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Research article

Sums of finite products of Chebyshev polynomials of two different types

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Abstract: In this paper, we consider sums of finite products of the second and third type Chebyshev polynomials, those of the second and fourth type Chebyshev polynomials and those of the third and fourth type Chebyshev polynomials, and represent each of them as linear combinations of Chebyshev polynomials of all types. Here the coefficients involve some terminating hypergeometric functions ${}_2F_1$. This problem can be viewed as a generalization of the classical linearization problems and is done by explicit computations.

Keywords: Chebyshev polynomials of the first, second, third and fourth kinds; terminating hypergeometric function
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1. Introduction

The linearization problem in general consists in determining the coefficients $c_{nm}(k)$ in the expansion of the product of two polynomials $q_n(x)$ and $r_m(x)$ in terms of an arbitrary polynomial sequence $\{p_k(x)\}_{k\geq 0}$:

$$q_n(x)r_m(x) = \sum_{k=0}^{n+m} c_{nm}(k)p_k(x).$$

A special problem of this is the case when $p_n(x) = q_n(x) = r_n(x)$, which is called either the standard linearization or Clebsch-Gordan-type problem.

Let $T_n(x), U_n(x), V_n(x)$ and $W_n(x)$ denote respectively the Chebyshev polynomials of the first, second, third and fourth kinds (see (1.7)–(1.18)). In this paper, we will consider the sums of finite products of Chebyshev polynomials of two different types $\alpha_{n,r,s}(x)$, $\beta_{n,r,s}(x)$ and $\gamma_{n,r,s}(x)$ in (1.23)–(1.25), and express each of them as linear combinations of Chebyshev polynomials of all kinds. Obviously, studying such sums of finite products can be viewed as a generalization of the linearization problem.

As another motivation for our study, we would like to mention the following. Let us put

$$\epsilon_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(x) B_{m-k}(x), \quad (m \geq 2),$$
$$\langle x \rangle = x - [x], \text{ for any real number } x,$$

where $B_n(x)$ are the Bernoulli polynomials given by $\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$.

Then, as it was noted in Introduction of [14], from the Fourier expansion of $\epsilon_m(\langle x \rangle)$ we can derive the following polynomial identity

$$\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(x) B_{m-k}(x) = \frac{2}{m} \sum_{k=0}^{m-2} \frac{1}{m-k} \binom{m}{k} B_{m-k} B_k(x) + \frac{2}{m} H_{m-1} B_m(x), \ (m \geq 2), \tag{1.1}$$

where $H_m = \sum_{j=1}^{m} \frac{1}{j}$ are the harmonic numbers.

Further, simple modification of (1.1) gives us

$$\begin{aligned} &\sum_{k=1}^{m-1} \frac{1}{2k(2m-2k)} B_{2k}(x) B_{2m-2k}(x) + \frac{2}{2m-1} B_1(x) B_{2m-1}(x) \\ &= \frac{1}{m} \sum_{k=1}^{m} \frac{1}{2k} \binom{2m}{2k} B_{2k} B_{2m-2k}(x) + \frac{1}{m} H_{2m-1} B_{2m}(x) + \frac{2}{2m-1} B_{2m-1} B_1(x), \quad (m \geq 2), \end{aligned} \tag{1.2}$$

Letting $x = \frac{1}{2}$ and $x = 0$ in (1.2) yield respectively the famous Faber-Pandharipande-Zagier identity (see [5]) and a slight variant of the well-known Miki's identity (see [4, 6, 17, 18]). Here, we emphasize that our methods are simple at the level of Fourier series expansions (see [1, 13, 14] and references therein), whereas others are quite involved. Indeed, Dunne-Schubert in [4] uses the asymptotic expansions of some special polynomials coming from quantum field theory computations, Gessel in [6] relies on two expansions for the Stirling numbers of the second kind, Miki in [17] utilizes a formula for the Fermat quotient $\frac{a^p-a}{p}$ modulo p^2, and Shiratani-Yokoyama in [18] employs p-adic analysis. A proof of the FPZ identity was also given by Zagier in the appendix of [5].

We will briefly mention some related previous works before we move on. Certain sums of finite products of special polynomials, like Bernoulli, Euler, Genocchi, all kinds of Chebyshev, Legendre, Laguerre, Fibonacci and Lucas polynomials, were represented in terms of Bernoulli polynomials (see [1, 13, 15] and references therein). All of these were derived from the Fourier series expansions of the functions closely related to these polynomials.

As to representations in terms of Chebyshev polynomials, some sums of finite products of special polynomials, such as all kinds of Chebyshev, Legendre, Laguerre, Fibonacci and Lucas polynomials, were expressed in terms of Chebyshev polynomials of all kinds (see [10, 12] and references therein). All of these were obtained by explicit computations, just as we will do in the present paper.

More recent related works are those on balancing and Lucas-balancing polynomials (see [8, 11] and references therein). Since the introduction about twenty years ago by Behera and Panda, the balancing numbers have been studied intensively and many interesting properties of them have been discovered. The balancing polynomials are natural generalizations of the balancing numbers. In [8], sums of finite products of balancing polynomials are represented in terms of nine orthogonal polynomials. Lucas-balancing numbers have close connection with balancing numbers and their natural extensions are Lucas-balancing polynomials. In [11], sums of finite products of Lucas-balancing polynomials are expressed in terms of nine orthogonal polynomials in two different ways each. The proofs of [8] and those of [11] are respectively based on a relation between balancing polynomials and Chebyshev polynomials of the second kind, and on that between Lucas-balancing polynomials and Chevbyshev polynomials of the first kind, which are observed by Frontczak.

We emphasize here that, whereas all of the results so far treated sums of finite products of some polynomials of single type, this paper takes care of those of two different types.

In the rest of this section, we will fix notations that will be used throughout this paper and recall some basic facts about Chebyshev polynomials. Then we will state our main results.

Let n be any nonnegative integer. Then the falling factorial polynomials $(x)_n$ and the rising factorial polynomials $\langle x\rangle_n$ are respectively defined by

$$(x)_n = x(x-1)\cdots(x-n+1), \quad (n \geq 1), \quad (x)_0 = 1, \tag{1.3}$$

$$\langle x\rangle_n = x(x+1)\cdots(x+n-1), \quad (n \geq 1), \quad \langle x\rangle_0 = 1. \tag{1.4}$$

It is immediate to see that the two factorial polynomials are related by

$$(-1)^n (x)_n = \langle -x\rangle_n, \quad (-1)^n \langle x\rangle_n = (-x)_n. \tag{1.5}$$

The Gauss hypergeometric function ${}_2F_1(a, b; c; x)$ are given by

$${}_2F_1(a, b : c : x) = \sum_{n=0}^{\infty} \frac{\langle a\rangle_n \langle b\rangle_n}{\langle c\rangle_n} \frac{x^n}{n!}, \quad (|x| < 1). \tag{1.6}$$

In below, we will briefly recall some very basic facts about Chebyshev polynomials of the first, second, third and fourth kinds. For further details on this family of orthogonal polynomials, we let the readers refer to the standard books [2, 3, 16].

The Chebyshev polynomials of the first, second, third and fourth kinds are respectively given by the generating functions as follows:

$$\frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x)t^n, \tag{1.7}$$

$$F(t, x) = \frac{1}{1-2xt+t^2} = \sum_{n=0}^{\infty} U_n(x)t^n, \tag{1.8}$$

$$\frac{1-t}{1-2xt+t^2} = \sum_{n=0}^{\infty} V_n(x)t^n, \tag{1.9}$$

$$\frac{1+t}{1-2xt+t^2} = \sum_{n=0}^{\infty} W_n(x)t^n, \tag{1.10}$$

They are also explicitly given in terms of Gauss hypergeometric functions as in the following:

$$T_n(x) = {}_2F_1(-n, n; \frac{1}{2}; \frac{1-x}{2}) \tag{1.11}$$
$$= \frac{n}{2}\sum_{l=0}^{[\frac{n}{2}]}(-1)^l \frac{1}{n-l}\binom{n-l}{l}(2x)^{n-2l}, \quad (n \geq 1),$$
$$U_n(x) = (n+1){}_2F_1(-n, n+2; \frac{3}{2}; \frac{1-x}{2}) \tag{1.12}$$
$$= \sum_{l=0}^{[\frac{n}{2}]}(-1)^l \binom{n-l}{l}(2x)^{n-2l}, \quad (n \geq 0),$$
$$V_n(x) = {}_2F_1(-n, n+1; \frac{1}{2}; \frac{1-x}{2}) \tag{1.13}$$
$$= \sum_{l=0}^{n}\binom{n+l}{2l}2^l(x-1)^l, \quad (n \geq 0),$$
$$W_n(x) = (2n+1){}_2F_1(-n, n+1; \frac{3}{2}; \frac{1-x}{2}) \tag{1.14}$$
$$= (2n+1)\sum_{l=0}^{n}\frac{2^l}{2l+1}\binom{n+l}{2l}(x-1)^l, \quad (n \geq 0).$$

In addition, the Chebyshev polynomials of the first, second, third and fourth kinds are given by the following Rodrigues' formulas:

$$T_n(x) = \frac{(-1)^n 2^n n!}{(2n)!}(1-x^2)^{\frac{1}{2}}\frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}}, \tag{1.15}$$
$$U_(x) = \frac{(-1)^n 2^n (n+1)!}{(2n+1)!}(1-x^2)^{-\frac{1}{2}}\frac{d^n}{dx^n}(1-x^2)^{n+\frac{1}{2}}, \tag{1.16}$$
$$(1-x)^{-\frac{1}{2}}(1+x)^{\frac{1}{2}}V_n(x) = \frac{(-1)^n 2^n n!}{(2n)!}\frac{d^n}{dx^n}(1-x)^{n-\frac{1}{2}}(1+x)^{n+\frac{1}{2}}, \tag{1.17}$$
$$(1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}W_n(x) = \frac{(-1)^n 2^n n!}{(2n)!}\frac{d^n}{dx^n}(1-x)^{n+\frac{1}{2}}(1+x)^{n-\frac{1}{2}}. \tag{1.18}$$

As is well known, they satisfy the following orthogonality relations with respect to various weight functions:

$$\int_{-1}^{1}(1-x^2)^{-\frac{1}{2}}T_n(x)T_m(x)dx = \frac{\pi}{\epsilon_n}\delta_{n,m}, \tag{1.19}$$
$$\int_{-1}^{1}(1-x^2)^{\frac{1}{2}}U_n(x)U_m(x)dx = \frac{\pi}{2}\delta_{n,m}, \tag{1.20}$$
$$\int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}V_n(x)V_m(x)dx = \pi\delta_{n,m}, \tag{1.21}$$
$$\int_{-1}^{1}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}}W_n(x)W_m(x)dx = \pi\delta_{n,m}, \tag{1.22}$$

where

$$\epsilon_n = \begin{cases} 1, & \text{if } n = 0, \\ 2, & \text{if } n \geq 1, \end{cases} \qquad \delta_{n,m} = \begin{cases} 0, & \text{if } n \neq m, \\ 1, & \text{if } n = m. \end{cases}$$

Let n, r, s be nonnegative integers with $r + s \geq 1$. Then we will consider the following sums of finite products of Chebyshev polynomials of two different types:

$$\alpha_{n,r,s}(x) = \sum_{i_1+\cdots i_r+j_1+\cdots+j_s=n} U_{i_1}(x)\cdots U_{i_r}(x)V_{j_1}(x)\cdots V_{j_s}(x), \tag{1.23}$$

$$\beta_{n,r,s}(x) = \sum_{i_1+\cdots i_r+j_1+\cdots+j_s=n} U_{i_1}(x)\cdots U_{i_r}(x)W_{j_1}(x)\cdots W_{j_s}(x), \tag{1.24}$$

$$\gamma_{n,r,s}(x) = \sum_{i_1+\cdots i_r+j_1+\cdots+j_s=n} V_{i_1}(x)\cdots V_{i_r}(x)W_{j_1}(x)\cdots W_{j_s}(x), \tag{1.25}$$

where the sums are over all nonnegative integers $i_1, \cdots, i_r, j_1, \cdots j_s$ with $i_1 + \cdots i_r + j_1 + \cdots j_s = n$.

We note here that $\alpha_{n,r,s}(x), \beta_{n,r,s}(x)$ and $\gamma_{n,r,s}(x)$ all have degree n.

The following three theorems are the main results of this paper.

Theorem 1. *Let n, r, s be nonnegative integers with $r + s \geq 1$. Then we have the following identities:*

$$\sum_{i_1+\cdots i_r+j_1+\cdots+j_s=n} U_{i_1}(x)\cdots U_{i_r}(x)V_{j_1}(x)\cdots V_{j_s}(x)$$

$$= \sum_{k=0}^{n} \frac{(-1)^{n-k}\epsilon_k}{(r+s-1)!} \sum_{j=0}^{[\frac{n-k}{2}]} \binom{s}{n-k-2j} \frac{(k+r+s+2j-1)!}{(k+j)!j!} \times {}_2F_1(-j, -j-k; 1-k-r-s-2j; 1)T_k(x) \tag{1.26}$$

$$= \sum_{k=0}^{n} \frac{(-1)^{n-k}(k+1)}{(r+s-1)!} \sum_{j=0}^{[\frac{n-k}{2}]} \binom{s}{n-k-2j} \frac{(k+r+s+2j-1)!}{(k+j+1)!j!} \times {}_2F_1(-j, -j-k-1; 1-k-r-s-2j; 1)U_k(x) \tag{1.27}$$

$$= \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(r+s-1)!} \sum_{j=0}^{n-k} \binom{s}{n-k-j} \frac{(-1)^j(k+r+s+j-1)!}{(k+[\frac{j+1}{2}])![\frac{j}{2}]!} \times {}_2F_1(-[\frac{j}{2}], -[\frac{j+1}{2}]-k; 1-k-r-s-j; 1)V_k(x) \tag{1.28}$$

$$= \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(r+s-1)!} \sum_{j=0}^{n-k} \binom{s}{n-k-j} \frac{(k+r+s+j-1)!}{(k+[\frac{j+1}{2}])![\frac{j}{2}]!} \times {}_2F_1(-[\frac{j}{2}], -[\frac{j+1}{2}]-k; 1-k-r-s-j; 1)W_k(x). \tag{1.29}$$

Theorem 2. *Let n, r, s be nonnegative integers with $r + s \geq 1$. Then we have the following representations:*

$$\sum_{i_1+\cdots i_r+j_1+\cdots+j_s=n} U_{i_1}(x)\cdots U_{i_r}(x)W_{j_1}(x)\cdots W_{j_s}(x)$$

$$= \sum_{k=0}^{n} \frac{\epsilon_k}{(r+s-1)!} \sum_{j=0}^{[\frac{n-k}{2}]} \binom{s}{n-k-2j} \frac{(k+r+s+2j-1)!}{(k+j)!j!} \tag{1.30}$$
$$\times {}_2F_1(-j,-j-k;1-k-r-s-2j;1)T_k(x)$$
$$= \sum_{k=0}^{n} \frac{(k+1)}{(r+s-1)!} \sum_{j=0}^{[\frac{n-k}{2}]} \binom{s}{n-k-2j} \frac{(k+r+s+2j-1)!}{(k+j+1)!j!} \tag{1.31}$$
$$\times {}_2F_1(-j,-j-k-1;1-k-r-s-2j;1)U_k(x)$$
$$= \sum_{k=0}^{n} \frac{1}{(r+s-1)!} \sum_{j=0}^{n-k} \binom{s}{n-k-j} \frac{(k+r+s+j-1)!}{(k+[\frac{j+1}{2}])![\frac{j}{2}]!} \tag{1.32}$$
$$\times {}_2F_1(-[\frac{j}{2}],-[\frac{j+1}{2}]-k;1-k-r-s-j;1)V_k(x)$$
$$= \sum_{k=0}^{n} \frac{1}{(r+s-1)!} \sum_{j=0}^{n-k} (-1)^j \binom{s}{n-k-j} \frac{(k+r+s+j-1)!}{(k+[\frac{j+1}{2}])![\frac{j}{2}]!} \tag{1.33}$$
$$\times {}_2F_1(-[\frac{j}{2}],-[\frac{j+1}{2}]-k;1-k-r-s-j;1)W_k(x).$$

Theorem 3. *Let n, r, s be nonnegative integers with $r + s \geq 1$. Then the following expressions are established:*

$$\sum_{i_1+\cdots i_r+j_1+\cdots+j_s=n} V_{i_1}(x)\cdots V_{i_r}(x)W_{j_1}(x)\cdots W_{j_s}(x)$$
$$= \sum_{k=0}^{n} \frac{\epsilon_k}{(r+s-1)!} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(k+r+s+2l-1)!}{(k+l)!l!} \tag{1.34}$$
$$\times {}_2F_1(-l,-l-k;1-k-r-s-2l;1) \sum_{j=0}^{n-k-2l} (-1)^j \binom{r}{j} \binom{s}{n-k-2l-j} T_k(x)$$
$$= \sum_{k=0}^{n} \frac{(k+1)}{(r+s-1)!} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(k+r+s+2l-1)!}{(k+l+1)!l!} \tag{1.35}$$
$$\times {}_2F_1(-l,-l-k-1;1-k-r-s-2l;1) \sum_{j=0}^{n-k-2l} (-1)^j \binom{r}{j} \binom{s}{n-k-2l-j} U_k(x)$$
$$= \sum_{k=0}^{n} \frac{1}{(r+s-1)!} \sum_{l=0}^{n-k} \frac{(k+r+s+l-1)!}{(k+[\frac{l+1}{2}])![\frac{l}{2}]!} \tag{1.36}$$
$$\times {}_2F_1(-[\frac{l}{2}],-[\frac{l}{2}]-k-1;1-k-r-s-l;1) \sum_{j=0}^{n-k-l} (-1)^j \binom{r}{j} \binom{s}{n-k-l-j} V_k(x)$$
$$= \sum_{k=0}^{n} \frac{1}{(r+s-1)!} \sum_{j=0}^{n-k} \frac{(-1)^l(k+r+s+l-1)!}{(k+[\frac{l+1}{2}])![\frac{l}{2}]!} \tag{1.37}$$

$$\times {}_2F_1(-[\frac{l}{2}], -[\frac{l}{2}] - k - 1; 1 - k - r - s - l; 1) \sum_{j=0}^{n-k-l} (-1)^j \binom{r}{j} \binom{s}{n-k-l-j} W_k(x).$$

2. Proofs of Theorems 1 and 2

In this section, we will prove Theorems 1 and 2. For this, we first state Propositions 4 and 5 which will be used in the proofs of Theorems 1, 2 and 3.

The facts (a) and (b) in Proposition 4 are stated respectively in Eqs (1.1) and (1.36) of [9], while (c) and (d) are respectively from Eqs (1.25) and (2.1) of [7]. We note that all of these facts follow from the orthogonality relations in (1.19)–(1.22), Rodrigues' formulas in (1.15)–(1.18) and integration by parts.

Proposition 4. *Let $q(x) \in \mathbb{R}[x]$ be a polynomial of degree n. Then the following hold.*

(a) $q(x) = \sum_{k=0}^{n} c_{k,1} T_k(x)$, *where* $c_{k,1} = \frac{(-1)^k 2^k k! \epsilon_k}{(2k)! \pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x^2)^{k-\frac{1}{2}} dx$,

(b) $q(x) = \sum_{k=0}^{n} c_{k,2} U_k(x)$, *where* $c_{k,2} = \frac{(-1)^k 2^{k+1} (k+1)!}{(2k+1)! \pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x^2)^{k+\frac{1}{2}} dx$,

(c) $q(x) = \sum_{k=0}^{n} c_{k,3} V_k(x)$, *where* $c_{k,3} = \frac{(-1)^k k! 2^k}{(2k)! \pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x)^{k-\frac{1}{2}} (1+x)^{k+\frac{1}{2}} dx$,

(d) $q(x) = \sum_{k=0}^{n} c_{k,4} W_k(x)$, *where* $c_{k,4} = \frac{(-1)^k k! 2^k}{(2k)! \pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x)^{k+\frac{1}{2}} (1+x)^{k-\frac{1}{2}} dx$.

The following proposition is shown in [10].

Proposition 5. *Let m, k be any nonnegative integers. Then we have the following.*

(a) $\int_{-1}^{1} (1-x^2)^{k-\frac{1}{2}} x^m dx = \begin{cases} 0, & \text{if } m \equiv 1 \pmod 2, \\ \frac{m!(2k)!\pi}{2^{m+2k} (\frac{m}{2}+k)! (\frac{m}{2})! k!}, & \text{if } m \equiv 0 \pmod 2, \end{cases}$

(b) $\int_{-1}^{1} (1-x^2)^{k+\frac{1}{2}} x^m dx = \begin{cases} 0, & \text{if } m \equiv 1 \pmod 2, \\ \frac{m!(2k+2)!\pi}{2^{m+2k+2} (\frac{m}{2}+k+1)! (\frac{m}{2})! (k+1)!}, & \text{if } m \equiv 0 \pmod 2, \end{cases}$

(c) $\int_{-1}^{1} (1-x)^{k-\frac{1}{2}} (1+x)^{k+\frac{1}{2}} x^m dx = \begin{cases} \frac{(m+1)!(2k)!\pi}{2^{m+2k+1} (\frac{m+1}{2}+k)! (\frac{m+1}{2})! k!}, & \text{if } m \equiv 1 \pmod 2, \\ \frac{m!(2k)!\pi}{2^{m+2k} (\frac{m}{2}+k)! (\frac{m}{2})! k!}, & \text{if } m \equiv 0 \pmod 2, \end{cases}$

(d) $\int_{-1}^{1} (1-x)^{k+\frac{1}{2}} (1+x)^{k-\frac{1}{2}} x^m dx = \begin{cases} -\frac{(m+1)!(2k)!\pi}{2^{m+2k+1} (\frac{m+1}{2}+k)! (\frac{m+1}{2})! k!}, & \text{if } m \equiv 1 \mod 2, \\ \frac{m!(2k)!\pi}{2^{m+2k} (\frac{m}{2}+k)! (\frac{m}{2})! k!}, & \text{if } m \equiv 0 \pmod 2. \end{cases}$

The next lemma is crucial in showing Theorem 1.

Lemma 6. *Let n, r, s be any nonnegative integers with $r + s \geq 1$. Then we have the following identity.*

$$\sum_{i_1+\cdots i_r+j_1+\cdots+j_s=n} U_{i_1}(x) \cdots U_{i_r}(x) V_{j_1}(x) \cdots V_{j_s}(x) = \frac{1}{(r+s-1)! 2^{r+s-1}} \sum_{l=0}^{n} \binom{s}{l} (-1)^l U_{n+r+s-l-1}^{(r+s-1)}(x), \tag{2.1}$$

where the sum is over all nonnegative integers $i_1, \cdots, i_r, j_1, \cdots, j_s$, with $i_1 + \cdots i_r + j_1 + \cdots + j_s = n$.

Proof. Let $F(t, x) = (1 - 2xt + t^2)^{-1}$. Then we observe that

$$\frac{\partial^{r+s-1}}{\partial x^{r+s-1}} F(t, x) = (r+s-1)! (2t)^{r+s-1} (1 - 2xt + t^2)^{-(r+s)}. \tag{2.2}$$

Now, by making use of (1.8), (1.9) and (2.2), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{i_1+\cdots i_r+j_1+\cdots+j_s=n} U_{i_1}(x)\cdots U_{i_r}(x)V_{j_1}(x)\cdots V_{j_s}(x)\right)t^n \\
&= \left(\sum_{i=0}^{\infty} U_i(x)t^i\right)\left(\sum_{j=0}^{\infty} V_j(x)t^j\right)^s \\
&= \left(\frac{1}{1-2xt+t^2}\right)^r\left(\frac{1-t}{1-2xt+t^2}\right)^s \\
&= (1-t)^s(1-2xt+t^2)^{-(r+s)} \\
&= \frac{(1-t)^s}{(r+s-1)!2^{r+s-1}}\frac{1}{t^{r+s-1}}\frac{\partial^{r+s-1}}{\partial x^{r+s-1}}F(t,x) \\
&= \frac{(1-t)^s}{(r+s-1)!2^{r+s-1}}\frac{1}{t^{r+s-1}}\sum_{m=0}^{\infty} U_{m+r+s-1}^{(r+s-1)}(x)t^{m+r+s-1} \\
&= \frac{1}{(r+s-1)!2^{r+s-1}}\sum_{l=0}^{\infty}\binom{s}{l}(-1)^l t^l\sum_{m=0}^{\infty} U_{m+r+s-1}^{(r+s-1)}(x)t^m \\
&= \frac{1}{(r+s-1)!2^{r+s-1}}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{s}{l}(-1)^l U_{n+r+s-l-1}^{(r+s-1)}(x)\right)t^n,
\end{aligned}$$

which completes the proof for (2.1). □

Here and in the following, we will assume

$$\binom{n}{r}=0, \quad \text{if } r<0 \text{ or } r>n. \tag{2.3}$$

Similarly to Lemma 6, the following lemma can be shown.

Lemma 7. *Let n, r, s be any nonnegative integers with $r+s\geq 1$. Then the following identity holds.*

$$\sum_{i_1+\cdots i_r+j_1+\cdots+j_s=n} U_{i_1}(x)\cdots U_{i_r}(x)W_{j_1}(x)\cdots W_{j_s}(x)=\frac{1}{(r+s-1)!2^{r+s-1}}\sum_{l=0}^{n}\binom{s}{l}U_{n+r+s-l-1}^{(r+s-1)}(x), \tag{2.4}$$

where the sum is over all nonnegative integers $i_1,\cdots,i_r,j_1,\cdots,j_s$, with $i_1+\cdots i_r+j_1+\cdots+j_s=n$.

From (1.12), we see that the r-th derivative of $U_n(x)$ is given by

$$U_n^{(r)}(x)=\sum_{m=0}^{[\frac{n-r}{2}]}(-1)^m\binom{n-m}{m}(n-2m)_r 2^{n-2m}x^{n-2m-r}. \tag{2.5}$$

Thus, from (2.5) we have

$$\begin{aligned}
U_{n+r+s-l-1}^{(r+s+k-1)}(x)=&\sum_{m=0}^{[\frac{n-k-l}{2}]}(-1)^m\binom{n+r+s-l-m-1}{m} \\
&\times(n+r+s-l-2m-1)_{r+s+k-1}2^{n+r+s-l-2m-1}x^{n-k-l-2m}.
\end{aligned} \tag{2.6}$$

Here, we will show only (1.26) and (1.28) of Theorem 1, and leave similar proofs for (1.27) and (1.29) as exercises to the readers.

With $\alpha_{n,r,s}(x)$ as in (1.23), we let

$$\alpha_{n,r,s}(x) = \sum_{k=0}^{n} c_{k,1} T_k(x). \tag{2.7}$$

Then, from (a) of Proposition 4, (2.1), (2.6) and integration by parts k times, we have

$$\begin{aligned}
c_{k,1} =& \frac{(-1)^k 2^k k! \epsilon_k}{(2k)!\pi} \int_{-1}^{1} \alpha_{n,r,s}(x) \frac{d^k}{dx^k}(1-x^2)^{k-\frac{1}{2}} dx \\
=& \frac{(-1)^k 2^k k! \epsilon_k}{(2k)!\pi (r+s-1)! 2^{r+s-1}} \sum_{l=0}^{n} \binom{s}{l} (-1)^l \int_{-1}^{1} U_{n+r+s-l-1}^{(r+s-1)}(x) \frac{d^k}{dx^k}(1-x^2)^{k-\frac{1}{2}} dx \\
=& \frac{2^k k! \epsilon_k}{(2k)!\pi (r+s-1)! 2^{r+s-1}} \sum_{l=0}^{n-k} \binom{s}{l} (-1)^l \int_{-1}^{1} U_{n+r+s-l-1}^{(r+s+k-1)}(x) (1-x^2)^{k-\frac{1}{2}} dx \\
=& \frac{2^k k! \epsilon_k}{(2k)!\pi (r+s-1)! 2^{r+s-1}} \sum_{l=0}^{n-k} \binom{s}{l} (-1)^l \sum_{m=0}^{[\frac{n-k-l}{2}]} (-1)^m \binom{n+r+s-l-m-1}{m} \\
& \times (n+r+s-l-2m-1)_{r+s+k-1} 2^{n+r+s-l-2m-1} \int_{-1}^{1} x^{n-k-l-2m} (1-x^2)^{k-\frac{1}{2}} dx.
\end{aligned} \tag{2.8}$$

Here we note from (a) of Proposition 5 that

$$\int_{-1}^{1} x^{n-k-l-2m}(1-x^2)^{k-\frac{1}{2}} dx = \begin{cases} 0, & \text{if } l \not\equiv n-k \pmod 2, \\ \frac{(n-k-l-2m)!(2k)!\pi}{2^{n+k-l-2m}\left(\frac{n+k-l}{2}-m\right)!\left(\frac{n-k-l}{2}-m\right)!k!}, & \text{if } l \equiv n-k \pmod 2. \end{cases} \tag{2.9}$$

Now, from (2.7)–(2.9) and after some simplifications, we get

$$\begin{aligned}
\alpha_{n,r,s}(x) =& \frac{1}{(r+s-1)!} \sum_{k=0}^{n} \sum_{\substack{0 \le l \le n-k \\ l \equiv n-k \pmod 2}} \sum_{m=0}^{[\frac{n-k-l}{2}]} \frac{\epsilon_k \binom{s}{l} (-1)^l (-1)^m (n+r+s-l-m-1)!}{m! \left(\frac{n+k-l}{2}-m\right)! \left(\frac{n-k-l}{2}-m\right)!} T_k(x) \\
=& \frac{1}{(r+s-1)!} \sum_{k=0}^{n} \sum_{j=0}^{[\frac{n-k}{2}]} \epsilon_k \binom{s}{n-k-2j} (-1)^{n-k} \sum_{m=0}^{j} \frac{(-1)^m (k+r+s+2j-m-1)!}{m!(k+j-m)!(j-m)!} T_k(x) \\
=& \frac{1}{(r+s-1)!} \sum_{k=0}^{n} \sum_{j=0}^{[\frac{n-k}{2}]} \epsilon_k \binom{s}{n-k-2j} (-1)^{n-k} \frac{(k+r+s+2j-1)!}{(k+j)!j!} \\
& \times \sum_{m=0}^{j} \frac{<-j>_m <-j-k>_m}{m! <1-k-r-s-2j>_m} T_k(x) \\
=& \sum_{k=0}^{n} \frac{(-1)^{n-k}\epsilon_k}{(r+s-1)!} \sum_{j=0}^{[\frac{n-k}{2}]} \binom{s}{n-k-2j} \frac{(k+r+s+2j-1)!}{(k+j)!j!} \\
& \times {}_2F_1(-j, -j-k; 1-k-r-s-2j; 1) T_k(x).
\end{aligned} \tag{2.10}$$

This shows (1.26) of Theorem 1.

Next, let us put

$$\alpha_{n,r,s}(x)=\sum_{k=0}^{n}c_{k,3}V_k(x). \tag{2.11}$$

Then, from (c) of Proposition 4, (2.1), (2.6), integration by parts k times and proceeding just as in (2.8), we see that

$$\begin{aligned} c_{k,3}=&\frac{k!2^k}{(2k)!\pi(r+s-1)!2^{r+s-1}}\sum_{l=0}^{n-k}\binom{s}{l}(-1)^l\sum_{m=0}^{[\frac{n-k-l}{2}]}(-1)^m\binom{n+r+s-l-m-1}{m}\\ &\times(n+r+s-l-2m-1)_{r+s+k-1}2^{n+r+s-l-2m-1}\int_{-1}^{1}x^{n-k-l-2m}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}}dx, \end{aligned} \tag{2.12}$$

where we note from (c) of Proposition 5 that

$$\begin{aligned} &\int_{-1}^{1}x^{n-k-l-2m}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}}dx\\ &=\begin{cases}\frac{(n-k-l-2m+1)!(2k)!\pi}{2^{n+k-l-2m+1}\left(\frac{n+k-l+1}{2}-m\right)!\left(\frac{n-k-l+1}{2}-m\right)!k!}, & \text{if } l\not\equiv n-k \pmod 2,\\ \frac{(n-k-l-2m)!(2k)!\pi}{2^{n+k-l-2m}\left(\frac{n+k-l}{2}-m\right)!\left(\frac{n-k-l}{2}-m\right)!k!}, & \text{if } l\equiv n-k \pmod 2.\end{cases} \end{aligned} \tag{2.13}$$

From (2.11)–(2.13), and after some simplifications, we have

$$\alpha_{n,r,s}(x)=\Sigma_1+\Sigma_2, \tag{2.14}$$

where

$$\begin{aligned} \Sigma_1=&\sum_{k=0}^{n}\sum_{\substack{0\le l\le n-k\\ l\not\equiv n-k \pmod 2}}\sum_{m=0}^{[\frac{n-k-l}{2}]}\frac{\binom{s}{l}(-1)^l(-1)^m(n+r+s-l-m-1)!(n-k-l-2m+1)}{2(r+s-1)!m!\left(\frac{n+k-l+1}{2}-m\right)!\left(\frac{n-k-l+1}{2}-m\right)!}V_k(x),\\ \Sigma_2=&\sum_{k=0}^{n}\sum_{\substack{0\le l\le n-k\\ l\equiv n-k \pmod 2}}\sum_{m=0}^{[\frac{n-k-l}{2}]}\frac{\binom{s}{l}(-1)^l(-1)^m(n+r+s-l-m-1)!}{(r+s-1)!m!\left(\frac{n+k-l}{2}-m\right)!\left(\frac{n-k-l}{2}-m\right)!}V_k(x). \end{aligned} \tag{2.15}$$

By proceeding analogously to the case of (2.10), we note from (2.15) that

$$\begin{aligned} \Sigma_1=&\sum_{k=0}^{n}\sum_{j=0}^{[\frac{n-k-1}{2}]}\frac{(-1)^{n-k-1}\binom{s}{n-k-2j-1}(k+r+s+2j)!}{(r+s-1)!(k+j+1)!j!}\sum_{m=0}^{j}\frac{(-1)^m(k+j+1)_m(j)_m}{m!(k+r+s+2j)_m}V_k(x)\\ =&\sum_{k=0}^{n}\sum_{j=0}^{[\frac{n-k-1}{2}]}\frac{(-1)^{n-k-1}\binom{s}{n-k-2j-1}(k+r+s+2j)!}{(r+s-1)!(k+j+1)!j!}{}_2F_1(-j,-j-k-1;-k-r-s-2j;1)V_k(x), \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} \Sigma_2=&\sum_{k=0}^{n}\sum_{j=0}^{[\frac{n-k}{2}]}\frac{(-1)^{n-k}\binom{s}{n-k-2j}(k+r+s+2j-1)!}{(r+s-1)!(k+j)!j!}\sum_{m=0}^{j}\frac{(-1)^m(k+j)_m(j)_m}{m!(k+r+s+2j-1)_m}V_k(x)\\ =&\sum_{k=0}^{n}\sum_{j=0}^{[\frac{n-k}{2}]}\frac{(-1)^{n-k}\binom{s}{n-k-2j}(k+r+s+2j-1)!}{(r+s-1)!(k+j)!j!}{}_2F_1(-j,-j-k;-k-r-s-2j+1;1)V_k(x). \end{aligned} \tag{2.17}$$

Now, the result in (1.28) follows from (2.14), (2.16) and (2.17). For Theorem 2, we note the following. From (2.1) and (2.4), we see that the only difference between $\alpha_{n,r,s}(x)$ and $\beta_{n,r,s}(x)$ (see (1.23) and (1.24)) are the alternating sign $(-1)^l$ in their sums which corresponds to $(-1)^{n-k}$ in (1.26)–(1.29). This remark gives the results in (1.30)–(1.33) of Theorem 2.

3. Proof of Theorem 3

In this section, we will show (1.35) and (1.36) in Theorem 3 and leave (1.34) and (1.37) as exercises to the readers. The next lemma can be shown just in the case of Lemma 6.

Lemma 8. *Let n, r, s be nonnegative integers with $r+s \geq 1$. Then the following identity holds true.*

$$\sum_{i_1+\cdots i_r+j_1+\cdots+j_s=n} V_{i_1}(x)\cdots V_{i_r}(x)W_{j_1}(x)\cdots W_{j_s}(x) = \frac{1}{(r+s-1)!2^{r+s-1}}\sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^j\binom{r}{j}\binom{s}{i-j}U^{(r+s-1)}_{n+r+s-i-1}(x), \tag{3.1}$$

where the sum is over all nonnegative integers $i_1,\cdots,i_r,j_1,\cdots,j_s$ with $i_1+\cdots i_r+j_1+\cdots+j_s=n$.

With $\gamma_{n,r,s}(x)$ as in (1.25), we let

$$\gamma_{n,r,s}(x) = \sum_{k=0}^{n} c_{k,2}U_k(x). \tag{3.2}$$

Then, by making use of (b) of Proposition 4, (3.1), (2.6) and integration by parts k times, we get

$$\begin{aligned} c_{k,2} =& \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi}\int_{-1}^{1}\gamma_{n,r,s}(x)\frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}}dx \\ =& \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!\pi(r+s-1)!2^{r+s-1}}\sum_{i=0}^{n}\sum_{j=0}^{i}(-1)^j\binom{r}{j}\binom{s}{i-j}\int_{-1}^{1}U^{(r+s-1)}_{n+r+s-i-1}(x)\frac{d^k}{dx^k}(1-x^2)^{k+\frac{1}{2}}dx \\ =& \frac{2^{k+1}(k+1)!}{(2k+1)!\pi(r+s-1)!2^{r+s-1}}\sum_{i=0}^{n-k}\sum_{j=0}^{i}(-1)^j\binom{r}{j}\binom{s}{i-j}\int_{-1}^{1}U^{(r+s+k-1)}_{n+r+s-i-1}(x)(1-x^2)^{k+\frac{1}{2}}dx \\ =& \frac{2^{k+1}(k+1)!}{(2k+1)!\pi(r+s-1)!2^{r+s-1}}\sum_{i=0}^{n-k}\sum_{j=0}^{i}(-1)^j\binom{r}{j}\binom{s}{i-j}\sum_{m=0}^{[\frac{n-k-i}{2}]}(-1)^m\binom{n+r+s-i-m-1}{m} \\ & \times(n+r+s-i-2m-1)_{r+s+k-1}2^{n+r+s-i-2m-1}\int_{-1}^{1}x^{n-k-i-2m}(1-x^2)^{k+\frac{1}{2}}dx. \end{aligned} \tag{3.3}$$

Here we need to observe from (b) of Proposition 5 that

$$\int_{-1}^{1}x^{n-k-i-2m}(1-x^2)^{k+\frac{1}{2}}dx = \begin{cases} 0, & \text{if } i \not\equiv n-k \pmod 2, \\ \frac{(n-k-i-2m)!(2k+2)!\pi}{2^{n+k-i-2m+2}\left(\frac{n+k-i}{2}-m+1\right)!\left(\frac{n-k-i}{2}-m\right)!(k+1)!}, & \text{if } i \equiv n-k \pmod 2. \end{cases} \tag{3.4}$$

Now, from (3.2)–(3.4) and after some simplifications, we have

$$\begin{aligned}\gamma_{n,r,s}(x) &= \sum_{k=0}^{n} \frac{(k+1)}{(r+s-1)!} \sum_{\substack{0\le i\le n-k \\ i\equiv n-k \pmod 2}} \sum_{j=0}^{i} \sum_{m=0}^{[\frac{n-k-i}{2}]} \frac{(-1)^j\binom{r}{j}\binom{s}{i-j}(-1)^m(n+r+s-i-m-1)!}{m!\left(\frac{n+k-i}{2}-m+1\right)!\left(\frac{n-k-i}{2}-m\right)!} U_k(x) \\
&= \sum_{k=0}^{n} \frac{(k+1)}{(r+s-1)!} \sum_{l=0}^{[\frac{n-k}{2}]} \sum_{j=0}^{n-k-2l} \sum_{m=0}^{l} \frac{(-1)^j\binom{r}{j}\binom{s}{n-k-2l-j}(-1)^m(k+r+s+2l-m-1)!}{m!(k+l+1-m)!(l-m)!} U_k(x) \\
&= \sum_{k=0}^{n} \frac{(k+1)}{(r+s-1)!} \sum_{l=0}^{[\frac{n-k}{2}]} \sum_{j=0}^{n-k-2l} \frac{(-1)^j\binom{r}{j}\binom{s}{n-k-2l-j}(k+r+s+2l-1)!}{(k+l+1)!l!} \\
&\quad \times \sum_{m=0}^{l} \frac{<-l>_m<-l-k-1>_m}{m!<1-k-r-s-2l>_m} U_k(x) \\
&= \sum_{k=0}^{n} \frac{(k+1)}{(r+s-1)!} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(k+r+s+2l-1)!}{(k+l+1)!l!} {}_2F_1(-l,-l-k-1;1-k-r-s-2l;1) \\
&\quad \times \sum_{j=0}^{n-k-2l} (-1)^j\binom{r}{j}\binom{s}{n-k-2l-j} U_k(x).
\end{aligned} \tag{3.5}$$

This shows (1.35) in Theorem 3.

Next, we set

$$\gamma_{n,r,s}(x) = \sum_{k=0}^{n} c_{k,3} V_k(x). \tag{3.6}$$

Then, from (c) of Proposition 4, (3.1), (2.6), integration by parts k times and proceeding just as in (3.3), we have

$$\begin{aligned} c_{k,3} =& \frac{k!2^k}{(2k)!\pi(r+s-1)!2^{r+s-1}} \sum_{i=0}^{n-k} \sum_{j=0}^{i} (-1)^j \binom{r}{j}\binom{s}{i-j} \sum_{m=0}^{[\frac{n-k-i}{2}]} (-1)^m \binom{n+r+s-i-m-1}{m} \\
&\times (n+r+s-i-2m-1)_{r+s+k-1} 2^{n+r+s-i-2m-1} \int_{-1}^{1} x^{n-k-i-2m}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}} dx. \end{aligned} \tag{3.7}$$

From (2.13), (3.6), (3.7) and after some simplifications, we obtain

$$\gamma_{n,r,s}(x) = \Sigma_1 + \Sigma_2, \tag{3.8}$$

where

$$\begin{aligned} \Sigma_1 &= \sum_{k=0}^{n} \sum_{\substack{0\le i\le n-k \\ i\not\equiv n-k \pmod 2}} \sum_{j=0}^{i} \sum_{m=0}^{[\frac{n-k-i}{2}]} \frac{(-1)^j\binom{r}{j}\binom{s}{i-j}(-1)^m(n+r+s-i-m-1)!(n-k-i-2m+1)}{2(r+s-1)!m!\left(\frac{n+k-i+1}{2}-m\right)!\left(\frac{n-k-i+1}{2}-m\right)!} V_k(x), \\
\Sigma_2 &= \sum_{k=0}^{n} \sum_{\substack{0\le i\le n-k \\ i\equiv n-k \pmod 2}} \sum_{j=0}^{i} \sum_{m=0}^{[\frac{n-k-i}{2}]} \frac{(-1)^j\binom{r}{j}\binom{s}{i-j}(-1)^m(n+r+s-i-m-1)!}{(r+s-1)!m!\left(\frac{n+k-i}{2}-m\right)!\left(\frac{n-k-i}{2}-m\right)!} V_k(x). \end{aligned} \tag{3.9}$$

By proceeding similarly to the case of (3.5), we see from (3.9) that

$$\begin{aligned}
\Sigma_1 &= \sum_{k=0}^{n} \sum_{l=0}^{[\frac{n-k-1}{2}]} \sum_{j=0}^{n-k-2l-1} \frac{(-1)^j \binom{r}{j}\binom{s}{n-k-2l-j-1}(k+r+s+2l)!}{(r+s-1)!(k+l+1)!l!} \sum_{m=0}^{l} \frac{(-1)^m (k+l+1)_m (l)_m}{m!(k+r+s+2l)_m} V_k(x) \\
&= \sum_{k=0}^{n} \frac{1}{(r+s-1)!} \sum_{l=0}^{[\frac{n-k-1}{2}]} \frac{(k+r+s+2l)!}{(k+l+1)!l!} {}_2F_1(-l,-l-k-1;-k-r-s-2l;1) \qquad (3.10) \\
&\times \sum_{j=0}^{n-k-2l-1} (-1)^j \binom{r}{j}\binom{s}{n-k-2l-j-1} V_k(x),
\end{aligned}$$

and

$$\begin{aligned}
\Sigma_2 &= \sum_{k=0}^{n} \sum_{l=0}^{[\frac{n-k}{2}]} \sum_{j=0}^{n-k-2l} \frac{(-1)^j \binom{r}{j}\binom{s}{n-k-2l-j}(k+r+s+2l-1)!}{(r+s-1)!(k+l)!l!} \sum_{m=0}^{l} \frac{(-1)^m (k+l)_m (l)_m}{m!(k+r+s+2l-1)_m} V_k(x) \\
&= \sum_{k=0}^{n} \frac{1}{(r+s-1)!} \sum_{l=0}^{[\frac{n-k}{2}]} \frac{(k+r+s+2l-1)!}{(k+l)!l!} {}_2F_1(-l,-l-k;1-k-r-s-2l;1) \qquad (3.11) \\
&\times \sum_{j=0}^{n-k-2l} (-1)^j \binom{r}{j}\binom{s}{n-k-2l-j} V_k(x).
\end{aligned}$$

Now, the result in (1.36) follows from (3.8), (3.10) and (3.11).

4. Conclusions

Let n, r, s be nonnegative integers with $r + s \geq 1$. Then we considered sums of finite products of the second and third type Chebyshev polynomials $\alpha_{n,r,s}(x)$ in (1.23), those of the second and fourth type Chebyshev polynomials $\beta_{n,r,s}(x)$ in (1.24), and those of the third and fourth type Chebyshev polynomials $\gamma_{n,r,s}(x)$ in (1.25), and represented each of them as linear combinations of Chebyshev polynomials of all types $T_k(x)$, $U_k(x)$, $V_k(x)$ and $W_k(x)$. We observed that the coefficients involve some terminating hypergeometric functions ${}_2F_1$. This is done by explicit computations.

We noticed that this problem can be viewed as a generalization of the classical linearization problems. Also, we explained in the Introduction that the standard linearization problem for Bernoulli polynomials (more precisely with some coefficients) yields the identity (1.2) which in turn gives the famous Faber-Pandharipande-Zagier identity and a slight variant of the well-known Miki's identity. We emphasized that, whereas all of the results so far treated sums of finite products of some polynomials of single type, this paper takes care of those of two different types.

In 1998, Faber and Pandharipande found that certain conjectural relations between Hodge integrals in Gromov-Witten theory require the FPZ identity. It is very pleasing that the FPZ identity is useful in Gromov-Witten theory which has to with string theory in physics. It would be nice if we had applications of our results to some disciplines outside of mathematics. As one of our future projects, we would like to continue to study problem of representing sums of finite products of some special polynomials by other special polynomials and to find their applications in physics, science and engineering as well as in mathematics.

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Conflicts of interest

The authors declare no conflict of interest.

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