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Research article

Collectively fixed point theorems in noncompact abstract convex spaces with applications

Haishu Lu*, **Kai Zhang and Rong Li**

School of Business, Jiangsu University of Technology, Changzhou 213001, Jiangsu, China

* **Correspondence:** Email: luhaishu@126.com; Tel: +86519 86953306; Fax: +8651986953300.

Abstract: In this paper, by using the KKM theory and the properties of Γ-convexity and $\Re\mathfrak{C}$-mapping, we investigate the existence of collectively fixed points for a family with a finite number of set-valued mappings on the product space of noncompact abstract convex spaces. Consequently, as applications, some existence theorems of generalized weighted Nash equilibria and generalized Pareto Nash equilibria for constrained multiobjective games, some nonempty intersection theorems with applications to the Fan analytic alternative formulation and the existence of Nash equilibria, and some existence theorems of solutions for generalized weak implicit inclusion problems in noncompact abstract convex spaces are given. The results obtained in this paper extend and generalize many corresponding results of the existing literature.

Keywords: abstract convex space; collectively fixed point; Nash equilibrium; nonempty intersection theorem; generalized weak implicit inclusion problem
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1. Introduction

Collectively fixed point theorems for a family of set-valued mappings play a key role in studying pure and applied mathematical problems, which can be seen as natural generalizations of fixed point theorems. In 1991, Tarafdar [1] first established a collectively fixed point theorem in the framework of nonempty compact convex subsets of Hausdorff topological vector spaces and then provided its applications in the existence problem of equilibrium points for abstract economies. Since then, many authors have investigated and developed this topic under different assumptions in Hausdorff topological vector spaces. See, for example, [2–7] and the references therein.

On the other hand, to broaden the application of the collectively fixed point theory, many authors have studied the collectively fixed point problem in the framework of Hausdorff topological spaces without linear structure. In 1992, Tarafdar [8] extended the collectively fixed point theorem in [1] to compact H-spaces and then gave some applications to the nonempty intersection problem of sets with H-convex sections and existence problem of equilibrium points for an abstract economy. In 1999, Park [9] proved a collectively fixed point theorem which generalizes the collectively fixed point theorems in [1, 8] to compact G-convex spaces. In 2003, Yu and Lin [10] generalized the collectively fixed point theorem in [9] to noncompact G-convex spaces. In 2007, Ding [11] and Zhang and Cheng [12] obtained some collectively fixed point theorems in noncompact FC-spaces. In 2010, Al-Homidan et al. [13] derived a collectively fixed point theorem and a maximal element theorem in noncompact topological semilattice spaces and presented applications to problems on generalized abstract economy, systems of vector quasi-equilibrium, and constrained Nash equilibrium. In 2011, Khanh et al. [14] proved some collectively fixed point theorems in noncompact GFC-spaces and gave applications to collectively coincidence point theorems and systems of variational relations. Recently, by means of the technique of partition of unity and Tikhonov fixed point theorem, Khanh and Quan [15] proved the existence of collectively fixed points for a family of set-valued mappings defined on the product set of nonempty sets which have topologically based structures and do not possess linear or convexity structures. Furthermore, they gave applications to coincidence points of a family of set-valued mappings and intersection points of a family of sets.

The abstract convex space is first introduced by Park [16], which includes the spaces mentioned above as special cases. So far, a small part of the literature discussed the problem of collectively fixed points in abstract convex spaces. In 2010, by using a Fan-Browder type fixed point theorem in [17], Park [18] obtained a collectively fixed point theorem for finite families of compact abstract convex spaces and then used this collectively fixed point theorem to obtain a Fan-type nonempty intersection theorem for sets with Γ-convex sections. Recently, Lu and Hu [19] proved a new collectively fixed point theorem for finite families of noncompact abstract convex spaces and gave its applications to equilibria for generalized abstract economies. It is needed to point out that the Hausdorffness of the spaces involved in the collectively fixed point theorems in [1–15] for a family of set-valued mappings is necessary since these theorems are proved based on the partition of unity argument. Note that the proofs of the collectively fixed point theorems in [18, 19] are based on the Fan-Browder-type fixed point theorem in abstract convex spaces whose Hausdorff separation property can be dropped. Thus, in this sense, the corresponding collectively fixed point theorems in these two cases cannot be deduced from each other.

Motivated and inspired by the work mentioned above, in this paper, the main goal of this paper is to prove the existence of collectively fixed points for a family with a finite number of set-valued mappings defined on the product space of noncompact abstract convex spaces. These obtained collectively fixed point theorems have two alternative coercivity conditions. Furthermore, as applications, in the framework of noncompact abstract convex spaces, some existence theorems of generalized weighted Nash equilibria and generalized Pareto Nash equilibria for constrained multiobjective games, some nonempty intersection theorems for sets with abstract convex sections, and some existence theorems of solutions for generalized weak implicit inclusion problems are established.

The rest of this paper is organized as follows. In Section 2, we introduce some notation, definitions, and lemmas for further investigations. Section 3 is devoted to theorems on collectively fixed points in noncompact abstract convex spaces. The following sections give applications of collectively fixed points in noncompact abstract convex spaces. Section 4 contains existence results for generalized weighted Nash equilibra and generalized Pareto Nash equilibria for constrained multiobjective games. In Section 5, we deal with some nonempty intersection theorems for sets with abstract convex sections and give applications to the Fan analytic alternative formulation and the existence of Nash equilibria for noncooperative games in noncompact abstract convex spaces. Finally, in Section 6, by using a maximal element theorem which is essentially equivalent to fixed point theorem, we obtain some existence results of solutions for generalized weak implicit inclusion problems in the setting of noncompact abstract convex spaces.

2. Preliminaries

In this section, we give some notation, definitions, and lemmas for later use.

Let $\mathbb{R}$ and $\mathbb{N}$ denote the set of the real numbers and the set of the natural numbers, respectively. For a nonempty set X, let 2^X and $\langle X\rangle$ denote by the family of all subsets of X and by the family of nonempty finite subsets of X, respectively. Let $T : X \to 2^Y$ be a set-valued mapping, where X and Y are two nonempty sets. Then the graph of T is defined by the set $\{(x, y) \in X \times Y : y \in T(x)\}$ and the set-valued mapping $T^{-1} : Y \to 2^X$ is defined by $T^{-1}(y) = \{x \in X : y \in T(x)\}$ for each $y \in Y$. For each $y \in Y$, we call $T^{-1}(y)$ the lower section of T. For every $X_0 \subseteq X$, $T(X_0) := \cup_{x\in X_0} T(x)$. If A and B are subsets of a topological space X such that $A \subseteq B$, then we denote the closure (respectively, interior) of A in B by $\mathrm{cl}_B A$ (respectively, $\mathrm{int}_B A$). When $B = X$, $\mathrm{cl}A$ (respectively, $\mathrm{int}A$) denotes the closure (respectively, interior) of A. A topological space X is said to be first-countable if for each $x \in X$, there exists a sequence $\{N_1, N_2, \ldots\}$ of neighbourhoods of x such that for any neighbourhood N of x, there exists an integer k such that $N_k \subseteq N$. The product of countable first-countable topological spaces is first-countable, although uncountable product needs not be. Let A be a subset of a first countable topological space X. Then $x \in \mathrm{cl}A$ if and only if there exists a sequence $\{x_n\}$ in A such that $x_n \to x$. We should point out that if A is a subset of a topological space X, then $x \in \mathrm{cl}A$ if and only if there exists a net $\{x_\alpha\}$ in A such that $x_\alpha \to x$.

Definition 2.1 ([20]). Let X and Y be two topological spaces. A set-valued mapping $T : X \to 2^Y$ is called to be:

(i) upper semicontinuous (respectively, lower semicontinuous) at $x \in X$ if for each open set U in Y with $T(x) \subseteq U$ (respectively, $T(x) \bigcap U \neq \emptyset$), there is a neighborhood $V(x)$ of x such that $T(x') \subseteq U$ (respectively, $T(x') \bigcap U \neq \emptyset$) for every $x' \in V(x)$;

(ii) upper semicontinuous (respectively, lower semicontinuous) on X if it is upper semicontinuous (respectively, lower semicontinuous) at every point $x \in X$;

(iii) continuous on X if it is both upper semicontinuous and lower semicontinuous on X;

(iv) closed if its graph $\mathrm{Gr}(T) = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$.

Lemma 2.1 ([20]). *Let $T : X \to 2^Y$ be a set-valued mapping, where X is a topological space and Y is a compact topological space. If the graph of T is closed in $X \times Y$, then T is upper semicontinuous.*

Lemma 2.2 ([21]). *Let X and Y be two topological spaces and $T : X \to 2^Y$ be a set-valued mapping. Then T is lower semicontinuous at $x \in X$ if and only if for each $y \in T(x)$ and each net $\{x_\alpha\} \subseteq X$ such that $x_\alpha \to x$, there is a net $\{y_\alpha\} \subseteq Y$ such that $y_\alpha \in T(x_\alpha)$ for every α and $y_\alpha \to y$.*

Lemma 2.3 ([21]). *Let X and Y be two topological spaces and $T : X \to 2^Y$ be a set-valued mapping. If either T is upper semicontinuous on X with compact values and Y is Hausdorff, or T is upper semicontinuous on X with closed values and Y is regular, then T is closed, that is, the graph of T is* *closed in* $X \times Y$.

Lemma 2.4 ([22]). *Let* X *and* Y *be two topological spaces and* $T : X \to 2^Y$ *be a set-valued mapping. If* T *has compact values, then* T *is upper semicontinuous at* $x \in X$ *if and only if for each net* $\{x_\alpha\} \subseteq X$ *such that* $x_\alpha \to x$ *and for each net* $\{y_\alpha\} \subseteq T(x_\alpha)$ *for every* α, *there exist* $y \in T(x)$ *and a subsbet* $\{y_\beta\}$ *of* $\{y_\alpha\}$ *such that* $y_\beta \to y$.

In what follows, we introduce some basic definitions and lemmas related to abstract convex spaces. For more details, the reader may refer to [16–18, 23, 27–29].

Definition 2.2 ([23]). If X is a topological space, Y is a nonempty set, and $\Gamma : \langle Y \rangle \to 2^X$ is a set-valued mapping with nonempty values $\Gamma_A := \Gamma(A)$ for every $A \in \langle Y \rangle$, then the family $(X, Y; \Gamma)$ is called to be an abstract convex space. When $X = Y$, we denote $(X, X; \Gamma)$ by $(X; \Gamma)$.

Remark 2.1. It is worthwhile noticing that abstract convex spaces contain convex spaces due to Lassonde [24], H-spaces introduced by Horvath [25], G-convex spaces due to Park and Kim [9], L-spaces due to Ben-El-Mechaiekh et al. [26], GFC-spaces due to Khanh et al. [14], FC-spaces due to Ding [11], and many other topological spaces with generalized convex structure (for example, see [18] and references therein).

Definition 2.3 ([23]). Given an abstract convex space $(X, Y; \Gamma)$ and a nonempty subset Y' of Y, we define the Γ-convex hull of Y' by $\mathrm{co}_\Gamma(Y') = \bigcup\{\Gamma_A : A \in \langle Y' \rangle\}$.

Definition 2.4 ([23]). Let $(X, Y; \Gamma)$ be an abstract convex space. A nonempty subset X' of X is said to be a Γ-convex subset of $(X, Y; \Gamma)$ relative to a nonempty subset Y' of Y if we have $\Gamma_N \subseteq X'$ for every $N \in \langle Y' \rangle$, that is, $\mathrm{co}_\Gamma(Y') \subseteq X'$. In case $X = Y$, a nonempty subset X' of X is said to be Γ-convex if $\mathrm{co}_\Gamma(X') \subseteq X'$, that is, X' is Γ-convex relative to itself.

Remark 2.2. Given an abstract convex space $(X, Y; \Gamma)$, by Definition 2.3, we can see that if a nonempty subset X' of X is a Γ-convex subset of $(X, Y; \Gamma)$ relative to a nonempty subset Y' of Y, then $(X', Y'; \Gamma|_{\langle Y' \rangle})$ itself is an abstract convex space which is called to be a subspace of $(X, Y; \Gamma)$.

Definition 2.5 ([23]). Let $(X, Y; \Gamma)$ be an abstract convex space and Z be a set. For a set-valued mapping $H : X \to 2^Z$ with nonempty values, if a set-valued mapping $G : Y \to 2^Z$ satisfies $H(\Gamma_A) \subseteq G(A)$ for every $A \in \langle Y \rangle$, then G is called to be a KKM mapping with respect to H. A KKM mapping $G : Y \to 2^X$ is a KKM mapping with respect to the identity mapping 1_X.

Definition 2.6 ([23]). Let $(X, Y; \Gamma)$ be an abstract convex space and Z be a topological space. A set-valued mapping $H : X \to 2^Z$ is called to be a $\mathfrak{RC}$-mapping, if for any closed-valued KKM mapping $G : Y \to 2^Z$ with respect to H, the family $\{G(y) : y \in Y\}$ has the finite intersection property. We denote $\mathfrak{RC}(X, Z) := \{H : X \to 2^Z | \, H \text{ is a } \mathfrak{RC}\text{-mapping}\}$.

Definition 2.7 ([27]). Let $(X, Y; \Gamma)$ be an abstract convex space. A function $f : X \to \mathbb{R}$ is said to be quasi-convex (respectively, quasi-concave) relative to a nonempty subset Y' of Y if the set $\{x \in X : f(x) < t\}$ (respectively, $\{x \in X : f(x) > t\}$) is Γ-convex relative to Y' for every $r \in \mathbb{R}$. In case $X = Y$, a function $f : X \to \mathbb{R}$ is said to be quasi-convex (respectively, quasi-concave) if the set $\{x \in X : f(x) < t\}$ (respectively, $\{x \in X : f(x) > t\}$) is Γ-convex for every $r \in \mathbb{R}$

Lemma 2.5 ([28]). *Let* $\{(X_i, Y_i; \Gamma_i)\}_{i \in I}$ *be a family of abstract convex spaces, where* I *is a finite (or infinite) index set. Let* $X := \prod_{i \in I} X_i$ *be equipped with the product topology and* $Y := \prod_{i \in I} Y_i$. *For each* $i \in I$, *let* $\pi_i : Y \to Y_i$ *be the projection. Define* $\Gamma = \prod_{i \in I} \Gamma_i : \langle Y \rangle \to 2^E$ *by* $\Gamma(A) := \prod_{i \in I} \Gamma_i(\pi_i(A))$ *for each* $A \in \langle Y \rangle$, *where* $\pi_i(A)$ *is the projection of* A *onto* X_i. *Then* $(X, Y; \Gamma)$ *is an abstract convex space.*

Lemma 2.6 ([29]). *Let* $(X, Y; \Gamma)$ *be an abstract convex space,* $(X', Y'; \Gamma|_{\langle Y' \rangle})$ *be a subspace of* $(X, Y; \Gamma)$, *and Z be a topological space. If* $H \in \mathfrak{RC}(X, Z)$*, then* $H|_{X'} \in \mathfrak{RC}(X', \mathrm{cl}(H(X')))$.

Let $(X;\Gamma)$ be an abstract convex space and C be a nonempty subset of X. We define the Γ-convex combination of C, denoted by $\Gamma\text{-co}(C)$ as follows.

$$\Gamma\text{-co}(C) = \bigcap\{D \subseteq X : D \text{ is } \Gamma\text{-convex and } C \subseteq D\}.$$

We can see that $\Gamma\text{-co}(C)$ is the smallest Γ-convex subset containing C. In fact, for any Γ-convex subset D of X with $C \subseteq D$, it follows from the definition of $\Gamma\text{-co}(C)$ that $\Gamma\text{-co}(C) \subseteq D$. Next, we show that $\Gamma\text{-co}(C)$ is Γ-convex. Indeed, let $A \in \langle \Gamma\text{-co}(C)\rangle$. Then for each Γ-convex subset D of X with $C \subseteq D$, we have $A \subseteq \Gamma\text{-co}(C) \subseteq D$. Since D is Γ-convex, it follows that $\Gamma_A \subseteq D$ and thus, $\Gamma_A \subseteq \Gamma\text{-co}(C)$ which implies that $\Gamma\text{-co}(C)$ is Γ-convex. It is obvious that C is Γ-convex if and only if $C = \Gamma\text{-co}(C)$.

Lemma 2.7. *Let* $(X;\Gamma)$ *be an abstract convex space and C be a nonempty subset of X. Then* $\Gamma\text{-co}(C) = \bigcup\{\Gamma\text{-co}(A) : A \in \langle C\rangle\}$.

Proof. Let $A \in \langle C\rangle$. Then by the fact that $\Gamma\text{-co}(A)$ is the smallest Γ-convex subset containing A and $\Gamma\text{-co}(C)$ is the smallest Γ-convex subset containing C, we have $\Gamma\text{-co}(A) \subseteq \Gamma\text{-co}(C)$. Therefore, $\bigcup\{\Gamma\text{-co}(A) : A \in \langle C\rangle\} \subseteq \Gamma\text{-co}(C)$. Next, we prove that $\Gamma\text{-co}(C) \subseteq \bigcup\{\Gamma\text{-co}(A) : A \in \langle C\rangle\}$. Since $\bigcup\{\Gamma\text{-co}(A) : A \in \langle C\rangle\} \supseteq C$, it suffices to show that $\bigcup\{\Gamma\text{-co}(A) : A \in \langle C\rangle\}$ is Γ-convex. Let $B = \{x_0, x_1, \dots, x_n\} \in \langle \bigcup\{\Gamma\text{-co}(A) : A \in \langle C\rangle\}\rangle$. Then there exist finite subsets $A_0, A_1, \dots, A_n$ of C such that $x_i \in \Gamma\text{-co}(A_i)$, $i = 0, 1, \dots, n$. Let $\widehat{A} = \bigcup_{i=0}^{n} A_i$. Then we have $\widehat{A} \in \langle C\rangle$ and $x_i \in \Gamma\text{-co}(\widehat{A})$, $i = 0, 1, \dots, n$. Therefore, by the fact that $\Gamma\text{-co}(\widehat{A})$ is Γ-convex, we get $\Gamma_B \subseteq \Gamma\text{-co}(\widehat{A}) \subseteq \bigcup\{\Gamma\text{-co}(A) : A \in \langle C\rangle\}$, which implies that $\bigcup\{\Gamma\text{-co}(A) : A \in \langle C\rangle\}$ is Γ-convex subset containing C. Hence, $\Gamma\text{-co}(C) \subseteq \bigcup\{\Gamma\text{-co}(A) : A \in \langle C\rangle\}$. This completes the proof. □

Remark 2.3. Lemma 2.7 extends Lemma 1 obtained by Tarafdar [30] in H-spaces, Lemma 2.1 by Tan and Zhang [31] in G-convex spaces, and Lemma 2.1 by Ding [32] in FC-spaces to abstract convex spaces.

Lemma 2.8. *Let* $(X;\Gamma)$ *be an abstract convex space, Y be a topological space, and* $F : Y \to 2^X$ *be a set-valued mapping such that* $F^{-1}(x)$ *is open in Y for every* $x \in X$*. Then the set-valued mapping* $\Gamma\text{-co}(F) : Y \to 2^X$ *defined by* $\Gamma\text{-co}(F)(y) = \Gamma\text{-co}(F(y))$ *for every* $y \in Y$*, has the property that* $(\Gamma\text{-co}(F))^{-1}(x)$ *is open in Y.*

Proof. Let $x \in X$ and $y \in (\Gamma\text{-co}(F))^{-1}(x)$ be any given. Then it suffices to find an open neighborhood O of y in Y such that $O \subseteq (\Gamma\text{-co}(F))^{-1}(x)$. Since $x \in \Gamma\text{-co}(F(y))$, it follows from Lemma 2.7 that there exists $A = \{x_0, \dots, x_n\} \in \langle F(y)\rangle$ such that $x \in \Gamma\text{-co}(A)$. Let $O = \bigcap_{i=0}^{n} F^{-1}(x_i)$. Since $F^{-1}(x_i)$ is open in Y and $y \in F^{-1}(x_i)$ for every $i = 0, \dots, n$, it follows that O is an open neighborhood of y in Y. We show that $O \subseteq (\Gamma\text{-co}(F))^{-1}(x)$. In fact, let $w \in O$ be any given. Then $x_i \in F(w)$ for all $i = 0, \dots, n$. Hence, we have $x \in \Gamma\text{-co}(A) \subseteq \Gamma\text{-co}(F(w))$ and so, $w \in (\Gamma\text{-co}(F))^{-1}(x)$. This implies that $(\Gamma\text{-co}(F))^{-1}(x)$ is open in Y for every $x \in X$. This completes the proof. □

Remark 2.4. Lemma 2.2 due to Ding [32] with underlying FC-spaces, Lemma 3.1 due to Ding [33] for a H-space setting, and Lemma 2.2 due to Tan and Zhang [31] for the framework of a G-convex space are special cases of Lemma 2.8.

3. Collectively fixed points

In this section, by using the KKM method, we obtain the following theorem which characterizes the existence of collectively fixed points for finite families of set-valued mappings in noncompact abstract convex spaces.

Theorem 3.1. *Let $\{(X_i;\Gamma_i)\}_{i\in I}$ be a family of abstract convex spaces such that $(X;\Gamma) := (\prod_{i\in I} X_i;\Gamma)$ is an abstract convex space defined as in Lemma 2.5, where I is a finite index set. Let K be a nonempty compact subset of X. For each $i \in I$, let $S_i,\ T_i : X \to 2^{X_i}$ be two set-valued mappings satisfying*

(i) *for each $x \in X$, $S_i(x) \subseteq T_i(x)$ and $T_i(x)$ is Γ_i-convex;*

(ii) *for each $u_i \in X_i$, $S_i^{-1}(u_i)$ is open in X;*

(iii) *for each $x \in K$, $S_i(x) \neq \emptyset$;*

(iv) *one of the following two conditions holds:*

$(\mathrm{iv})_1$ *for each $N_i \in \langle X_i\rangle$, there exists a compact Γ_i-convex subset L_{N_i} of $(X_i;\Gamma_i)$ containing N_i, such that for $L := \prod_{i\in I} L_{N_i}$, we have*

$$L \setminus K \subseteq \bigcup_{u\in L} \mathrm{int}_L\Big(\bigcap_{i\in I} T_i^{-1}(u_i)\bigcap L\Big);$$

$(\mathrm{iv})_2$ *there exists $u_0 \in X$ such that $\mathrm{cl}(X \setminus \bigcap_{i\in I} T_i^{-1}(u_{0i})) \subseteq K$.*

If $(X;\Gamma)$ satisfies $1_X \in \mathfrak{RC}(X,X)$, then there exists $\overline{x} = (\overline{x}_i)_{i\in I} \in X$ such that $\overline{x}_i \in T_i(\overline{x})$ for every $i \in I$.

Proof. Define two set-valued mappings $S, T : X \to 2^X$ by $S(x) = \prod_{i\in I} S_i(x)$ and $T(x) = \prod_{i\in I} T_i(x)$ for every $x \in X$, respectively. We distinguish the following two cases for proving the conclusion that there exists $\overline{x} \in X$ such that $\overline{x} \in T(\overline{x})$.

Case I. If $(\mathrm{iv})_1$ holds, then we suppose contrary to the assertion that $x \notin T(x)$ for every $x \in X$. Define two set-valued mappings $\widetilde{S}, \widetilde{T} : X \to 2^X$ by $\widetilde{S}(u) = (X \setminus S^{-1}(u)) \bigcap K$ and $\widetilde{T}(u) = \mathrm{cl}(X \setminus T^{-1}(u)) \bigcap K$ for every $u \in X$, respectively. We show that the family $\{\widetilde{T}(u) : u \in X\}$ has the finite intersection property. Indeed, let $N \in \langle X\rangle$ be any given and let π_i be the projection from X to X_i for every $i \in I$. Then for each $i \in I$, we have $\pi_i(N) = N_i \in \langle X_i\rangle$ and thus, it follows from $(\mathrm{iv})_1$ that there is a compact Γ_i-convex subset L_{N_i} of $(X_i;\Gamma_i)$ containing N_i such that $L = \prod_{i\in I} L_{N_i}$. Further, let us define two set-valued mappings $S', T' : L \to 2^L$ by $S'(u) = L \setminus S^{-1}(u)$ and $T'(u) = \mathrm{cl}_L(L \setminus T^{-1}(u))$ for every $u \in L$, respectively. For each $u \in X$, by the definition of S, we have

$$\begin{aligned} S^{-1}(u) &= \Big\{x \in X : u \in S(x)\Big\} \\ &= \Big\{x \in X : u \in \prod_{i\in I} S_i(x)\Big\} \\ &= \Big\{x \in X : u_i \in S_i(x), \forall i \in I\Big\} \\ &= \Big\{x \in X : x \in S_i^{-1}(u_i), \forall i \in I\Big\} \\ &= \bigcap_{i\in I} S_i^{-1}(u_i). \end{aligned}$$

Similarly, we have $T^{-1}(u) = \bigcap_{i\in I} T_i^{-1}(u_i)$ for every $u \in X$. Since I is a finite index set, it follows from (ii) that $S^{-1}(u)$ is open in X for every $u \in X$. By (i), we can see that $T'(u) \subseteq S'(u)$ for every $u \in X$. Now, we check that the set-valued mapping $T'' : L \to 2^L$ defined by $T''(u) = L \setminus T^{-1}(u)$ for every $u \in L$, is a KKM mapping. In fact, if this were not, then there exist $A \in \langle L\rangle$ and $x_0 \in \Gamma(A) \subseteq L$ such that

$$x_0 \notin \bigcup_{u\in A} T''(u) = L \setminus \Big(\bigcap_{u\in A} T^{-1}(u)\Big),$$

which implies that $x_0 \in \bigcap_{u\in A} T^{-1}(u)$ and thus, $A \subseteq T(x_0)$. By (i) again, we can deduce that $T(x_0)$ is Γ-convex. Therefore, we have $x_0 \in \Gamma(A) \subseteq T(x_0)$, which contradicts our assumption that $x \notin T(x)$ for every $x \in X$. Hence, $T'' : L \to 2^L$ is a KKM mapping and so is T'. Since L is Γ-convex, it follows from Remark 2.2 that $(L;\Gamma|_{\langle L\rangle})$ be a subspace of $(X;\Gamma)$. So, by Lemma 2.6 and the fact that $1_X \in \Re\mathfrak{C}(X,X)$, we have $1_L \in \Re\mathfrak{C}(L,L)$. Since T' is a KKM mapping with closed compact values and $(\text{iv})_1$ holds, it follows that $\emptyset \neq \bigcap_{u\in L} T'(u) = \bigcap_{u\in L} \text{cl}_L(L \setminus T^{-1}(u)) \subseteq L \bigcap K$. Let $x_0 \in \bigcap_{u\in L} T'(u)$. Then we have

$$\begin{aligned} x_0 &\in \bigcap_{u\in L} T'(u) \\ &\subseteq \bigcap_{u\in N} \Big(T'(u) \bigcap K\Big) \\ &\subseteq \bigcap_{u\in N} \widetilde{T}(u). \end{aligned}$$

This implies that the family $\{\widetilde{T}(u) : u \in X\}$ has the finite intersection property. By the compactness of K, we obtain $\bigcap_{u\in X} \widetilde{T}(u) \neq \emptyset$. Since $\widetilde{T}(u) \subseteq \widetilde{S}(u)$ for every $u \in X$, we have

$$\begin{aligned} \emptyset &\neq \bigcap_{u\in X} \widetilde{S}(u) \\ &= \bigcap_{u\in X} \Big(X \setminus S^{-1}(u)\Big) \bigcap K \\ &= K \setminus \bigcup_{u\in X} S^{-1}(u), \end{aligned}$$

which implies that there exists $x^* \in K$ such that $S(x^*) = \emptyset$. By the definition of S again, there exists $i_0 \in I$ such that $S_{i_0}(x^*) = \emptyset$, which contradicts (iii). Therefore, there exists $\overline{x} \in X$ such that $\overline{x} \in T(\overline{x})$. By the definition of T, we have $\overline{x}_i \in T_i(\overline{x})$ for every $i \in I$. This completes the proof.

Case II. Assume that $(\text{iv})_2$ hold. Suppose to the contrary that $x \notin T(x)$ for every $x \in X$. Define two set-valued mappings $\widetilde{S}, \widetilde{T} : X \to 2^X$ by $\widetilde{S}(u) = (X \setminus S^{-1}(u))$ and $\widetilde{T}(u) = \text{cl}(X \setminus T^{-1}(u))$ for every $u \in X$, respectively. By (i), (ii), and the expressions of $S^{-1}(u)$ and $T^{-1}(u)$ in Case I, we have $\widetilde{T}(u) \subseteq \widetilde{S}(u)$ for every $u \in X$. We show that $\Gamma(A) \subseteq \bigcup_{u\in A} \widetilde{T}(u)$ for every $A \in \langle X\rangle$, that is, $\widetilde{T}$ is a KKM mapping. Otherwise, there exist $A \in \langle X\rangle$ and a point $x_0 \in \Gamma(A)$ such that $x_0 \notin \bigcup_{u\in A} \widetilde{T}(u) = X \setminus \bigcap_{u\in A} \text{int}T^{-1}(u)$. It follows that $x_0 \in \bigcap_{u\in A} T^{-1}(u)$. Therefore, we have $A \subseteq T(x_0)$. According to (i) and the definition of T, we can see that $T(x_0)$ is Γ-convex and thus, $x_0 \in \Gamma(A) \subseteq T(x_0)$. This creates a contradiction. Hence, $\widetilde{T}$ is a KKM mapping. Since $1_X \in \Re\mathfrak{C}(X,X)$ and $\widetilde{T}(u)$ is closed in X for every $u \in X$, it follows that the family $\{\widetilde{T}(u) : u \in X\}$ has the finite intersection property. By $(\text{iv})_2$, there exists $u_0 \in X$ such that

$$\begin{aligned} \widetilde{T}(u_0) &= \text{cl}(X \setminus T^{-1}(u_0)) \\ &= \text{cl}\Big(X \setminus \bigcap_{i\in I} T_i^{-1}(u_{0i})\Big) \\ &\subseteq K, \end{aligned}$$

which implies that $\widetilde{T}(u_0)$ is compact. Consequently, the intersection of the family $\{\widetilde{T}(u) : u \in X\}$ is nonempty. Let $x_0 \in \bigcap_{u\in X} \widetilde{T}(u)$. Then we have $x_0 \in K \bigcap(\bigcap_{u\in X} \widetilde{S}(u))$. Thus, we get $S(x_0) = \emptyset$. It follows from the definition of S that there exists $i_0 \in I$ such that $S_{i_0}(x_0) = \emptyset$, which contradicts (iii). Therefore, there exists $\overline{x} \in X$ such that $\overline{x} \in T(\overline{x})$. By the definition of T again, we have $\overline{x}_i \in T_i(\overline{x})$ for every $i \in I$. The proof is complete. □

Remark 3.1. (1) Unlike Theorem 6.1 obtained by Park [18], the abstract convex spaces involved in Theorem 3.1 is not required to be compact.

(2) Theorem 3.1 cannot be regarded as a special case of Theorem 10 due to Lu and Hu [19]. Although (i)–(iii) of Theorem 3.1 are stronger than the corresponding conditions of Theorem 10 in Lu and Hu [19], Theorem 3.1 has two coercive conditions to be selected, and both the first coercive condition of Theorem 3.1 and the corresponding coercive condition of Theorem 10 in Lu and Hu [19] are independent of each other. Thus, Theorem 3.1 and Theorem 10 obtained by Lu and Hu [19] cannot be deduced from each other. In addition, the methods of proving these two theorems are also different. The proof of our theorem is based on KKM theory in abstract convex spaces, and the proof of Theorem 10 in Lu and Hu [19] is to use a fixed point theorem in abstract convex spaces.

Theorem 3.2. *Let $\{(X_i;\Gamma_i)\}_{i\in I}$ be a family of abstract convex spaces such that $(X;\Gamma) := (\prod_{i\in I} X_i;\Gamma)$ is an abstract convex space defined as in Lemma 2.5, where I is a finite index set. Let K be a nonempty compact subset of X. For each $i \in I$, let $S_i,\ T_i : X \to 2^{X_i}$ be two set-valued mappings satisfying*

(i) *for each $x \in X$, $S_i(x) \subseteq \Gamma\text{-co}(T_i(x))$;*

(ii) *for each $u_i \in X_i$, $S_i^{-1}(u_i)$ is open in X;*

(iii) *for each $x \in K$, $S_i(x) \neq \emptyset$;*

(iv) *one of the following two conditions holds*:

(iv)$_1$ *for each $N_i \in \langle X_i\rangle$, there exists a compact Γ_i-convex subset L_{N_i} of $(X_i;\Gamma_i)$ containing N_i, such that for $L := \prod_{i\in I} L_{N_i}$, we have*

$$L \setminus K \subseteq \bigcup_{u\in L} \operatorname{int}_L\Big(\bigcap_{i\in I} T_i^{-1}(u_i) \bigcap L\Big);$$

(iv)$_2$ *there exists $u_0 \in X$ such that* $\operatorname{cl}(X \setminus \bigcap_{i\in I} T_i^{-1}(u_{0i})) \subseteq K$.

If $(X;\Gamma)$ satisfies $1_X \in \mathfrak{RC}(X,X)$, then there exists $\overline{x} = (\overline{x}_i)_{i\in I} \in X$ such that $\overline{x}_i \in \Gamma\text{-co}(T_i(\overline{x}))$ for every $i \in I$.

Proof. For each $i \in I$, we define a set-valued mapping $\widetilde{T}_i : X \to 2^{X_i}$ by $\widetilde{T}_i(x) = \Gamma\text{-co}(T_i(x))$ for every $x \in X$. By (i) the definition of Γ-convex combination, we can see that $S_i(x) \subseteq \widetilde{T}_i(x)$ and $\widetilde{T}_i(x)$ is Γ_i-convex for every $i \in I$ and every $x \in X$. From (iv) and the definition of $\widetilde{T}_i$, one can see that one of the following two conditions holds:

- for each $N_i \in \langle X_i\rangle$, there exists a compact Γ_i-convex subset L_{N_i} of $(X_i;\Gamma_i)$ containing N_i, such that for $L := \prod_{i\in I} L_{N_i}$, we have

$$L \setminus K \subseteq \bigcup_{u\in L} \operatorname{int}_L\Big(\bigcap_{i\in I} \widetilde{T}_i^{-1}(u_i) \bigcap L\Big);$$

- there exists $u_0 \in X$ such that $\operatorname{cl}(X \setminus \bigcap_{i\in I} \widetilde{T}_i^{-1}(u_{0i})) \subseteq K$.

So far, combined with (ii) and (iii), we can see that all the conditions of Theorem 3.1 are fulfilled. Thus, by Theorem 3.1, there exists $\overline{x} = (\overline{x}_i)_{i\in I} \in X$ such that $\overline{x}_i \in \Gamma\text{-co}(T_i(\overline{x}))$ for every $i \in I$. This completes the proof.

Remark 3.2. Theorem 3.1 is equivalent to Theorem 3.2. In fact, we only need to show that theorem 3.2 implies Theorem 3.1. By (i) of Theorem 3.1 and the definition of Γ-convex combination, we have $T_i(x) = \Gamma\text{-co}(T_i(x))$ for every $i \in I$ and every $x \in X$. Therefore, it follows from Theorem 3.2 that there exists $\overline{x} = (\overline{x}_i)_{i\in I} \in X$ such that $\overline{x}_i \in \Gamma\text{-co}(T_i(\overline{x})) = T_i(x)$ for every $i \in I$.

Let I in Theorem 3.1 be a singleton. Then we have the following fixed point theorem.

Theorem 3.3. *Let $(X;\Gamma)$ be an abstract convex space, K be a nonempty compact subset of X, and $S,\ T : X \to 2^X$ be two set-valued mappings such that*

(i) *for each $x \in X$, $S(x) \subseteq T(x)$ and $T(x)$ is Γ-convex*;

(ii) *for each $u \in X$, $S^{-1}(u)$ is open in X*;

(iii) *for each $x \in K$, $S(x) \neq \emptyset$*;

(iv) *one of the following two conditions holds*:

$(\text{iv})_1$ *for each $N \in \langle X\rangle$, there exists a compact Γ-convex subset L_N of $(X;\Gamma)$ containing N such that*

$$L_N \setminus K \subseteq \bigcup_{u\in L_N} \text{int}_{L_N}\Big(T^{-1}(u) \bigcap L_N\Big);$$

$(\text{iv})_2$ *there exists $u_0 \in X$ such that $\text{cl}(X \setminus T^{-1}(u_0)) \subseteq K$.*

If $(X;\Gamma)$ satisfies $1_X \in \mathfrak{RC}(X, X)$, then there exists $\overline{x} \in X$ such that $\overline{x} \in T(\overline{x})$.

Remark 3.3. Theorem 3.3 extends the famous Fan-Browder fixed point theorem due to Browder [34], Corollary 1 obtained by Horvath and Ciscar [35], Theorem 3.2 by Yannelis and Prabhakar [36], Corollary 1 by Ansari and Yao [3], Corollary 3.1 by Al-Homidan and Ansari [37], Theorem 2.4 by Luo [38], and several other fixed point theorems in the literature to noncompact abstract convex spaces (see Park [18] and the references therein).

When I is a singleton and $S = T$, it is obvious that the following maximal element theorem can be obtained from Theorem 3.1 (or Theorem 3.3). We omit the proof.

Theorem 3.4. *Let $\{(X;\Gamma^1)\}$ and $\{(Y;\Gamma^2)\}$ be two abstract convex spaces such that $(X \times Y;\Gamma^1 \times \Gamma^2)$ is an abstract convex space defined as in Lemma 2.5. Let K be a nonempty compact subset of $X \times Y$. Let $T : X \times Y \to 2^{X\times Y}$ be a set-valued mapping satisfying*

(i) *for each $(x, y) \in X \times Y$, $T(x, y)$ is $\Gamma^1 \times \Gamma^2$-convex*;

(ii) *for each $(u, v) \in X \times Y$, $T^{-1}(u, v)$ is open in $X \times Y$*;

(iii) *for each $(x, y) \in X \times Y$, $(x, y) \notin T(x, y)$*;

(iv) *one of the following two conditions holds*:

$(\text{iv})_1$ *for each $N_0 \times N_1 \in \langle X \times Y\rangle$, there exist a compact Γ^1-convex subset L_{N_0} of $(X;\Gamma^1)$ containing N_0 and a compact Γ^2-convex subset L_{N_1} of $(Y;\Gamma^2)$ containing N_1 such that for $L := L_{N_0} \times L_{N_1}$, one has $L \setminus K \subseteq \bigcup_{(u,v)\in L} T^{-1}(u, v)$*;

$(\text{iv})_2$ *there exists $(u_0, v_0) \in X \times Y$ such that $X \times Y \setminus T^{-1}(u_0, v_0) \subseteq K$.*

If $(X\times Y;\Gamma^1\times\Gamma^2)$ satisfies $1_{X\times Y} \in \mathfrak{RC}(X\times Y, X\times Y)$, then there exists $(\overline{x}, \overline{y}) \in K$ such that $T(\overline{x}, \overline{y}) = \emptyset$.

Remark 3.4. (1) It is obvious that $(\text{iv})_1$ of Theorem 3.4 is equivalent to the following condition:

$(\text{iv})_1{}'$ for each $N_0 \times N_1 \in \langle X \times Y\rangle$, there exist a compact Γ^1-convex subset L_{N_0} of $(X;\Gamma^1)$ containing N_0 and a compact Γ^2-convex subset L_{N_1} of $(Y;\Gamma^2)$ containing N_1 such that for $L := L_{N_0} \times L_{N_1}$, one has $L \setminus K \subseteq \bigcup_{(u,v)\in L}(T^{-1}(u, v) \bigcap L)$.

(2) If we drop (i) of Theorem 3.4, then (iii) of Theorem 3.4 can be replaced by the following stronger condition:

$$(x, y) \notin \Gamma^1 \times \Gamma^2\text{-co}(T(x, y)),\ \forall (x, y) \in X \times Y. \tag{3.1}$$

In fact, we can show that the conclusion of Theorem 3.4 still holds when (3.1) is satisfied. Define a set-valued mapping $\widetilde{T} : X \times Y \to 2^{X\times Y}$ by $\widetilde{T}(x, y) = \Gamma^1 \times \Gamma^2\text{-co}(T(x, y))$ for every $(x, y) \in X \times Y$. It is obvious that $\widetilde{T}(x, y)$ is $\Gamma^1 \times \Gamma^2$-convex for every $(x, y) \in X \times Y$. By Lemma 2.8, $\widetilde{T}^{-1}(u, v)$ is open in

$X\times Y$ for every $(u,v)\in X\times Y$. It follows from (3.1) that $(x,y)\notin \widetilde{T}(x,y)$ for every $(x,y)\in X\times Y$. Finally, by (iv), we can see that one of the following two conditions holds:

- for each $N_0\times N_1\in\langle X\times Y\rangle$, there exist a compact Γ^1-convex subset L_{N_0} of $(X;\Gamma^1)$ containing N_0 and a compact Γ^2-convex subset L_{N_1} of $(Y;\Gamma^2)$ containing N_1 such that for $L:=L_{N_0}\times L_{N_1}$, one has

$$L\setminus K\subseteq \bigcup_{(u,v)\in L} T^{-1}(u,v)\subseteq \bigcup_{(u,v)\in L}\widetilde{T}^{-1}(u,v);$$

- there exists $(u_0,v_0)\in X\times Y$ such that

$$X\times Y\setminus \widetilde{T}^{-1}(u_0,v_0)\subseteq X\times Y\setminus T^{-1}(u_0,v_0)\subseteq K.$$

Thus, all the hypotheses of Theorem 3.4 are satisfied. Therefore, by Theorem 3.4, there exists $(\overline{x},\overline{y})\in K$ such that $\widetilde{T}(\overline{x},\overline{y})=\emptyset$ and so, $T(\overline{x},\overline{y})=\emptyset$.

(3) Combining the above arguments in (2), we can see that Theorem 3.4 generalizes Lemma 2.1 of Balaj and Lin [39] in the following aspects: (a) from noncompact topological vector spaces to noncompact abstract convex spaces; (b) the Hausdorffness of the topological spaces in Theorem 3.4 is redundant, while the topological spaces in Lemma 2.1 of Balaj and Lin [39] are assumed to be Hausdorff; (c) from one coercivity condition to two alternative coercivity conditions; (d) the conclusion of our Theorem 3.4 is stronger than that of Lemma 2.1 of Balaj and Lin [39] since the maximal elements of T can be found in K instead of X.

4. Weighted Nash equilibria and Pareto equilibria

In this section, we shall consider the constrained multiobjective game in its strategic form $\Theta := ((X_i;\Gamma_i),U^i,A_i,B_i)_{i\in I}$, where $I=\{1,2,\dots,n\}$ is a finite set of player. For each $i\in I$, X_i is the strategy set of player i such that $(X_i;\Gamma_i)$ is an abstract convex space, $A_i,B_i : X=\prod_{i\in I}X_i\to 2^{X_i}$ are two constraint set-valued mappings of the ith player, and $U^i : X=\Pi_{i\in I}X_i\to\mathbb{R}^{k_i}$ is the payoff function of the ith player, where $k_i\in\mathbb{N}$. For each $i\in I$, we denote $X_{\widehat{i}} := \prod_{j\in I\setminus i}X_j$. If $x=(x_1,x_2,\dots,x_n)\in X$, then we write $x_{\widehat{i}} := (x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$ for every $i\in I$. If $x_i\in X_i$, $z_i\in X_i$ and $x_{\widehat{i}}\in X_{\widehat{i}}$, then we use the notation $(x_{\widehat{i}},x_i) := (x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_n)=x\in X$ and the natation $(x_{\widehat{i}},z_i) := (x_1,\dots,x_{i-1},z_i,x_{i+1},\dots,x_n)\in X$. If a choice $x=(x_1,\dots,x_n)$ is played, each player i is trying to find his/her vector payoff function $U^i(x) := (u^i_1(x),\dots,u^i_{k_i}(x))$ consisting of non-commensurable outcomes. Each player i has a preference $\geq_i$ over the outcome space $\mathbb{R}^{k_i}$. For each $i\in I$, the ith player's preference $\geq_i$ is defined by

$$z^1\geq_i z^2 \text{ if and only if } z^1_j\geq z^2_j \text{ for each } j=1,\dots,k_i,$$

where $z^1=(z^1_1,\dots,z^1_{k_i})\in\mathbb{R}^{k_i}$ and $z^2=(z^2_1,\dots,z^2_{k_i})\in\mathbb{R}^{k_i}$. The players' preference relations induce the preferences on X which is defined by $x\geq_i y \Leftrightarrow U^i(x)\geq_i U^i(y)$ for each player i and their choices $x=(x_1,\dots,x_n),\ y=(y_1,\dots,y_n)\in X$.

If $A_(x)=B_i(x)\neq X_i$ for every $i\in I$ and every $x\in X$, then the model of constrained multiobjective games with two constrained set-valued mappings reduces to the model of constrained multiobjective games with one constrained set-valued mapping considered by Ding [40] and Kim and Ding [41]. If $A_(x)=B_i(x)=X_i$ for every $i\in I$ and every $x\in X$, then the constrained multiobjective game model reduces to the multiobjective game model studied by Wang [42], Yuan and Tarafdar [43], and Yu and Yuan [44].

We need to point out that the constrained multiobjective game model in this paper is a non-cooperative game model, which implies that there is no communicating between players and so, players act as free agents, and each player is trying to minimize his/her own payoff function according to his/her preference.

For a multiobjective game, as it is well known, in general, there does not exist a strategy $\widehat{x} \in X$ to minimize all $u^i_j s$ for each player $i \in I$; see, for example, Yu [45] and the references therein. Hence, we need to give some solution concepts for the multicriteria games with constraint set-valued mappings.

Throughout this paper, for each $m \in \mathbb{N}$, we shall denote by $\mathbb{R}^m_+ := \{q := (q_1, \dots, q_m) \in \mathbb{R}^m : q_j \geq 0, \forall j = 1, \dots, m\}$ and $\text{int}\mathbb{R}^m_+ := \{q := (q_1, \dots, q_m) \in \mathbb{R}^m : q_j > 0, \forall j = 1, \dots, m\}$ the nonnegative orthant of $\mathbb{R}^m$ and the nonempty interior with the topology induced by the Euclidean metric, respectively. For each $u, v \in \mathbb{R}^m$, $u \cdot v$ denotes the standard Euclidian inner product.

Let $\widehat{x} = (\widehat{x_1}, \dots, \widehat{x_n}) \in X$. Now, we have the following definitions.

Definition 4.1. A strategy $\widehat{x_i} \in X_i$ of player i is said to be a generalized Pareto efficient strategy (respectively, a generalized weak Pareto efficient strategy) of the constrained multiobjective game $\Theta = ((X_i; \Gamma_i), U^i, A_i, B_i)_{i \in I}$ with respect to $\widehat{x}$ if $\widehat{x_i} \in B_i(\widehat{x})$ and there is no strategy $x_i \in A_i(\widehat{x})$ such that

$$U^i(\widehat{x}) - U^i(\widehat{x_{\hat{i}}}, x_i) \in \mathbb{R}^{k_i}_+ \setminus \{0\} \text{ (respectively, } U^i(\widehat{x}) - U^i(\widehat{x_{\hat{i}}}, x_i) \in \text{int}\mathbb{R}^{k_i}_+).$$

Definition 4.2. A strategy $\widehat{x} \in X$ is said to be a generalized Pareto equilibrium (respectively, a generalized weak Pareto equilibrium) of the constrained multiobjective game $\Theta = ((X_i; \Gamma_i), U^i, A_i, B_i)_{i \in I}$ if for each player i, $\widehat{x_i} \in X_i$ is a generalized Pareto efficient strategy (respectively, a generalized weak Pareto efficient strategy) of the constrained multiobjective game $\Theta := ((X_i; \Gamma_i), U^i, A_i, B_i)_{i \in I}$ with respect to $\widehat{x}$.

Remark 4.1. The above two definitions generalize the corresponding definitions in [42–44]. It is clear that every generalized Pareto equilibrium is a generalized weak Pareto equilibrium, but the converse is not always true.

Definition 4.3. A strategy $\widehat{x} \in X$ is said to be a generalized weighted Nash equilibrium with respect to the weight vector $W = (W_i)_{i \in I}$ with $W_i = (W_{i,1}, W_{i,2} \dots, W_{i,k_i}) \in \mathbb{R}^{k_i}_+$ of the constrained multiobjective game $\Theta = ((X_i; \Gamma_i), U^i, A_i, B_i)_{i \in I}$ if for each player i, we have

(i) $\widehat{x_i} \in B_i(\widehat{x})$;

(ii) $W_i \in \mathbb{R}^{k_i}_+ \setminus \{0\}$;

(iii) $W_i \cdot U^i(\widehat{x}) \leq W_i \cdot U^i(\widehat{x_{\hat{i}}}, x_i)$ for every $x_i \in A_i(\widehat{x})$, where $\cdot$ denotes the inner product in $\mathbb{R}^{k_i}$.

Remark 4.2. When $W_i \in \mathbb{R}^{k_i}_+$ with $\sum_{j=1}^{k_i} W_{ij} = 1$ for every $i \in I$, the strategy $\widehat{x} \in X$ is said to a normalized form of generalized weighted Nash equilibrium with respect to the weight vector W. In addition, it follows from the above definition that $\widehat{x} \in X$ is a generalized weighted Nash equilibrium with respect to the weight vector $W = (W_i)_{i \in I}$ of the constrained multiobjective game $\Theta = ((X_i; \Gamma_i), U^i, A_i, B_i)_{i \in I}$ if and only if $\widehat{x} \in X$ is a solution of the constrained optimization problem as follows: find $\widehat{x} \in X$ such that for each $i \in I$, $\widehat{x_i} \in B_i(\widehat{x})$ and $\min_{y_i \in A_i(\widehat{x})} W_i \cdot U^i(\widehat{x_{\hat{i}}}, y_i) = W_i \cdot U^i(\widehat{x})$.

The following lemma shows that the existence problem of generalized weak Pareto equilibrium (respectively, generalized Pareto equilibrium) for a constrained multiobjective game can be reduced to the existence problem of generalized weighted Nash equilibrium under certain conditions.

Lemma 4.1. *Let $\Theta = ((X_i; \Gamma_i), U^i, A_i, B_i)_{i \in I}$ be a constrained multiobjective game. Then a normalized form of generalized weighted Nash equilibrium $\widehat{x} \in X$ with respect to a weight $W = (W_1, \dots, W_n)$,*

$W_i \in \mathbb{R}_+^{k_i} \setminus \{0\}$ *(respectively,* $W_i \in \text{int}\mathbb{R}_+^{k_i}$*) and* $\sum_{j=1}^{k_i} W_{i,j} = 1$ *for every* $i \in I$*, is a generalized weak Pareto equilibrium (respectively, a generalized Pareto equilibrium) of the game* Θ*.*

Proof. Suppose to the contrary that $\widehat{x}$ is not a generalized weak Pareto equilibrium. Then by Definitions 4.1 and 4.2, there exists some $i_0 \in I$ such that $\widehat{x}_{i_0} \notin B_{i_0}(\widehat{x})$ or there exists an $x_{i_0} \in A_{i_0}(\widehat{x})$ such that

$$U^i(\widehat{x_{\widehat{i_0}}}, \widehat{x}_{i_0}) - U^i(\widehat{x_{\widehat{i_0}}}, x_{i_0}) \in \text{int}\mathbb{R}_+^{k_{i_0}}.$$

It is obvious that $\widehat{x}_{i_0} \notin B_{i_0}(\widehat{x})$ contradicts the the assumption that $\widehat{x}$ is a normalized generalized weighted Nash equilibrium with respect to the weight $W = (W_1, \ldots, W_n)$. Thus, we only consider the second case that there exists an $x_{i_0} \in A_{i_0}(\widehat{x})$ such that $U^i(\widehat{x_{\widehat{i_0}}}, \widehat{x}_{i_0}) - U^i(\widehat{x_{\widehat{i_0}}}, x_{i_0}) \in \text{int}\mathbb{R}_+^{k_{i_0}}$. In fact, since $W_{i_0} \in \mathbb{R}_+^{k_{i_0}} \setminus \{0\}$ with $\sum_{j=1}^{k_{i_0}} W_{i_0,j} = 1$, it follows that $W_{i_0} \cdot U^i(\widehat{x_{\widehat{i_0}}}, \widehat{x}_{i_0}) > W_{i_0} \cdot U^i(\widehat{x_{\widehat{i_0}}}, x_{i_0})$, which also contradicts the fact that $\widehat{x}$ is a normalized form of generalized weighted Nash equilibrium with respect to the weight $W = (W_1, \ldots, W_n)$. Therefore, $\widehat{x}$ is a generalized weak Pareto equilibrium. Now, we suppose that $W_i \in \text{int}\mathbb{R}_+^{k_i}$ and $\sum_{j=1}^{k_i} W_{i,j} = 1$ for every $i \in I$. We show that $\widehat{x}$ is a generalized Pareto equilibrium by contradiction. If this was not the case, then by Definitions 5.1 and 5.2, there exists $i_0 \in I$ such that $\widehat{x}_{i_0} \notin B_{i_0}(\widehat{x})$ or there exists an $x_{i_0} \in A_{i_0}(\widehat{x})$ such that

$$U^i(\widehat{x_{\widehat{i_0}}}, \widehat{x}_{i_0}) - U^i(\widehat{x_{\widehat{i_0}}}, x_{i_0}) \in \mathbb{R}_+^{k_{i_0}} \setminus \{0\}.$$

By using the same argument as in the above, we get contradictions. Therefore, $\widehat{x}$ is a generalized Pareto equilibrium. This completes the proof. □

Remark 4.3. It should be noted that the conclusion of Lemma 4.1 still holds if $\widehat{x} \in X$ is a generalized weighted Nash equilibrium with respect to a weight $W = (W_1, \ldots, W_n)$ satisfying $W_i \in \mathbb{R}_+^{k_i} \setminus \{0\}$ (respectively, $W_i \in \text{int}\mathbb{R}_+^{k_i}$) for every $i \in I$. Also, we point out that a generalized Pareto equilibrium is not necessarily a generalized weighted Nash equilibrium.

Lemma 4.2 ([41]). *Let* X *and* Y *be two topological spaces. Let* $T : X \to 2^Y$ *be a continuous set-valued mapping such that* $T(x)$ *is nonempty compact subset of* Y *for every* $x \in X$*. Suppose that* $f : X \times Y \to \mathbb{R}$ *is a continuous function. Then the function* $\xi : X \to \mathbb{R}$ *defined by* $\xi(x) := min_{y \in T(x)} f(x, y)$ *for every* $x \in X$*, is a continuous function on* X*.*

Now, as applications of Theorems 3.1 and 3.3, we have the following existence theorems of generalized weighted Nash equilibria and generalized Pareto equilibria for constrained multiobjective games.

Theorem 4.1. *Let* $\Theta = ((X_i; \Gamma_i), U^i, A_i, B_i)_{i \in I}$ *be a constrained multiobjective game such that* $(X; \Gamma) := (\prod_{i \in I} X_i; \Gamma)$ *is an abstract convex space defined as in Lemma 2.5 and* K *is a nonempty compact subset of* X*, where* I *is a finite index set. For each* $i \in I$ *and each* $u_i \in X_i$*,* $A_i^{-1}(u_i)$ *is open in* X*. Assume that there exists a weight vector* $W = (W_1, \ldots, W_n)$ *with* $W_i \in \mathbb{R}_+^{k_i} \setminus \{0\}$ *such that for each* $i \in I$*, the following conditions are satisfied*:

(i) *for each* $x \in X$*,* $\emptyset \neq A_i(x) \subseteq B_i(x)$*, and* $B_i(x)$ *is* Γ_i*-convex*;

(ii) *for each* $x \in X$*, the set* $\{u_i \in X_i : W_i \cdot U^i(x_{\widehat{i}}, u_i) < W_i \cdot U^i(x_{\widehat{i}}, x_i)\}$ *is* Γ_i*-convex*;

(iii) *for each* $u_i \in X_i$*, the set* $\{x \in X : W_i \cdot U^i(x_{\widehat{i}}, u_i) < W_i \cdot U^i(x_{\widehat{i}}, x_i)\}$ *is open in* X;

(iv) *the set* $\mathfrak{F}_i = \{x \in X :$ *there exists* $u_i \in A_i(x)$ *such that* $W_i \cdot U^i(x_{\widehat{i}}, u_i) < W_i \cdot U^i(x_{\widehat{i}}, x_i)\}$ *is a closed subset of* X;

(v) *one of the following conditions holds*:

(v)$_1$ *for each $N_i \in \langle X_i \rangle$, there exists a compact Γ_i-convex subset L_{N_i} of $(X_i;\Gamma_i)$ containing N_i such that $L \setminus K \subseteq \bigcup_{u\in L} \mathrm{int}_L(\bigcap_{i\in I}((X\setminus \mathfrak{F}_i)\bigcap B_i^{-1}(u_i))\bigcap L)$, where $L := \prod_{i\in I} L_{N_i}$;*

(v)$_2$ *there exists $u_0 \in X$ such that $\mathrm{cl}(X \setminus \bigcap_{i\in I}((X\setminus \mathfrak{F}_i)\bigcap B_i^{-1}(u_{0i}))) \subseteq K$.*

If $(X;\Gamma)$ satisfies $1_X \in \mathfrak{RC}(X,X)$, then the game Θ has a generalized weighted Nash equilibrium $\widehat{x} \in X$ with respect to the weight vector $W = (W_i)_{i\in I}$ and hence it has a generalized weak Pareto equilibrium. Further, if $W_i \in \mathrm{int}\mathbb{R}_+^{k_i}$ with $\sum_{j=1}^{k_i} W_{i,j} = 1$ for every $i \in I$, then Θ has a generalized Pareto equilibrium.

Proof. We shall prove this theorem by considering the following two cases:

Case I. Suppose that the set $\mathfrak{F}_i = \{x \in X :$ there exists $u_i \in A_i(x)$ such that $W_i \cdot U^i(x_{\widehat{i}}, u_i) < W_i \cdot U^i(x_{\widehat{i}}, x_i)\}$ is empty for every $i \in I$. Then we have $W_i \cdot U^i(x_{\widehat{i}}, u_i) \geq W_i \cdot U^i(x_{\widehat{i}}, x_i)$ for every $i \in I$, $x \in X$, and every $u_i \in A_i(x)$. By (v), we know that the one of the following conditions holds:

- for each $N_i \in \langle X_i \rangle$, there exists a compact Γ_i-convex subset L_{N_i} of $(X_i;\Gamma_i)$ containing N_i such that $L \setminus K \subseteq \bigcup_{u\in L} \mathrm{int}_L(\bigcap_{i\in I}((X\setminus \mathfrak{F}_i)\bigcap B_i^{-1}(u_i))\bigcap L) \subseteq \bigcup_{u\in L} \mathrm{int}_L(\bigcap_{i\in I} B_i^{-1}(u_i)\bigcap L)$, where $L := \prod_{i\in I} L_{N_i}$.
- there exists $u_0 \in X$ such that $\mathrm{cl}(X \setminus \bigcap_{i\in I}(B_i^{-1}(u_{0i}))) \subseteq \mathrm{cl}(X \setminus \bigcap_{i\in I}((X\setminus \mathfrak{F}_i)\bigcap B_i^{-1}(u_{0i}))) \subseteq K$.

By combining (i) and the fact that $A_i^{-1}(u_i)$ is open in X for every $u_i \in X_i$, we can see that all the hypotheses of Theorem 3.1 are satisfied. Thus, by Theorem 3.1, there exists $\widehat{x} \in X$ such that $\widehat{x_i} \in B_i(\widehat{x})$ for every $i \in I$. Therefore, for each $i \in I$, $\widehat{x_i} \in B_i(\widehat{x})$ and $W_i \cdot U^i(\widehat{x}) \leq W_i \cdot U^i(\widehat{x_{\widehat{i}}}, x_i)$ for every $x_i \in A_i(\widehat{x})$, which implies that $\widehat{x} \in X$ is a generalized weighted Nash equilibrium of the game Θ with respect to the weight vector $W = (W_i)_{i\in I}$. It follows from Lemma 4.1 that $\widehat{x} \in X$ is also a generalized weak Pareto equilibrium of Θ, and a generalized Pareto equilibrium of Θ if $W_i \in \mathrm{int}\mathbb{R}_+^{k_i}$ with $\sum_{j=1}^{k_i} W_{i,j} = 1$ for every $i \in I$.

Case II. Suppose that the set $\mathfrak{F}_i = \{x \in X :$ there exists $u_i \in A_i(x)$ such that $W_i \cdot U^i(x_{\widehat{i}}, u_i) < W_i \cdot U^i(x_{\widehat{i}}, x_i)\}$ is nonempty for every $i \in I$. Define a set-valued mapping $Q_i : X \to 2^{X_i}$ by

$$Q_i(x) = \{u_i \in X_i : W_i \cdot U^i(x_{\widehat{i}}, u_i) < W_i \cdot U^i(x_{\widehat{i}}, x_i)\}, \ \forall i \in I \text{ and } x \in X. \tag{4.1}$$

By (4.1), we get

$$x_i \notin Q_i(x), \ \forall i \in I \text{ and } x \in X. \tag{4.2}$$

Further, for each $i \in I$, we define two set-valued mappings $S_i, T_i : X \to 2^{X_i}$ by setting, for each $x \in X$,

$$S_i(x) = \begin{cases} Q_i(x)\bigcap A_i(x), & \text{if } x \in \mathfrak{F}_i, \\ A_i(x), & \text{if } x \in X \setminus \mathfrak{F}_i, \end{cases}$$

$$T_i(x) = \begin{cases} Q_i(x)\bigcap B_i(x), & \text{if } x \in \mathfrak{F}_i, \\ B_i(x), & \text{if } x \in X \setminus \mathfrak{F}_i. \end{cases}$$

It follows from (i), (ii), and the definitions of $\mathfrak{F}_i$ and Q_i that $S_i(x) \subseteq T_i(x)$, $T_i(x)$ is Γ_i-convex, and $S_i(x) \neq \emptyset$ for every $i \in I$ and every $x \in X$. For each $i \in I$ and each $u_i \in X_i$, we have

$$\begin{aligned} S_i^{-1}(u_i) &= \Big\{x \in X : u_i \in S_i(x)\Big\} \\ &= \Big\{x \in \mathfrak{F}_i : u_i \in Q_i(x)\bigcap A_i(x)\Big\}\bigcup\Big\{x \in X \setminus \mathfrak{F}_i : u_i \in A_i(x)\Big\} \\ &= \Big((X\setminus \mathfrak{F}_i)\bigcap A_i^{-1}(u_i)\Big)\bigcup\Big(\mathfrak{F}_i \bigcap Q_i^{-1}(u_i)\bigcap A_i^{-1}(u_i)\Big) \end{aligned}$$

$$= \Big((X \setminus \mathfrak{F}_i) \bigcap A_i^{-1}(u_i)\Big) \bigcup \Big(Q_i^{-1}(u_i) \bigcap A_i^{-1}(u_i)\Big).$$

Then by (iii), (iv), and the definition of Q_i, we can see that $S_i^{-1}(u_i)$ is open in X. Similarly, we get

$$T_i^{-1}(u_i) = \Big((X \setminus \mathfrak{F}_i) \bigcap B_i^{-1}(u_i)\Big) \bigcup \Big(Q_i^{-1}(u_i) \bigcap B_i^{-1}(u_i)\Big).$$

Next, we show that (iv) of Theorem 3.1 is fulfilled. Indeed, by (v) and the expression of $T_i^{-1}(u_i)$, we can see that one of the following conditions holds:

- for each $N_i \in \langle X_i \rangle$, there exists a compact Γ_i-convex subset L_{N_i} of $(X_i; \Gamma_i)$ containing N_i such that for $L := \prod_{i \in I} L_{N_i}$, we have

$$\begin{aligned} L \setminus K &\subseteq \bigcup_{u \in L} \text{int}_L \Big(\bigcap_{i \in I} ((X \setminus \mathfrak{F}_i) \bigcap B_i^{-1}(u_i)) \bigcap L \Big) \\ &\subseteq \bigcup_{u \in L} \text{int}_L \Big(\bigcap_{i \in I} T_i^{-1}(u_i) \bigcap L \Big). \end{aligned}$$

- there exists $u_0 \in X$ such that

$$\begin{aligned} \text{cl}\Big(X \setminus \bigcap_{i \in I} T_i^{-1}(u_{0i}) \Big) &\subseteq \text{cl}\Big(X \setminus \bigcap_{i \in I} ((X \setminus \mathfrak{F}_i) \bigcap B_i^{-1}(u_{0i})) \Big) \\ &\subseteq K. \end{aligned}$$

Thus, we can see that all the conditions of Theorem 3.1 are satisfied. Therefore, it follows from Theorem 3.1 that there exists $\widehat{x} \in X$ such that $\widehat{x}_i \in T_i(\widehat{x})$ for every $i \in I$. If $\widehat{x}_i \in \mathfrak{F}_i$ for some $i \in I$, then it follows from the definition of T_i that $\widehat{x}_i \in Q_i(\widehat{x}) \bigcap B_i(\widehat{x})$. Hence, $\widehat{x}_i \in Q_i(\widehat{x})$, which contradicts (4.2). Therefore, we have $\widehat{x}_i \in X \setminus \mathfrak{F}_i$ for every $i \in I$. By the definitions of Q_i, $\mathfrak{F}_i$, and T_i, we can deduce that for each $i \in I$, $\widehat{x}_i \in B_i(\widehat{x})$ and $Q_i(\widehat{x}) \cap A_i(\widehat{x}) = \emptyset$, that is, for each $i \in I$, $\widehat{x}_i \in B_i(\widehat{x})$ and $W_i \cdot U^i(\widehat{x}) \leq W_i \cdot U^i(\widehat{x}_{\hat{i}}, x_i)$ for every $x_i \in A_i(\widehat{x})$, which implies that $\widehat{x} \in X$ is a generalized weighted Nash equilibrium of the game Θ with respect to the weight vector $W = (W_i)_{i \in I}$. By Lemma 4.1, one can see that $\widehat{x} \in X$ is also a generalized weak Pareto equilibrium of Θ, and a generalized Pareto equilibrium of Θ if $W_i \in \text{int}\mathbb{R}_+^{k_i}$ with $\sum_{j=1}^{k_i} W_{i,j} = 1$ for every $i \in I$. This completes the proof. □

Theorem 4.2. *Let $\Theta = ((X_i; \Gamma_i), U^i, A_i, B_i)_{i \in I}$ be a constrained multiobjective game such that $(X; \Gamma) := (\prod_{i \in I} X_i; \Gamma)$ is a compact abstract convex space defined as in Lemma 2.5, where I is a finite index set. For each $i \in I$, the graph of B_i is closed in $X \times X_i$ and A_i is a continuous set-valued mapping such that each $A_i(x)$ is a Γ_i-convex subset of X_i. Assume that there exists a weight vector $W = (W_1, \ldots, W_n)$ with $W_i \in \mathbb{R}_+^{k_i} \setminus \{0\}$ such that for each $i \in I$, the following conditions are satisfied*:

(i) *for each $x \in X$, $\emptyset \neq A_i(x) \subseteq B_i(x)$, and $B_i(x)$ is Γ_i-convex*;

(ii) *for each $u_i \in X_i$, $B_i^{-1}(u_i)$ is open in X*;

(iii) *the function $(x, u) \mapsto W_i \cdot U^i(x_{\hat{i}}, u_i)$ is jointly continuous on $X \times X$*;

(iv) *for each $x \in X$, the function $u \mapsto W_i \cdot U^i(x_{\hat{i}}, u_i)$ is quasi-convex on X.*

If $(X; \Gamma)$ satisfies $1_X \in \mathfrak{RC}(X, X)$, then the game Θ has a generalized weighted Nash equilibrium $\widehat{x} \in X$ with respect to the weight vector $W = (W_i)_{i \in I}$ and hence it has a generalized weak Pareto equilibrium. Further, if $W_i \in \text{int}\mathbb{R}_+^{k_i}$ with $\sum_{j=1}^{k_i} W_{i,j} = 1$ for every $i \in I$, then Θ has a generalized Pareto equilibrium.

Proof. For each $m \in \mathbb{N}$, define a set-valued mapping $T_m : X \to 2^X$ as follows:

$$T_m(x) = \prod_{i\in I} B_i(x) \bigcap \prod_{i\in I} \Big(\{u_i \in X_i : W_i \cdot U^i(x_{\widehat{i}}, u_i) < \min_{y_i\in A_i(x)} W_i \cdot U^i(x_{\widehat{i}}, y_i) + \frac{1}{m}\}\Big), \ \forall x \in X.$$

Thus, we have $T_m(x) = \prod_{i\in I}\{u_i \in B_i(x) : W_i \cdot U^i(x_{\widehat{i}}, u_i) < \min_{y_i\in A_i(x)} W_i \cdot U^i(x_{\widehat{i}}, y_i) + \frac{1}{m}\}$ for every $x \in X$. By (i) and (iv), we can see that $T_m(x)$ is a nonempty Γ-convex subset of X for every $x \in X$. Note that for each $u \in X$, we have

$$\begin{aligned}
T_m^{-1}(u) &= \Big\{x \in X : u \in T_m(x)\Big\} \\
&= \Big\{x \in X : u \in \prod_{i\in I}\{u_i \in B_i(x) : W_i \cdot U^i(x_{\widehat{i}}, u_i) < \min_{y_i\in A_i(x)} W_i \cdot U^i(x_{\widehat{i}}, y_i) + \frac{1}{m}\}\Big\} \\
&= \Big\{x \in X : u_i \in B_i(x) \text{ and } W_i \cdot U^i(x_{\widehat{i}}, u_i) < \min_{y_i\in A_i(x)} W_i \cdot U^i(x_{\widehat{i}}, y_i) + \frac{1}{m}, \ \forall i \in I\Big\} \\
&= \Big(\bigcap_{i\in I} B_i^{-1}(u_i)\Big) \bigcap \Big(\bigcap_{i\in I}\{x \in X : W_i \cdot U^i(x_{\widehat{i}}, u_i) < \min_{y_i\in A_i(x)} W_i \cdot U^i(x_{\widehat{i}}, y_i) + \frac{1}{m}\}\Big).
\end{aligned}$$

By (ii), (iii), and Lemma 4.2, we have that $T_m^{-1}(u)$ is open in X for every $u \in X$. Therefore, by Theorem 3.2 with $K = X$ and $S = T$, T_m has a fixed point $x(m) \in X$. Then it follows from the definition of T_m that $W_i \cdot U^i(x_{\widehat{i}}(m), x_i(m)) < \min_{y_i\in A_i(x(m))} W_i \cdot U^i(x_{\widehat{i}}(m), y_i) + \frac{1}{m}$ for every $i \in I$. Since X is compact, we may assume that $x(m) \to \widehat{x} \in X$ without loss of generality. Since $x_i(m) \in B_i(x(m))$ and the graph of B_i is closed in $X \times X_i$, we have $\widehat{x_i} \in B_i(\widehat{x})$. By (iii) and Lemma 4.2 again, we have

$$\begin{aligned}
W_i \cdot U^i(\widehat{x_{\widehat{i}}}, \widehat{x_i}) &= \lim_{m\to\infty} W_i \cdot U^i(x_{\widehat{i}}(m), x_i(m)) \\
&\le \lim_{m\to\infty} \min_{y_i\in A_i(x(m))} W_i \cdot U^i(x_{\widehat{i}}(m), y_i) \\
&= \min_{y_i\in A_i(\widehat{x})} W_i \cdot U^i(\widehat{x_{\widehat{i}}}, y_i) \\
&\le \min_{y_i\in B_i(\widehat{x})} W_i \cdot U^i(\widehat{x_{\widehat{i}}}, y_i).
\end{aligned}$$

Since $\widehat{x_i} \in B_i(\widehat{x})$ for every $i \in I$, we have $W_i \cdot U^i(\widehat{x_{\widehat{i}}}, \widehat{x_i}) = \min_{y_i\in A_i(\widehat{x})} W_i \cdot U^i(\widehat{x_{\widehat{i}}}, y_i)$, which implies that $\widehat{x} \in X$ is a generalized weighted Nash equilibrium of the game Θ with respect to the weight vector $W = (W_i)_{i\in I}$. By Lemma 4.1, we can see that $\widehat{x} \in X$ is also a generalized weak Pareto equilibrium of Θ, and a generalized Pareto equilibrium of Θ if $W_i \in \mathrm{int}\mathbb{R}_+^{k_i}$ with $\sum_{j=1}^{k_i} W_{i,j} = 1$ for every $i \in I$. This completes the proof.

Remark 4.4. Theorem 4.2 generalizes Theorem 2 due to Kim and Ding [41] in the following aspects: (a) from topological vector spaces to abstract convex spaces without any linear and convex structure; (b) the topological spaces in Theorem 4.2 need not possess Hausdorff property; (c) from constrained multiobjective games with one constrained set-valued mapping to constrained multiobjective games with two constrained set-valued mappings. Theorem 4.2 also generalizes Theorem 3.1 due to Wang [42] and Theorem 1 due to Yu and Yuan [44] to abstract convex spaces under much weaker assumptions.

If $A_i = B_i$ for every $i \in I$, then by Theorems 4.1 and 4.2, we have the following two theorems.

Theorem 4.3. *Let $\Theta = ((X_i; \Gamma_i), U^i, A_i)_{i\in I}$ be a constrained multiobjective game such that $(X; \Gamma) := (\prod_{i\in I} X_i; \Gamma)$ is an abstract convex space defined as in Lemma 2.5 and K is a nonempty compact subset* *of X, where I is a finite index set. For each $i \in I$ and each $u_i \in X_i$, $A_i^{-1}(u_i)$ is open in X. Assume that there exists a weight vector $W = (W_1, \dots, W_n)$ with $W_i \in \mathbb{R}_+^{k_i} \setminus \{0\}$ such that for each $i \in I$, the following conditions are satisfied*:

(i) *for each $x \in X$, $A_i(x)$ is nonempty Γ_i-convex*;

(ii) *for each $x \in X$, the set $\{u_i \in X_i : W_i \cdot U^i(x_{\widehat{i}}, u_i) < W_i \cdot U^i(x_{\widehat{i}}, x_i)\}$ is Γ_i-convex*;

(iii) *for each $u_i \in X_i$, the set $\{x \in X : W_i \cdot U^i(x_{\widehat{i}}, u_i) < W_i \cdot U^i(x_{\widehat{i}}, x_i)\}$ is open in X*;

(iv) *the set $\mathfrak{F}_i = \{x \in X$: there exists $u_i \in A_i(x)$ such that $W_i \cdot U^i(x_{\widehat{i}}, u_i) < W_i \cdot U^i(x_{\widehat{i}}, x_i)\}$ is a nonempty closed subset of X*;

(v) *one of the following conditions holds*:

$(v)_1$ *for each $N_i \in \langle X_i \rangle$, there exists a compact Γ_i-convex subset L_{N_i} of $(X_i; \Gamma_i)$ containing N_i such that $L \setminus K \subseteq \bigcup_{u \in L} \mathrm{int}_L(\bigcap_{i \in I}((X \setminus \mathfrak{F}_i) \bigcap A_i^{-1}(u_i)) \bigcap L)$, where $L := \prod_{i \in I} L_{N_i}$*;

$(v)_2$ *there exists $u_0 \in X$ such that $\mathrm{cl}(X \setminus \bigcap_{i \in I}((X \setminus \mathfrak{F}_i) \bigcap A_i^{-1}(u_{0i}))) \subseteq K$.*

If $(X; \Gamma)$ satisfies $1_X \in \mathfrak{RC}(X, X)$, then the game Θ has a generalized weighted Nash equilibrium $\widehat{x} \in X$ with respect to the weight vector $W = (W_i)_{i \in I}$ and hence it has a generalized weak Pareto equilibrium. Further, if $W_i \in \mathrm{int}\mathbb{R}_+^{k_i}$ with $\sum_{j=1}^{k_i} W_{i,j} = 1$ for every $i \in I$, then Θ has a generalized Pareto equilibrium.

Theorem 4.4. *Let $\Theta = ((X_i; \Gamma_i), U^i, A_i)_{i \in I}$ be a constrained multiobjective game such that $(X; \Gamma) := (\prod_{i \in I} X_i; \Gamma)$ is a compact abstract convex space defined as in Lemma 2.5, where I is a finite index set. For each $i \in I$, the graph of A_i is closed in $X \times X_i$. Assume that there exists a weight vector $W = (W_1, \dots, W_n)$ with $W_i \in \mathbb{R}_+^{k_i} \setminus \{0\}$ such that for each $i \in I$, the following conditions are satisfied*:

(i) *for each $x \in X$, $A_i(x)$ is nonempty Γ_i-convex*;

(ii) *for each $u_i \in X_i$, $A_i^{-1}(u_i)$ is open in X*;

(iii) *the function $(x, u) \mapsto W_i \cdot U^i(x_{\widehat{i}}, u_i)$ is jointly continuous on $X \times X$*;

(iv) *for each $x \in X$, the function $u \mapsto W_i \cdot U^i(x_{\widehat{i}}, u_i)$ is quasi-convex on X.*

If $(X; \Gamma)$ satisfies $1_X \in \mathfrak{RC}(X, X)$, then the game Θ has a generalized weighted Nash equilibrium $\widehat{x} \in X$ with respect to the weight vector $W = (W_i)_{i \in I}$ and hence it has a generalized weak Pareto equilibrium. Further, if $W_i \in \mathrm{int}\mathbb{R}_+^{k_i}$ with $\sum_{j=1}^{k_i} W_{i,j} = 1$ for every $i \in I$, then Θ has a generalized Pareto equilibrium.

Proof. It suffices to prove that A_i is a continuous set-valued mapping for every $i \in I$. In fact, since the graph of A_i is closed in $X \times X_i$ and X_i is compact topological space for every $i \in I$, it follows from Lemma 2.1 that A_i is an upper semicontinuous set-valued mapping. We note that each A_i has open lower sections and so, A_i is a lower semicontinuous set-valued mapping. Therefore, A_i is a continuous set-valued mapping. Let $A_i = B_i$ for every $i \in I$. Then by Theorem 4.2, the conclusion of Theorem 4.4 holds. This completes the proof.

By setting $A_i(x) \equiv X_i$ for every $i \in I$ and every $x \in X$, we have the following corollaries from Theorems 4.3-4.4. These two corollaries characterize the existence of weighted Nash equilibria for the multiobjective games without constrained set-valued mappings.

Corollary 4.1. *Let $\Theta = ((X_i; \Gamma_i), U^i)_{i \in I}$ be a multiobjective game such that $(X; \Gamma) := (\prod_{i \in I} X_i; \Gamma)$ is an abstract convex space defined as in Lemma 2.5 and K is a nonempty compact subset of X, where I is a finite index set. Assume that there exists a weight vector $W = (W_1, \dots, W_n)$ with $W_i \in \mathbb{R}_+^{k_i} \setminus \{0\}$ such that for each $i \in I$, the following conditions are satisfied*:

(i) *for each $x \in X$, the set $\{u_i \in X_i : W_i \cdot U^i(x_{\widehat{i}}, u_i) < W_i \cdot U^i(x_{\widehat{i}}, x_i)\}$ is Γ_i-convex*;

(ii) *for each $u_i \in X_i$, the set $\{x \in X : W_i \cdot U^i(x_{\widehat{i}}, u_i) < W_i \cdot U^i(x_{\widehat{i}}, x_i)\}$ is open in X*;

(iii) *the set* $\mathfrak{F}_i = \{x \in X :$ *there exists* $u_i \in X_i$ *such that* $W_i \cdot U^i(x_{\widehat{i}}, u_i) < W_i \cdot U^i(x_{\widehat{i}}, x_i)\}$ *is closed in* X;

(iv) *one of the following conditions holds*:

(iv)$_1$ *for each* $N_i \in \langle X_i \rangle$*, there exists a compact* Γ_i*-convex subset* L_{N_i} *of* $(X_i; \Gamma_i)$ *containing* N_i *such that* $L \setminus K \subseteq \bigcup_{u \in L} \text{int}_L(\bigcap_{i \in I}(X \setminus \mathfrak{F}_i) \bigcap L)$*, where* $L := \prod_{i \in I} L_{N_i}$;

(iv)$_2$ *there exists* $u_0 \in X$ *such that* $\text{cl}(X \setminus \bigcap_{i \in I}(X \setminus \mathfrak{F}_i)) \subseteq K$.

If $(X; \Gamma)$ *satisfies* $1_X \in \mathfrak{RC}(X, X)$*, then the game* Θ *has a weighted Nash equilibrium* $\widehat{x} \in X$ *with respect to the weight vector* $W = (W_i)_{i \in I}$ *and hence it has a weak Pareto equilibrium. Further, if* $W_i \in \text{int}\mathbb{R}^{k_i}_+$ *with* $\sum_{j=1}^{k_i} W_{i,j} = 1$ *for every* $i \in I$*, then* Θ *has a Pareto equilibrium.*

Remark 4.5. If $\{(X_i; \Gamma_i)\}_{i \in I}$ is a family of abstract convex spaces such that X_i is a first-countable topological space for every $i \in I$, then (iii) of Corollary 4.1 can be replaced with the following condition:

(iii)$'$ for each $i \in I$, the graph of the set-valued mapping $Q_i : X \to 2^{X_i}$ defined by $Q_i(x) = \{u_i \in X_i : W_i \cdot U^i(x_{\widehat{i}}, u_i) < W_i \cdot U^i(x_{\widehat{i}}, x_i)\}$ for each $x \in X$, is closed in $X \times X_i$ and for each compact subset $Z \subseteq X$, the set $Q_i(Z)$ is compact subset of X_i.

In fact, let $i \in I$ be fixed. For each $x \in \text{cl}(\{x \in X : Q_i(x) \neq \emptyset\})$, since each X_i is a first-countable topological space, it follows that $X = \prod_{i \in I} X_i$ is a first-countable topological space. By Theorem 2.40 due to Aliprantis and Border [21], there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq \{x \in X : Q_i(x) \neq \emptyset\}$ such that $x_n \to x \in X$. Thus, we have $Q_i(x_n) \neq \emptyset$ and thus, for every $n \in \mathbb{N}$, there exists $u_{in} \in X_i$ such that $u_{in} \in Q_i(x_n)$. Let $L = \{x_n\}_{n \in \mathbb{N}} \cup \{x\}$. Then by Theorem 2.38 due to Aliprantis and Border [21], L is compact subset of X. By (iii)$'$, the set $Q_i(L) = \cup_{x \in L} Q_i(x)$ is compact subset of X_i. Since $\{u_{in}\}_{n \in \mathbb{N}} \subseteq Q_i(L)$, it follows that $\{u_{in}\}_{n \in \mathbb{N}}$ has a convergent subnet with limit u_i^*. Without loss of generality, we may assume that $u_{in} \to u_i$. Since the graph of Q_i is closed, we have $u_i \in Q_i(x)$, which implies that

$$x \in \{x \in X : Q_i(x) \neq \emptyset\}.$$

Therefore, the set $\{x \in X : Q_i(x) \neq \emptyset\} = \{x \in X :$ there exists $u_i \in X_i$ such that $W_i \cdot U^i(x_{\widehat{i}}, u_i) < W_i \cdot U^i(x_{\widehat{i}}, x_i)\}$ is closed in X.

Corollary 4.2. *Let* $\Theta = ((X_i; \Gamma_i), U^i)_{i \in I}$ *be a multiobjective game such that* $(X; \Gamma) := (\prod_{i \in I} X_i; \Gamma)$ *is a compact abstract convex space defined as in Lemma 2.5, where* I *is a finite index set. Assume that there exists a weight vector* $W = (W_1, \ldots, W_n)$ *with* $W_i \in \mathbb{R}^{k_i}_+ \setminus \{0\}$ *such that for each* $i \in I$*, the following conditions are satisfied*:

(i) *the function* $(x, u) \mapsto W_i \cdot U^i(x_{\widehat{i}}, u_i)$ *is jointly continuous on* $X \times X$;

(ii) *for each* $x \in X$*, the function* $u \mapsto W_i \cdot U^i(x_{\widehat{i}}, u_i)$ *is quasi-convex on* X.

If $(X; \Gamma)$ *satisfies* $1_X \in \mathfrak{RC}(X, X)$*, then the game* Θ *has a weighted Nash equilibrium* $\widehat{x} \in X$ *with respect to the weight vector* $W = (W_i)_{i \in I}$ *and hence it has a weak Pareto equilibrium. Further, if* $W_i \in \text{int}\mathbb{R}^{k_i}_+$ *with* $\sum_{j=1}^{k_i} W_{i,j} = 1$ *for every* $i \in I$*, then* Θ *has a Pareto equilibrium..*

Remark 4.6. Corollary 4.2 is different from Corollary 4.1 in the following aspects: (a) the topological spaces in Corollary 4.1 may be noncompact, while the topological spaces in Corollary 4.2 need to be compact; (b) (i) and (ii) of Corollary 4.1 are respectively weaker than (i) and (ii) of Corollary 4.2; (c) in order to guarantee the conclusion of Corollary 4.1 holds, the closeness condition of the set $\mathfrak{F}_i$ and the coercive condition, that is, (iii) and (iv) of Corollary 4.1 must be satisfied, but Corollary 4.2 does not need theses conditions.

5. Sets with abstract convex sections

In this section, by using Theorems 3.1 and 3.2, we establish some new nonempty intersection theorems for sets with abstract convex sections. Furthermore, as applications of nonempty intersection property for sets with abstract convex sections, we obtain an analytic alternative formulation and two existence results of Nash equilibria for noncooperative games in noncompact abstract convex spaces.

Theorem 5.1. *Let $\{(X_i;\Gamma_i)\}_{i\in I}$ be a family of abstract convex spaces such that $(X;\Gamma) := (\prod_{i\in I} X_i;\Gamma)$ is an abstract convex space defined as in Lemma 2.5 and $K = \prod_{i\in I} K_i$ is a nonempty compact subset of X, where I is a finite index set. For each $i \in I$, let P_i and Q_i be two subsets of X satisfying the following conditions:*

(i) *for each $x_{\hat{i}} \in X_{\hat{i}}$, $\{y_i \in X_i : (x_{\hat{i}}, y_i) \in P_i\} \subseteq \{y_i \in X_i : (x_{\hat{i}}, y_i) \in Q_i\}$ and $\{y_i \in X_i : (x_{\hat{i}}, y_i) \in Q_i\}$ is Γ_i-convex*;

(ii) *for each $u_i \in X_i$, $\{x_{\hat{i}} \in X_{\hat{i}} : (x_{\hat{i}}, u_i) \in P_i\}$ is open in $X_{\hat{i}}$*;

(iii) *for each $x_{\hat{i}} \in K_{\hat{i}}$, $\{y_i \in X_i : (x_{\hat{i}}, y_i) \in P_i\} \neq \emptyset$*;

(iv) *one of the following two conditions holds*:

$(\mathrm{iv})_1$ *for each $N_i \in \langle X_i\rangle$, there exists a compact Γ_i-convex subset L_{N_i} of $(X_i;\Gamma_i)$ containing N_i, such that for $L := \prod_{i\in I} L_{N_i}$, we have*

$$L \setminus K \subseteq \bigcup_{u\in L} \mathrm{int}_L\Big((\bigcap_{i\in I}(\{x_{\hat{i}} \in X_{\hat{i}} : (x_{\hat{i}}, u_i) \in Q_i\} \times X_i)) \bigcap L\Big);$$

$(\mathrm{iv})_2$ *there exists $u_0 = (u_{0i})_{i\in I} \in X$ such that* $\mathrm{cl}(X \setminus \bigcap_{i\in I}(\{x_{\hat{i}} \in X_{\hat{i}} : (x_{\hat{i}}, u_{0i}) \in Q_i\} \times X_i)) \subseteq K$.

If $(X;\Gamma)$ satisfies $1_X \in \mathfrak{RC}(X, X)$, then $\bigcap_{i\in I} Q_i \neq \emptyset$.

Proof. For each $i \in I$, let us define two set-valued mappings $S_i, T_i : X \to 2^{X_i}$ by $S_i(x) = \{y_i \in X_i : (x_{\hat{i}}, y_i) \in P_i\}$ and $T_i(x) = \{y_i \in X_i : (x_{\hat{i}}, y_i) \in Q_i\}$ for every $x = (x_i)_{i\in I} \in X$. Then by (i), we have $S_i(x) \subseteq T_i(x)$ and $T_i(x)$ is Γ_i-convex for every $i \in I$ and every $x \in X$. For each $i \in I$ and each $u_i \in X_i$, we have $S_i^{-1}(u_i) = \{x_{\hat{i}} \in X_{\hat{i}} : (x_{\hat{i}}, u_i) \in P_i\} \times X_i$ which is an open subset of X by (ii) and the definition of S_i. For each $i \in I$, it follows from (iii) and the definition of S_i that $S_i(x) \neq \emptyset$ for every $x \in K$. Finally, we show that (iv) of Theorem 3.1 is fulfilled. Indeed, by (iv) and the fact that $T_i^{-1}(u_i) = \{x_{\hat{i}} \in X_{\hat{i}} : (x_{\hat{i}}, u_i) \in Q_i\} \times X_i$, one can see that one of the following conditions holds:

- for each $N_i \in \langle X_i\rangle$, there exists a compact Γ_i-convex subset L_{N_i} of $(X_i;\Gamma_i)$ containing N_i such that for $L := \prod_{i\in I} L_{N_i}$, we have

$$\begin{aligned} L \setminus K &\subseteq \bigcup_{u\in L} \mathrm{int}_L\Big((\bigcap_{i\in I}(\{x_{\hat{i}} \in X_{\hat{i}} : (x_{\hat{i}}, u_i) \in Q_i\} \times X_i)) \bigcap L\Big) \\ &\subseteq \bigcup_{u\in L} \mathrm{int}_L\Big(\bigcap_{i\in I} T_i^{-1}(u_i) \bigcap L\Big). \end{aligned}$$

- there exists $u_0 = (u_{0i})_{i\in I} \in X$ such that

$$\begin{aligned} \mathrm{cl}\Big(X \setminus \bigcap_{i\in I} T_i^{-1}(u_{0i})\Big) &= \mathrm{cl}(X \setminus \bigcap_{i\in I}(\{x_{\hat{i}} \in X_{\hat{i}} : (x_{\hat{i}}, u_{0i}) \in Q_i\} \times X_i)) \\ &\subseteq K. \end{aligned}$$

Thus, we can see that all the conditions of Theorem 3.1 are satisfied. Therefore, it follows from Theorem 3.1 that there exists $\widehat{x} \in X$ such that $\widehat{x_i} \in T_i(\widehat{x}) = \{y_i \in X_i : (\widehat{x_{\hat{i}}}, y_i) \in Q_i\}$ for every $i \in I$, that is, $\widehat{x} = (\widehat{x_{\hat{i}}}, \widehat{x_i}) \in Q_i$ for every $i \in I$ and thus, $\bigcap_{i\in I} Q_i \neq \emptyset$. Our proof is complete. □

Remark 5.1. Theorem 5.1 extends Theorem 7.1 in Park [18], Theorem 22 in Park [23], Theorem 4.15 in Bielawski [46], and Theorem 5.2 in Kirk et al. [47] to noncompact abstract convex spaces.

Theorem 5.2. *Let $\{(X_i;\Gamma_i)\}_{i\in I}$ be a family of abstract convex spaces such that $(X;\Gamma) := (\prod_{i\in I} X_i;\Gamma)$ is an abstract convex space defined as in Lemma 2.5 and $K = \prod_{i\in I} K_i$ is a nonempty compact subset of X, where I is a finite index set. For each $i \in I$, let P_i and Q_i be two subsets of X satisfying the following conditions:*

(i) *for each $x_{\widehat{i}} \in X_{\widehat{i}}$, $\Gamma\text{-co}(\{y_i \in X_i : (x_{\widehat{i}}, y_i) \in P_i\}) \subseteq \{y_i \in X_i : (x_{\widehat{i}}, y_i) \in Q_i\}$;*

(ii) *for each $u_i \in X_i$, $\{x_{\widehat{i}} \in X_{\widehat{i}} : (x_{\widehat{i}}, u_i) \in P_i\}$ is open in $X_{\widehat{i}}$;*

(iii) *for each $x_{\widehat{i}} \in K_{\widehat{i}}$, $\{y_i \in X_i : (x_{\widehat{i}}, y_i) \in P_i\} \neq \emptyset$;*

(iv) *one of the following two conditions holds*:

(iv)$_1$ *for each $N_i \in \langle X_i\rangle$, there exists a compact Γ_i-convex subset L_{N_i} of $(X_i;\Gamma_i)$ containing N_i, such that for $L := \prod_{i\in I} L_{N_i}$, we have*

$$L \setminus K \subseteq \bigcup_{u\in L} \text{int}_L\Big((\bigcap_{i\in I}(\{x_{\widehat{i}} \in X_{\widehat{i}} : (x_{\widehat{i}}, u_i) \in P_i\} \times X_i)) \bigcap L\Big);$$

(iv)$_2$ *there exists $u_0 = (u_{0i})_{i\in I} \in X$ such that $\text{cl}(X \setminus \bigcap_{i\in I}(\{x_{\widehat{i}} \in X_{\widehat{i}} : (x_{\widehat{i}}, u_{0i}) \in P_i\} \times X_i)) \subseteq K$.*

If $(X;\Gamma)$ satisfies $1_X \in \mathfrak{RC}(X,X)$, then $\bigcap_{i\in I} Q_i \neq \emptyset$.

Proof. For each $i \in I$, we define two set-valued mappings $S_i, \widetilde{S}_i : X \to 2^{X_i}$ by $S_i(x) = \{y_i \in X_i : (x_{\widehat{i}}, y_i) \in P_i\}$ and $\widetilde{S}_i(x) = \Gamma\text{-co}(\{y_i \in X_i : (x_{\widehat{i}}, y_i) \in P_i\}) = \Gamma\text{-co}(S_i(x))$ for every $x = (x_i)_{i\in I} \in X$. It is obvious that $\Gamma\text{-co}(S_i(x))$ is Γ_i-convex for all $i \in I$ and all $x = (x_i)_{i\in I} \in X$. From (ii) and the definition of S_i, it follows that $S_i^{-1}(u_i) = \{x_{\widehat{i}} \in X_{\widehat{i}} : (x_{\widehat{i}}, u_i) \in P_i\} \times X_i$ is an open subset of X for every $i \in I$ and every $u_i \in X_i$. Thus, by Lemma 2.8, $\widetilde{S}_i^{-1}(u_i)$ is also an open subset of X for every $i \in I$ and every $u_i \in X_i$. By (iii), we have $\widetilde{S}_i(x) \supseteq S_i(x) \neq \emptyset$ for every $i \in I$ and every $x \in K$. Since $S_i^{-1}(u_i) = \{x_{\widehat{i}} \in X_{\widehat{i}} : (x_{\widehat{i}}, u_i) \in P_i\} \times X_i \subseteq \widetilde{S}_i^{-1}(u_i)$ for every $i \in I$ and every $u_i \in X_i$, it follows from (iv) that that one of the following conditions holds:

- for each $N_i \in \langle X_i\rangle$, there exists a compact Γ_i-convex subset L_{N_i} of $(X_i;\Gamma_i)$ containing N_i such that for $L := \prod_{i\in I} L_{N_i}$, we have

$$\begin{aligned} L \setminus K &\subseteq \bigcup_{u\in L} \text{int}_L\Big((\bigcap_{i\in I}(\{x_{\widehat{i}} \in X_{\widehat{i}} : (x_{\widehat{i}}, u_i) \in P_i\} \times X_i)) \bigcap L\Big) \\ &= \bigcup_{u\in L} \text{int}_L\Big(\bigcap_{i\in I} S_i^{-1}(u_i) \bigcap L\Big) \\ &\subseteq \bigcup_{u\in L} \text{int}_L\Big(\bigcap_{i\in I} \widetilde{S}_i^{-1}(u_i) \bigcap L\Big). \end{aligned}$$

- there exists $u_0 = (u_{0i})_{i\in I} \in X$ such that

$$\begin{aligned} \text{cl}\Big(X \setminus \bigcap_{i\in I} \widetilde{S}_i^{-1}(u_{0i})\Big) &\subseteq \text{cl}\Big(X \setminus \bigcap_{i\in I} S_i^{-1}(u_{0i})\Big) \\ &= \text{cl}(X \setminus \bigcap_{i\in I}(\{x_{\widehat{i}} \in X_{\widehat{i}} : (x_{\widehat{i}}, u_{0i}) \in P_i\} \times X_i)) \\ &\subseteq K. \end{aligned}$$

Thus, we can see that all the conditions of Theorem 3.1 with $S_i = T_i$ are satisfied. Therefore, we know that there exists $\widehat{x} \in X$ such that $\widehat{x}_i \in \widetilde{S}_i(\widehat{x}) = \Gamma\text{-co}(S_i(\widehat{x})) = \Gamma\text{-co}(\{y_i \in X_i : (\widehat{x}_{\widehat{i}}, y_i) \in P_i\})$ for every

$i \in I$. For this $\widehat{x}$, by (i), we have $\widehat{x_i} \in \Gamma\text{-co}(\{y_i \in X_i : (\widehat{x_{\hat{i}}}, y_i) \in P_i\}) \subseteq \{y_i \in X_i : (\widehat{x_{\hat{i}}}, y_i) \in Q_i\}$ for every $i \in I$, which implies that $\widehat{x} = (\widehat{x_{\hat{i}}}, \widehat{x_i}) \in Q_i$ for every $i \in I$. Therefore, we get $\bigcap_{i\in I} Q_i \neq \emptyset$. This completes the proof.

Remark 5.2. Except that the condition that the index set of Theorem 5.2 is finite is stronger than the condition that the index set of Theorem 16 due to Fan [48] is arbitrary, Theorem 5.2 partially generalizes Theorem 16 of Fan [48] in the following aspects: (a) from compact topological vector spaces to noncompact abstract convex spaces without any linear and convex structure; (b) there is no Hausdorff separation requirement for the abstract convex spaces involved Theorem 5.3. The topological vector spaces in Theorem 16 of Fan [48] need to meet the Hausdorff separation requirement because the continuous unity partition theory is used in the proof of this theorem; (c) even if we strengthen the abstract convex spaces in Theorem 5.2 to be topological vector spaces, (iii) of Theorem 5.2 is still weaker than the first half of (b) of Theorem 16 due to Fan [48].

Theorem 5.3. *Let $\{(X_i; \Gamma_i)\}_{i\in I}$ be a family of abstract convex spaces such that $(X; \Gamma) := (\prod_{i\in I} X_i; \Gamma)$ is an abstract convex space defined as in Lemma 2.5 and $K = \prod_{i\in I} K_i$ is a nonempty compact subset of X, where I is a finite index set. For each $i \in I$, let P_i and Q_i be two subsets of X satisfying the following conditions:*

(i) *for each $x_{\hat{i}} \in X_{\hat{i}}$, $\{y_i \in X_i : (x_{\hat{i}}, y_i) \in P_i\} \subseteq \Gamma\text{-co}(\{y_i \in X_i : (x_{\hat{i}}, y_i) \in Q_i\})$;*

(ii) *for each $u_i \in X_i$, $\{x_{\hat{i}} \in X_{\hat{i}} : (x_{\hat{i}}, u_i) \in P_i\}$ is open in $X_{\hat{i}}$;*

(iii) *for each $x_{\hat{i}} \in K_{\hat{i}}$, $\{y_i \in X_i : (x_{\hat{i}}, y_i) \in P_i\} \neq \emptyset$;*

(iv) *one of the following two conditions holds*:

$(\text{iv})_1$ *for each $N_i \in \langle X_i\rangle$, there exists a compact Γ_i-convex subset L_{N_i} of $(X_i; \Gamma_i)$ containing N_i, such that for $L := \prod_{i\in I} L_{N_i}$, we have*

$$L \setminus K \subseteq \bigcup_{u\in L} \text{int}_L\Big(\big(\bigcap_{i\in I} (\{x_{\hat{i}} \in X_{\hat{i}} : (x_{\hat{i}}, u_i) \in Q_i\} \times X_i)\big) \bigcap L \Big);$$

$(\text{iv})_2$ *there exists $u_0 = (u_{0i})_{i\in I} \in X$ such that* $\text{cl}(X \setminus \bigcap_{i\in I}(\{x_{\hat{i}} \in X_{\hat{i}} : (x_{\hat{i}}, u_{0i}) \in Q_i\} \times X_i)) \subseteq K$.

If $(X; \Gamma)$ satisfies $1_X \in \mathfrak{RC}(X, X)$, then there exists $\widehat{x} \in X$ such that $\widehat{x_i} \in \Gamma\text{-co}(\{y_i \in X_i : (\widehat{x_{\hat{i}}}, y_i) \in Q_i\})$ for every $i \in I$.

Proof. For each $i \in I$, define two set-valued mappings $S_i, T_i : X \to 2^{X_i}$ by $S_i(x) = \{y_i \in X_i : (x_{\hat{i}}, y_i) \in P_i\}$ and $T_i(x) = \{y_i \in X_i : (x_{\hat{i}}, y_i) \in Q_i\}$ for every $x = (x_i)_{i\in I} \in X$. Then it is easy to verify that S_i and T_i satisfy all the requirements of Theorem 3.2. Therefore, by Theorem 3.2, there exists $\widehat{x} \in X$ such that $\widehat{x_i} \in \Gamma\text{-co}(T_i(\widehat{x})) = \Gamma\text{-co}(\{y_i \in X_i : (\widehat{x_{\hat{i}}}, y_i) \in Q_i\})$ for every $i \in I$. This completes the proof. □

Remark 5.3. We can compare Theorem 5.3 and Theorem 2.3 obtained by Lan and Webb [2] from the following aspects: (a) Theorem 5.3 is based on noncompact abstract convex spaces without any linear and convex structure. The Hausdorffness of the abstract convex spaces involved Theorem 5.3 is redundant. Theorem 2.3 due to by Lan and Webb [2] is established in the framework of Hausdorff topological vector spaces; (b) Theorem 5.3 has two coercive conditions to be available, and Theorem 2.3 obtained by Lan and Webb [2] has only one coercive condition; (c) there are two families of subsets of X in Theorem 5.3. In Theorem 2.3 obtained by Lan and Webb [2], there is only one family of subsets of X; (d) even the abstract convex spaces in Theorem 5.3 are strengthened to be topological vector spaces, (iii) of Theorem 5.3 is weaker than (S_1) of Theorem 2.3 due to Lan and Webb [2]; (e) Theorem 5.3 deals with nonempty intersection of finite number of sets with abstract convex sections, and Theorem 2.3 in Lan and Webb [2] concerns on nonempty intersection of arbitrary number of sets with convex sections.

Theorem 5.4. *Suppose that all the requirements of Theorem 5.3 are satisfied. For each $i \in I$, let V_i be a subset of X such that for each $x \in X$, there is a subset $I(x)$ of I such that Γ-co$(\{y_i \in X_i : (x_{\widehat{i}}, y_i) \in Q_i\}) \subseteq \{y_i \in X_i : (x_{\widehat{i}}, y_i) \in V_i\}$ for every $i \in I(x)$. Then there exists $\widehat{x} \in X$ such that $\bigcap_{i \in I(\widehat{x})} V_i \neq \emptyset$.*

Proof. By Theorem 5.3, there exists $\widehat{x} \in X$ such that $\widehat{x}_i \in \Gamma$-co$(\{y_i \in X_i : (\widehat{x_{\widehat{i}}}, y_i) \in Q_i\})$ for every $i \in I$. Therefore, for this $\widehat{x}$, we have $\widehat{x}_i \in \{y_i \in X_i : (\widehat{x_{\widehat{i}}}, y_i) \in V_i\}$ for every $i \in I(\widehat{x})$, which implies that there exists a point $\widehat{x} \in X$ such that $\bigcap_{i \in I(\widehat{x})} V_i \neq \emptyset$. This completes the proof.

Now, we present the following analytical formulation of Theorem 5.3.

Theorem 5.5. *Let $\{(X_i; \Gamma_i)\}_{i \in I}$ be a family of abstract convex spaces such that $(X; \Gamma) := (\prod_{i \in I} X_i; \Gamma)$ is an abstract convex space defined as in Lemma 2.5 and $K = \prod_{i \in I} K_i$ is a nonempty compact subset of X, where I is a finite index set. For each $i \in I$, let ξ_i, ρ_i, $\upsilon_i : X \to \mathbb{R}$ be three real-valued functions and let t_i be a real number satisfying the following conditions:*

(i) *for each $x \in X$, $\xi_i(x) \leq \rho_i(x) \leq \upsilon_i(x)$;*

(ii) *for each $u_i \in X_i$, $\xi_i(., u_i)$ is lower semicontinuous on $X_{\widehat{i}}$;*

(iii) *for each $x_{\widehat{i}} \in X_{\widehat{i}}$, $\upsilon_i(x_{\widehat{i}}, .)$ is quasiconcave on X_i;*

(iv) *one of the following two conditions holds*:

(iv)$_1$ *for each $N_i \in \langle X_i \rangle$, there exists a compact Γ_i-convex subset L_{N_i} of $(X_i; \Gamma_i)$ containing N_i, such that for $L := \prod_{i \in I} L_{N_i}$, we have*

$$L \setminus K \subseteq \bigcup_{u \in L} \mathrm{int}_L\Big((\bigcap_{i \in I} (\{x_{\widehat{i}} \in X_{\widehat{i}} : \rho_i(x_{\widehat{i}}, u_i) > t_i\} \times X_i)) \bigcap L \Big);$$

(iv)$_2$ *there exists $u_0 = (u_{0i})_{i \in I} \in X$ such that* $\mathrm{cl}(X \setminus \bigcap_{i \in I}(\{x_{\widehat{i}} \in X_{\widehat{i}} : \rho_i(x_{\widehat{i}}, u_{0i}) > t_i\} \times X_i)) \subseteq K$.

If $(X; \Gamma)$ satisfies $1_X \in \mathfrak{RC}(X, X)$, then either there exist an $i \in I$ and an $x_{\widehat{i}} \in K_{\widehat{i}}$ such that $\xi_i(x_{\widehat{i}}, y_i) \leq t_i$ for every $y_i \in X_i$ or there exists $\widehat{x} \in X$ such that $\upsilon_i(\widehat{x}) > t_i$ for every $i \in I$.

Proof. Suppose that for each $i \in I$ and each $x_{\widehat{i}} \in K_{\widehat{i}}$, there is $y_i \in X_i$ satisfying $\xi_i(x_{\widehat{i}}, y_i) > t_i$. For each $i \in I$, we define $P_i = \{x \in X : \xi_i(x) > t_i\}$, $Q_i = \{x \in X : \rho_i(x) > t_i\}$, and $V_i = \{x \in X : \upsilon_i(x) > t_i\}$. Then by (i), for each $i \in I$ and each $x_{\widehat{i}} \in X_{\widehat{i}}$, we have

$$\begin{aligned} \{y_i \in X_i : (x_{\widehat{i}}, y_i) \in P_i\} &\subseteq \{y_i \in X_i : (x_{\widehat{i}}, y_i) \in Q_i\} \\ &\subseteq \Gamma\text{-co}(\{y_i \in X_i : (x_{\widehat{i}}, y_i) \in Q_i\}). \end{aligned}$$

By (ii), it follows that the set $\{x_{\widehat{i}} \in X_{\widehat{i}} : (x_{\widehat{i}}, u_i) \in P_i\}$ is an open subset of $X_{\widehat{i}}$ for every $u_i \in X_i$. From the beginning of the proof, we can see that $\{y_i \in X_i : (x_{\widehat{i}}, y_i) \in P_i\} \neq \emptyset$ for all $i \in I$ and all $x_{\widehat{i}} \in K_{\widehat{i}}$. By (iv), one of the following two conditions holds:

- for each $N_i \in \langle X_i \rangle$, there exists a compact Γ_i-convex subset L_{N_i} of $(X_i; \Gamma_i)$ containing N_i such that for $L := \prod_{i \in I} L_{N_i}$, we have

$$\begin{aligned} L \setminus K &\subseteq \bigcup_{u \in L} \mathrm{int}_L\Big((\bigcap_{i \in I} (\{x_{\widehat{i}} \in X_{\widehat{i}} : \rho_i(x_{\widehat{i}}, u_i) > t_i\} \times X_i)) \bigcap L \Big) \\ &= \bigcup_{u \in L} \mathrm{int}_L\Big((\bigcap_{i \in I} (\{x_{\widehat{i}} \in X_{\widehat{i}} : (x_{\widehat{i}}, u_i) \in Q_i\} \times X_i)) \bigcap L \Big). \end{aligned}$$

- there exists $u_0 = (u_{0i})_{i\in I} \in X$ such that

$$\begin{aligned} K &\supseteq \operatorname{cl}(X \setminus \bigcap_{i\in I}(\{x_{\widehat{i}} \in X_{\widehat{i}} : \rho_i(x_{\widehat{i}}, u_{0i}) > t_i\} \times X_i)) \\ &= \operatorname{cl}(X \setminus \bigcap_{i\in I}(\{x_{\widehat{i}} \in X_{\widehat{i}} : (x_{\widehat{i}}, u_{0i}) \in Q_i\} \times X_i)). \end{aligned}$$

Therefore, it follows from Theorem 5.3 that there exists there exists $\widehat{x} \in X$ such that $\widehat{x_i} \in \Gamma\text{-co}(\{y_i \in X_i : (\widehat{x_{\widehat{i}}}, y_i) \in Q_i\})$ for every $i \in I$. By (iii) and the fact that $\rho_i(x) \leq \upsilon_i(x)$ for every $x \in X$, we have $\widehat{x} \in \bigcap_{i\in I} V_i$, which implies that there exists $\widehat{x} \in X$ such that $\upsilon_i(\widehat{x}) > t_i$ for every $i \in I$. The proof is finished.

Remark 5.4. Theorem 5.5 generalizes Theorem 8.1 of Park [18] in the following two aspects: (a) from compact abstract convex spaces to noncompact abstract convex spaces; (b) from two families of real-valued functions to three families of real-valued functions.

Theorem 5.6. *Let $\{(X_i;\Gamma_i)\}_{i\in I}$ be a family of abstract convex spaces such that $(X;\Gamma) := (\prod_{i\in I} X_i;\Gamma)$ is the abstract convex space defined as in Lemma 2.5 and $K = \prod_{i\in I} K_i$ is a nonempty compact subset of X, where I is a finite index set. For each $i \in I$, let $\xi_i, \rho_i, \upsilon_i : X \to \mathbb{R}$ be three real-valued functions satisfying the following conditions:*

(i) *for each $x \in X$, $\xi_i(x) \leq \rho_i(x) \leq \upsilon_i(x)$;*

(ii) *for each $u_i \in X_i$, $\xi_i(., u_i)$ is lower semicontinuous on $X_{\widehat{i}}$;*

(iii) *for each $x_{\widehat{i}} \in X_{\widehat{i}}$, $\upsilon_i(x_{\widehat{i}}, .)$ is quasiconcave on X_i;*

(iv) *for each $x_{\widehat{i}} \in X_{\widehat{i}}$, $\xi_i(x_{\widehat{i}}, .)$ is bounded on X_i and for any $\varepsilon > 0$, suppose that one of the following two conditions holds*:

(iv)$_1$ *for each $N_i \in \langle X_i\rangle$, there exists a compact Γ_i-convex subset L_{N_i} of $(X_i;\Gamma_i)$ containing N_i, such that for $L := \prod_{i\in I} L_{N_i}$, we have*

$$L \setminus K \subseteq \bigcup_{u\in L} \operatorname{int}_L\Big((\bigcap_{i\in I}(\{x_{\widehat{i}} \in X_{\widehat{i}} : \rho_i(x_{\widehat{i}}, u_i) > \sup_{y_i\in X_i} \xi_i(x_{\widehat{i}}, y_i) - \varepsilon\} \times X_i)) \bigcap L\Big);$$

(iv)$_2$ *there exists $u_0 = (u_{0i})_{i\in I} \in X$ such that*

$$\operatorname{cl}(X \setminus \bigcap_{i\in I}(\{x_{\widehat{i}} \in X_{\widehat{i}} : \rho_i(x_{\widehat{i}}, u_{0i}) > \sup_{y_i\in X_i} \xi_i(x_{\widehat{i}}, y_i) - \varepsilon\} \times X_i)) \subseteq K.$$

If $(X;\Gamma)$ satisfies $1_X \in \mathfrak{RC}(X,X)$, then there exists $\widehat{x^\varepsilon} = (\widehat{x^\varepsilon_{\widehat{i}}}, \widehat{x^\varepsilon_i}) \in X$ such that $\upsilon_i(\widehat{x^\varepsilon}) > \sup_{y_i\in X_i} \xi_i(\widehat{x^\varepsilon_{\widehat{i}}}, y_i) - \varepsilon$ for every $i \in I$.

Proof. Set $t_i := \sup_{y_i\in X_i} \xi_i(x_{\widehat{i}}, y_i) - \varepsilon \in \mathbb{R}$ for all $i \in I$ and all $x_{\widehat{i}} \in X_{\widehat{i}}$. Then it is easy to see that for each $i \in I$ and each $x_{\widehat{i}} \in X_{\widehat{i}}$, there exists $y_i \in X_i$ such that $\xi_i(x_{\widehat{i}}, y_i) > t_i$. Thus, it follows from Theorem 5.5 that there exists $\widehat{x^\varepsilon} = (\widehat{x^\varepsilon_{\widehat{i}}}, \widehat{x^\varepsilon_i}) \in X$ such that $\upsilon_i(\widehat{x^\varepsilon}) > t_i = \sup_{y_i\in X_i} \xi_i(\widehat{x^\varepsilon_{\widehat{i}}}, y_i) - \varepsilon$ for every $i \in I$. This completes the proof.

Remark 5.5. Under the conditions of Theorem 9.1 due to Park [18], only the conclusion similar to that of Theorem 5.6 can be obtained. This is because $\widehat{x} \in X$ varies with ε and the conditions of Theorem 9.1 in Park [18] are not sufficient to guarantee the continuity of the function $x_{\widehat{i}} \mapsto \max_{y_i\in X_i} f_i(x_{\widehat{i}}, y_i)$. Thus, from this perspective, Theorem 5.6 generalizes Theorem 9.1 of Park [18] in the following aspects: (a) from compact abstract convex spaces to noncompact abstract convex spaces; (b) from two families of real-valued functions to three families of real-valued functions; (c) the condition that $\xi_i(x_{\widehat{i}}, .)$ is bounded on X_i for every $x_{\widehat{i}} \in X_{\widehat{i}}$, is weaker than (9.2) of Theorem 9.1 due to Park [18].

From Theorem 5.6 for $\xi_i = \rho_i = \upsilon_i$, we can derive the following existence theorem of ε-Nash equilibria for noncooperative games in noncompact abstract convex spaces.

Theorem 5.7. *Let $\{(X_i; \Gamma_i)\}_{i\in I}$ be a family of abstract convex spaces such that $(X; \Gamma) := (\prod_{i\in I} X_i; \Gamma)$ is an abstract convex space defined as in Lemma 2.5 and $K = \prod_{i\in I} K_i$ is a nonempty compact subset of X, where I is a finite index set. For each $i \in I$, let $\xi_i : X \to \mathbb{R}$ be a real-valued function satisfying the following conditions:*

(i) *for each $u_i \in X_i$, $\xi_i(., u_i)$ is lower semicontinuous on $X_{\widehat{i}}$;*

(ii) *for each $x_{\widehat{i}} \in X_{\widehat{i}}$, $\xi_i(x_{\widehat{i}}, .)$ is quasiconcave on X_i;*

(iii) *for each $x_{\widehat{i}} \in X_{\widehat{i}}$, $\xi_i(x_{\widehat{i}}, .)$ is bounded on X_i and for any $\varepsilon > 0$, suppose that one of the following two conditions holds:*

$(\text{iii})_1$ *for each $N_i \in \langle X_i\rangle$, there exists a compact Γ_i-convex subset L_{N_i} of $(X_i; \Gamma_i)$ containing N_i, such that for $L := \prod_{i\in I} L_{N_i}$, we have*

$$L \setminus K \subseteq \bigcup_{u\in L} \text{int}_L\Big((\bigcap_{i\in I}(\{x_{\widehat{i}} \in X_{\widehat{i}} : \xi_i(x_{\widehat{i}}, u_i) > \sup_{y_i\in X_i} \xi_i(x_{\widehat{i}}, y_i) - \varepsilon\} \times X_i)) \bigcap L\Big);$$

$(\text{iii})_2$ *there exists $u_0 = (u_{0i})_{i\in I} \in X$ such that*

$$\text{cl}(X \setminus \bigcap_{i\in I}(\{x_{\widehat{i}} \in X_{\widehat{i}} : \xi_i(x_{\widehat{i}}, u_{0i}) > \sup_{y_i\in X_i} \xi_i(x_{\widehat{i}}, y_i) - \varepsilon\} \times X_i)) \subseteq K.$$

If $(X; \Gamma)$ satisfies $1_X \in \mathfrak{RC}(X, X)$, then there exists $\widehat{x^\varepsilon} = (\widehat{x^\varepsilon_{\widehat{i}}}, \widehat{x^\varepsilon_i}) \in X$ such that $\xi_i(\widehat{x^\varepsilon}) > \sup_{y_i\in X_i} \xi_i(\widehat{x^\varepsilon_{\widehat{i}}}, y_i) - \varepsilon$ for every $i \in I$.

By using a special case of Theorem 5.7, we have the following existence theorem of Nash equilibria for noncooperative games in compact abstract convex spaces.

Corollary 5.1. *Let $\{(X_i; \Gamma_i)\}_{i\in I}$ be a family of compact abstract convex spaces such that $(X; \Gamma) := (\prod_{i\in I} X_i; \Gamma)$ is an abstract convex space defined as in Lemma 2.5, where I is a finite index set. For each $i \in I$, let $\xi_i : X \to \mathbb{R}$ be a real-valued function such that:*

(i) *ξ_i is upper semicontinuous on X;*

(ii) *for each $u_i \in X_i$, $\xi_i(., u_i)$ is lower semicontinuous on $X_{\widehat{i}}$;*

(iii) *for each $x_{\widehat{i}} \in X_{\widehat{i}}$, $\xi_i(x_{\widehat{i}}, .)$ is quasiconcave on X_i.*

If $(X; \Gamma)$ satisfies $1_X \in \mathfrak{RC}(X, X)$, then there exists $\widehat{x} \in X$ such that $\xi_i(\widehat{x}) = \max_{y_i\in X_i} \xi_i(\widehat{x_{\widehat{i}}}, y_i)$ for every $i \in I$.

Proof. Let $\varepsilon > 0$. Then by Theorem 5.7 with each X_i being a compact abstract convex space, it follows there exists $\widehat{x^\varepsilon} = (\widehat{x^\varepsilon_{\widehat{i}}}, \widehat{x^\varepsilon_i}) \in X$ such that $\xi_i(\widehat{x^\varepsilon}) > \max_{y_i\in X_i} \xi_i(\widehat{x^\varepsilon_{\widehat{i}}}, y_i) - \varepsilon$. Let $\varepsilon \to 0$. By the compactness of X and $\{\widehat{x^\varepsilon}\} \subseteq X$, we assume that $\widehat{x^\varepsilon} \to \widehat{x}$ without loss of generality. By (i) and (ii), it follows from Lemma 2 of Yu and Yuan [44] that the function $x_{\widehat{i}} \mapsto \max_{y_i\in X_i} \xi_i(x_{\widehat{i}}, y_i)$ is continuous. Immediately using (i) again, we get $\xi_i(\widehat{x_{\widehat{i}}}, \widehat{x_i}) \ge \overline{\lim}_{\varepsilon\to 0} \xi_i(\widehat{x^\varepsilon_{\widehat{i}}}, \widehat{x^\varepsilon_i}) \ge \overline{\lim}_{\varepsilon\to 0} \max_{y_i\in X_i} \xi_i(\widehat{x^\varepsilon_{\widehat{i}}}, y_i) = \max_{y_i\in X_i} \xi_i(\widehat{x_{\widehat{i}}}, y_i)$. Thus, we have $\xi_i(\widehat{x}) = \max_{y_i\in X_i} \xi_i(\widehat{x_{\widehat{i}}}, y_i)$ for every $i \in I$. This completes the proof.

6. Generalized weak implicit inclusion problems

In this section, we use Theorem 3.4 to establish some existence results of solutions for generalized weak implicit inclusion problems in noncompact abstract convex spaces. We first formulate the problems in the following.

Let $(X;\Gamma^1)$ and $(Y;\Gamma^2)$ be two abstract convex spaces and let Z be a nonempty set. Let $A, B : X \to 2^X$, $F : X \to 2^Y$, $G : X \to 2^Z$, and $H : Y \times Z \to 2^X$ be five set-valued mappings. We consider the F-generalized weak implicit inclusion problem denoted by (FGWIIP): find $(\widehat{x}, \widehat{y}) \in X \times Y$ such that $\widehat{x} \in A(\widehat{x})$, $\widehat{y} \in F(\widehat{x})$, and for each $u \in B(\widehat{x})$, there exists $z \in G(\widehat{x})$ for which $u \in H(\widehat{y}, z)$ and the S-generalized weak implicit inclusion problem denoted by (SGWIIP): find $(\widehat{x}, \widehat{y}) \in X \times Y$ such that $\widehat{x} \in A(\widehat{x})$, $\widehat{y} \in F(\widehat{x})$, and $u \in H(\widehat{y}, z)$ for every $u \in B(\widehat{x})$ and every $z \in G(\widehat{x})$. If $X = Z$ and G is the identity mapping on X, then (FGWIIP) coincides with (SGWIIP).

Note that if $X = Y$ and F is the identity mapping on X, then (FGWIIP) reduces to the generalized weak implicit inclusion problem denoted by (GWIIP): find $\widehat{x} \in X$ such that $\widehat{x} \in A(\widehat{x})$ and for each $u \in B(\widehat{x})$, there exists $z \in G(\widehat{x})$ for which $u \in H(\widehat{x}, z)$. If $A(x) = B(x) \equiv X$ for every $x \in X$, then (GWIIP) reduces to the generalized implicit inclusion problem denoted by (GIIP): find $\widehat{x} \in X$ such that for each $u \in X$, there exists $z \in G(\widehat{x})$: $u \in H(\widehat{x}, z)$, which was discussed by Wang and Huang [49] under the condition that X and Z are two Hausdorff topological vector spaces. If $X = Z$ and G is the identity mapping on X, then (GWIIP) reduces to the extended weak inclusion problem denoted by (EWIP): find $\widehat{x} \in X$ such that $\widehat{x} \in A(\widehat{x})$ and $B(\widehat{x}) \subseteq H(\widehat{x}, \widehat{x})$. If $A(x) = B(x) \equiv X$ for every $x \in X$, then (EWIP) reduces to the extended inclusion problem (for short, EIP): find $\widehat{x} \in X$ such that $X \subseteq H(\widehat{x}, \widehat{x})$, which was studied by Fang and Huang [50] under the condition that X is a real Banach space. If $X = Z$, G is the identity mapping on X, and $H(x, z) = H(x)$ for every $(x, z) \in X \times X$, then (GIIP) reduces to the inclusion problem denoted by (IP): find $\widehat{x} \in X$ such that $X \subseteq H(\widehat{x})$, which was investigated by Di Bella [51] when X is a Hausdorff topological vector space.

From these special cases, we can see that (FGWIIP) extends and unifies the corresponding models in [49–51].

Definition 6.1. Let $(X;\Gamma)$ be an abstract convex space and let Y and Z be two nonempty sets. Let $F : X \to 2^Y$ and $G : X \to 2^Z$ be two set-valued mappings. A set-valued mapping $H : Y \times Z \to 2^X$ is said to be Γ-quasiconvex-like with respect to F and G if for each $N = \{u_0, u_1, \dots, u_n\} \in \langle X\rangle$, each $x \in \Gamma\text{-co}(N)$, and for each $y \in F(x)$, there exist $j \in \{0, 1, \dots, n\}$ and $z \in G(x)$ such that $u_j \in H(y, z)$.

Definition 6.2. Let $(X;\Gamma)$ be an abstract convex space and let Y and Z be two nonempty sets. Let $F : X \to 2^Y$ and $G : X \to 2^Z$ be two set-valued mappings. A set-valued mapping $H : Y \times Z \to 2^X$ is said to be strong Γ-quasiconvex-like with respect to F and G if for each $N = \{u_0, u_1, \dots, u_n\} \in \langle X\rangle$ and for each $x \in \Gamma\text{-co}(N)$ and each $y \in F(x)$, there exists $j \in \{0, 1, \dots, n\}$ such that $u_j \in H(y, z)$ for every $z \in G(x)$.

Definition 6.3. Let $(X;\Gamma)$ be an abstract convex space, Y be a topological vector space, $C \subseteq Y$ be a nonempty convex cone, and $\eta : X \times X \to X$ be a single-valued mapping. A set-valued mapping $F : X \to 2^Y$ is said to be C-Γ-quasiconvex in the second argument of η if for each $x \in X$, each $A = \{y_0, y_1, \dots, y_n\} \in \langle X\rangle$ and each $z \in \Gamma(A)$, there exists $j \in \{0, 1, \dots, n\}$ such that $F(\eta(x, z)) \subseteq F(\eta(x, y_j)) - C$.

Theorem 6.1. *Let $(X;\Gamma^1)$ and $(Y;\Gamma^2)$ be two abstract convex spaces such that $(X \times Y;\Gamma^1 \times \Gamma^2)$ is an abstract convex space defined as in Lemma 2.5. Let K be a nonempty compact subset of $X \times Y$ and Z be a nonempty set. Let $A, B : X \to 2^X$, $F : X \to 2^Y$, $G : X \to 2^Z$, and $H : Y \times Z \to 2^X$ be five set-valued mappings satisfying*

(i) *for each $x \in X$, $B(x) \subseteq A(x)$;*

(ii) *B and F have nonempty Γ^1-convex and Γ^2-convex values and open lower sections*;

(iii) *the set* $\mathfrak{F} = \{(x, y) \in X \times Y : x \in A(x)$ *and* $y \in F(x)\}$ *is closed in* $X \times Y$;

(iv) *for each* $u \in X$, *the set* $\{(x, y) \in X \times Y : u \notin H(y, z)$ *for every* $z \in G(x)\}$ *is open in* $X \times Y$;

(v) *for each* $x \in X$ *and each* $y \in F(x)$, $x \notin \Gamma^1\text{-co}(\{u \in X : u \notin H(y, z) \text{ for every } z \in G(x)\})$;

(vi) *one of the following conditions holds*:

$(\text{vi})_1$ *for each* $N_0 \times N_1 \in \langle X \times Y\rangle$, *there exist a compact* Γ^1*-convex subset* L_{N_0} *of* $(X; \Gamma^1)$ *containing* N_0 *and a compact* Γ^2*-convex subset* L_{N_1} *of* $(Y; \Gamma^2)$ *containing* N_1 *such that for* $L := L_{N_0} \times L_{N_1}$ *and for each* $(x, y) \in L \setminus K$, *there exists* $(u, v) \in L$ *such that* $u \in B(x)$, $v \in F(x)$, *and* $u \notin H(y, z)$ *for every* $z \in G(x)$;

$(\text{vi})_2$ *there exists* $(u_0, v_0) \in X \times Y$ *such that for each* $(x, y) \in X \times Y \setminus K$, *one has* $u_0 \in B(x)$, $v_0 \in F(x)$, *and* $u_0 \notin H(y, z)$ *for every* $z \in G(x)$.

If $(X \times Y; \Gamma^1 \times \Gamma^2)$ *satisfies* $1_{X\times Y} \in \mathfrak{RC}(X \times Y, X \times Y)$, *then (FGWIIP) is solvable, that is, there exists* $(\widehat{x}, \widehat{y}) \in K$ *such that* $\widehat{x} \in A(\widehat{x})$, $\widehat{y} \in F(\widehat{x})$, *and for each* $u \in B(\widehat{x})$, *there exists* $z \in G(\widehat{x})$ *for which* $u \in H(\widehat{y}, z)$.

Proof. By (ii), B has nonempty Γ^1-convex values and open lower sections. Let $\pi(K)$ denotes the projection of K onto X. Then it is clear that $\pi(K)$ is a compact subset of X. For each $N_0 \times N_1 \in \langle X \times Y\rangle$, it follows from $(\text{vi})_1$ that there exist a compact Γ^1-convex subset L_{N_0} of $(X; \Gamma^1)$ containing N_0 and a compact Γ^2-convex subset L_{N_1} of $(Y; \Gamma^2)$ containing N_1. Let $x \in L_{N_0} \setminus \pi(K)$ and $y \in L_{N_1}$ be given arbitrarily. Then we have $(x, y) \in L \setminus K$. By $(\text{vi})_1$ again, for each $x \in L_{N_0} \setminus \pi(K)$, there exists $u \in L_{N_0}$ such that $u \in B(x)$, which implies that $L_{N_0} \setminus \pi(K) \subseteq \bigcup_{u\in L_{N_0}}(B^{-1}(u) \bigcap L_{N_0})$. Similarly, let $x \in X \setminus \pi(K)$ and $y \in Y$ be any given. Then we have $(x, y) \in X \times Y \setminus K$ and thus, by $(\text{vi})_2$, there exists $u_0 \in X$ such that for each $x \in X \setminus \pi(K)$, we have $u_0 \in B(x)$, which implies that $X \setminus B^{-1}(u_0) \subseteq \pi(K)$. Therefore, all the conditions of Corollary 3.1 with $S = T$ are fulfilled and thus, it follows that there exists $x_0 \in X$ such that $x_0 \in B(x_0) \subseteq A(x_0)$. Then we have $x_0 \times F(x_0) \subseteq \mathfrak{F}$ and hence, the set $\mathfrak{F}$ is nonempty.

Define a set-valued mapping $T : X \times Y \to 2^{X\times Y}$ by setting, for each $(x, y) \in X \times Y$,

$$T(x, y) = \begin{cases} (B(x) \bigcap J(x, y)) \times F(x), & \text{if } (x, y) \in \mathfrak{F}, \\ B(x) \times F(x), & \text{if } (x, y) \in X \times Y \setminus \mathfrak{F}, \end{cases}$$

where $J : X \times Y \to 2^X$ is defined by $J(x, y) = \{u \in X : u \notin H(y, z) \text{ for every } z \in G(x)\}$ for every $(x, y) \in X \times Y$. For each $(u, v) \in X \times Y$, we have

$$\begin{aligned} T^{-1}(u, v) \;&=\; \Big((X \times Y \setminus \mathfrak{F}) \bigcap (B^{-1}(u) \times Y) \bigcap (F^{-1}(v) \times Y)\Big) \\ &\bigcup\; \Big(J^{-1}(u) \bigcap (B^{-1}(u) \times Y) \bigcap (F^{-1}(v) \times Y)\Big). \end{aligned}$$

Since $J^{-1}(u) = \{(x, y) \in X \times Y : u \notin H(y, z) \text{ for every } z \in G(x)\}$ for every $u \in X$, it follows from (iv) that $J^{-1}(u)$ is open in $X \times Y$. By (ii) and (iii), one can see that $T^{-1}(u, v)$ is open in $X \times Y$ for every $(u, v) \in X \times Y$. By (v) and the definition of J, we have

$$(x, y) \notin \Gamma^1\text{-co}(B(x) \bigcap J(x, y)) \times F(x) = \Gamma^1\text{-co}(B(x) \bigcap J(x, y)) \times \Gamma^2\text{-co}(F(x)), \;\; \forall (x, y) \in \mathfrak{F}.$$

Since $\Gamma^1\text{-co}(B(x) \bigcap J(x, y)) \times \Gamma^2\text{-co}(F(x))$ is a $\Gamma^1 \times \Gamma^2$-convex subset, $(B(x) \bigcap J(x, y)) \times F(x) \subseteq \Gamma^1\text{-co}(B(x) \bigcap J(x, y)) \times \Gamma^2\text{-co}(F(x))$, and $\Gamma^1 \times \Gamma^2\text{-co}((B(x) \bigcap J(x, y)) \times F(x))$ is the smallest $\Gamma^1 \times \Gamma^2$-convex subset containing $(B(x) \bigcap J(x, y)) \times F(x)$, it follows that $(x, y) \notin \Gamma^1 \times \Gamma^2\text{-co}((B(x) \bigcap J(x, y)) \times F(x))$

for every $(x, y) \in \mathfrak{F}$. It is easy to see that $(x, y) \notin B(x) \times F(x) = \Gamma^1 \times \Gamma^2\text{-co}(B(x) \times F(x))$ for every $(x, y) \in X \times Y \setminus \mathfrak{F}$. Therefore, in both two cases, we have $(x, y) \notin \Gamma^1 \times \Gamma^2\text{-co}(T(x, y))$ for every $(x, y) \in X \times Y$. By (vi), we know that one of the following two conditions holds:

- for each $N_0 \times N_1 \in \langle X \times Y \rangle$, there exist a compact Γ^1-convex subset L_{N_0} of $(X; \Gamma^1)$ containing N_0 and a compact Γ^2-convex subset L_{N_1} of $(Y; \Gamma^2)$ containing N_1 such that for $L := L_{N_0} \times L_{N_1}$, we have $L \setminus K \subseteq \bigcup_{(u,v)\in L} T^{-1}(u, v)$.
- there exists $(u_0, v_0) \in X \times Y$ such that $X \times Y \setminus T^{-1}(u_0, v_0) \subseteq K$.

Thus, by Theorem 3.4 and Remark 3.4, there exists $(\widehat{x, y}) \in K$ such that $T(\widehat{x, y}) = \emptyset$. Since B and F have nonempty values, we can conclude that $(\widehat{x, y}) \in \mathfrak{F}$. Thus, $\widehat{x} \in A(\widehat{x})$, $\widehat{y} \in F(\widehat{x})$, and $B(\widehat{x}) \bigcap J(\widehat{x, y}) = \emptyset$. Therefore, for each $u \in B(\widehat{x})$, there exists $z \in G(\widehat{x})$ for which $u \in H(\widehat{y}, z)$. This completes the proof.

Remark 6.1. (1) (v) of Theorem 6.1 can be replaced by the following stronger condition:

(v)′ H is Γ^1-quasiconvex-like with respect to F and G.

In fact, suppose to the contrary that there exist $x \in X$ and $y \in F(x)$ such that $x \in \Gamma^1\text{-co}(\{u \in X : u \notin H(y, z) \text{ for every } z \in G(x)\})$. Then by Lemma 2.7, there exists $\{u_0, u_1, \ldots, u_n\} \in \langle \{u \in X : u \notin H(y, z) \text{ for every } z \in G(x)\} \rangle$ such that $x \in \Gamma^1\text{-co}(\{u_0, u_1, \ldots, u_n\})$. By (v)′, there exists $j \in \{0, 1, \ldots, n\}$ and $z \in G(x)$ such that $u_j \in H(y, z)$, which contradicts that $u_j \notin H(y, z)$ for every $z \in G(x)$. Therefore, (v)′ implies (v) of Theorem 6.1.

(2) the following two conditions imply that (v)′ holds.

(a) for each $x \in X$ and each $y \in F(x)$, the set $\{u \in X : u \notin H(y, z) \text{ for every } z \in G(x)\}$ is Γ^1-convex.

(b) for each $x \in X$ and each $y \in F(x)$, there exists $z \in G(x)$ such that $x \in H(y, z)$.

Indeed, by way of contradiction, suppose that for some $N = \{u_0, u_1, \ldots, u_n\} \in \langle X \rangle$, some $x \in \Gamma\text{-co}(N)$, and for some $y \in F(x)$, $u_j \notin H(y, z)$ for every $j \in \{0, 1, \ldots, n\}$ and every $z \in G(x)$. By (a), we have $x \notin H(y, z)$, which contradicts (b).

(3) If we assume that X has Hausdorff property and Z is a topological space, then (iv) of Theorem 6.1 can be replaced by the following condition:

(iv)′ G and H are two upper semicontinuous set-valued mappings with compact values.

In fact, it suffices to prove that the set $\{(x, y) \in X \times Y : \text{there exists } z \in G(x) \text{ such that } u \in H(y, z)\}$ is closed in $X \times Y$ for every $u \in X$. Let $\{(x_\alpha, y_\alpha)\} \subseteq \{(x, y) \in X \times Y : \text{there exists } z \in G(x) \text{ such that } u \in H(y, z)\}$ be an arbitrary net such that $(x_\alpha, y_\alpha) \to (x_0, y_0)$. Then for each α, there exists $z_\alpha \in G(x_\alpha)$ such that $u \in H(y_\alpha, z_\alpha)$. Since G is an upper semicontinuous set-valued mapping with compact values, it follows from Lemma 2.4 that there exist $z_0 \in G(x_0)$ and a subnet $\{z_\beta\}$ of $\{z_\alpha\}$ such that $z_\beta \to z_0$. Since H is an upper semicontinuous set-valued mapping with compact values, it follows from Lemma 2.3 that H is closed. In addition, for each β, we have $u \in H(y_\beta, z_\beta)$ and $(y_\beta, z_\beta) \to (y_0, z_0)$, so, $u \in H(y_0, z_0)$. Therefore, $(x_0, y_0) \in \{(x, y) \in X \times Y : \text{there exists } z \in G(x) \text{ such that } u \in H(y, z)\}$, which implies that the set $\{(x, y) \in X \times Y : \text{there exists } z \in G(x) \text{ such that } u \in H(y, z)\}$ is closed in $X \times Y$ for every $u \in X$ and thus, the set $\{(x, y) \in X \times Y : u \notin H(y, z) \text{ for every } z \in G(x)\}$ is open in $X \times Y$ for every $u \in X$.

By using the similar arguments as in Theorem 6.1, we have the following existence result of solutions for (GWIIP). We omit the proof.

Theorem 6.2. *Let $(X; \Gamma)$ be an abstract convex space, K be a nonempty compact subset of X, and Z be a nonempty set. Let $A, B : X \to 2^X$, $G : X \to 2^Z$, and $H : X \times Z \to 2^X$ be four set-valued mappings satisfying*

(i) *for each $x \in X$, $B(x) \subseteq A(x)$*;

(ii) *B has nonempty Γ-convex values and open lower sections*;

(iii) *the set* $\mathfrak{F} = \{x \in X : x \in A(x)\}$ *is closed in* X;

(iv) *for each* $u \in X$*, the set* $\{x \in X : u \notin H(x,z)$ *for every* $z \in G(x)\}$ *is open in* X;

(v) *for each* $x \in X$*,* $x \notin \Gamma\text{-co}(\{u \in X : u \notin H(x,z) \text{ for every } z \in G(x)\})$;

(vi) *one of the following conditions holds*:

(vi)$_1$ *for each* $N_0 \in \langle X \rangle$*, there exists a compact* Γ*-convex subset* L_{N_0} *of* $(X;\Gamma)$ *containing* N_0 *such that for each* $x \in L_{N_0} \setminus K$*, there exists* $u \in L_{N_0}$ *such that* $u \in B(x)$ *and* $u \notin H(x,z)$ *for every* $z \in G(x)$;

(vi)$_2$ *there exists* $u_0 \in X$ *such that for each* $x \in X \setminus K$*, one has* $u_0 \in B(x)$ *and* $u_0 \notin H(x,z)$ *for every* $z \in G(x)$.

If $(X;\Gamma)$ *satisfies* $1_X \in \mathfrak{RC}(X,X)$*, then (GWIIP) is solvable, that is, there exists* $\widehat{x} \in K$ *such that* $\widehat{x} \in A(\widehat{x})$ *and for each* $u \in B(\widehat{x})$*, there exists* $z \in G(\widehat{x})$ *for which* $u \in H(\widehat{x},z)$.

Remark 6.2. Theorem 6.2 generalizes Theorem 3.1 of Wang and Huang [49] in the following aspects: (a) from noncompact Hausdorff topological vector spaces to noncompact abstract convex spaces without any linear and convex structure. In fact, for X in Theorem 3.1 of Wang and Huang [49], let $\Gamma_A = \text{co}(A)$ for every $A \in \langle X \rangle$, where $\text{co}(A)$ denotes the convex hull of A. Then $(X;\Gamma)$ forms an abstract convex space. Further, we can prove that $1_X \in \mathfrak{RC}(X,X)$ (for details, see the proof of the following Theorem 6.3); (b) from two set-valued mappings to four set-valued mappings; (c) from one coercivity condition to two alternative coercivity conditions. And K in Theorem 6.2 only needs to be compact, while D in Theorem 3.1 of Wang and Huang [49] needs to be compact convex; (d) (v) of Theorem 6.2 is weaker than (i) and (ii) of Theorem 3.1 due to Wang and Huang [49]. In such an abstract convex space perspective, it is easy to see that (i) and (ii) of Theorem 3.1 due to Wang and Huang [49] can deduce (v) of Theorem 6.2; (e) concerns on the more general set Z without any topological and linear structure instead of the nonempty set Y in Theorem 3.1 of Wang and Huang [49], which is a subset of a Hausdorff topological vector space. In addition, the proof of Theorem 6.2 originates from the existence of maximal elements in noncompact abstract convex spaces, while Theorem 3.1 of Wang and Huang [49] is proved based on the famous FKKM theorem. Therefore, the proof method of Theorem 6.2 is different from that of Theorem 3.1 of Wang and Huang [49].

In Theorem 6.2, if X is a Banach space, then the compactness of L_{N_0} can be weakened to weak compactness.

Theorem 6.3. *Let* X *be a real Banach space,* K *be a nonempty weak compact subset of* X*, and* Z *be a nonempty set. Let* $A, B : X \to 2^X$*,* $G : X \to 2^Z$*, and* $H : X \times Z \to 2^X$ *be four set-valued mappings satisfying*

(i) *for each* $x \in X$*,* $B(x) \subseteq A(x)$;

(ii) *B has nonempty convex values and weakly open lower sections*;

(iii) *the set* $\mathfrak{F} = \{x \in X : x \in A(x)\}$ *is weakly closed in* X;

(iv) *for each* $u \in X$*, the set* $\{x \in X : u \notin H(x,z)$ *for every* $z \in G(x)\}$ *is weakly open in* X;

(v) *for each* $x \in X$*,* $x \notin \text{co}(\{u \in X : u \notin H(x,z) \text{ for every } z \in G(x)\})$;

(vi) *one of the following conditions holds*:

(vi)$_1$ *for each* $N_0 \in \langle X \rangle$*, there exists a weak compact convex subset* L_{N_0} *of* $(X;\Gamma)$ *containing* N_0 *such that for each* $x \in L_{N_0} \setminus K$*, there exists* $u \in L_{N_0}$ *such that* $u \in B(x)$ *and* $u \notin H(x,z)$ *for every* $z \in G(x)$;

(vi)$_2$ *there exists* $u_0 \in X$ *such that for each* $x \in X \setminus K$*, one has* $u_0 \in B(x)$ *and* $u_0 \notin H(x,z)$ *for every* $z \in G(x)$.

If $(X;\Gamma)$ *satisfies* $1_X \in \mathfrak{RC}(X,X)$*, then (GWIIP) is solvable, that is, there exists* $\widehat{x} \in K$ *such that*

$\widehat{x} \in A(\widehat{x})$ *and for each* $u \in B(\widehat{x})$*, there exists* $z \in G(\widehat{x})$ *for which* $u \in H(\widehat{x}, z)$.

Proof. Let $\Gamma : \langle X \rangle \to 2^X$ be a set-valued mapping defined by $\Gamma_A = \mathrm{co}(A)$ for every $A \in \langle X \rangle$, where $\mathrm{co}(A)$ denotes the convex hull of A. Endowing X with the weak topology, we can see that $(X; \Gamma)$ forms an abstract convex space and (i)-(vi) of Theorem 6.2 are satisfied. Now, we show that $1_X \in \mathfrak{RC}(X, X)$. In fact, let $G : X \to 2^X$ is a KKM mapping with respect to the identity mapping 1_X such that each $G(x)$ is weakly closed in X. Then for each $A = \{x_0, x_1, \ldots, x_n\} \in \langle X \rangle$, we have $\Gamma_A = \mathrm{co}(\{x_0, x_1, \ldots, x_n\}) \subseteq \bigcup_{i=0}^n G(x_i)$ and further, we can define a mapping $\sigma : \Delta_n \to \mathrm{co}(\{x_0, x_1, \ldots, x_n\})$ by $\sigma(t) = \sum_{i=0}^n t_i x_i$ for every $t = (t_0, t_1, \ldots, t_n) \in \Delta_n$ with $\sum_{i=0}^n t_i = 1$ and $t_i \geq 0$, where Δ_n denotes the standard n-dimensional simplex with vertices $\{e_0, e_1, \ldots, e_n\}$. Considering the norm topology on $\mathrm{co}(\{x_0, x_1, \ldots, x_n\})$, we can see that the continuity of σ can be guaranteed by the fact that $\|\sigma(t_1) - \sigma(t_2)\| \leq \sum_{i=0}^n |t_{i1} - t_{i2}| \|x_i\|$ for every $t_1 = (t_{01}, t_{11}, \ldots, t_{n1}) \in \Delta_n$ with $\sum_{i=0}^n t_{i1} = 1$, $t_{i1} \geq 0$ and every $t_2 = (t_{02}, t_{12}, \ldots, t_{n2}) \in \Delta_n$ with $\sum_{i=0}^n t_{i2} = 1$, $t_{i2} \geq 0$. For every $i \in \{0, 1, \ldots, n\}$, let $E_i = \sigma^{-1}(\mathrm{co}(\{x_0, x_1, \ldots, x_n\}) \bigcap G(x_i))$. Since each $G(x_i)$ is weakly closed in X, it follows that $G(x_i)$ is closed in X. Thus, we can see that $\mathrm{co}(\{x_0, x_1, \ldots, x_n\}) \bigcap G(x_i)$ is a closed subset of $\mathrm{co}(\{x_0, x_1, \ldots, x_n\})$. By the continuity of σ, each E_i is closed in Δ_n. Next, let us prove that $\mathrm{co}(\{e_i : i \in I\}) \subseteq \bigcup_{i \in I} E_i$ for every $I = \{i_1, i_2, \ldots, i_k\} \in \langle \{0, 1, \ldots, n\} \rangle$. In fact, let $t = \sum_{j=1}^k t_{i_j} e_{i_j} \in \mathrm{co}(\{e_i : i \in I\})$ be any given such that $\sum_{j=1}^k t_{i_j} = 1$ and $t_{i_j} \geq 0$. By the definition of σ and the hypothesis that G is a KKM mapping with respect to the identity mapping 1_X, we have $\sigma(t) \in \mathrm{co}\{x_{i_1}, x_{i_2}, \ldots, x_{i_k}\} \subseteq \bigcup_{j=1}^k G(x_{i_j})$. Thus, there exists $j \in \{1, 2, \ldots, k\}$ such that $\sigma(t) \in \mathrm{co}(\{x_0, x_1, \ldots, x_n\}) \bigcap G(x_{i_j})$ and consequently, $t \in E_{i_j}$. By applying the classical KKM principle to the family $\{E_i\}_{i=0}^n$, there exists $t_0 \in \mathrm{co}(\{e_0, e_1, \ldots, e_n\})$ such that $t_0 \in \bigcap_{i=0}^n E_i$ and so, $\sigma(t_0) \in \bigcap_{i=0}^n G(x_i)$, which implies that the family $\{G(x) : x \in X\}$ has the finite intersection property. Therefore, as a consequence of Theorem 6.2, (GWIIP) is solvable, that is, there exists $\widehat{x} \in K$ such that $\widehat{x} \in A(\widehat{x})$ and for each $u \in B(\widehat{x})$, there exists $z \in G(\widehat{x})$ for which $u \in H(\widehat{x}, z)$. This completes the proof.

In Theorem 6.2, if $A(x) = B(x) \equiv X$ for every $x \in X$, then we have the following existence result of solutions for (GIIP).

Theorem 6.4. *Let* $(X; \Gamma)$ *be an abstract convex space,* K *be a nonempty compact subset of* X*, and* Z *be a nonempty set. Let* $G : X \to 2^Z$ *and* $H : X \times Z \to 2^X$ *be two set-valued mappings satisfying*

(i) *for each* $u \in X$*, the set* $\{x \in X : u \notin H(x, z)$ *for every* $z \in G(x)\}$ *is open in* X;

(ii) *for each* $x \in X$, $x \notin \Gamma$-co$(\{u \in X : u \notin H(x, z)$ for every $z \in G(x)\})$;

(iii) *one of the following conditions holds*:

(iii)$_1$ *for each* $N_0 \in \langle X \rangle$*, there exists a compact* Γ*-convex subset* L_{N_0} *of* $(X; \Gamma)$ *containing* N_0 *such that for each* $x \in L_{N_0} \setminus K$*, there exists* $u \in L_{N_0}$ *such that* $u \notin H(x, z)$ *for every* $z \in G(x)$;

(iii)$_2$ *there exists* $u_0 \in X$ *such that for each* $x \in X \setminus K$*, one has* $u_0 \notin H(x, z)$ *for every* $z \in G(x)$.

If $(X; \Gamma)$ *satisfies* $1_X \in \mathfrak{RC}(X, X)$*, then (GIIP) is solvable, that is, there exists* $\widehat{x} \in K$ *such that for each* $u \in X$*, there exists* $z \in G(\widehat{x})$ *for which* $u \in H(\widehat{x}, z)$.

In Theorem 6.2, if $X = Z$ and G is the identity mapping on X, then we obtain the following existence theorem of solutions for (EWIP).

Theorem 6.4. *Let* $(X; \Gamma)$ *be an abstract convex space,* K *be a nonempty compact subset of* X*, and let* $A, B : X \to 2^X$*, and* $H : X \times X \to 2^X$ *be three set-valued mappings satisfying*

(i) *for each* $x \in X$, $B(x) \subseteq A(x)$;

(ii) B *has nonempty* Γ*-convex values and open lower sections*;

(iii) *the set* $\mathfrak{F} = \{x \in X : x \in A(x)\}$ *is closed in* X;

(iv) *for each* $u \in X$*, the set* $\{x \in X : u \notin H(x, x)\}$ *is open in* X;

(v) *for each* $x \in X$, $x \notin \Gamma\text{-co}(\{u \in X : u \notin H(x, x)\})$;

(vi) *one of the following conditions holds*:

$(\text{vi})_1$ *for each* $N_0 \in \langle X \rangle$, *there exists a compact* Γ*-convex subset* L_{N_0} *of* $(X; \Gamma)$ *containing* N_0 *such that for each* $x \in L_{N_0} \setminus K$, *there exists* $u \in L_{N_0}$ *such that* $u \in B(x)$ *and* $u \notin H(x, x)$;

$(\text{vi})_2$ *there exists* $u_0 \in X$ *such that for each* $x \in X \setminus K$, *one has* $u_0 \in B(x)$ *and* $u_0 \notin H(x, x)$.

If $(X; \Gamma)$ *satisfies* $1_X \in \mathfrak{RC}(X, X)$, *then (EWIP) is solvable, that is, there exists* $\widehat{x} \in K$ *such that* $\widehat{x} \in A(\widehat{x})$ *and* $B(\widehat{x}) \subseteq H(\widehat{x}, \widehat{x})$.

In Theorem 6.4, by setting $A(x) = B(x) \equiv X$ for every $x \in X$, we have the following existence result of solutions for (EIP).

Corollary 6.1. *Let* $(X; \Gamma)$ *be an abstract convex space, K be a nonempty compact subset of X, and let* $H : X \times X \to 2^X$ *be a set-valued mapping satisfying*

(i) *for each* $u \in X$, *the set* $\{x \in X : u \notin H(x, x)\}$ *is open in X*;

(ii) *for each* $x \in X$, $x \notin \Gamma\text{-co}(\{u \in X : u \notin H(x, x)\})$;

(iii) *one of the following conditions holds*:

$(\text{iii})_1$ *for each* $N_0 \in \langle X \rangle$, *there exists a compact* Γ*-convex subset* L_{N_0} *of* $(X; \Gamma)$ *containing* N_0 *such that for each* $x \in L_{N_0} \setminus K$, *there exists* $u \in L_{N_0}$ *such that* $u \notin H(x, x)$;

$(\text{iii})_2$ *there exists* $u_0 \in X$ *such that for each* $x \in X \setminus K$, *one has* $u_0 \notin H(x, x)$.

If $(X; \Gamma)$ *satisfies* $1_X \in \mathfrak{RC}(X, X)$, *then (EIP) is solvable, that is, there exists* $\widehat{x} \in K$ *such that* $X \subseteq H(\widehat{x}, \widehat{x})$.

Remark 6.3. (1) Corollary 6.1 generalizes Theorem 3.4 of Wang and Huang [49] in the following aspects: (a) from noncompact Hausdorff topological vector spaces to noncompact abstract convex spaces without any linear and convex structure; (b) from one coercivity condition to two alternative coercivity conditions; (c) (ii) of Corollary 6.1 is weaker than (i) and (ii) of Theorem 3.4 due to Wang and Huang [49]. In addition, the proof of Corollary 6.1 is different from that of Theorem 3.4 due to Wang and Huang [49]. In fact, Corollary 6.1 is proved based on the existence of maximal elements in noncompact abstract convex spaces, while Theorem 3.4 of Wang and Huang [49] is proved using the famous FKKM theorem.

(2) Corollary 6.1 is different from Theorem 2.3 of Fang and Huang [50] in the following two ways: (a) X needs not be a Banach space; (b) the proof technique is different. Corollary 6.1 is established based on the existence of maximal elements in noncompact abstract convex spaces, while the proof of Theorem 2.3 of Fang and Huang [50] is proved by using the Kakutani-Fan-Glicksberg fixed point theorem.

By Corollary 6.1, we have the following corollary which is an existence result of solutions for (IP).

Corollary 6.2. *Let* $(X; \Gamma)$ *be an abstract convex space, K be a nonempty compact subset of X, and let* $H : X \to 2^X$ *be a set-valued mapping satisfying*

(i) *for each* $u \in X$, *the set* $\{x \in X : u \notin H(x)\}$ *is open in X*;

(ii) *for each* $x \in X$, $x \notin \Gamma\text{-co}(\{u \in X : u \notin H(x)\})$;

(iii) *one of the following conditions holds*:

$(\text{iii})_1$ *for each* $N_0 \in \langle X \rangle$, *there exists a compact* Γ*-convex subset* L_{N_0} *of* $(X; \Gamma)$ *containing* N_0 *such that for each* $x \in L_{N_0} \setminus K$, *there exists* $u \in L_{N_0}$ *such that* $u \notin H(x)$;

$(\text{iii})_2$ *there exists* $u_0 \in X$ *such that for each* $x \in X \setminus K$, *one has* $u_0 \notin H(x)$.

If $(X; \Gamma)$ *satisfies* $1_X \in \mathfrak{RC}(X, X)$, *then (IP) is solvable, that is, there exists* $\widehat{x} \in K$ *such that* $X \subseteq H(\widehat{x})$.

Proof. Define a set-valued mapping $\widetilde{H} : X \times X \to 2^X$ by $\widetilde{H}(x, z) = H(x)$ for every $(x, z) \in X \times X$. Then we can see that all the conditions of Corollary 6.1 are fulfilled. Thus, it follows from Corollary 6.1 that there exists $\widehat{x} \in K$ such that $X \subseteq \widetilde{H}(\widehat{x}, \widehat{x}) = H(\widehat{x})$, that is, (IP) is solvable. This completes the proof.

Now, as applications of Theorem 6.2, we have the following existence theorems of solutions for generalized set-valued implicit Stampacchia-type vector equilibrium problems and generalized set-valued implicit weak vector equilibrium problems in the framework of noncompact abstract convex spaces.

Theorem 6.5. *Let $(X; \Gamma)$ be an abstract convex space, K be a nonempty compact subset of X, Y be a topological vector space, and Z be a nonempty set. Let $A, B : X \to 2^X$, $C : X \to 2^Y$, $G : X \to 2^Z$, and $F : X \times Z \times X \to 2^Y$ be five set-valued mappings satisfying*

(i) *for each $x \in X$, $C(x)$ is a convex cone*;

(ii) *for each $x \in X$, $B(x) \subseteq A(x)$*;

(iii) *B has nonempty Γ-convex values and open lower sections*;

(iv) *the set $\mathfrak{F} = \{x \in X : x \in A(x)\}$ is closed in X*;

(v) *for each $u \in X$, the set $\{x \in X : F(x, z, u) \subseteq -C(x) \setminus \{0\}$ for every $z \in G(x)\}$ is open in X*;

(vi) *for each $x \in X$, $x \notin \Gamma\text{-co}(\{u \in X : F(x, z, u) \subseteq -C(x) \setminus \{0\}$ for every $z \in G(x)\})$*;

(vii) *one of the following conditions holds*:

$(\text{vii})_1$ *for each $N_0 \in \langle X \rangle$, there exists a compact Γ-convex subset L_{N_0} of $(X; \Gamma)$ containing N_0 such that for each $x \in L_{N_0} \setminus K$, there exists $u \in L_{N_0}$ such that $u \in B(x)$ and $F(x, z, u) \subseteq -C(x) \setminus \{0\}$ for every $z \in G(x)$*;

$(\text{vii})_2$ *there exists $u_0 \in X$ such that for each $x \in X \setminus K$, one has $u_0 \in B(x)$ and $F(x, z, u_0) \subseteq -C(x) \setminus \{0\}$ for every $z \in G(x)$.*

If $(X; \Gamma)$ satisfies $1_X \in \mathfrak{RC}(X, X)$, then the generalized set-valued implicit Stampacchia-type vector equilibrium problem is solvable, that is, there exists $\widehat{x} \in K$ such that $\widehat{x} \in A(\widehat{x})$ and for each $u \in B(\widehat{x})$, there exists $z \in G(\widehat{x})$ for which $F(\widehat{x}, z, u) \nsubseteq -C(\widehat{x}) \setminus \{0\}$.

Proof. Define a set-valued mapping $H : X \times Z \to 2^X$ by $H(x, z) = \{u \in X : F(x, z, u) \nsubseteq -C(x) \setminus \{0\}\}$ for every $(x, z) \in X \times Z$. Then it is easy to see that all the conditions of Theorem 6.2 are satisfied. Thus, by Theorem 6.2, there exists $\widehat{x} \in K$ such that $\widehat{x} \in A(\widehat{x})$ and for each $u \in B(\widehat{x})$, there exists $z \in G(\widehat{x})$ for which $u \in H(\widehat{x}, z)$, that is, there exists $\widehat{x} \in K$ such that $\widehat{x} \in A(\widehat{x})$ and for each $u \in B(\widehat{x})$, there exists $z \in G(\widehat{x})$ for which $F(\widehat{x}, z, u) \nsubseteq -C(\widehat{x}) \setminus \{0\}$. This completes the proof.

Remark 6.4. Theorem 6.5 generalizes Theorem 4.6 of Wang and Huang [49] in the following aspects: (a) from three set-valued mappings to five set-valued mappings; (b) from one coercivity condition to two alternative coercivity conditions. And the K in Theorem 6.5 only needs to be compact, while the D in Theorem 4.6 of Wang and Huang [49] needs to be compact convex; (c) (v) of Theorem 6.5 is weaker than (ii) and (iii) of Theorem 4.6 due to Wang and Huang [49]; (d) concerns on the more general set Z without any topological and linear structure instead of the nonempty set Y in Theorem 4.6 of Wang and Huang [49], which is a subset of a Hausdorff topological vector space. In addition, the proof of Theorem 6.5 is based on the existence of maximal elements in noncompact abstract convex spaces, while Theorem 4.6 of Wang and Huang [49] is proved using the famous FKKM theorem. Therefore, the proof technique of Theorem 6.5 is different from that of Theorem 4.6 of Wang and Huang [49].

Theorem 6.6. *Let $(X; \Gamma)$ be an abstract convex space, K be a nonempty compact subset of X, and Y be a topological vector space. Let $A, B : X \to 2^X$, $C : X \to 2^Y$, and $F : X \times X \to 2^Y$ be four set-valued mappings satisfying*

(i) *for each $x \in X$, $C(x)$ is a convex cone*;

(ii) *for each* $x \in X$, $B(x) \subseteq A(x)$;

(iii) *B has nonempty* Γ*-convex values and open lower sections*;

(iv) *the set* $\mathfrak{F} = \{x \in X : x \in A(x)\}$ *is closed in* X;

(v) *for each* $u \in X$, *the set* $\{x \in X : F(x,u) \subseteq -C(x) \setminus \{0\}\}$ *is open in* X;

(vi) *for each* $x \in X$, $x \notin \Gamma\text{-co}(\{u \in X : F(x,u) \subseteq -C(x) \setminus \{0\}\})$;

(vii) *one of the following conditions holds*:

$(\text{vii})_1$ *for each* $N_0 \in \langle X \rangle$, *there exists a compact* Γ*-convex subset* L_{N_0} *of* $(X;\Gamma)$ *containing* N_0 *such that for each* $x \in L_{N_0} \setminus K$, *there exists* $u \in L_{N_0}$ *such that* $u \in B(x)$ *and* $F(x,u) \subseteq -C(x) \setminus \{0\}$;

$(\text{vii})_2$ *there exists* $u_0 \in X$ *such that for each* $x \in X \setminus K$, *one has* $u_0 \in B(x)$ *and* $F(x,u_0) \subseteq -C(x) \setminus \{0\}$.

If $(X;\Gamma)$ *satisfies* $1_X \in \mathfrak{RC}(X,X)$, *then the generalized set-valued Stampacchia-type vector equilibrium problem is solvable, that is, there exists* $\widehat{x} \in K$ *such that* $\widehat{x} \in A(\widehat{x})$ *and* $F(\widehat{x},u) \nsubseteq -C(\widehat{x}) \setminus \{0\}$ *for every* $u \in B(\widehat{x})$.

Proof. Let $Z = X$. Then we define two set-valued mappings $\widetilde{F} : X \times Z \times X \to 2^Y$ and $G : X \to 2^Z$ by $\widetilde{F}(x,z,u) = F(x,u)$ for every $(x,z,u) \in X \times Z \times X$ and $G(x) = \{x\}$ for every $x \in X$, respectively. It is easy to see that all the requirements of Theorem 6.5 are fulfilled. Therefore, it follows from Theorem 6.5 that the conclusion of Theorem 6.6 holds. This completes the proof.

Remark 6.5. Theorem 6.6 generalizes Theorem 2.1 of Kazmi and Khan [52] in the following aspects: (a) from real Banach spaces to noncompact abstract convex spaces without any linear and convex structure; (b) from a single-valued mapping to four set-valued mappings; (c) concerns the more general generalized set-valued Stampacchia-type vector equilibrium problems with movable convex cones instead of the generalized system problems with a fixed solid, pointed, closed and convex cone with apex at the origin; (d) in Theorem 6.6, the topological spaces X and Y need not to be Hausdorff spaces, while the spaces X and Y in Theorem 2.1 of Kazmi and Khan [52] have Hausdorff property. In fact, It can be seen from the proof of Theorem 2.1 of Kazmi and Khan [52] that the Hausdorff property of X is indispensable. In addition, the proof of Theorem 6.6 is essentially based on the existence of maximal elements in noncompact abstract convex spaces, while Theorem 2.1 of Kazmi and Khan [52] is proved by using the famous Brouwer's fixed point theorem. Thus, the proof method of Theorem 6.6 is different from that of Theorem 2.1 of Kazmi and Khan [52].

Theorem 6.7. *Let* $(X;\Gamma)$ *be an abstract convex space, K be a nonempty compact subset of X, Y be a topological vector space, and Z be a topological space. Let* $A, B : X \to 2^X$, $C : X \to 2^Y$, $G : X \to 2^Z$, *and* $F : X \times X \to 2^Y$ *be five set-valued mappings. Let* $\zeta : X \times Z \to X$ *be a continuous mapping and* $\eta : X \times X \to X$ *be a continuous mapping in the first argument. Suppose that*:

(i) *for each* $x \in X$, $C(x)$ *is a convex cone with* $\text{int}C(x) \neq \emptyset$ *and the set-valued mapping* $W : X \to 2^Y$ *defined by* $W(x) = Y \setminus \{-\text{int}C(x)\}$ for every $x \in X$, *is closed*;

(ii) *G and F are two upper semicontinuous set-valued mappings with compact values*;

(iii) *for each* $x \in X$, $B(x) \subseteq A(x)$;

(iv) *B has nonempty* Γ*-convex values and open lower sections*;

(v) *the set* $\mathfrak{F} = \{x \in X : x \in A(x)\}$ *is closed in* X;

(vi) *for each* $x \in X$, $x \notin \Gamma\text{-co}(\{u \in X : F(\zeta(x,z),\eta(x,u)) \subseteq -\text{int}C(x)$ for every $z \in G(x)\})$;

(vii) *one of the following conditions holds*:

$(\text{vii})_1$ *for each* $N_0 \in \langle X \rangle$, *there exists a compact* Γ*-convex subset* L_{N_0} *of* $(X;\Gamma)$ *containing* N_0 *such that for each* $x \in L_{N_0} \setminus K$, *there exists* $u \in L_{N_0}$ *such that* $u \in B(x)$ *and* $F(\zeta(x,z),\eta(x,u)) \subseteq -\text{int}C(x)$ *for every* $z \in G(x)$;

(vii)$_2$ *there is $u_0 \in X$ such that for each $x \in X \setminus K$, one has $u_0 \in B(x)$ and $F(\zeta(x,z),\eta(x,u_0)) \subseteq -\mathrm{int}C(x)$ for every $z \in G(x)$.*

If $(X;\Gamma)$ satisfies $1_X \in \mathfrak{RC}(X,X)$, then the generalized set-valued implicit weak vector equilibrium problem is solvable, that is, there exists $\widehat{x} \in K$ such that $\widehat{x} \in A(\widehat{x})$ and for each $u \in B(\widehat{x})$, there exists $z \in G(\widehat{x})$ for which $F(\zeta(\widehat{x},z),\eta(\widehat{x},u)) \nsubseteq -\mathrm{int}C(\widehat{x})$.

Proof. Define a set-valued mapping $H : X \times Z \to 2^X$ by $H(x,z) = \{u \in X : F(\zeta(x,z),\eta(x,u)) \nsubseteq -\mathrm{int}C(x)\}$ for every $(x,z) \in X \times Z$. By (vi), we get $x \notin \Gamma\text{-co}(\{u \in X : u \notin H(x,z) \text{ for every } z \in G(x)\})$ for every $x \in X$. Now, we show that the set $\{x \in X : \text{there exists } z \in G(x) \text{ such that } u \in H(x,z)\} = \{x \in X : \text{there exists } z \in G(x) \text{ such that } F(\zeta(x,z),\eta(x,u)) \nsubseteq -\mathrm{int}C(x)\}$ is closed in X for every $u \in X$. In fact, let $\{x_\alpha\}$ be an arbitrary net of $\{x \in X : \text{there exists } z \in G(x) \text{ such that } F(\zeta(x,z),\eta(x,u)) \nsubseteq -\mathrm{int}C(x)\}$ such that $x_\alpha \to x_0$. Then for each α, there exists $z_\alpha \in G(x_\alpha)$ such that $F(\zeta(x_\alpha,z_\alpha),\eta(x_\alpha,u)) \nsubseteq -\mathrm{int}C(x_\alpha)$ and thus, for each α, there exists $\vartheta_\alpha \in F(\zeta(x_\alpha,z_\alpha),\eta(x_\alpha,u))$ such that $\vartheta_\alpha \notin -\mathrm{int}C(x_\alpha)$, which implies that $\vartheta_\alpha \in Y \setminus \{-\mathrm{int}C(x_\alpha)\} = W(x_\alpha)$. Since G is upper semicontinuous with compact vales by (ii), it follows from Lemma 2.4 that there exist $z_0 \in G(x_0)$ and a subnet $\{z_\beta\}$ of $\{z_\alpha\}$ such that $z_\beta \to z_0$. Further, Since F is upper semicontinuous with compact vales by (ii) again, ζ is continuous and η is continuous in the first argument, by Lemma 2.4 again, there exist $\vartheta_0 \in F(\zeta(x_0,z_0),\eta(x_0,u))$ and a subnet $\{\vartheta_\gamma\}$ of $\{\vartheta_\beta\}$ such that $\vartheta_\gamma \to \vartheta_0$. Therefore, we have $(x_\gamma,\vartheta_\gamma) \to (x_0,\vartheta_0)$ and $\vartheta_\gamma \in W(x_\vartheta)$ for every γ. Since the graph of W is closed in $X \times Y$ by (i), it follows that $\vartheta_0 \in W(x_0) = Y \setminus \{-\mathrm{int}C(x_0)\}$. Combining the fact that $\vartheta_0 \in F(\zeta(x_0,z_0),\eta(x_0,u))$, we know that $F(\zeta(x_0,z_0),\eta(x_0,u)) \nsubseteq -\mathrm{int}C(x_0)$. Thus, we have

$$x_0 \in \{x \in X : \text{there exists } z \in G(x) \text{ such that } u \in H(x,z)\},$$

which implies that the set $\{x \in X : \text{there exists } z \in G(x) \text{ such that } u \in H(x,z)\}$ is closed in X for every $u \in X$. Thus, (iv) of Theorem 6.2 is satisfied. By (vii) and the definition of H, we know that one of the following two conditions holds:

- for each $N_0 \in \langle X \rangle$, there exist a compact Γ-convex subset L_{N_0} of $(X;\Gamma)$ containing N_0 such that for each $x \in L_{N_0} \setminus K$, there exists $u \in L_{N_0}$ such that $u \in B(x)$ and $u \notin H(x,z)$ for every $z \in G(x)$.
- there exists $u_0 \in X$ such that for each $x \in X \setminus K$, one has $u_0 \in B(x)$ and $u_0 \notin H(x,z)$ for every $z \in G(x)$.

Combining (iii)-(v), we can see that all the requirements of Theorem 6.2 are fulfilled. Thus, it follows from Theorem 6.2 that there exists $\widehat{x} \in K$ such that $\widehat{x} \in A(\widehat{x})$ and for each $u \in B(\widehat{x})$, there exists $z \in G(\widehat{x})$ for which $u \in H(\widehat{x},z)$, that is, $\widehat{x} \in A(\widehat{x})$ and for each $u \in B(\widehat{x})$, there exists $z \in G(\widehat{x})$ for which $F(\zeta(\widehat{x},z),\eta(\widehat{x},u)) \nsubseteq -\mathrm{int}C(\widehat{x})$. This completes the proof.

Remark 6.6. Wang and Huang [49] studied the implicit set-valued weak vector equilibrium problem in the setting of Hausdorff topological vector spaces. Under some linear and convex assumptions, Wang and Huang [49] obtained an existence theorem of solutions for the implicit set-valued weak vector equilibrium problem. However, in the setting of noncompact abstract convex spaces without any linear and convex structure, Theorem 6.7 characterizes the existence of solutions for the generalized set-valued implicit weak vector equilibrium problem which is more general than the implicit set-valued weak vector equilibrium problem studied by Wang and Huang [49].

Remark 6.7. (vi) of Theorem 6.7 can be replaced by the following two conditions:

(vi)$'$ for each $x \in X$ and each $z \in G(x)$, $F(\zeta(x,z),\cdot)$ is $C(x)$-Γ-quasiconvex in the second argument of η.

(vi)$''$ for each $x \in X$, there exists $z \in G(x)$ such that $F(\zeta(x,z),\eta(x,x)) \nsubseteq -\mathrm{int}C(x)$.

Indeed, we first show that the set $D = \{u \in X : F(\zeta(x,z), \eta(x,u)) \subseteq -\text{int}C(x) \text{ for every } z \in G(x)\}$ is Γ-convex for every $x \in X$. In fact, let $A = \{u_0, u_1, \ldots, u_n\} \in \langle D \rangle$ and $u \in \Gamma(A)$ be given arbitrarily. Then by (vi)$'$, there exists $j \in \{0, 1, \ldots, n\}$ such that for each $x \in X$ and each $z \in G(x)$, we have

$$\begin{aligned} F(\zeta(x,z), \eta(x,u)) &\subseteq F(\zeta(x,z), \eta(x,u_j)) - C(x) \\ &\subseteq -\text{int}C(x) - C(x) \\ &\subseteq -\text{int}C(x), \end{aligned}$$

which implies that

$$\Gamma(A) \subseteq D.$$

Then it follows that the set $D = \{u \in X : F(\zeta(x,z), \eta(x,u)) \subseteq -\text{int}C(x) \text{ for every } z \in G(x)\}$ is Γ-convex for every $x \in X$. Secondly, by this fact and (vi)$''$, we have $x \notin \{u \in X : F(\zeta(x,z), \eta(x,u)) \subseteq -\text{int}C(x) \text{ for every } z \in G(x)\} = \Gamma\text{-co}(\{u \in X : F(\zeta(x,z), \eta(x,u)) \subseteq -\text{int}C(x) \text{ for every } z \in G(x)\})$ for every $x \in X$.

Finally, by using Theorem 3.4 and the same arguments as in Theorem 6.1, we obtain the following existence theorem of solutions for (SGWIIP).

Theorem 6.8. *Let $(X; \Gamma^1)$ and $(Y; \Gamma^2)$ be two abstract convex spaces such that $(X \times Y; \Gamma^1 \times \Gamma^2)$ is an abstract convex space defined as in Lemma 2.5. Let K be a nonempty compact subset of $X \times Y$ and Z be a nonempty set. Let $A, B : X \to 2^X$, $F : X \to 2^Y$, $G : X \to 2^Z$, and $H : Y \times Z \to 2^X$ be five set-valued mappings satisfying*

(i) *for each $x \in X$, $B(x) \subseteq A(x)$;*

(ii) *B and F have nonempty Γ^1-convex and Γ^2-convex values and open lower sections;*

(iii) *the set $\mathfrak{F} = \{(x,y) \in X \times Y : x \in A(x) \text{ and } y \in F(x)\}$ is closed in $X \times Y$;*

(iv) *for each $u \in X$, the set $\{(x,y) \in X \times Y : u \notin H(y,z) \text{ for some } z \in G(x)\}$ is open in $X \times Y$;*

(v) *for each $x \in X$ and each $y \in F(x)$, $x \notin \Gamma^1\text{-co}(\{u \in X : u \notin H(y,z) \text{ for some } z \in G(x)\})$;*

(vi) *one of the following conditions holds:*

(vi)$_1$ *for each $N_0 \times N_1 \in \langle X \times Y \rangle$, there exist a compact Γ^1-convex subset L_{N_0} of $(X; \Gamma^1)$ containing N_0 and a compact Γ^2-convex subset L_{N_1} of $(Y; \Gamma^2)$ containing N_1 such that for $L := L_{N_0} \times L_{N_1}$ and for each $(x,y) \in L \setminus K$, there exists $(u,v) \in L$ such that $u \in B(x)$, $v \in F(x)$, and $u \notin H(y,z)$ for some $z \in G(x)$;*

(vi)$_2$ *there exists $(u_0, v_0) \in X \times Y$ such that for each $(x,y) \in X \times Y \setminus K$, one has $u_0 \in B(x)$, $v_0 \in F(x)$, and $u_0 \notin H(y,z)$ for some $z \in G(x)$.*

If $(X \times Y; \Gamma^1 \times \Gamma^2)$ satisfies $1_{X \times Y} \in \mathfrak{RC}(X \times Y, X \times Y)$, then (SGWIIP) is solvable, that is, there exists $(\widehat{x}, \widehat{y}) \in K$ such that $\widehat{x} \in A(\widehat{x})$, $\widehat{y} \in F(\widehat{x})$, and $u \in H(\widehat{y}, z)$ for every $u \in B(\widehat{x})$ and every $z \in G(\widehat{x})$.

Proof. By using the same arguments as in Theorem 6.1, we can show that the set $\mathfrak{F}$ is nonempty. Define a set-valued mapping $J : X \times Y \to 2^X$ is defined by $J(x,y) = \{u \in X : u \notin H(y,z) \text{ for some } z \in G(x)\}$ for every $(x,y) \in X \times Y$. Further, let us define a set-valued mapping $T : X \times Y \to 2^{X \times Y}$ by setting, for each $(x,y) \in X \times Y$,

$$T(x,y) = \begin{cases} (B(x) \bigcap J(x,y)) \times F(x), & \text{if } (x,y) \in \mathfrak{F}, \\ B(x) \times F(x), & \text{if } (x,y) \in X \times Y \setminus \mathfrak{F}. \end{cases}$$

For each $(u,v) \in X \times Y$, we have

$$T^{-1}(u,v) = \Big((X \times Y \setminus \mathfrak{F}) \bigcap (B^{-1}(u) \times Y) \bigcap (F^{-1}(v) \times Y) \Big)$$

$$\bigcup \left(J^{-1}(u)\bigcap(B^{-1}(u)\times Y)\bigcap(F^{-1}(v)\times Y)\right).$$

By (iv), the set $J^{-1}(u)$ is open in $X\times Y$for every $u\in X$. Thus, it follows from (ii) and (iii) that $T^{-1}(u,v)$ is open in $X\times Y$ for every $(u,v)\in X\times Y$. By (v) and using the same arguments as in Theorem 6.1, we can deduce that $(x,y)\notin\Gamma^1\times\Gamma^2$-co$(T(x,y))$ for every $(x,y)\in X\times Y$. By (vi), it follows that one of the following two conditions holds:

- for each $N_0\times N_1\in\langle X\times Y\rangle$, there exist a compact Γ^1-convex subset L_{N_0} of $(X;\Gamma^1)$ containing N_0 and a compact Γ^2-convex subset L_{N_1} of $(Y;\Gamma^2)$ containing N_1 such that for $L:=L_{N_0}\times L_{N_1}$, we have $L\setminus K\subseteq\bigcup_{(u,v)\in L}T^{-1}(u,v)$.
- there exists $(u_0,v_0)\in X\times Y$ such that $X\times Y\setminus T^{-1}(u_0,v_0)\subseteq K$.

Thus, by Theorem 3.4 and Remark 3.4, there exists $(\widehat{x},\widehat{y})\in K$ such that $T(\widehat{x},\widehat{y})=\emptyset$. Since B and F have nonempty values, we can conclude that $(\widehat{x},\widehat{y})\in\mathfrak{F}$. Thus, $\widehat{x}\in A(\widehat{x})$, $\widehat{y}\in F(\widehat{x})$, and $B(\widehat{x})\bigcap J(\widehat{x},\widehat{y})=\emptyset$. Therefore, $u\in H(\widehat{y},z)$ for every $u\in B(\widehat{x})$ and every $z\in G(\widehat{x})$. This completes the proof.

Remark 6.8. (1) (v) of Theorem 6.8 can be replaced by the following stronger condition:

(v)′ H is strong Γ^1-quasiconvex-like with respect to F and G.

In fact, suppose to the contrary that there exist $x\in X$ and $y\in F(x)$ such that $x\in\Gamma^1$-co$(\{u\in X: u\notin H(y,z)$ for some $z\in G(x)\})$. Then it follows from Lemma 2.7 that there exists $\{u_0,u_1,\dots,u_n\}\in\langle\{u\in X: u\notin H(y,z)$ for some $z\in G(x)\}\rangle$ such that $x\in\Gamma^1$-co$(\{u_0,u_1,\dots,u_n\})$. By (v)′, there exists $j\in\{0,1,\dots,n\}$ and $u_j\in H(y,z)$ for every $z\in G(x)$, which contradicts that $u_j\notin H(y,z)$ for some $z\in G(x)$. Therefore, (v)′ implies (v) of Theorem 6.5.

(2) the following two conditions imply that (v)′ holds.

(a) for each $x\in X$ and each $y\in F(x)$, the set $\{u\in X: u\notin H(y,z)$ for some $z\in G(x)\}$ is Γ^1-convex.

(b) for each $x\in X$ and each $y\in F(x)$, $x\in H(y,z)$ for every $z\in G(x)$.

Indeed, by way of contradiction, suppose that for some $N=\{u_0,u_1,\dots,u_n\}\in\langle X\rangle$, some $x\in\Gamma$-co(N), there exists a point $y\in F(x)$ such that for each $j\in\{0,1,\dots,n\}$, $u_j\notin H(y,z)$ for some $z\in G(x)$. By (a), we have $x\notin H(y,z)$, which contradicts (b).

(3) If we assume that Z is a topological space, then (iv) of Theorem 6.8 can be replaced by the following condition:

(iv)′ G is a lower semicontinuous set-valued mapping and H is closed.

In fact, it is sufficient to prove that the set $\{(x,y)\in X\times Y:\ u\in H(y,z)$ for every $z\in G(x)\}$ is closed in $X\times Y$ for every $u\in X$. Let $(x^*,y^*)\in$ cl$(\{(x,y)\in X\times Y:\ u\in H(y,z)$ for every $z\in G(x)\})$ any given. Then there is a net $\{(x_\alpha,y_\alpha)\}\subseteq\{(x,y)\in X\times Y:\ u\in H(y,z)$ for every $z\in G(x)\}$ such that $(x_\alpha,y_\alpha)\to(x^*,y^*)$. Therefore, we have $u\in H(y_\alpha,z_\alpha)$ for every $z'\in G(x_\alpha)$. Since G is a lower semicontinuous set-valued mapping, it follows from Lemma 2.2 that for each $z\in G(x^*)$, there exists $z_\alpha\in G(x_\alpha)$ such that $z_\alpha\to z$. Since H is closed, we have $u\in H(y^*,z)$. This shows that $(x^*,y^*)\in\{(x,y)\in X\times Y:\ u\in H(y,z)$ for every $z\in G(x)\}$ and so, the set $\{(x,y)\in X\times Y:\ u\in H(y,z)$ for every $z\in G(x)\}$ is closed in $X\times Y$ for every $u\in X$. Thus, the set $\{(x,y)\in X\times Y: u\notin H(y,z)$ for some $z\in G(x)\}$ is open in $X\times Y$ for every $u\in X$.

7. Conclusions

In this paper, based on the KKM theory and the properties of Γ-convexity and $\mathfrak{R}\mathfrak{C}$-mapping, we have dealt with the existence of collectively fixed points in the framework of noncompact abstract convex spaces and provided applications to some existence theorems of generalized weighted Nash equilibria and generalized Pareto Nash equilibria for constrained multiobjective games, some nonempty intersection theorems for sets with abstract convex sections, and some existence theorems of solutions for generalized weak implicit inclusion problems. In our view, future research should focus on considering how to further generalize and improve the collectively fixed point theorems obtained in this paper in the framework of noncompact abstract convex spaces. Furthermore, on this basis, the existence of generalized weighted Nash equilibria and generalized Pareto Nash equilibria for constrained multiobjective games with infinite players and the existence of solutions for systems of generalized vector quasi-variational equilibrium problems should be investigated.

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Conflict of interest

The authors declare that they have no competing interests.

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