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Research article

Local geometric properties of the lightlike Killing magnetic curves in de Sitter 3-space

Xiaoyan Jiang and Jianguo Sun*

School of Science, China University of Petroleum (east China), Qingdao, 266555, China

* **Correspondence:** Email: Sunjg616@upc.edu.cn; Tel: +8653286983375; Fax: +8653286983375.

Abstract: In this article, we mainly discuss the local differential geometrical properties of the lightlike Killing magnetic curve $\boldsymbol{\gamma}(s)$ in $\mathbb{S}^3_1$ with a magnetic field $\boldsymbol{V}$. Here, a new Frenet frame $\{\boldsymbol{\gamma}, \boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ is established, and we obtain the local structure of $\boldsymbol{\gamma}(s)$. Moreover, the singular properties of the binormal lightlike surface of the $\boldsymbol{\gamma}(s)$ are given. Finally, an example is used to understand the main results of the paper.

Keywords: lightlike Killing magnetic curve; De Sitter 3-space; local structure; binormal lightlike surface; singularities
Mathematics Subject Classification: 35A53, 58C20

1. Introduction

Since Einstein discovered the general theory of relativity in 1905, many scientists have studied Minkowski space systematically. The existence of the lightlike vector is the principal difference between Minkowski space and Euclidean space. Meanwhile, in Minkowski space, a lightlike curve has some special properties. In physic, the geometric particle model was constructed by using the lightlike curve [1]. A. Ferrandez et al. [2] considered the equations of the particles in 3-dimensional lightlike curves. The second author and D. Pei [3] studied the differential geometric properties of the lightlike curves on Λ^3_1. Also, D. Pei etc. [3] pointed out that de Sitter 3-space was a crucial model of the physical universe in Minkowski space. Some properties of spacelike curves in $\mathbb{S}^3_1$ were studied by T. Fusho and S. Izumiya in [4]. Y. Li and Z. Wang [5] studied the geometric properties of lightlike tangent developables.

Following the action of the Lorentz force produced through the magnetic field $\boldsymbol{F}$, the trajectory of the charged particle is called the magnetic curve. Under certain conditions, the magnetic curve is regarded as the extension of the geodesic [6]. Magnetic curves describe the movement of charged particles in several physical scenarios and form magnetic flux in the background magnetic field [7].

In recent years, many researchers have studied magnetic curves in different spaces [8–12]. In a Riemannian 3-space $(\boldsymbol{M}_3, g)$, Z. Bozkurt [8] used a new variational method to research the magnetic flow with the Killing magnetic field. The results of classification for the Killing magnetic trajectories on the Minkowski 3-space was obtained in [9]. In 3-dimensional massive gravity, G. Clémen [10] considered a black hole with a lightlike Killing vector. M. I. Munteanu [6] introduced the magnetic curves in Euclidean space and used different methods to study the corresponding Killing magnetic curves.

With the deepening of theoretical research, the application of the singularity theory is more and more extensive [13–22]. Z. Wang [13,20,21] considered the singularity classifications of ruled lightlike surfaces in $\mathbb{S}^3_1$. However, very little has been researched about the differential geometric properties of the lightlike Killing magnetic curves. The second author [14, 15] studied the singularity types of the Killing magnetic curves and the lightlike Killing magnetic curves in $\mathbb{R}^3_1$. Here, the classifications of the singularity of the lightlike Killing magnetic curves are considered in $\mathbb{S}^3_1$.

The content of the article is summarized as follows. Firstly, the second part defines $\boldsymbol{\gamma}(s)$ (in the following text, we use $\boldsymbol{\gamma}(s)$ to represent the lightlike Killing magnetic curve), the related concepts of the magnetic curve, and Frenet formulas of $\boldsymbol{\gamma}(s)$. Section 3 shows the major results of the paper (Theorem 3.1), which gives the singularity classification of $\boldsymbol{\gamma}(s) \in \mathbb{S}^3_1$ with $\boldsymbol{V}$. In the fourth section, the height function of $\boldsymbol{\gamma}(s)$ is used to obtain the singularity classification (Proposition 4.1). Section 5 introduces the unfolding of the height function and proves the Theorem 3.1. To enrich the local theory, the local structure of $\boldsymbol{\gamma}(s) \in \mathbb{S}^3_1$ with $\boldsymbol{V}$ is given in the sixth section. In the last section, to better understand the main results of this article, an example of $\boldsymbol{\gamma}(s)$ with $\boldsymbol{V}$ is given.

2. Preliminaries

The relevant definitions of the $\mathbb{R}^4_1$ and $\mathbb{S}^3_1$ are described in [13]. In this section, some definitions related to magnetic curves are introduced. We establish the Frenet frame $\{\boldsymbol{\gamma}, \boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ and obtain the Frenet-Sernet formula.

We define $\boldsymbol{V}$ as a Killing vector field and $F_V = \iota_V dv_g$, where ι is an inner product. By

$$\phi(X) = V \wedge X,$$

we can get the Lorentz force of the F_V. Thus, we obtain the Lorentz force equation defined as

$$\nabla_{\gamma'}\gamma' = V \wedge \gamma',$$

$\boldsymbol{\gamma}(s)$ is called a *Killing magnetic curve* [6–12].

Definition 2.1. *For a Killing magnetic curve* $\boldsymbol{\gamma}(s) \in \mathbb{S}^3_1$*, if* $\langle \boldsymbol{\gamma}'(s), \boldsymbol{\gamma}'(s)\rangle = 0$*, we call* $\boldsymbol{\gamma}(s)$ *the lightlike Killing magnetic curve.*

In the following, we suppose $\boldsymbol{\gamma}(s)$ as a lightlike Killing curve with Killing field $\boldsymbol{V}$.

Since $\boldsymbol{\gamma}(s) \in \mathbb{S}^3_1$, $\langle \boldsymbol{\gamma}(s), \boldsymbol{\gamma}(s)\rangle = 1$, so $\langle \boldsymbol{\gamma}(s), \boldsymbol{\gamma}'(s)\rangle = 0$. We now define

$$\boldsymbol{T}(s) = \boldsymbol{\gamma}'(s), \boldsymbol{N}(s) = \frac{V \wedge \boldsymbol{\gamma}'(s)}{\|V \wedge \boldsymbol{\gamma}'(s)\|},$$

then

$$\langle \boldsymbol{N}(s), \boldsymbol{N}(s)\rangle = 1,$$
$$\langle \boldsymbol{T}(s), \boldsymbol{T}(s)\rangle = 0,$$
$$\langle \boldsymbol{T}(s), \boldsymbol{N}(s)\rangle = \langle \boldsymbol{\gamma}(s), \boldsymbol{T}(s)\rangle = \langle \boldsymbol{\gamma}(s), \boldsymbol{N}(s)\rangle = 0.$$

There exists a lightlike transversal vector $\boldsymbol{B}(s)$, satisfying

$$\langle \boldsymbol{T}(s), \boldsymbol{B}(s)\rangle = 1,$$

$$\langle \boldsymbol{\gamma}(s), \boldsymbol{B}(s)\rangle = \langle \boldsymbol{N}(s), \boldsymbol{B}(s)\rangle = \langle \boldsymbol{B}(s), \boldsymbol{B}(s)\rangle = 0.$$

Therefore, we obtain the Frenet-Sernet formula of $\boldsymbol{\gamma}(s)$ as follows:

$$\begin{cases} \boldsymbol{\gamma}'(s) = \boldsymbol{T}(s) \\ \boldsymbol{T}'(s) = k_1(s)\boldsymbol{N}(s) \\ \boldsymbol{N}'(s) = -k_1(s)\boldsymbol{B}(s) + k_2(s)\boldsymbol{T}(s) \\ \boldsymbol{B}'(s) = -k_2(s)\boldsymbol{N}(s) - \boldsymbol{\gamma}(s), \end{cases} \tag{2.1}$$

where $k_1(s) = \langle \boldsymbol{T}'(s), \boldsymbol{N}(s)\rangle = -\langle \boldsymbol{N}'(s), \boldsymbol{T}(s)\rangle = \|V \wedge \boldsymbol{\gamma}'(s)\|$, $k_2(s) = \langle \boldsymbol{N}'(s), \boldsymbol{B}(s)\rangle = -\langle \boldsymbol{B}'(s), \boldsymbol{N}(s)\rangle$.

Remark 2.2. *If $k_1(s) = 0$, then $\boldsymbol{\gamma}(s)$ is a straight line, and we omit it here.*

3. The classification of singularity

Here, we define the tangent indicatrix of $\boldsymbol{\gamma}(s)$ as $\Phi : I \to \mathbb{S}^3_1$ given by

$$\Phi(s) = \varepsilon\frac{k_1(s)k_2'(s) - k_1'(s)k_2(s)}{\varrho(s)}\boldsymbol{\gamma}(s) + \varepsilon\frac{k_1(s)k_2(s)}{\varrho(s)}\boldsymbol{T}(s),$$

where $\varepsilon = \pm 1$,

$$\varrho(s) = \sqrt{2k_1^3(s)k_2(s) + (k_1(s)k_2'(s) - k_1'(s)k_2(s))^2}.$$

We define a surface

$$\mathcal{BNS} : I \times \mathbb{R} \to \mathbb{S}^3_1$$

by

$$\mathcal{BNS}(s, \mu) = \Phi(s) + \mu B(s),$$

we call $\mathcal{BNS}(s, \mu)$ the *binormal lightlike surface* of the tangent indicatrix of $\boldsymbol{\gamma}(s)$.

For any $\boldsymbol{v}_0 \in \mathbb{S}^3_1$, we call the set :

$$\mathcal{NMB}(\boldsymbol{v}_0) = \{\boldsymbol{u} \in \mathbb{S}^3_1 \mid \langle \boldsymbol{u} - \boldsymbol{v}_0, V \wedge \boldsymbol{u}'\rangle = 0\}$$

the *lightlike magnetic bundle* through $\boldsymbol{v}_0$.

Also, a geometric invariant $\sigma(s)$ of $\boldsymbol{\gamma}(s)$ is given in $\mathbb{S}^3_1$ with $\boldsymbol{V}$ by

$$\sigma(s) = \varepsilon\frac{1}{\varrho}(5k_1^2k_2'^2 - 5k_1k_1'k_2k_2' + k_1^4 + 3k_1^2k_1'k_2 + k_1^3k_2'' - k_1^2k_1''k_2 - 3k_1k_1'^2k_2).$$

By definition in [13], we can get the major results.

Theorem 3.1. *For a curve $\boldsymbol{\gamma}(s)$ in $\mathbb{S}^3_1$ with $\boldsymbol{V}$, when $\varrho(s_0) \neq 0$, giving a vector $\boldsymbol{v}_0 = \mathcal{BNS}(s_0, \mu_0)$ and the magnetic bundle $\mathcal{NMB}(\boldsymbol{v}_0)$, we have the following conclusions:*
(1) As for $\boldsymbol{\gamma}(s)$ and $\mathcal{NMB}(\boldsymbol{v}_0)$, there are at lowest 2-point contact at s_0.
(2) As for $\boldsymbol{\gamma}(s)$ and $\mathcal{NMB}(\boldsymbol{v}_0)$, there are at lowest 3-point contact at s_0 if and only if

$$\boldsymbol{v} = \varepsilon(\frac{k_1(s_0)k_2'(s_0) - k_1'(s_0)k_2(s_0)}{\varrho(s_0)}\boldsymbol{\gamma}(s_0) + \frac{k_2(s_0)k_1(s_0)}{\varrho(s_0)}\boldsymbol{T}(s_0) + \frac{k_1^2(s_0)}{\varrho(s_0)}\boldsymbol{B}(s_0))$$

where $\varepsilon = \pm 1$ and $\sigma(s_0) \neq 0$. With the above conclusions, the germ of image $\mathcal{BNS}(s, \mu)$ at (s_0, μ_0) is locally diffeomorphic to to cuspidal edge $(C \times \mathbb{R})$ (see Figure 1).
(3) As for $\boldsymbol{\gamma}(s)$ and $\mathcal{NMB}(\boldsymbol{v}_0)$, there are at lowest 4-point contact at s_0 if and only if

$$\boldsymbol{v} = \varepsilon(\frac{k_1(s_0)k_2'(s_0) - k_1'(s_0)k_2(s_0)}{\varrho(s_0)}\boldsymbol{\gamma}(s_0) + \frac{k_2(s_0)k_1(s_0)}{\varrho(s_0)}\boldsymbol{T}(s_0) + \frac{k_1^2(s_0)}{\varrho(s_0)}\boldsymbol{B}(s_0))$$

and $\sigma(s_0) = 0, \sigma'(s_0) \neq 0$. With the above conclusions, the germ of image $\mathcal{BNS}(s, \mu)$ at (s_0, μ_0) is locally diffeomorphic to swallowtail (SW)(see Figure 2).

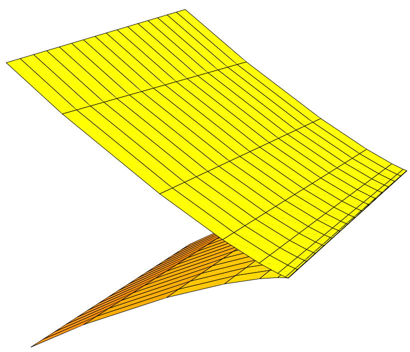

Figure 1. cuspidal edge.

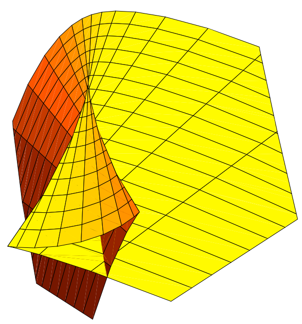

Figure 2. swallowtail.

The cuspidal edge is defined as $C \times \mathbb{R} = \{(x_1, x_2, x_3) \mid x_1 = u, x_2 = \pm v^{1/2}, x_3 = v^{1/3}\}$, and the swallowtail is defined as $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2 v, x_2 = 4u^3 + 2uv, x_3 = v\}$.

4. The height function

Here, we define a function on $\boldsymbol{\gamma} : I \to \mathbb{S}^3_1$, and get a geometric invariant $\sigma(s)$ of the tangent indicatrix of $\boldsymbol{\gamma}(s)$ in $\mathbb{S}^3_1$ with $\boldsymbol{V}$.

For $\boldsymbol{\gamma} : I \to \mathbb{S}^3_1$, we call the function

$$H : I \times \mathbb{S}^3_1 \to \mathbb{R}$$

by

$$H(s, \boldsymbol{v}) = \langle \boldsymbol{\gamma}(s) - \boldsymbol{v}, V \wedge \boldsymbol{\gamma}'(s) \rangle,$$

as a height function of $\boldsymbol{\gamma}(s)$. Giving a vector $\boldsymbol{v_0} \in \mathbb{S}^3_1$, we define $h(s) = H_{v_0}(s) = H(s, \boldsymbol{v}_0)$. Then we can draw the following conclusions:

Proposition 4.1. *For a lightlike Killing magnetic curve $\boldsymbol{\gamma}(s)$ in $\mathbb{S}^3_1$ with $\boldsymbol{V}$, when*

$$\varrho(s_0) = \sqrt{2k_1^3(s_0)k_2(s_0) + (k_1(s_0)k_2'(s_0) - k_1'(s_0)k_2(s_0)^2} \neq 0,$$

then
(1) $h(s_0) = 0$ *if and only if there exist* $a, b, d \in \mathbb{R}$*, such that* $\boldsymbol{v} = a\boldsymbol{\gamma}(s_0) + b\boldsymbol{T}(s_0) + d\boldsymbol{B}(s_0)$ *and* $a^2 + 2bd = 1$.
(2) $h(s_0) = \nabla_{\gamma'(s)} h(s_0) = 0$ *if and only if*

$$\boldsymbol{v} = \varepsilon \sqrt{1 - \frac{2k_2(s_0)}{k_1(s_0)} d^2} \boldsymbol{\gamma}(s_0) + \frac{k_2(s_0)}{k_1(s_0)} d\boldsymbol{T}(s_0) + d\boldsymbol{B}(s_0).$$

(3) $h(s_0) = \nabla_{\gamma'(s)} h(s_0) = \nabla_{\gamma'(s)} \nabla_{\gamma'(s)} h(s_0) = 0$ *if and only if* $d = \varepsilon \frac{k_1^2(s_0)}{\varrho(s_0)}$ *, then*

$$\boldsymbol{v} = \varepsilon(\frac{k_1(s_0)k_2'(s_0) - k_1'(s_0)k_2(s_0)}{\varrho(s_0)} \boldsymbol{\gamma}(s_0) + \frac{k_2(s_0)k_1(s_0)}{\varrho(s_0)} \boldsymbol{T}(s_0) + \frac{k_1^2(s_0)}{\varrho(s_0)} \boldsymbol{B}(s_0)).$$

(4) $h(s_0) = \nabla_{\gamma'(s)} h(s_0) = \nabla_{\gamma'(s)} \nabla_{\gamma'(s)} h(s_0) = \nabla^{(3)}_{\gamma'(s)} h(s_0) = 0$ *if and only if*

$$\boldsymbol{v} = \varepsilon(\frac{k_1(s_0)k_2'(s_0) - k_1'(s_0)k_2(s_0)}{\varrho(s_0)} \boldsymbol{\gamma}(s_0) + \frac{k_2(s_0)k_1(s_0)}{\varrho(s_0)} \boldsymbol{T}(s_0) + \frac{k_1^2(s_0)}{\varrho(s_0)} \boldsymbol{B}(s_0)),$$

and $\sigma(s_0) = 0$.
(5) $h(s_0) = \nabla_{\gamma'(s)} h(s_0) = \nabla_{\gamma'(s)} \nabla_{\gamma'(s)} h(s_0) = \nabla^{(3)}_{\gamma'(s)} h(s_0) = \nabla^{(4)}_{\gamma'(s)} h(s_0) = 0$ *if and only if*

$$\boldsymbol{v} = \varepsilon(\frac{k_1(s_0)k_2'(s_0) - k_1'(s_0)k_2(s_0)}{\varrho(s_0)} \boldsymbol{\gamma}(s_0) + \frac{k_2(s_0)k_1(s_0)}{\varrho(s_0)} \boldsymbol{T}(s_0) + \frac{k_1^2(s_0)}{\varrho(s_0)} \boldsymbol{B}(s_0)),$$

and $\sigma(s_0) = \sigma'(s_0) = 0$.

Proof. We assume that $\boldsymbol{v} = a\boldsymbol{\gamma}(s) + b\boldsymbol{T}(s) + c\boldsymbol{N}(s) + d\boldsymbol{B}(s) = 1$, where a, b, c, d in $\mathbb{R}$. Since $\boldsymbol{v} \in \mathbb{S}^3_1$, $a^2 + 2bd + c^2 = 1$. By using the Frenet formula (2.1), we have the following results:
(1) When $h(s_0) = 0$, we obtion

$$\begin{aligned}\langle \boldsymbol{\gamma} - \boldsymbol{v}, V \wedge \boldsymbol{\gamma}' \rangle &= \langle \boldsymbol{\gamma} - \boldsymbol{v}, k_1 \boldsymbol{N} \rangle \\ &= -ck_1 \\ &= 0,\end{aligned}$$

where $k_1(s_0) \neq 0$, then $c = 0$, $\boldsymbol{v} = a\boldsymbol{\gamma}(s_0) + b\boldsymbol{T}(s_0) + d\boldsymbol{B}(s_0)$.
(2) When $h(s_0) = \nabla_{\gamma'} h(s_0) = 0$, we have

$$\begin{aligned}\nabla_{\gamma'} h(s_0) &= \nabla_{\gamma'} \langle \boldsymbol{\gamma} - \boldsymbol{v}, V \wedge \boldsymbol{\gamma}' \rangle \\ &= \langle \boldsymbol{\gamma}', V \wedge \boldsymbol{\gamma}' \rangle + \langle \boldsymbol{\gamma} - \boldsymbol{v}, (V \wedge \boldsymbol{\gamma}')' \rangle \\ &= \langle \boldsymbol{\gamma}', V \wedge \boldsymbol{\gamma}' \rangle + \langle (1-a)\boldsymbol{\gamma} - b\boldsymbol{T} - d\boldsymbol{B}, k_1'\boldsymbol{N} + k_1\boldsymbol{N}' \rangle \\ &= k_1^2 b - k_1 k_2 d \\ &= 0,\end{aligned}$$

then $b = \frac{k_2(s_0)}{k_1(s_0)} d$, $a = \varepsilon \sqrt{1 - \frac{2k_2(s_0)}{k_1(s_0)} d^2}$, and

$$\boldsymbol{v} = \varepsilon \sqrt{1 - \frac{2k_2(s_0)}{k_1(s_0)} d^2}\, \boldsymbol{\gamma}(s_0) + \frac{k_2(s_0)}{k_1(s_0)} d\boldsymbol{T}(s_0) + d\boldsymbol{B}(s_0).$$

(3) When $h(s_0) = \nabla_{\gamma'} h(s_0) = \nabla_{\gamma'} \nabla_{\gamma'} h(s_0) = 0$, we obtion

$$\begin{aligned}\nabla_{\gamma'} \nabla_{\gamma'} h(s_0) &= \nabla_{\gamma'} \nabla_{\gamma'} \langle \boldsymbol{\gamma} - \boldsymbol{v}, V \wedge \boldsymbol{\gamma}' \rangle \\ &= \nabla_{\gamma'} \langle \boldsymbol{\gamma} - \boldsymbol{v}, (k_1\boldsymbol{N})' \rangle \\ &= \langle \boldsymbol{\gamma}', (k_1\boldsymbol{N})' \rangle + \langle \boldsymbol{\gamma} - \boldsymbol{v}, (k_1\boldsymbol{N})'' \rangle \\ &= -\varepsilon k_1^2 \sqrt{1 - \tfrac{2k_2}{k_1} d^2} + (k_1 k_2' - k_1' k_2) d \\ &= 0,\end{aligned}$$

we can obtain $d = \varepsilon \frac{k_1^2(s_0)}{\varrho(s_0)}$, where

$$\varrho(s_0) = \sqrt{2k_1^3(s_0)k_2(s_0) + (k_1(s_0)k_2'(s_0) - k_1'(s_0)k_2(s_0)^2)} \neq 0.$$

Hence,

$$\boldsymbol{v} = \varepsilon\Big(\frac{k_1(s_0)k_2'(s_0) - k_1'(s_0)k_2(s_0)}{\varrho(s_0)} \boldsymbol{\gamma}(s_0) + \frac{k_2(s_0)k_1(s_0)}{\varrho(s_0)} \boldsymbol{T}(s_0) + \frac{k_1^2(s_0)}{\varrho(s_0)} \boldsymbol{B}(s_0)\Big).$$

(4) When $h(s_0) = \nabla_{\gamma'} h(s_0) = \nabla_{\gamma'} \nabla_{\gamma'} h(s_0) = \nabla^{(3)}_{\gamma'} h(s_0) = 0$, we have

$$\begin{aligned}\nabla^{(3)}_{\gamma'} h_v(s_0) &= \nabla_{\gamma'} \nabla_{\gamma'} \nabla_{\gamma'} \langle \boldsymbol{\gamma} - \boldsymbol{v}, V \wedge \boldsymbol{\gamma}' \rangle \\ &= \nabla_{\gamma'} \langle \boldsymbol{\gamma}', (k_1\boldsymbol{N})' \rangle + \langle \boldsymbol{\gamma}(s) - \boldsymbol{v}, (k_1\boldsymbol{N})'' \rangle \\ &= \nabla_{\gamma'} (\langle \boldsymbol{\gamma}', k_1'\boldsymbol{N} - k_1^2\boldsymbol{B} + k_1k_2\boldsymbol{T} \rangle + \langle \boldsymbol{\gamma}(s) - \boldsymbol{v}, k_1^2\boldsymbol{\gamma} + (2k_1'k_2 + k_1k_2')\boldsymbol{T} + (k_1'' + 2k_1^2k_2)\boldsymbol{N} - 3k_1k_1'\boldsymbol{B} \rangle) \\ &= \langle k_1\boldsymbol{N}, k_1'\boldsymbol{N} - k_1^2\boldsymbol{B} + k_1k_2\boldsymbol{T} \rangle + 2\langle \boldsymbol{\gamma}', k_1^2\boldsymbol{\gamma} + (2k_1'k_2 + k_1k_2')\boldsymbol{T} + (k_1'' + 2k_1^2k_2)\boldsymbol{N} - 3k_1k_1'\boldsymbol{B} \rangle) \\ &+ \langle \boldsymbol{\gamma} - \boldsymbol{v}, 5k_1k_1'\boldsymbol{\gamma} + (k_1^2 + 3k_1''k_2 + 3k_1'k_2' + k_1k_2'' + 2k_1^2k_2^2)\boldsymbol{T} \rangle \\ &+ \langle \boldsymbol{\gamma} - \boldsymbol{v}, (9k_1k_1'k_2 + 3k_1^2k_2' + 2k_1''')\boldsymbol{N} - (2k_1^3k_2 + 4k_1k_1'' + 3k_1'^2)\boldsymbol{B} \rangle \\ &= \varepsilon[-5k_1k_2' \tfrac{k_1k_2' - k_1'k_2}{\varrho} - \tfrac{1}{\varrho}(k_1^4 + 3k_1^2k_1'k_2 + k_1^3k_2'' - k_1^2k_1''k_2 - 3k_1k_1'^2k_2)] \\ &= 0,\end{aligned}$$

we can obtain

$$\boldsymbol{v} = \varepsilon(\frac{k_1(s_0)k_2'(s_0) - k_1'(s_0)k_2(s_0)}{\varrho(s_0)}\boldsymbol{\gamma}(s_0) + \frac{k_2(s_0)k_1(s_0)}{\varrho(s_0)}\boldsymbol{T}(s_0) + \frac{k_1^2(s_0)}{\varrho(s_0)}\boldsymbol{B}(s_0)),$$

$\sigma(s_0) = 0$ and $\sigma'(s_0) \neq 0$.
(5) When $h(s_0) = \nabla_{\gamma'}h(s_0) = \nabla_{\gamma'}\nabla_{\gamma'}h(s_0) = \nabla^{(3)}_{\gamma'}h(s_0) = \nabla^{(4)}_{\gamma'}h(s_0) = 0$,

$$\begin{aligned}
&\nabla^{(4)}_{\gamma'}h_v(s_0) = \nabla_{\gamma'}\nabla_{\gamma'}\nabla_{\gamma'}\nabla_{\gamma'}\langle\boldsymbol{\gamma} - \boldsymbol{v}, V \wedge \boldsymbol{\gamma}'\rangle = \langle k_1'\boldsymbol{N} - k_1^2\boldsymbol{B} + k_1k_2\boldsymbol{T}, k_1'\boldsymbol{N} - k_1^2\boldsymbol{B} + k_1k_2\boldsymbol{T}\rangle \\
&+3\langle k_1\boldsymbol{N}, k_1^2\boldsymbol{\gamma} + (2k_1'k_2 + k_1k_2')\boldsymbol{T} + (k_1'' + 2k_1^2k_2)\boldsymbol{N} - 3k_1k_1'\boldsymbol{B}\rangle \\
&+3\langle\boldsymbol{\gamma}'(s), (9k_1k_1'k_2 + 3k_1^2k_2' + 2k_1''')\boldsymbol{N} - (2k_1^3k_2 + 4k_1k_1'' + 3k_1'^2)\boldsymbol{B}\rangle \\
&+\langle\boldsymbol{\gamma} - \boldsymbol{v}, (8k_1'^2 + 9k_1k_1'' + 2k_1^3k_2)\boldsymbol{\gamma}(s) + (7k_1k_1' + 5k_1'''k_2 + 6k_1''k_2' + 4k_1'k_2'' + k_1k_2''' + 13k_1k_2k_2^2 + 7k_1^2k_2k_2')\boldsymbol{T}\rangle \\
&+\langle\boldsymbol{\gamma} - \boldsymbol{v}, (k_1^3 + 18k_1k_1''k_2 + 12k_1k_1'k_2' + 4k_1^2k_2'' + 4k_1^3k_2^2 + 12k_1'^2k_2 + 2k_1^{(4)})\boldsymbol{N}\rangle \\
&-\langle\boldsymbol{\gamma} - \boldsymbol{v}, (15k_1^2k_1'k_2 + 6k_1k_1''' + 5k_1^3k_2' + 10k_1k_1'')\boldsymbol{B}\rangle \\
&= -\varepsilon\tfrac{k_1k_2' - k_1'k_2}{\varrho(s_0)} - \varepsilon\tfrac{k_1^2}{\varrho(s_0)}(7k_1k_1' + 5k_1'''k_2 + 6k_1''k_2' + 4k_1'k_2'' + k_1k_2''' + 13k_1k_2k_2^2 + 7k_1^2k_2k_2') \\
&+\varepsilon\tfrac{k_2k_1}{\varrho(s_0)}(15k_1^2k_1'k_2 + 6k_1k_1''' + 5k_1^3k_2' + 10k_1k_1'') \\
&= 0,
\end{aligned}$$

we can obtain

$$\boldsymbol{v} = \varepsilon(\frac{k_1(s_0)k_2'(s_0) - k_1'(s_0)k_2(s_0)}{\varrho(s_0)}\boldsymbol{\gamma}(s_0) + \frac{k_2(s_0)k_1(s_0)}{\varrho(s_0)}\boldsymbol{T}(s_0) + \frac{k_1^2(s_0)}{\varrho(s_0)}\boldsymbol{B}(s_0)),$$

and $\sigma(s_0) = \sigma'(s_0) = 0$. □

5. Unfolding of functions

Based on the unfolding theory of the height function germ, we prove Theorem 3.1. It is described in detail in the book [17, 20].

Here, a significant set concerning the unfolding is given. The *discriminant set* of F is the set

$$\mathfrak{D}_F = \{x \in \mathbb{R}^r | \text{ there exists } s \text{ with } F = \frac{\partial F}{\partial s} = 0 \text{ at } (s, x)\}.$$

We get the vital result [17]. This result is the singular classification theorem, which is the same as Theorem 8. By Proposition 4.1, the discriminant set of $H(s, \boldsymbol{v}) = \langle\boldsymbol{\gamma}(s) - \boldsymbol{v}, V \wedge \boldsymbol{\gamma}'(s)\rangle$ is the set

$$\mathfrak{D}_H = \{\boldsymbol{v} = \phi(s) + \mu\boldsymbol{B}(s) | s, \mu \in \mathbb{R}\}.$$

Then, the following results are proved.

Theorem 5.1. *For a curve $\boldsymbol{\gamma}(s)$, when $\varrho(s_0) \neq 0$, $H(s, \boldsymbol{v}) : I \times \mathbb{S}^3_1 \to \mathbb{R}$ be the height function of $\boldsymbol{\gamma}(s)$ and $\boldsymbol{v} \in \mathfrak{D}_H$. $H(s, \boldsymbol{v})$ is a versal unfolding of h_{v_0}, when h_{v_0} has A_k-singularity at s ($k = 1, 2, 3$).*

Proof. Assuming that $\boldsymbol{\gamma}(s) = \{\gamma_1(s), \gamma_2(s), \gamma_3(s), \gamma_4(s)\}$,
$\boldsymbol{N}(s) = \{N_1, N_2, N_3, N_4\}, \boldsymbol{v}(s) = \{v_1, v_2, v_3, v_4\}$ in $\mathbb{S}^3_1$, where

$$v_1 = \varepsilon\sqrt{v_2^2 + v_3^2 + v_4^2 - 1},$$

then

$H(s,\boldsymbol{v}) = \langle \boldsymbol{\gamma} - \boldsymbol{v}, V \wedge \boldsymbol{\gamma}' \rangle = \langle -\boldsymbol{v}, V \wedge \boldsymbol{\gamma}' \rangle = -\langle \boldsymbol{v}, k_1 \boldsymbol{N} \rangle = -k_1(-\varepsilon\sqrt{v_2^2 + v_3^2 + v_4^2 - 1}N_1 + v_2N_2 + v_3N_3 + v_4N_4).$

We have

$$\frac{\partial H(s,\boldsymbol{v})}{\partial v_i} = k_1(\varepsilon \frac{v_i}{v_1} N_1 - N_i),$$

where $i = 2, 3, 4$, thus

$$\frac{\partial}{\partial s}(\frac{\partial H(s,\boldsymbol{v})}{\partial v_i}) = \varepsilon \frac{v_i}{v_1}(k_1' N_1 + k_1 N') - k_1' N_i - k_1 N_i',$$

$$\frac{\partial^2}{\partial s^2}(\frac{\partial H(s,\boldsymbol{v})}{\partial v_i}) = \varepsilon \frac{v_i}{v_1}(k_1'' N_1 + 2k_1' N_1' + k_1 N_1'') - (k_1'' N_i + 2k_1' N_i' + k_1 N_i''),$$

therefore, we have the 2-jet of $\frac{\partial H(s,\boldsymbol{v})}{\partial v_i}$ at s_0 as follows:

$$k_1(\varepsilon \tfrac{v_i}{v_1} N_1 - N_i) + (\varepsilon \tfrac{v_i}{v_1}(k_1' N_1 + k_1 N_1') - k_1' N_i - k_1 N_i')(s - s_0) + \tfrac{1}{2!}(\varepsilon \tfrac{v_i}{v_1}(k_1'' N_1 + 2k_1' N_1' + k_1 N_1'') \\ -(k_1'' N_i + 2k_1' N_i' + k_1 N_i''))(s - s_0)^2.$$

We now define

$$A = \begin{pmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{pmatrix},$$

where $a_i = k_1(\varepsilon \frac{v_i}{v_1} N_1 - N_i)$, $b_i = \varepsilon \frac{v_i}{v_1}(k_1' N_1 + k_1 N_1') - k_1' N_i - k_1 N_i'$, $c_i = \varepsilon \frac{v_i}{v_1}(k_1'' N_1 + 2k_1' N_1' + k_1 N_1'') - (k_1'' N_i + 2k_1' N_i' + k_1 N_i'')$, $i = 2, 3, 4$.
Then,

$$\begin{aligned} det(A) &= k_1^3 \begin{vmatrix} \varepsilon \frac{v_2}{v_1} N_1 - N_2 & \varepsilon \frac{v_3}{v_1} N_1 - N_3 & \varepsilon \frac{v_4}{v_1} N_1 - N_4 \\ \varepsilon \frac{v_2}{v_1} N_1' - N_2' & \varepsilon \frac{v_3}{v_1} N_1' - N_3' & \varepsilon \frac{v_4}{v_1} N_1' - N_4' \\ \varepsilon \frac{v_2}{v_1} N_1'' - N_2'' & \varepsilon \frac{v_3}{v_1} N_1'' - N_3'' & \varepsilon \frac{v_4}{v_1} N_1'' - N_4'' \end{vmatrix} \\ &= -\varepsilon \frac{k_1^3}{v_1}(v, \boldsymbol{N} \wedge \boldsymbol{N}' \wedge \boldsymbol{N}'') \\ &= -\varepsilon \frac{k_1^3}{\sqrt{v_2^2 + v_3^2 + v_4^2 - 1}}(\boldsymbol{v}, \boldsymbol{N} \wedge \boldsymbol{N}' \wedge \boldsymbol{N}''). \end{aligned}$$

Since

$$\boldsymbol{N} \wedge \boldsymbol{N}' \wedge \boldsymbol{N}'' = (-k_1 k_2' + k_1' k_2)\boldsymbol{\gamma}(s) - k_1^2 \boldsymbol{T}(\boldsymbol{s}) - k_2 k_1 \boldsymbol{B}(\boldsymbol{s}),$$

and

$$\boldsymbol{v} = \varepsilon(\frac{k_1 k_2' - k_1' k_2}{\varrho(s)}\boldsymbol{\gamma}(s) + \frac{k_2 k_1}{\varrho(s)}\boldsymbol{T}(s) + \frac{k_1^2}{\varrho}\boldsymbol{B}(s)),$$

where $\boldsymbol{v} \in \mathfrak{D}_H$ is a singular point. Thus

$$det(A) = k_1^3 \frac{(-k_1 k_2' + k_1' k_2)^2 + k_1^4 + k_1^2 k_2^2}{v_1 \varrho} = k_1^3 \frac{(-k_1 k_2' + k_1' k_2)^2 + k_1^4 + k_1^2 k_2^2}{\varrho\sqrt{v_2^2 + v_3^2 + v_4^2 - 1}} \neq 0,$$

then, the rank of A is equal to 3. So the theorem holds. □

Proof of Theorem 3.1. For the curve $\boldsymbol{\gamma}(s)$, when $\varrho(s_0) \neq 0$. Suppose $\boldsymbol{v}_0 = \mathcal{BNS}(s_0, \boldsymbol{u}_0)$, A function $\mathfrak{H} : \mathcal{NMB}(\boldsymbol{v}_0) \to \mathbb{R}$ is defined by $\mathfrak{H}(u) = \langle \boldsymbol{u} - \boldsymbol{v}_0, V \wedge \boldsymbol{u} \rangle$. Thus, we get $h_{\boldsymbol{v}_0} = \mathfrak{H}(\boldsymbol{\gamma}(s))$. Because 0 and $\mathfrak{H}^{-1}(0) = \mathcal{NMB}(\boldsymbol{v}_0)$ is a regular value of $\mathfrak{H}$, as for $\boldsymbol{\gamma}(s)$ and $\mathcal{NMB}(\boldsymbol{v}_0)$, there are $(k+1)$-point contact at s_0 ($k = 1, 2, 3$) if and only if $h_{\boldsymbol{v}_0}$ has A_k-singularity at s_0. Apply the singularity theory [17], we complete the proof by the conclusion of Proposition 4.1 and Theorem 5.1.

6. The local structure of the lightlike magnetic curve

Here, the local structure of curve $\boldsymbol{\gamma}(s)$ in $\mathbb{S}^3_1$ with $\boldsymbol{V}$ is considered. The Taylor's formula of $\boldsymbol{\gamma}(s_0)$ at $s = s_0$ is given by

$$\boldsymbol{\gamma}(s_0 + \triangle s) - \boldsymbol{\gamma}(s_0) = \boldsymbol{\gamma}'(s_0)\triangle s + \tfrac{1}{2!}\boldsymbol{\gamma}''(s_0)(\triangle s)^2 + \tfrac{1}{3!}\boldsymbol{\gamma}'''(s_0)(\triangle s)^3 + \tfrac{1}{4!}(\boldsymbol{\gamma}''''(s_0) + \boldsymbol{\varepsilon})(\triangle s)^4,$$

where $\lim\limits_{\triangle s \to 0} \boldsymbol{\varepsilon} = \mathbf{0}$.

Since

$$\begin{aligned}
\boldsymbol{\gamma}'(s) &= \boldsymbol{T}(s),\\
\boldsymbol{\gamma}''(s) &= k_1(s)\boldsymbol{N}(s),\\
\boldsymbol{\gamma}'''(s) &= k_1'(s)\boldsymbol{N}(s) + k_1(s)(-k_1(s)\boldsymbol{B}(s) + k_2(s)\boldsymbol{T}(s) = k_1(s)k_2(s)\boldsymbol{T}(s) + k_1'(s)\boldsymbol{N}(s) - k_1^2(s)\boldsymbol{B}(s),\\
\boldsymbol{\gamma}''''(s) &= k_1^2(s)\boldsymbol{\gamma}(s) + (2k_1'(s)k_2(s) + k_1(s)k_2'(s))\boldsymbol{T}(s) + (k_1''(s) + 2k_1^2(s)k_2(s))\boldsymbol{N}(s) - 3k_1(s)k_1'(s)\boldsymbol{B}(s).
\end{aligned}$$

Then

$$\begin{aligned}
&\boldsymbol{\gamma}(s_0 + \triangle s) - \boldsymbol{\gamma}(s_0)\\
&= \boldsymbol{T}(s_0)\triangle s + \tfrac{1}{2!}k_1(s_0)\boldsymbol{N}(s_0)(\triangle s)^2 + \tfrac{1}{3!}(k_1(s_0)k_2(s_0)\boldsymbol{T}(s_0) + k_1'(s_0)\boldsymbol{N}(s_0) - k_1^2\boldsymbol{B}(s_0))(\triangle s)^3\\
&+\tfrac{1}{4!}(k_1^2(s_0)\boldsymbol{\gamma}(s_0) + (2k_1'(s_0)k_2(s_0) + k_1(s_0)k_2'(s_0))\boldsymbol{T}(s_0) + (k_1''(s_0) + 2k_1^2(s_0)k_2(s_0))\boldsymbol{N}(s_0)\\
&-3k_1(s_0)k_1'(s_0)\boldsymbol{B}(s_0) + \boldsymbol{\varepsilon})(\triangle s)^4,
\end{aligned}$$

where $\boldsymbol{\varepsilon} = \varepsilon_1\boldsymbol{\gamma}(s_0) + \varepsilon_2\boldsymbol{T}(s_0) + \varepsilon_3\boldsymbol{N}(s_0) + \varepsilon_4\boldsymbol{B}(s_0)$.

Therefore, we have

$$\begin{aligned}
\boldsymbol{\gamma}(s_0 + \triangle s) - \boldsymbol{\gamma}(s_0) &= \tfrac{1}{24}(k_1^2(s_0) + \varepsilon_1)(\triangle s)^4\boldsymbol{\gamma}(s_0)\\
&+(\triangle s + \tfrac{1}{6}k_1(s_0)k_2(s_0)(\triangle s)^3 + \tfrac{1}{24}(2k_1'(s_0)k_2(s_0) + k_1(s_0)k_2'(s_0) + \varepsilon_2)(\triangle s)^4)\boldsymbol{T}(s_0)\\
&+(\tfrac{1}{2}k_1(s_0)(\triangle s)^2 + \tfrac{1}{6}k_1'(s_0)(\triangle s)^3 + \tfrac{1}{24}(k_1''(s_0) + 2k_1^2(s_0)k_2(s_0) + \varepsilon_3)(\triangle s)^4)\boldsymbol{N}(s_0)\\
&+(-\tfrac{1}{6}k_1^2(s_0)(\triangle s)^3 + \tfrac{1}{24}(-3k_1(s_0)k_1'(s_0) + \varepsilon_4)(\triangle s)^4)\boldsymbol{B}(s_0).
\end{aligned}$$

When $\boldsymbol{\varepsilon} \to 0$, taking the first term in each of the coefficients of $\boldsymbol{T}(s_0), \boldsymbol{N}(s_0)$ and $\boldsymbol{B}(s_0)$, we obtain

$$\boldsymbol{\gamma}(s_0 + \triangle s) - \boldsymbol{\gamma}(s_0) = \frac{1}{24}k_1^2(s_0)(\triangle s)^4\boldsymbol{\gamma}(s_0) + \triangle s\boldsymbol{T}(s_0) + \frac{1}{2}k_1(s_0)(\triangle s)^2\boldsymbol{N}(s_0) - \frac{1}{6}k_1^2(s_0)(\triangle s)^3\boldsymbol{B}(s_0).$$

We suppose $\{\alpha, \xi, \zeta, \eta\}$ be the coordinates adjacent to $\boldsymbol{\gamma}(s_0)$, then

$$\begin{cases} \alpha = \frac{1}{24}k_1^2 s^4\\ \xi = s\\ \zeta = \frac{1}{2}k_1 s^2\\ \eta = -\frac{1}{6}k_1^2 s^3, \end{cases} \tag{6.1}$$

which can be regarded as an approximate equation for the structure of the curve $\boldsymbol{\gamma} = \boldsymbol{\gamma}(s)$ near the point $\boldsymbol{\gamma}(s_0)$. We can obtain the shape of the curve near the point s_0 is completely determined by $k_1(s_0)$.

We consider the local structure of curve $\boldsymbol{\gamma}(s)$ at $\boldsymbol{\gamma}(s_0)$:

(1) the local structure of $\boldsymbol{\gamma}(s)$ at $\boldsymbol{\gamma}(s_0)$ onto the tangent space ($\alpha = 0$) is

$$(\alpha = 0), \xi = s, \zeta = \frac{1}{2}k_1 s^2, \eta = -\frac{1}{6}k_1^2 s^3,$$

as shown in the figure (see Figure 3).

(2) the local structure of $\boldsymbol{\gamma}(s)$ at $\boldsymbol{\gamma}(s_0)$ onto the principal normal vector space ($\xi = 0$) is

$$(\xi = 0), \alpha = \frac{1}{24}k_1^2 s^4, \zeta = \frac{1}{2}k_1 s^2, \eta = -\frac{1}{6}k_1^2 s^3,$$

as shown in the figure (see Figure 4).

(3) the local structure of $\boldsymbol{\gamma}(s)$ at $\boldsymbol{\gamma}(s_0)$ onto the 1st binormal normal vector space ($\zeta = 0$) is

$$(\zeta = 0), \alpha = \frac{1}{24}k_1^2 s^4, \xi = s, \eta = -\frac{1}{6}k_1^2 s^3,$$

as shown in the figure (see Figure 5).

(4) the local structure of $\boldsymbol{\gamma}(s)$ at $\boldsymbol{\gamma}(s_0)$ onto the 2nd binormal vector space ($\eta = 0$) is

$$(\eta = 0), \alpha = \frac{1}{24}k_1^2 s^4, \xi = s, \zeta = \frac{1}{2}k_1 s^2,$$

as shown in the figure (see Figure 6).

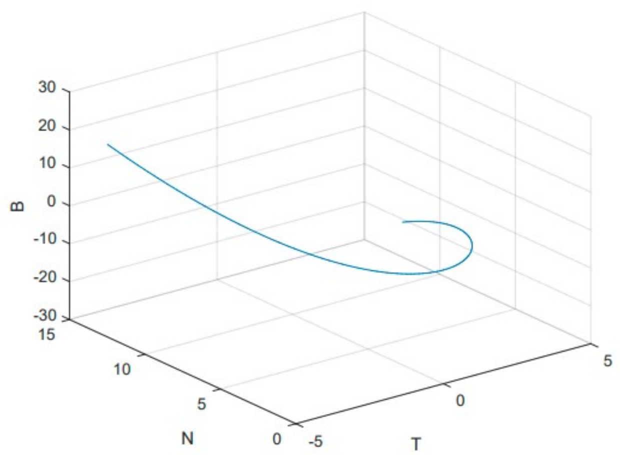

Figure 3. the projection of the tangent space $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$.

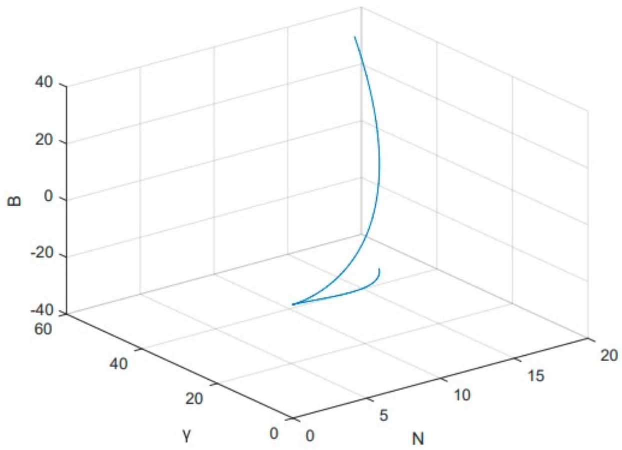

Figure 4. the projection of the principal normal vector space $\{\boldsymbol{\gamma}, \boldsymbol{N}, \boldsymbol{B}\}$.

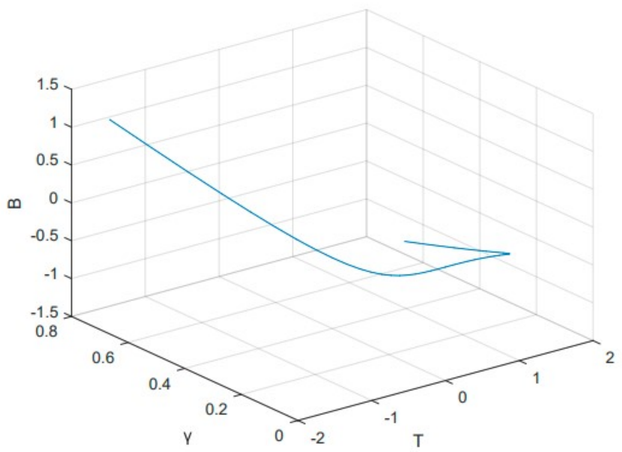

Figure 5. the projection of the 1st binormal normal vector space $\{\boldsymbol{\gamma}, \boldsymbol{T}, \boldsymbol{B}\}$.

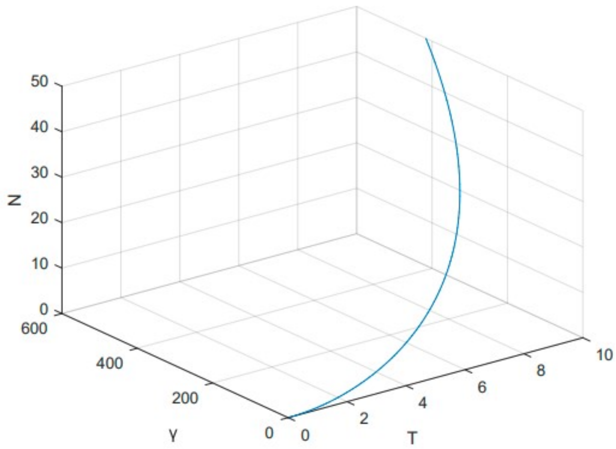

Figure 6. the projection of the 2nd binormal normal vector space $\{\boldsymbol{\gamma}, \boldsymbol{T}, \boldsymbol{N}\}$.

7. Examples

Here, we consider an example of $\boldsymbol{\gamma}(s)$ in $\mathbb{S}_1^3$ with $\boldsymbol{V}$ to show the major results of the paper. The graphics of $\boldsymbol{\gamma}(s)$, the tangent indicatrix of $\boldsymbol{\gamma}(s)$, the binormal lightlike surface, and the singularities are given in the following text.

Since it's impossible to draw a graphic in four dimensions, we only show projections to the 3-dimension tangent space generated by $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$.

Example 7.1. *Let $\boldsymbol{\gamma} : I \to \mathbb{S}_1^3$ be a lightlike Killing magnetic curve with* **V** *as follows:*

$$\boldsymbol{\gamma}(s) = \{\frac{\sqrt{2}}{2}\sinh s, \frac{\sqrt{2}}{2}\sin s, \frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}\cosh s\},$$

and

$$\boldsymbol{\gamma}'(s) = \{\frac{\sqrt{2}}{2}\cosh s, \frac{\sqrt{2}}{2}\cos s, -\frac{\sqrt{2}}{2}\sin s, \frac{\sqrt{2}}{2}\sinh s\},$$

$$\boldsymbol{\gamma}''(s) = \{\frac{\sqrt{2}}{2}\sinh s, -\frac{\sqrt{2}}{2}\sin s, -\frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}\cosh s\},$$

$$\boldsymbol{B}(s) = \{-\sqrt{2}\cosh s, \sqrt{2}\sin s, \sqrt{2}\cos s, -\sqrt{2}\sinh s\}.$$

By calculation, we can get $k_1 = 1, k_2 = 1$, then

$$\boldsymbol{T}(s) = \{\frac{\sqrt{2}}{2}\cosh s, \frac{\sqrt{2}}{2}\cos s, -\frac{\sqrt{2}}{2}\sin s, \frac{\sqrt{2}}{2}\sinh s\},$$

$$N(s) = \{\frac{\sqrt{2}}{2}\sinh s, -\frac{\sqrt{2}}{2}\sin s, -\frac{\sqrt{2}}{2}\cos s, \frac{\sqrt{2}}{2}\cosh s\}.$$

Hence, the tangent indicatrix of $\boldsymbol{\gamma}(s)$:

$$\Phi(s) = \varepsilon\frac{1}{2}\{\cosh s, \cos s, -\sin s, \sinh s\},$$

and the binormal lightlike surface :

$$\mathcal{BNS}(s,\mu) = \{(-\sqrt{2}\mu + \varepsilon\frac{1}{2})\cosh s,\ \sqrt{2}\mu\sin s + \varepsilon\frac{1}{2}\cos s,\ \sqrt{2}\mu\cos s - \varepsilon\frac{1}{2}\sin s, (-\sqrt{2}\mu + \varepsilon\frac{1}{2})\sinh s\},$$

then the projection of $\boldsymbol{\gamma}(s)$ (see Figure 7), the tangent indicatrix of $\boldsymbol{\gamma}(s)$ (see Figure 8), and the binormal lightlike surface (see Figure 9) onto the 3-dimension tangent space as follows.

The singular locus of the binormal lightlike surface onto the 3-dimension tangent space (see Figure 10) is

$$\varepsilon\{-\frac{1}{2}\cosh s, \sin s + \frac{1}{2}\cos s, \cos s - \frac{1}{2}\sin s, -\frac{1}{2}\sinh s\}.$$

By calculation, we can obtain the geometric invariant $\sigma(s) = \pm\frac{\sqrt{2}}{2}$, and $\sigma'(s) = 0$ for all s.

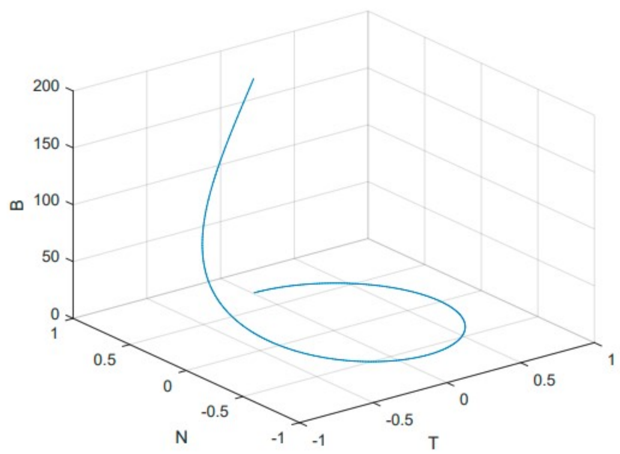

Figure 7. the projection of $\boldsymbol{\gamma}(s)$ tothe 3-dimension tangent space $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$.

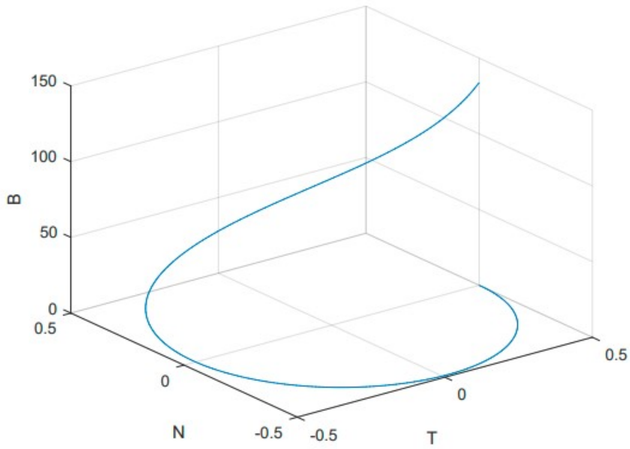

Figure 8. the tangent indicatrix of $\gamma(s)$ onto the 3-dimension tangent space $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$.

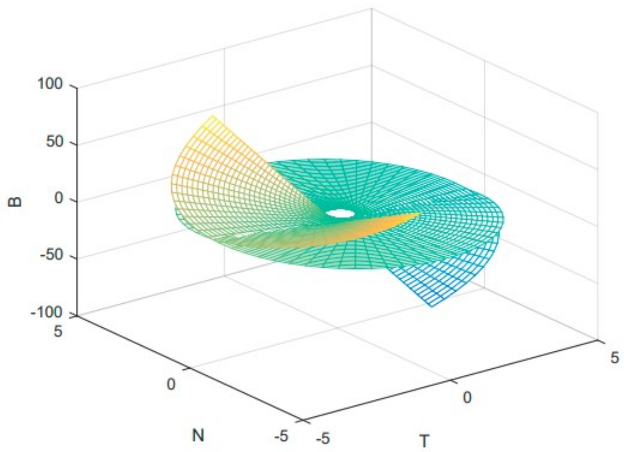

Figure 9. the binormal lightlike surface onto the 3-dimension tangent space $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$.

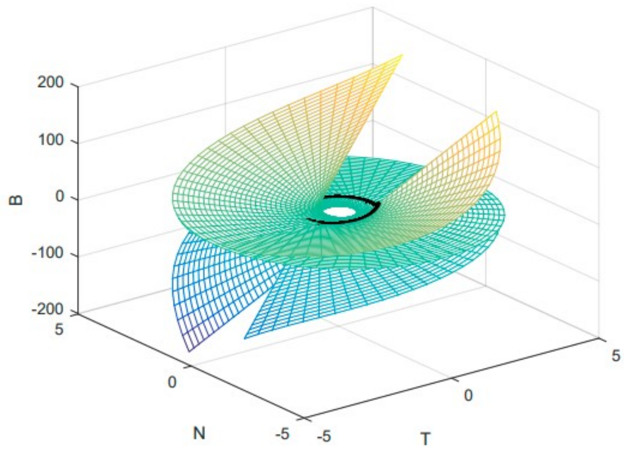

Figure 10. the binormal lightlike surface and its singular locus (black curve) onto the tangent space $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$.

8. Conclusions

In the previous paper [14, 15], we had obtained the singularities of the Killing and null Killing magnetic curves in Minkowski space. In this paper, we observed the singularity properties of the lightlike Killing magnetic curves in $\mathbb{S}^3_1$, which were degenerate curves. By considering the lightlike tangent vector, we constructed a new Frenet equations of a lightlike Killing magnetic curve $\gamma(s)$ using the transversality theorem. Under the view of the contact theory, we obtain some local geometric properties of the lightlike Killing magnetic curves in $\mathbb{S}^3_1$. We made a profound research on the contact between rectifying surface and basic sphere space by the height functions. And we rightfully obtained the geometric invariants of a lightlike Killing magnetic curve, which is used to describe the properties of lightlike Killing magnetic curve. In the following research, some other local geometrical properties of the lightlike Killing magnetic curve in nullcone will be considered.

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Conflict of interest

The authors declare no conflict of interest.

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