$$

Thus, we obtain

$$\begin{aligned}
&|{}^H D^{\mu}_{1^+}(Tu)(t_2) - {}^H D^{\mu}_{1^+}(Tu)(t_1)| \\
\le&\frac{C}{\Gamma(\alpha-\mu)}((\ln t_2)^{\alpha-\mu-1} - (\ln t_1)^{\alpha-\mu-1}) + \frac{CB_2}{\Gamma(\alpha-\mu)}((\ln t_2)^{\alpha-\mu-\epsilon} - (\ln t_1)^{\alpha-\mu-\epsilon}),\ \forall u \in V,
\end{aligned}$$

where $B_2 = \int_1^e(\ln\frac{e}{s})^{\alpha-\mu-1}(\ln s)^{-\epsilon}\frac{ds}{s}$. From above and the uniform continuity of $(\ln t)^{\alpha-\mu-\epsilon}, (\ln t)^{\alpha-\mu-1}$, we can derive that TV is relatively compact in $C^1[1,e]$, and so we get that $T: K \to K$ is completely continuous.

□

3. Main result

For convenience, we denote

$$\varpi = \min\left\{\left(\int_1^e J(s)(\ln s)^{-\epsilon}\frac{ds}{s}\right)^{-1}, \left(\int_1^e \overline{J}(s)(\ln s)^{-\overline{\epsilon}}\frac{ds}{s}\right)^{-1}\right\}. \tag{3.1}$$

Theorem 3.1. *Assume that* (H_0) *hold, and*

(H_1) $(\ln t)^\epsilon \phi_q(f(t, x_0, x_1))$ *is continuous and nondecreasing on* x_0, x_1;

(H_2) *For any* $t \times x_0 \times x_1 \in [1, e] \times \mathbb{R}^+ \times \mathbb{R}^+$, *there exists* $d > 0$ *such that* $(\ln t)^\epsilon \phi_q(f(t, d, d)) \leq \varpi d$ *hold. Then the boundary value problem* (1.1, 1.2) *has the maximal and minimal positive solutions* $u^\star$ *and* $v^\star$ *on* $[1, e]$, *such that*

$$0 < \|u^\star\| \leq d,\ 0 < \|v^\star\| \leq d.$$

Moreover, for initial values $u_0(t) = d(\ln t)^{\alpha-1},\ v_0(t) = 0,\ t \in [1, e]$, *define the iterative sequences* $\{u_n\}$ *and* $\{v_n\}$ *by*

$$u_n = Tu_{n-1} = T^n u_0,\ v_n = Tv_{n-1} = T^n v_0, \tag{3.2}$$

then

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} T^n u_0 = u^\star,\ \lim_{n\to\infty} v_n = \lim_{n\to\infty} T^n v_0 = v^\star. \tag{3.3}$$

Proof. By Lemma 2.7, we know that $T : K \to K$ is completely continuous. Now we show T is nondecreasing. For any $u_1, u_2, {}^H D^\mu_{1^+} u_1, {}^H D^\mu_{1^+} u_2 \in K$ and $u_1 \leq u_2, {}^H D^\mu_{1^+} u_1 < {}^H D^\mu_{1^+} u_2$, according to the definition T and (H_2), we know that $Tu_1 \leq Tu_2$. Let $\overline{K}_d = \{x \in K : \|x\| \leq d\}$. Next we prove that $T : \overline{K}_d \to \overline{K}_d$. If $u \in \overline{K}_d$, then $\|u\| \leq d$, i.e. $\|u\|_0 \leq d, \|{}^H D^\mu_{1^+} u\|_0 \leq d$, by (4) of Lemma 2.5 and $(H_1), (H_2)$, we have

$$\begin{aligned}
(Tu)(t) &= \int_1^e G(t, s)\frac{\phi_q(f(s, u(s), {}^H D^\mu_{1^+} u(s)))}{s}ds \\
&\leq \int_1^e J(s)(\ln s)^{-\epsilon}(\ln s)^\epsilon \frac{\phi_q(f(s, d, d))}{s}ds \\
&\leq \varpi d \int_1^e J(s)(\ln s)^{-\epsilon}\frac{ds}{s} \\
&= d,\ t \in [1, e],
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
D^\mu_{1^+}(Tu)(t) &= \int_1^e {}^H D^\mu_{1^+} G(t, s)\frac{\phi_q(f(s, u(s), {}^H D^\mu_{1^+} u(s)))}{s}ds \\
&\leq \int_1^e \overline{J}(s)(\ln s)^{-\epsilon}(\ln s)^\epsilon \frac{\phi_q(f(s, d, d))}{s}ds \\
&\leq \varpi d \int_1^e \overline{J}(s)(\ln s)^{-\epsilon}\frac{ds}{s} \\
&= d,\ t \in [1, e],
\end{aligned} \tag{3.5}$$

then (3.4) *and* (3.5) show that $\|Tu\| = \max\left\{\max_{t\in[1,e]} |Tu(t)|, \max_{t\in[1,e]} {}^H D^\mu_{1^+} |Tu(t)|\right\} \leq d$, hence $A(K_d) \subseteq K_d$.

Let $u_0(t) = d(\ln t)^{\alpha-1},\ t \in [1, e]$, then $u_0(t) \in \overline{K}_d$. Let $u_1 = Tu_0,\ u_2 = T^2 u_0$, then we have $u_1,\ u_2 \in \overline{K}_d$. We denote $u_{n+1} = Tu_n = T^n u_0\ (n = 0, 1, 2, \cdots)$. In view of the fact that $T : K_d \to K_d$, it follows that $u_n \in T(K_d) \subseteq K_d\ (n = 1, 2, \cdots)$. Since T is completely continuous, we assert that the sequence $\{u_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} u_{n_k} = u^\star \in K_d$.

Since $u_1 = Tu_0 \in K_d$, by Lemma 2.5 and (H_2), we get

$$\begin{aligned} Tu_0(t) &= \int_1^e G(t,s)(\ln s)^{-\epsilon}(\ln s)^{\epsilon}\frac{\phi_q(f(s,u_0(s),{}^H D_{1^+}^{\mu} u_0(s)))}{s}ds \\ &\le \varpi d(\ln t)^{\alpha-1}\int_1^e J(s)(\ln s)^{-\epsilon}\frac{ds}{s} \\ &= d(\ln t)^{\alpha-1} = u_0(t),\ t\in[1,e], \end{aligned} \tag{3.6}$$

which implies $u_1 \le u_0$. Hence, by (H_1),

$$\begin{aligned} u_2(t) = Tu_1(t) &= \int_1^e G(t,s)(\ln s)^{-\epsilon}(\ln s)^{\epsilon}\frac{\phi_q(f(s,u_1(s),{}^H D_{1^+}^{\mu} u_1(s)))}{s}ds \\ &\le \int_1^e G(t,s)(\ln s)^{-\epsilon}(\ln s)^{\epsilon}\frac{\phi_q(f(s,u_0(s),{}^H D_{1^+}^{\mu} u_0(s)))}{s}ds = Tu_0(t) = u_1(t),\ t\in[1,e]. \end{aligned}$$

By the induction, we have $u_{n+1} \le u_n\ (n = 0, 1, 2, \cdots)$. Therefore, $\lim_{n\to\infty} u_n = u^\star$. Using the continuity of T and taking the limit $n \to \infty$ in $u_{n+1} = Tu_n$ yields $Tu^\star = u^\star$.

Let $v_0(t) = 0,\ t \in [1, e]$, apparently $v_0(t) \in \overline{K}_d$. Let $v_1 = Tv_0,\ v_2 = T^2 v_0$, then we have $v_1 \in \overline{K}_d,\ v_2 \in \overline{K}_d$. Let $v_n = Tv_{n-1} = T^n v_0\ (n = 0, 1, 2, \cdots)$, and since $T : \overline{K}_d \to \overline{K}_d$, we have $v_n \in T(\overline{K}_d) \subseteq \overline{K}_d\ (n = 1, 2, 3, \cdots)$. It follows from the complete continuity of T that $\{v_n\}_{n=1}^{\infty}$ is a sequentially compact set. Since $v_1 = Tv_0 \in \overline{K}_d$, we get

$$v_1(t) = Tv_0(t) = (T0)(t) \ge 0,\ t \in [1, e].$$

Hence, we obtain

$$v_2(t) = Tv_1(t) \ge (T0)(t) = v_1(t),\ t \in [1, e].$$

By induction, we have $v_{n+1} \ge v_n\ (n = 0, 1, 2, \cdots),\ 1 \le t < e$. Hence, there exists $v^\star \in \overline{K}_d$ such that $v_n \to v^\star$ as $n \to \infty$. Applying the continuity of T and $v_{n+1} = Tv_n$, we have that $Tv^\star = v^\star$.

If $f(t, 0) \not\equiv 0,\ 1 \le t \le e$, then the zero function is not the solution of BVP (1.1, 1.2). Hence, $v^\star$ is a positive solution of BVP (1.1, 1.2).

It is well known that each fixed point of T in K is a solution of BVP (1.1, 1.2), so by above proof, we get that $u^\star$ and $v^\star$ are positive solutions of the BVP (1.1, 1.2) on $[1, e]$. □

Remark 3.1. The iterative sequence in Theorem 3.1 begins with a simple function which is useful for computational purpose.

Remark 3.2. $u^\star$ and $v^\star$ are the maximal and minimal solutions of the BVP (1.1, 1.2) in $\overline{K}_d$, but $u^\star$ and $v^\star$ may be coincident, and when $u^\star$ and $v^\star$ are coincident, the boundary value problem (1.1, 1.2) will have a unique solution in $\overline{K}_d$.

4. Numerical examples

We consider the problem (1.1,1.2) with $\alpha = \frac{7}{2}, \mu = \frac{3}{2}, p = \frac{3}{2}, p_2 = \frac{1}{2}, p_1 = \frac{3}{2}, \eta_j = \frac{1}{2j^2}, \xi_j = e^{\frac{1}{j}}, \epsilon = \frac{1}{2}$, $f(t, x, y) = (\ln t)^{-\frac{1}{4}} x^{\frac{1}{2}} + y^{\frac{3}{2}}$, obviously, $(\ln t)^{\frac{1}{2}} \phi_p((\ln t)^{-\frac{1}{4}} x^{\frac{1}{2}} + y^{\frac{3}{2}}) = ((\ln t)^{\frac{3}{4}} x^{\frac{1}{2}} + \ln t y^{\frac{3}{2}})^{\frac{1}{2}}$,

$$\Delta = \frac{\Gamma(\alpha)}{\Gamma(\alpha - p_1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha - p_2)} \sum_{i=1}^{\infty} \eta_j (\ln \xi_j)^{\alpha - p_2 - 1} = \frac{\Gamma(\frac{7}{2})}{\Gamma(2)} - \frac{\Gamma(\frac{7}{2})}{\Gamma(3)} \sum_{j=1}^{\infty} \frac{1}{2j^2} \ln e^{\frac{1}{j}} = 0.7295\Gamma(\frac{7}{2}).$$

By simple calculation, we have f satisfies $(H_0), (H_1)$. Take $d = 1.5$,

$$\varpi = \min \left\{ \left(\int_1^e J(s)(\ln s)^{-\epsilon} \frac{ds}{s} \right)^{-1}, \left(\int_1^e \overline{J}(s)(\ln s)^{-\overline{\epsilon}} \frac{ds}{s} \right)^{-1} \right\} \approx 2.5722,$$

then we have $(\ln t)^{\frac{1}{2}} \phi_{\frac{3}{2}}((\ln t)^{-\frac{1}{4}} 1.5^{\frac{1}{2}} + 1.5^{\frac{3}{2}}) = 3.1200 < 2.5722 \cdot 1.5$, so the assumptions$(H_2)$ hold. So the assumptions of Theorem 3.1 are all satisfied. Hence, we deduce that the problem has a unique solution, which can be obtained by the iteration algorithm given in the Theorem 3.1, for the initial values $u_0(t) = 2.5(\ln t)^{\frac{5}{2}},\ v_0(t) = 0,\ t \in [1, e]$, we obtain the iterative sequence $\{u_k, v_k\}(k = 1, 2, \ldots)$ on $[1, e]$ by (3.2).

5. Conclusions

The maximal and minimal iterative positive solutions are investigated for a singular Hadamard fractional differential equation boundary value problem in this paper. Moreover, iterative schemes are established for approximating the maximal and minimal positive solutions based upon these properties. The iterative sequence in Theorem 3.1 begins with a simple function which is useful for computational purpose.

Conflict of interest

The author declares that there is no conflict of interests regarding the publication of this paper.

Authors' contribution

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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