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Research article

A free boundary problem with a Stefan condition for a ratio-dependent predator-prey model

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Abstract: In this paper we study a ratio-dependent predator-prey model with a free boundary caused by predator-prey interaction over a one dimensional habitat. We study the long time behaviors of the two species and prove a spreading-vanishing dichotomy; namely, as t goes to infinity, both prey and predator successfully spread to the whole space and survive in the new environment, or they spread within a bounded area and eventually die out. The criteria governing spreading and vanishing are obtained. Finally, when spreading occurs we provide some estimates to the asymptotic spreading speed of the moving boundary $h(t)$.

Keywords: free boundary; ratio-dependent model; spreading-vanishing dichotomy; criteria; asymptotic speed
Mathematics Subject Classification: 35K20, 35R35

1. Introduction

In this paper, we consider the following ratio-dependent predator-prey model,

$$\begin{cases} u_t - u_{xx} = \lambda u - u^2 - \frac{buv}{u+mv}, & t > 0,\ 0 < x < h(t), \\ v_t - dv_{xx} = \nu v - v^2 + \frac{cuv}{u+mv}, & t > 0,\ 0 < x < h(t), \\ u_x = v_x = 0, & t \geq 0,\ x = 0, \\ u = v = 0,\ h'(t) = -\mu(u_x + \rho v_x), & t \geq 0,\ x = h(t), \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & 0 \leq x \leq h_0, \\ h(0) = h_0, \end{cases} \tag{1.1}$$

where λ, b, m, d, ν, c, μ, ρ, h_0 are given positive constants, u and v stand for the prey and predator densities, respectively. The function $x = h(t)$ is the moving boundary determined by $u(t,x)$ and $v(t,x)$

which is the free boundary to be solved. The initial functions $u_0(x)$ and $v_0(x)$ satisfy the conditions

$$u_0, v_0 \in C^2([0, h_0]),\ u_0(x), v_0(x) > 0, x \in [0, h_0),$$

$$u_0'(0) = u_0(h_0) = v_0'(0) = v_0(h_0) = 0.$$

According to the classic Lotka-Volterra type predator-prey theory, there exist a “paradox of enrichment” stating that enriching the prey’s environment always leads to an unstable predator-prey system, and a “biological control paradox” which states that a low and stable prey equilibrium density does not exist. These two situations are inconsistent with the real world. In numerous settings, especially when predators have to search, share and compete for food, many mathematicians and biologists have confirmed that a ratio-dependent predator-prey model is more reasonable than the prey-dependent model (see [2, 5, 3, 1, 4]).

The equation $h'(t) = -\mu(u_x + \rho v_x)$ governing the free boundary is a special case of the two-phase Stefan condition; here, we assume that the expanding front propagates at a rate that is proportional to the magnitudes of the prey and predator population gradients. This is in line with tendency for both predator and prey to constantly move outward from some unknown boundary (free boundary). Suppose that the predator only lives on this prey as a result of the features of partial eclipse, picky eaters and the restraint of external environment. In order to survive the predator should follow the same trajectory as prey, and so is roughly consistent with the move curve (free boundary) model. This model can be used to study the following two common phenomenons: (i) The effect of controlling pest species (prey) by introducing a natural enemy (predator); (ii) the impact of a new or invasive species (predator) on a native species (prey).

The Stefan condition arises from the study of melting ice in water [6], but has come to be widely applied to other problems; for example, the Stefan condition was applied to the modeling of wound healing [7] and the presence of oxygen in muscles [8]. For population models, Du et al. [13, 10, 14, 11, 9, 12] have studied a series of nonlinear diffusion problems with free boundary on the one-phase Stefan condition where they addressed many critical problems such as the long time behavior of species, the conditions for spreading and vanishing and the asymptotic spreading speed of the front. Of particular note, they show that if the nonlinear term is a general monostable type, then a spreading-vanishing dichotomy stands. Wang et al. have investigated a succession of free boundary problems on diverse Stefan conditions of multispecies models and derived many useful conclusions (see [20, 19, 18, 21, 16, 15, 17, 22, 23]).

In reference [20], Wang studied the same free boundary problem for the classical Lotka-Volterra type predator-prey model. A spreading-vanishing dichotomy was proved, and the long time behavior of solutions and criteria for spreading and vanishing were obtained; moreover, when spreading was successful, an upper bound for the spreading speed was provided. The manuscript [24] studied a ratio-dependent predator-prey problem with a different free boundary in which the spreading front was only caused by prey. In that paper, the author studied the spreading behaviors of the two species and provided an accurate limit of the spreading speed as time increases.

In this paper, we focus on the research problem (1.1) and understand the asymptotic behaviors of both prey and predator via such a free boundary caused by their mutual interaction. We will always assume that (u, v, h) is the solution to problem (1.1). For the global existence, uniqueness and estimates of the positive solution (u, v, h), we establish the following theorem which can be proved in a similar manner as those found for Theorem 2.1, Lemma 2.1 and Theorem 2.2 in [15]:

Theorem 1.1. *For any* $0 < \alpha < 1$*, there exists* $T > 0$ *such that*

$$(u, v, h) \in [C^{\frac{1+\alpha}{2}, 1+\alpha}(\overline{D}_T)]^2 \times C^{1+\frac{\alpha}{2}}([0, T]),$$

where

$$D_T = \{(t, x) \in \mathbb{R}^2 : t \in (0, T], x \in (0, h(t))\};$$

furthermore, for $(t, x) \in (0, \infty) \times (0, h(t))$ *there exists a positive constant M such that*

$$0 < u(t, x), v(t, x), h'(t) \leq M.$$

The organization of this paper is as follows: In section 2, we provide some comparison principles which are needed for subsequent arguments. In Section 3, we analyze waves of finite length to construct a lower solution and obtain a spreading-vanishing dichotomy. Section 4 is devoted to the study of criteria governing spreading and vanishing. In Section 5, an estimate of asymptotic spreading speed is obtained. We end in Section 6 with a brief discussion.

2. Comparison principles

In this section, we provide some comparison principles with free boundaries which are critical to the subsequent development.

Lemma 2.1. *Define* $\Omega = \{(t, x) : t > 0, 0 < x < \overline{h}(t)\}$*. Let* $\overline{u}$*,* $\overline{v} \in C(\overline{\Omega}) \bigcap C^{1,2}(\Omega)$*,* $\overline{h} \in C^1([0, \infty))$ *and* $\overline{h}(t) > 0$ *for* $t \geq 0$*. If* $(\overline{u}, \overline{v}, \overline{h})$ *satisfies*

$$\begin{cases} \overline{u}_t - \overline{u}_{xx} \geq \lambda \overline{u} - \overline{u}^2, & t > 0,\ 0 < x < \overline{h}(t), \\ \overline{v}_t - d\overline{v}_{xx} \geq (\nu + c)\overline{v} - \overline{v}^2, & t > 0,\ 0 < x < \overline{h}(t), \\ \overline{u}_x(t, 0) \leq 0,\ \overline{v}_x(t, 0) \leq 0, & t > 0, \\ \overline{u}((t, \overline{h}(t))) = \overline{v}(t, \overline{h}(t)) = 0, & t \geq 0, \\ \overline{h}'(t) \geq -\mu[\overline{u}_x(t, \overline{h}(t)) + \rho \overline{v}_x(t, \overline{h}(t))], & t > 0, \\ \overline{u}(0, x) \geq u_0(x),\ \overline{v}(0, x) \geq v_0(x), & 0 \leq x \leq \overline{h}_0, \\ \overline{h}(0) \geq h_0, \end{cases}$$

then we have the inequalities

$$u \leq \overline{u},\ v \leq \overline{v} \ on\ D,\ h(t) \leq \overline{h}(t)\ for\ t \geq 0,$$

where $D := \{(t, x) : t \geq 0, 0 \leq x \leq h(t)\}$*.*

Define $\Omega_1 = \{(t, x) : t > 0, 0 < x < \underline{h}(t)\}$ and let $\underline{h} \in C^1([0, \infty))$ with $0 < \underline{h}(0) < h_0$. Similar to the above Lemma 2.1, we present a lower solution of (u, h) and (v, h), respectively.

Lemma 2.2. *Let* $\underline{u} \in C(\overline{\Omega}_1) \bigcap C^{1,2}(\Omega_1)$*. If* $(\underline{u}, \underline{h})$ *satisfies*

$$\begin{cases} \underline{u}_t - \underline{u}_{xx} \leq (\lambda - \frac{b}{m})\underline{u} - \underline{u}^2, & t > 0,\ 0 < x < \underline{h}(t), \\ \underline{u}_x(t, 0) = \underline{u}(t, \underline{h}(t)) = 0, & t > 0, \\ \underline{h}'(t) \leq -\mu \underline{u}_x(t, \underline{h}(t)), & t > 0, \\ 0 \leq \underline{u}(0, x) \leq u_0(x), & 0 \leq x \leq \underline{h}(0), \\ \underline{h}(0) \leq h(0), \end{cases}$$

then we have the inequalities

$$h(t) \geq \underline{h}(t), t \geq 0;\ u(t,x) \geq \underline{u}(t,x) \text{ on } \overline{\Omega}_1.$$

Lemma 2.3. *Let* $\underline{v} \in C(\overline{\Omega}_1) \bigcap C^{1,2}(\Omega_1)$. *If* $(\underline{v}, \underline{h})$ *satisfies*

$$\begin{cases} \underline{v}_t - \underline{v}_{xx} \leq \nu \underline{v} - \underline{v}^2, & t > 0,\ 0 < x < \underline{h}(t), \\ \underline{v}_x(t,0) = \underline{v}(t, \underline{h}(t)) = 0, & t > 0, \\ \underline{h}'(t) \leq -\mu\rho \underline{v}_x(t, \underline{h}(t)), & t > 0, \\ 0 \leq \underline{v}(0,x) \leq v_0(x), & 0 \leq x \leq \underline{h}(0), \\ \underline{h}(0) \leq h_0, \end{cases}$$

then we have the inequalities

$$h(t) \geq \underline{h}(t), t \geq 0;\ v(t,x) \geq \underline{v}(t,x) \text{ on } \overline{\Omega}_1.$$

Remark 2.1. *We also can define an upper solution to* (u, h) *and* (v, h) *by reversing all the inequalities in Lemmas 2.2 and 2.3.*

3. Waves of finite length and the spreading-vanishing dichotomy

In this section, we study the long time behavior of (u, v). Since $h(t)$ is monotonic increasing, then either $h(t) < \infty$ (vanishing case) or $h(t) \to \infty$ (spreading case) as $t \to \infty$.

3.1. Spreading case ($h_\infty = \infty$)

Assume that $h_\infty = \infty$, then (1.1) becomes

$$\begin{cases} u_t - u_{xx} = \lambda u - u^2 - \frac{buv}{u+mv}, & t > 0,\ x > 0, \\ v_t - dv_{xx} = \nu v - v^2 + \frac{cuv}{u+mv}, & t > 0,\ x > 0, \\ u_x(t,0) = v_x(t,0) = 0, & t > 0, \\ u(0,x) = u_0(x), v(0,x) = v_0(x), & x \geq 0, \end{cases} \tag{3.1}$$

and its stationary problem is

$$\begin{cases} -u_{xx} = \lambda u - u^2 - \frac{buv}{u+mv}, & x > 0, \\ -dv_{xx} = \nu v - v^2 + \frac{cuv}{u+mv}, & x > 0, \\ u(x) = u_0(x), v(x) = v_0(x), & x \geq 0. \end{cases} \tag{3.2}$$

The proof of the following theorem proceeds in precisely the same manner as that of Theorem 3.2 in [25].

Theorem 3.1. *Assume* $h_\infty = \infty$.

(i) *If* $m\lambda > b$, *then the solution* (u, v) *satisfies*

$$\underline{u} \le \liminf_{t\to\infty} u(t,x) \le \limsup_{t\to\infty} u(t,x) \le \overline{u},$$
$$\underline{v} \le \liminf_{t\to\infty} v(t,x) \le \limsup_{t\to\infty} v(t,x) \le \overline{v}$$

uniformly on the compact subset of $[0,\infty)$*, where* $\overline{u}$*,* $\underline{u}$*,* $\overline{v}$*,* $\underline{v}$ *are determined by*

$$\lambda - \underline{u} - \frac{b\overline{v}}{\underline{u}+m\overline{v}} = 0,\ \ \lambda - \overline{u} - \frac{b\underline{v}}{\overline{u}+m\underline{v}} = 0,$$

$$\nu - \overline{v} + \frac{c\overline{u}}{\overline{u}+m\overline{v}} = 0,\ \ \nu - \underline{v} + \frac{c\underline{u}}{\underline{u}+m\underline{v}} = 0.$$

(ii) *If* $0 < m\lambda - b < b\nu/c$*, then*

$$\lim_{t\to\infty} u(t,x) = u^* := \frac{A+\sqrt{\Delta_1}}{2(b+cm^2)},\ \ \lim_{t\to\infty} v(t,x) = v^* := \frac{u^*(\lambda-u^*)}{b-m(\lambda-u^*)},$$

where $A = \lambda(2cm^2+b) - mb(\nu+2c)$*,* $\Delta_1 = A^2 + 4(b+cm^2)[(b(\nu+c)-mc\lambda)](m\lambda-b)$*; moreover,* (u^*,v^*) *is the stationary solution of (3.2).*

3.2. Vanishing case

In this section, we concentrate on the vanishing case. In order to get sufficient conditions for vanishing, we will construct a suitable lower solution to (1.1) with respect to v by a phase plane analysis of the Eq (3.3).

3.2.1. Waves of finite length

In this section, we consider the solution $(s, q(z))$ of the following problem for $Z \in (0,\infty)$

$$\begin{cases} dq'' - sq' + f(q) = 0, & z\in[0,Z], \\ q(0)=0,\ q'(Z)=0,\ q(z)>0, & z\in[0,Z], \end{cases} \tag{3.3}$$

where for any fixed $u \ge 0$, $f(q) := \nu q - q^2 + \frac{cuq}{u+mq}$. Denote $q' = dq/dz$. We can rewrite the first equation of (3.3) into the equivalent form

$$\begin{cases} q' = p, \\ dp' = sp - f(q), \end{cases} \tag{3.4}$$

or

$$d\cdot\frac{dp}{dq} = s - \frac{f(q)}{p},\ \textit{when}\ p \ne 0. \tag{3.5}$$

For each $s \ge 0$ and $\eta > 0$, we denote by $p^s(q;\eta)$ the unique solution of (3.5) with initial condition $p^s(q)|_{q=0} = \eta$, where $\eta > 0$. We are most interested in the cases $s = 0$ and small $s > 0$.

When $s = 0$. A simple calculation deduces that

$$p^0(q;\eta) = \sqrt{\eta^2 - \frac{2}{d}\int_0^q f(\tau)d\tau},\ \ q\in[0,q^\eta), \tag{3.6}$$

where q^η is given by

$$\eta^2 = \frac{2}{d}\int_0^{q^\eta} f(\tau)d\tau. \tag{3.7}$$

Denote $\theta := \nu^*$, where ν^* is defined by Theorem 3.1. It follows that $q^\eta < \theta\ (< \nu + c)$ if and only if $0 < \eta < \eta^*$ where we have labelled

$$\eta^* = \sqrt{\frac{2}{d}\int_0^{\theta} f(\tau)d\tau}.$$

It follows that q^η is strictly increasing in the interval $\eta \in (0, \eta^*)$ and $q^\eta \to 0$ as $\eta \to 0$.

The positive solution $p^0(q;\eta)$ of (3.4) corresponds to a trajectory $(q_0(z;\eta), p_0(z;\eta))$ (with $s = 0$) that passes through $(0, \eta)$ at $z = 0$ and approaches $(q^\eta, 0)$ as z goes to z^η (see Figure 1). It follows from (3.4) with $s = 0$ and (3.6) and (3.7) that

$$z = \int_0^{q_0(z;\eta)} \frac{dr}{\sqrt{\frac{2}{d}\int_r^{q^\eta} f(\tau)d\tau}}.$$

So

$$z^\eta = \int_0^{q^\eta} \frac{dr}{\sqrt{\frac{2}{d}\int_r^{q^\eta} f(\tau)d\tau}}.$$

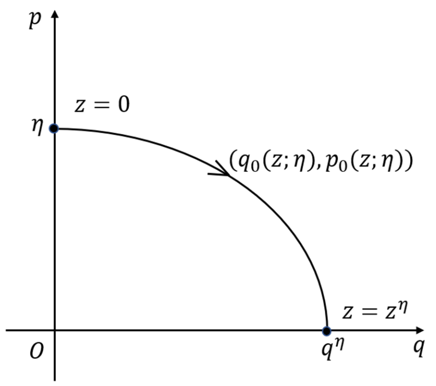

Figure 1. qp-plane of (3.4) when $s = 0$.

Note that $f(0) = 0$ and $q^\eta < \mu + c$. Recall that $q^\eta \to 0$ as $\eta \to 0$ from which we conclude

$$\begin{aligned} 2\int_r^{q^\eta} f(\tau)d\tau &= 2\int_r^{q^\eta} [f'(0)\tau + o(\tau)]d\tau \\ &= 2f'(0)\int_r^{q^\eta} \tau d\tau + o(1) \\ &= f'(0)((q^\eta)^2 - r^2) + o(1). \end{aligned}$$

Then

$$
\begin{aligned}
z^{\eta} &= \int_0^{q^{\eta}} \frac{\sqrt{d} + o(1)}{\sqrt{f'(0)((q^{\eta})^2 - r^2)}} dr \\
&= \sqrt{\frac{d}{f'(0)}} \arcsin \frac{r}{q^{\eta}} \Big|_0^{q^{\eta}} + o(1) \\
&= \frac{\pi}{2} \sqrt{\frac{d}{f'(0)}} + o(1).
\end{aligned}
$$

Define

$$
Z^* := \frac{\pi}{2} \sqrt{\frac{d}{f'(0)}}.
$$

According to the above discussions, we have the following result.

Lemma 3.1. *If $Z > Z^*$, then the elliptic boundary value problem*

$$
\begin{cases} dv_{xx} + f(v) = 0, \ x \in (0, Z), \\ v'(0) = v(Z) = 0 \end{cases} \tag{3.8}
$$

has at least one positive solution v_Z.

Proof. Since $Z > Z^*$, there exists $\eta_* \in (0, \eta^*)$ and correspondingly $q_* := q^{\eta_*} \in (0, \theta)$ such that $z_* := z^{\eta_*} \in (Z^*, Z)$. Let $(q(z), p(z))$ be the trajectory of (3.4) (with $s = 0$) that connects $(0, \eta_*)$ at $z = 0$ and $(q_*, 0)$ as z goes to z_*. Then $q(z)$ satisfies

$$
\begin{cases} dq'' + f(q) = 0, \ z \in (0, z_*), \\ q(0) = q'(z_*) = 0. \end{cases}
$$

Define

$$
\underline{v}(x) := \begin{cases} q(-x + z_*), & x \in (0, z_*], \\ 0, & x \in (z_*, Z]. \end{cases}
$$

Then $\underline{v}$ is a (weak) lower solution of (3.8). On the other hand, a sufficiently large constant $C \gg v + c$ is an upper solution of (3.8). We can conclude (3.8) has at least one positive solution by the standard upper-lower solution argument. □

Remark 3.1. *The positive solution v_Z of (3.8) corresponds to a trajectory $(q(z), p(z)) := (v_Z(Z - z), -v_Z'(Z - z))$ (with $s = 0$) passing through $(0, \eta) := (0, -v_Z'(Z))$ at $z = 0$ and approaching $(q^{\eta}, 0) := (v_Z(0), 0)$ as z goes to Z.*

Now we study (3.4) for small $s > 0$ as a perturbation of the case $s = 0$. For some small $s > 0$, (3.5) with initial data $p^s(q)|_{q=0} = \eta \in (0, \eta^*)$ has a solution $p^s(q; \eta)$ defined on $[0, q^{s,\eta}]$ for some $q^{s,\eta} > q^{\eta}$. Let $(q_s(z; \eta), p_s(z; \eta))$ be the trajectory of (3.4) (with small $s > 0$) that pass through $(0, \eta)$ at $z = 0$ and approaches $(q^{s,\eta}, 0)$ as z goes to $z^{s,\eta}$ (See Figure 2). We state the following results.

Lemma 3.2. *Fix $\eta \in (0, \eta^*)$. For any $\varepsilon > 0$, there exists some small $\delta > 0$ such that*

(i) if $s \in (0, \delta)$, then $q^{s,\eta} \in (q^\eta, q^\eta + \varepsilon)$ and $z^{s,\eta} \in (z^\eta - \varepsilon, z^\eta + \varepsilon)$;

(ii) $p^0(q; \eta) \leq p^s(q; \eta) \leq p^0(q; \eta) + \varepsilon$ for $q \in [0, q^\eta]$;

(iii) $q_0(z; \eta) \leq q_s(z; \eta) \leq q_0(z; \eta) + \varepsilon$ for $z \in [0, \min\{z^\eta, z^{s,\eta}\}]$.

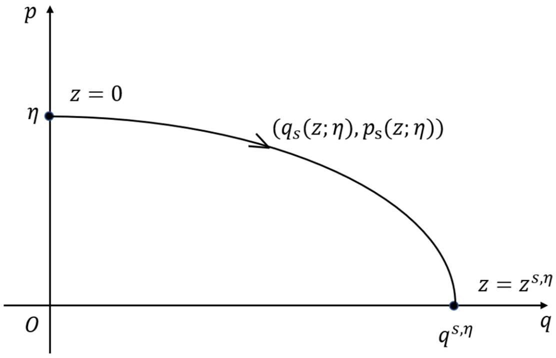

Figure 2. qp-plane of (3.4) when a small $s > 0$.

3.2.2. Vanishing case

In order to discuss the long-term behavior of (u, v), we first give two important propositions.

Proposition 3.1. *If $h_\infty < \infty$, then there exists a positive constant M such that*

$$\|u(t, \cdot), v(t, \cdot)\|_{C^1[0,h(t)]} \leq M, \ \forall t > 1.$$

and

$$\lim_{t\to\infty} h'(t) = 0.$$

Proof. Similar to the proof of Theorem 4.1 of [15] and so we omit it. □

Proposition 3.2. ([20]) *Let d, θ, β, g_0, C be positive constants. Suppose that $w \in C^{\frac{1+\alpha}{2}, 1+\alpha}([0, \infty) \times [0, g(t)])$ and $g \in C^{1+\frac{\alpha}{2}}([0, \infty))$ for some $\alpha > 0$ and satisfies $w(t, x) > 0$, $g(t) > 0$ for all $0 \leq t < \infty$ and $0 < x < g(t)$. Assume that $w_0 \in C^2([0, g_0])$ and satisfies $w_0'(0) = 0$, $w_0(g_0) = 0$ and $w_0(x) > 0$ in $(0, g_0)$; furthermore, suppose that*

$$\lim_{t\to\infty} g(t) = g_\infty < \infty, \ \lim_{t\to\infty} g'(t) = 0, \ \|w(t, \cdot)\|_{C[0,g(t)]} \leq \tilde{M}, \ \ \forall t > 1.$$

If (w, g) satisfies

$$\begin{cases} w_t - dw_{xx} \geq w(C - w), & t > 0, 0 < x < g(t), \\ w_x = 0, & t > 0, x = 0, \\ w = 0, \ g'(t) \geq -\beta w_x, & t > 0, x = g(t), \\ w(0, x) = w_0(x), & 0 \leq x \leq g_0, \\ g(0) = 0, & \end{cases}$$

then

$$\lim_{t\to\infty} \max_{0\leq x\leq g(t)} w(t, x) = 0.$$

Lemma 3.3. *Let (u, v, h) be a solution of the problem (1.1). If $h_\infty < \infty$, then*

$$\lim_{t\to\infty} \|u(t,\cdot), v(t,\cdot)\|_{C([0,h(t)])} = 0; \tag{3.9}$$

moreover,

$$h_\infty \le \frac{\pi}{2} \min\{\sqrt{m/(m\lambda - b)}, \sqrt{d/\nu}\}. \tag{3.10}$$

Proof. We first prove (3.9). Since (u, h) satisfies

$$\begin{cases} u_t - u_{xx} \ge u(\lambda - b/m - u), & t > 0,\ 0 < x < h(t), \\ u_x = 0, & t > 0, x = 0, \\ u = 0,\ h'(t) \ge -\mu u_x, & t > 0, x = h(t), \\ u(0, x) = u_0(x), & 0 \le x \le h_0, \\ h(0) = h_0, \end{cases}$$

by Propositions 3.1 and 3.2 we have $\lim_{t\to\infty} \|u(t,\cdot)\|_{C([0,h(t)])} = 0$. On the other hand, (v, h) satisfies

$$\begin{cases} v_t - dv_{xx} \ge v(\nu - v), & t > 0,\ 0 < x < h(t), \\ v_x = 0, & t > 0, x = 0, \\ v = 0,\ h'(t) \ge -\mu\rho v_x, & t > 0, x = h(t), \\ v(0, x) = v_0(x),\ h(0) = h_0, & 0 \le x \le h_0. \end{cases}$$

Similarly, we conclude that $\lim_{t\to\infty} \|v(t,\cdot)\|_{C([0,h(t)])} = 0$.

Now we proof (3.10) and first assert that $h_\infty \le \frac{\pi}{2}\sqrt{m/(m\lambda - b)}$; otherwise, there exists $\tau \gg 1$ such that

$$h(\tau) > \max\{h_0, \frac{\pi}{2}\sqrt{m/(m\lambda - b)}\}.$$

Let $l = h(\tau)$, and note $l > \frac{\pi}{2}\sqrt{m/(m\lambda - b)}$. Suppose $w(t, x)$ is the unique solution of the following problem

$$\begin{cases} w_t - w_{xx} = w(\lambda - b/m - w), & t > \tau,\ 0 < x < l, \\ w_x(t, 0) = w(t, l) = 0, & t > \tau, \\ w(\tau, x) = u(\tau, x), & 0 \le x \le l. \end{cases}$$

By using the comparison principle, we have

$$w(t, x) \le u(t, x), \quad t \ge \tau,\ 0 \le x \le l.$$

In view of $\lambda - b/m > (\frac{\pi}{2l})^2$, it is well known that $w(t, x) \to w^*(x)$ as $t \to \infty$ uniformly in a compact subset of $[0, l)$, where w^* is the unique positive solution of

$$\begin{cases} -w_{xx} = w(\lambda - b/m - w), & 0 < x < l, \\ w_x(0) = w(l) = 0. \end{cases}$$

It must be that $\varliminf_{t\to\infty} u(t, x) \ge \lim_{t\to\infty} w(t, x) = w^*(x) > 0$ which contradicts (3.9). Similarly, we have $h_\infty \le \frac{\pi}{2}\sqrt{d/\nu}$. The proof of (3.10) is finished. □

The following lemma gives a more precise upper bound of h_∞ by use of conclusions of Section 3.2.1 when vanishing occurs.

Lemma 3.4. *If $h_\infty < \infty$, then*

$$h_\infty \le Z^* := \frac{\pi}{2}\sqrt{\frac{d}{f'(0)}}; \tag{3.11}$$

that is to say, $h_\infty \le \frac{\pi}{2}\sqrt{d/(\nu+c)}$.

Proof. If this is not the case, we can find $t_0 > 0$ such that $h(t_0) > Z^*$. For a small $s < \mu\rho\eta$, we want to use $q_s(z;\eta)$ to construct a lower solution of (1.1). Define

$$k(t) := z^{s,\eta} + st, \ \textit{where}\ z^{s,\eta} \le Z^*,$$

$$w(t,x) := \begin{cases} q_s(z^{s,\eta};\eta), & x \in [0, st], \\ q_s(k(t)-x;\eta), & x \in [st, k(t)]. \end{cases}$$

Then $w_t \le w_{xx} + f(w)$ and $w_x(t,0) = w(t,k(t)) = 0$ for $t > 0$, $x \in (0, k(t))$; in addition, we see that

$$k(0) = z^{s,\eta} \le Z^* < h(t_0),$$

$$k'(t) = s < \mu\rho\eta = -\mu\rho w_x(t,k(t)).$$

Now we assert that

$$v(t_0,x) > w(0,x) := q_s(z^{s,\eta}-x;\eta), \ x \in [0, z^{s,\eta}] \tag{3.12}$$

holds. According to Lemma 3.1, problem (3.8) with right boundary $h(t_0)$ replacing Z has a positive solution $v_{h(t_0)} =: v_{t_0}$ which is a stationary solution. By the standard comparison principle we have

$$v(t,x) > v_{t_0}(x), \ x \in [0, h(t_0)], \ t > 0.$$

So there exists a small $\varepsilon > 0$ such that for $t \ge t_0$ we have

$$v(t,x) > v_{t_0}(x) + \varepsilon, \ x \in [0, h(t_0)]$$

and

$$v(t,x) > v_{t_0}(0) + \varepsilon, \ x \in [0, \varepsilon].$$

By Remark 3.1 and Lemma 3.2, we can find a small $s > 0$ such that

$$q_s(z^{s,\eta}-z;\eta) < q_0(z^{s,\eta}-z;\eta) + \varepsilon/2 < q_0(z^\eta - z;\eta) + \varepsilon, \ z \in [\varepsilon, z^{s,\eta}].$$

Due to the property that $q_0(z;\eta)$ and $q_s(z;\eta)$ increases monotonically with respect to z, we find that

$$q_0(z^\eta - z;\eta) < q_0(h(t_0)-z;\eta) = v_{t_0}(z), \ z \in [\varepsilon, z^{s,\eta}],$$

$$q_s(z^{s,\eta}-z;\eta) < q^{s,\eta} < q^\eta + \varepsilon = v_{t_0}(0) + \varepsilon, \ z \in (0, \varepsilon]$$

from which it follows that

$$v(t,x) > q_s(z^{s,\eta}-x;\eta), \ t \ge t_0, x \in [0, z^{s,\eta}].$$

If we let $t = t_0$, then (3.12) is proved.

Applying Lemma 2.3 we obtain that

$$h(t+t_0) \geq k(t),\ v(t+t_0,x) \geq w(t,x),\ t>0, x\in[0,k(t)]$$

which implies $h_\infty = \infty$. By Theorem 3.1 we have $\lim_{t\to\infty} v(t,x) = v^* > 0$ which yields a contradiction to Theorem 3.3. The inequality (3.11) is established. □

Combining Lemma 3.3 with Lemma 3.4, we have the following theorem directly.

Theorem 3.2. *Define*

$$\Lambda := \frac{\pi}{2}\min\left\{\sqrt{\frac{m}{m\lambda - b}}, \sqrt{\frac{d}{\nu + c}}\right\}.$$

If $h_\infty < \infty$, then $h_\infty \leq \Lambda$.

Remark 3.2. *Theorem 3.2 shows that if the prey and predator populations cannot spread into infinity, then they will never break through Λ and will vanish eventually.*

4. The criteria governing spreading and vanishing

In this section, we study the criteria of spreading and vanishing for problem (1.1). Recall that $h'(t) > 0$ for $t > 0$, then the next result is obtained directly by Theorem 3.2.

Theorem 4.1. *If $h_0 \geq \Lambda$, then $h_\infty = \infty$.*

Next we mainly discuss the case $h_0 < \Lambda$.

Lemma 4.1. *Suppose $h_0 < \Lambda$. If*

$$\mu \geq \mu^0 := \min\{\mu^*, \mu^{**}\},$$

where

$$\mu^* := \max\left\{1, \frac{m\|u_0\|_\infty}{m\lambda - b}\right\}\left(\frac{\pi}{2}\sqrt{\frac{m}{m\lambda - b}} - h_0\right)\left(\int_0^{h_0} u_0(x)dx\right)^{-1},$$

$$\mu^{**} := \max\left\{1, \frac{\|v_0\|_\infty}{\nu}\right\}\frac{d}{\nu}\left(\frac{\pi}{2}\sqrt{\frac{d}{\nu + c}} - h_0\right)\left(\int_0^{h_0} v_0(x)dx\right)^{-1},$$

then $h_\infty = \infty$.

Proof. First, consider the following auxiliary problem

$$\begin{cases} \underline{u}_t - \underline{u}_{xx} = (\lambda - \frac{b}{m})\underline{u} - \underline{u}^2, & t > 0, 0 < x < \underline{h}(t), \\ \underline{u}_x(t,0) = \underline{u}(t,\underline{h}(t)) = 0, & t > 0, \\ \underline{h}'(t) = -\mu\underline{u}_x(t,\underline{h}(t)), & t > 0, \\ \underline{u}(0,x) = u_0(x), & 0 \leq x \leq h_0, \\ \underline{h}(0) = h_0. \end{cases} \tag{4.1}$$

For the case $||u_0||_\infty \le \lambda - \frac{b}{m}$. Direct calculations yield

$$\begin{aligned}\frac{d}{dt}\int_0^{\underline{h}(t)} \underline{u}(t,x)dx &= \int_0^{\underline{h}(t)} \underline{u}_t(t,x)dx + \underline{h}'(t)\underline{u}(t,\underline{h}(t))\\ &= \int_0^{\underline{h}(t)} u_{xx}dx + \int_0^{\underline{h}(t)} [(\lambda - \frac{b}{m})\underline{u} - \underline{u}^2]dx\\ &= -\frac{\underline{h}'(t)}{\mu} + \int_0^{\underline{h}(t)} [(\lambda - \frac{b}{m})\underline{u} - \underline{u}^2]dx.\end{aligned}$$

Then we integrate 0 to t and derive

$$\begin{aligned}\int_0^{\underline{h}(t)} \underline{u}(t,x)dx &= (\int_0^{\underline{h}_0} u_0(x)dx + \frac{h_0 - \underline{h}(t)}{\mu}) + \int_0^t \int_0^{\underline{h}(s)} [(\lambda - \frac{b}{m})\underline{u} - \underline{u}^2]dxds\\ &:= I + II.\end{aligned}$$

Notice that $0 < \underline{u}(t,x) < \lambda - \frac{b}{m}$ for all $t > 0$ and $x \in [0, \underline{h}(t)]$, and so we have $II > 0$ for $t > 0$.

Assume that $h_\infty \ne \infty$. By Theorem 3.2 and Lemma 2.2, we have $\underline{h}_\infty := \lim_{t\to\infty} \underline{h}(t) \le \frac{\pi}{2}\sqrt{\frac{m}{m\lambda - b}}$ and $\lim_{t\to\infty} ||\underline{u}(t,\cdot)||_{C([0,h(t)])} = 0$; thus, $\int_0^{\underline{h}(t)} \underline{u}(t,x)dx \to 0$ implying $I < 0$ as $t \to \infty$ which is a contradiction with our assumption $\mu \ge \mu^*$. We see that it is the case that if $\mu > \mu^*$, then $h_\infty = \infty$.

For the case $||u_0||_\infty > \lambda - \frac{b}{m}$, we can replace u_0 with $\underline{u}_0 = \frac{(m\lambda - b)u_0(x)}{m||u_0||_\infty}$ in (4.1). Then we also have $\underline{u}(t,x) \le u(t,x)$ and $\underline{h} \le h$ for $t > 0$ and $x \in [0, \underline{h}(t)]$ by Lemma 2.2. From what we proved above for the case $||u_0||_\infty \le \lambda - \frac{b}{m}$, we also have $h_\infty = \infty$ if $\mu > \mu^*$.

We now consider the following auxiliary problem

$$\begin{cases}\underline{v}_t - d\underline{v}_{xx} = \nu\underline{v} - \underline{v}^2, & t > 0, 0 < x < \underline{h}(t),\\ \underline{v}_x(t,0) = \underline{v}(t,h(t)) = 0, & t > 0,\\ \underline{h}'(t) = -\mu\rho\underline{v}_x(t,h(t)), & t > 0,\\ \underline{v}(0,x) = v_0(x), & 0 \le x \le h_0,\\ \underline{h}(0) = h_0.\end{cases}$$

Note that $h_0 \le \max\{\frac{\pi}{2}\sqrt{\frac{d}{\nu}}, \frac{\pi}{2}\sqrt{\frac{d}{\nu + c}}\}$. Proceeding similarly as in the above discussion, we see that if

$$\begin{aligned}\mu &\ge \max\{1, \frac{||v_0||_\infty}{\nu}\} \cdot \frac{d}{\nu} \cdot (\min\{\frac{\pi}{2}\sqrt{\frac{d}{\nu}}, \frac{\pi}{2}\sqrt{\frac{d}{\nu + c}}\} - h_0)(\int_0^{h_0} v_0(x)dx)^{-1}\\ &= \mu^{**}\end{aligned}$$

then $\underline{h}_\infty = \infty$. We can then conclude by Lemma 2.3 that $h_\infty = \infty$. The proof is finished. □

Lemma 4.2. *Assume $h_0 < \Lambda$. There exists $\mu_0 > 0$ depending on u_0 and v_0 such that $h_\infty < \infty$ if $\mu \le \mu_0$.*

Proof. We will use Lemma 2.1 and construct a suitable upper solution of (1.1) to derive the desired conclusion. The approach is inspired by [9, 20]. Define

$$\sigma(t) = h_0(1 + \delta - \frac{\delta}{2}e^{-\beta t}),\ t \ge 0;\ V(y) = \cos(\frac{\pi y}{2}),\ 0 \le y \le 1,$$

$$\overline{u}(t,x) = \overline{v}(t,x) = Me^{-\beta t}V\left(\frac{x}{\sigma(t)}\right),\ t \ge 0,\ 0 \le x \le \sigma(t),$$

where β, δ and M are positive constants to be specified later.

Evaluating the definitions we have

$$\sigma(0) = h_0(1+\frac{\delta}{2}) > h_0,\ h_0(1+\frac{\delta}{2}) \le \sigma(t) \le h_0(1+\delta),$$

$$\overline{u}_x(t,0) = \overline{u}(t,\sigma(t)) = \overline{v}_x(t,0) = \overline{v}(t,\sigma(t)) = 0,\ \forall t \ge 0.$$

Let $M \gg 1$ such that $\overline{u}(0,x) \ge u_0(x)$, $\overline{v}(0,x) \ge v_0(x)$ for $x \in [0,h_0]$ and take $\beta = \frac{1}{2}(\frac{\pi}{2})^2 h_0^{-2}(1+\delta)^{-2} - \frac{1}{2}\max\{\lambda, \nu + c\}$. Then direct computations yield

$$\begin{aligned}
&\overline{u}_t - \overline{u}_{xx} - \overline{u}(\lambda - \overline{u})\\
=&\overline{u}\left(-\beta + \frac{\pi}{2}x\sigma^{-2}\sigma' \tan(\frac{\pi}{2}\frac{x}{\sigma(t)}) + (\frac{\pi}{2})^2\sigma^{-2} - \lambda + \overline{u}\right)\\
\ge&\overline{u}\left(-\beta + (\frac{\pi}{2})^2\sigma^{-2} - \lambda\right)\\
>&0, \qquad t > 0,\ 0 \le x \le \sigma(t).
\end{aligned}$$

Similarly, we have

$$\overline{v}_t - \overline{v}_{xx} - \overline{v}(\nu + c - \overline{v}) > 0, \quad t > 0,\ 0 \le x \le \sigma(t).$$

If we choose $\mu_0 = \frac{\delta\beta h_0^2}{2\pi M(1+\rho)}$, then for any $0 \le \mu \le \mu_0$ we have

$$\sigma'(t) + \mu(\overline{u}_x + \rho\overline{v}_x)|_{x=\sigma(t)} = \frac{e^{-\beta t}}{2}\left(\delta\beta h_0 - \frac{\pi M\mu(1+\rho)}{\sigma(t)}\right) > 0.$$

By virtue of Lemma 2.1, we have $\sigma(t) \ge h(t)$. If we take $t \to \infty$, then we conclude $h_\infty \le \sigma(\infty) = h_0(1+\delta) < \infty$. The proof is finished. □

Theorem 4.2. *Assume that $h_0 < \Lambda$. Then there exist $\overline{\mu} \ge \underline{\mu} > 0$ depending on u_0, v_0 and h_0, such that $h_\infty \le \Lambda$ if $\mu \le \underline{\mu}$ and $h_\infty = \infty$ if $\mu > \overline{\mu}$.*

Proof. The proof is similar to that of Theorem 3.9 in [9]. To emphasize the dependence (u,v,h) on μ, we write it as (u_μ, v_μ, h_μ). Define

$$\Sigma^* := \{\mu > 0 : h_{\mu,\infty} \le \Lambda\} \text{ and } \overline{\mu} := \sup \Sigma^*.$$

So $h_{\mu,\infty} = \infty$ if $\mu > \overline{\mu}$ by Theorem 3.2; thus, $\Sigma^* \subset (0,\overline{\mu}]$. We assert that $\overline{\mu} \in \Sigma^*$; otherwise, we have $h_{\overline{\mu},\infty} = \infty$. Then there exists $T > 0$ such that $h_{\overline{\mu}}(T) > \Lambda$. In view of the dependence of (u_μ, v_μ, h_μ) on μ, there exists $\varepsilon > 0$ such that $h_\mu(T) > \Lambda$ for $\mu \in (\overline{\mu} - \varepsilon, \overline{\mu} + \varepsilon)$. We conclude $(\overline{\mu} - \varepsilon, \overline{\mu} + \varepsilon) \bigcap \Sigma^* = \emptyset$ and $\sup \Sigma^* \le \overline{\mu} - \varepsilon$ which contradicts the definition of μ^*. This proves the assertion $\overline{\mu} \in \Sigma^*$.

Let

$$\Sigma_* := \{\mu : \mu \ge \mu_0 \textit{ such that } h_{\mu,\infty} \le \Lambda\} \textit{ and } \underline{\mu} := \sup \Sigma_*.$$

Then $\underline{\mu} \le \overline{\mu}$ and $(0,\underline{\mu}) \subset \Sigma_*$. In the same way as above, we can prove that $\underline{\mu} \in \Sigma_*$. This completes the proof. □

5. Asymptotic spreading speed

In this section, we give some estimates of $h(t)$ to understand the asymptotic spreading speed (if spreading happens). We first introduce a vital result which can easily be deduced by Theorem 6.2 of [12] in order to obtain an upper bound for $\limsup_{t\to\infty} \frac{h(t)}{t}$.

Proposition 5.1. *Let d, s, θ are positive constants. For any given $s > 2\sqrt{\theta d}$, the following problem*

$$\begin{cases} dq'' - sq' + q(\theta - q) = 0, & z \in [0, \infty), \\ q(0) = 0, \ q(\infty) = \theta, \\ q(z) > 0, \ q'(z) > 0, & z \in [0, \infty) \end{cases}$$

has a unique solution.

Remark 5.1. *For any given $s > 2\max\{\sqrt{\lambda}, \sqrt{d(\nu + c)}\}$, the problem*

$$\begin{cases} \phi'' - s\phi' + \phi(\lambda - \phi) = 0, d\psi'' - s\psi' + \psi(\nu + c - \psi) = 0 & \text{in } [0, \infty), \\ (\phi, \psi)(0) = (0, 0), (\phi, \psi)(\infty) = (\lambda, \nu + c), \\ \phi > 0, \psi > 0, \phi' > 0, \psi' > 0, & \text{in } [0, \infty) \end{cases} \tag{5.1}$$

has a unique solution (ϕ, ψ).

Theorem 5.1. *Suppose that $h_\infty = \infty$. Then we have*

$$\limsup_{t\to\infty} \frac{h(t)}{t} \le 2\max\{\sqrt{\lambda}, \sqrt{d(\nu + c)}\}.$$

Proof. The idea of the proof is inspired by [9]. Let $s > 2\max\{\sqrt{\lambda}, \sqrt{d(\nu + c)}\}$ and $(\phi(\xi), \psi(\xi))$ be the solution of (5.1). Recall that $\limsup_{t\to\infty} u(t, x) \le \lambda$ and $\limsup_{t\to\infty} v(t, x) \le \nu + c$ for $x \ge 0$. Then for any small $\varepsilon > 0$ there exists $T = T_\varepsilon > 0$ such that

$$u(t, x) \le (1 - \varepsilon)^{-1}\lambda, \ v(t, x) \le (1 - \varepsilon)^{-1}(\nu + c), \ \forall t \ge T, x \ge 0.$$

Since $\phi(\xi) \to \lambda$ and $\psi(\xi) \to \nu + c$ as $\xi \to \infty$, there exists $\xi_0 > 0$ such that

$$\phi(\xi_0) > (1 - \varepsilon)\lambda, \ \psi(\xi_0) > (1 - \varepsilon)(\nu + c).$$

Now define

$$k(t) = (1 - \varepsilon)^{-2}st + \xi_0 + h(T), \ t \ge 0,$$

$$\overline{u}(t, x) = (1 - \varepsilon)^{-2}\phi(k(t) - x), \ t \ge 0, \ 0 \le x \le \xi(t),$$

$$\overline{v}(t, x) = (1 - \varepsilon)^{-2}\psi(k(t) - x), \ t \ge 0, \ 0 \le x \le \xi(t),$$

where

$$s > \mu[\phi'(0) + \rho\psi'(0)]. \tag{5.2}$$

Clearly, we have

$$\overline{u}(t, k(t)) = \overline{v}(t, k(t)) = 0,$$

$$\overline{u}_x(t,0) = -(1-\varepsilon)^{-2}\phi'(k(t)) < 0,\ \overline{v}_x(t,0) = -(1-\varepsilon)^{-2}\psi'(k(t)) < 0.$$

For $x \in [0, h(T)]$, we have the inequality

$$\begin{aligned}\overline{u}(0,x) &= (1-\varepsilon)^{-2}\phi(\xi_0 + h(T) - x)\\ &\geq (1-\varepsilon)^{-1}\lambda\\ &\geq u(T,x)\end{aligned}$$

and similarity we have $\overline{v}(0,x) \geq v(T,x)$. Direct calculations deduce that

$$\begin{aligned}&\overline{u}_t - \overline{u}_{xx} - \overline{u}(\lambda - \overline{u})\\ =&(1-\varepsilon)^{-2}\left[(1-\varepsilon)^{-2}s\phi' - \phi'' - \phi(\lambda - (1-\varepsilon)^{-2}\phi)\right]\\ \geq&(1-\varepsilon)^{-2}\left[s\phi' - \phi'' - \phi(\lambda - \phi)\right]\\ =&0, \quad t > 0,\ 0 < x < \xi(t),\end{aligned}$$

and in the same way we derive $\overline{v}_t - d\overline{v}_{xx} - \overline{v}(\nu + c - \overline{v}) \geq 0$ for $t > 0$, $0 < x < \xi(t)$. It follows from (5.2) that

$$\begin{aligned}k'(t) &= (1-\varepsilon)^{-2}s\\ &> (1-\varepsilon)^{-2}\mu\left[\phi'(0) + \rho\psi'(0)\right]\\ &= -\mu\left[\overline{u}_x(t,k(t)) + \rho\overline{v}_x(t,k(t))\right];\end{aligned}$$

additionally, since $h'(t) > 0$, we have $k(0) = \xi_0 + h(T) > h_0$. By Lemma 2.1 we have $k(t) \geq h(t+T)$; therefore,

$$\limsup_{t\to\infty}\frac{h(t)}{t} \leq \lim_{t\to\infty}\frac{k(t-T)}{t} = (1-\varepsilon)^{-2}s,$$

from which it follows that

$$\limsup_{t\to\infty}\frac{h(t)}{t} \leq 2\max\left\{\sqrt{\lambda},\ \sqrt{d(\nu+c)}\right\}$$

by the arbitrariness of ε and $s > 2\max\{\sqrt{\lambda},\ \sqrt{d(\nu+c)}\}$. □

Remark 5.2. *Theorem 5.1 shows that when spreading occurs, the asymptotic spreading speed of* $h(t)$ *cannot be faster than* $2\max\{\sqrt{\lambda},\ \sqrt{d(\nu+c)}\}$.

Theorem 5.2. *Assume that* $s_i(\infty)$, $k_i(\infty) = \infty$ $(i = 1,2)$. *Let* (ϕ_i, s_i), (ψ_i, k_i) *be solutions of the free boundary problems*

$$\begin{cases}\phi_{1t} - \phi_{1xx} = \lambda\phi_1 - \phi_1^2, & t > 0, 0 < x < s_1(t),\\ \phi_{1x}(t,0) = \phi_1(t,s_1(t)) = 0, & t > 0,\\ s_1'(t) = -\kappa_1\phi_1(t,s_2(t)), & t > 0,\\ \phi_1(0,x) = \phi_{10}, & x \in [0, s_{10}],\\ s_1(0) = s_{10},\end{cases}$$

$$\begin{cases}\phi_{2t} - \phi_{2xx} = (\lambda - \frac{b}{m})\phi_2 - \phi_2^2, & t > 0, 0 < x < s_2(t),\\ \phi_{2x}(t,0) = \phi_2(t,s_2(t)) = 0, & t > 0,\\ s_2'(t) = -\kappa_2\phi_2(t,s_2(t)), & t > 0,\\ \phi_2(0,x) = \phi_{20}, & x \in [0, s_{20}],\\ s_2(0) = s_{20},\end{cases}$$

$$\begin{cases} \psi_{1t} - d\psi_{1xx} = (\nu + c)\psi_1 - \psi_1^2, & t > 0, 0 < x < k_1(t), \\ \psi_{1x}(t, 0) = \psi_1(t, k_1(t)) = 0, & t > 0, \\ k_1'(t) = -\tau_1\psi_1(t, k_1(t)), & t > 0, \\ \psi_1(0, x) = \psi_{10}, & x \in [0, k_{10}], \\ k_1(0) = k_{10}, \end{cases}$$

$$\begin{cases} \psi_{2t} - d\psi_{2xx} = \nu\psi_2 - \psi_2^2, & t > 0, 0 < x < k_2(t), \\ \psi_{2x}(t, 0) = \psi_2(t, k_2(t)) = 0, & t > 0, \\ k_2'(t) = -\tau_2\psi_2(t, k_2(t)), & t > 0, \\ \psi_2(0, x) = \psi_{20}, & x \in [0, k_{20}], \\ k_2(0) = k_{20}, \end{cases}$$

respectively, where κ_i, s_{i0}, τ_i, k_{i0} are positive constants. By Theorem 4.2 of [9], there exist positive constants s^*, s_*, k^*, k_* respectively such that

$$\lim_{t\to\infty} \frac{s_1(t)}{t} = s^*, \ \lim_{t\to\infty} \frac{s_2(t)}{t} = s_*, \ \lim_{t\to\infty} \frac{k_1(t)}{t} = k^*, \ \lim_{t\to\infty} \frac{k_2(t)}{t} = k_*.$$

Suppose that $\kappa_1 \geq \mu$, $\kappa_2 \leq \mu$, $\tau_1 \geq \mu\rho$, $\tau_2 \leq \mu\rho$ and

$$\phi_{10} \geq u_0, \ s_{10} \geq h_0, \ \phi_{20} \leq u_0, \ s_{20} \leq h_0,$$

$$\psi_{10} \geq v_0, \ k_{10} \geq h_0, \ \psi_{20} \leq v_0, \ k_{20} \leq h_0.$$

As a result of Lemmas 2.2, 2.3 and Remark 2.1, we have $s_1(t), k_1(t) \leq h(t) \leq s_2(t), k_2(t)$; therefore,

$$\max\{s_*, k_*\} \leq \liminf_{t\to\infty} \frac{h(t)}{t}, \ \limsup_{t\to\infty} \frac{h(t)}{t} \leq \min\{s^*, k^*\}.$$

6. Conclusions

In this paper, we studied a ratio-dependent predator-prey model with a Neumann boundary on the left side representing that the left boundary is fixed, and a free boundary $x = h(t)$ concerned with both prey and predator on the right side which describes the movement process for both prey and predator species. A spreading-vanishing dichotomy and the criteria for spreading and vanishing are established which are summarized below:

(i) (Spreading case) If the size of the initial habitat of prey and predator is equal to or more than $\Lambda := \frac{\pi}{2}\min\{\sqrt{\frac{m}{m\lambda - b}}, \sqrt{\frac{d}{\nu + c}}\}$, or less than Λ but the moving coefficient μ of the free boundary is greater than some positive constant $\overline{\mu}$ which depends on u_0, v_0 and h_0, then both species will spread successfully. In addition, as t goes to infinity, the prey and predator populations go to their stationary solutions u^* and v^*, respectively.

(ii) (Vanishing case) If the size of the initial habitat is less than Λ and the moving coefficient μ of the free boundary $h(t)$ is not greater than the constant $\underline{\mu}$ which also depends on u_0, v_0 and h_0, then the two species will eventually vanish. In addition, as $t \to \infty$ the free boundary is limited to Λ.

When spreading occurs, we estimated the asymptotic spread speed of the free boundary $x = h(t)$. We provided an upper bound for $\limsup_{t\to\infty} \frac{h(t)}{t}$ which is $2\max\{\sqrt{\lambda}, \sqrt{d(v+c)}\}$ (Theorem 5.1), and gave the scope of $\frac{h(t)}{t}$ which is bounded below by $\max\{s_*, k_*\}$ and bounded above by $\min\{s^*, k^*\}$ (Theorem 5.2).

The positive constant "Λ" is a vital threshold to determine whether spreading occurs (for more explanations see [9]). In order to get a more accurate number, we studied the waves of finite length to construct a lower solution of (1.1) and derived a smaller number $\frac{\pi}{2}\sqrt{\frac{d}{v+c}}$ than the previous number $\frac{\pi}{2}\sqrt{\frac{d}{v}}$.

When vanishing occurs in the setting model studied in this paper, both prey and predator will eventually die out, while in [24] only the prey population will vanish. This is an important difference between $h(t)$ depending on both prey and predator and the cases of dependence on prey only. In the natural world, predators that only live on this prey will not be able to survive if the prey population goes extinct; intuitively, the results in this paper seems to be closer to reality.

The above conclusions are instructive for us. Assume that a predator v only survives on a prey u. Then two species co-exist; that is, when a new or an invasive species invades, either the two species v and u die out eventually or if the local species can escape to the whole space, then the invasive species will become widespread throughout the whole space. In order to protect the local species, we can (i) enlarge the initial habitat of the local species, (ii) increase the coefficient of the free boundary. I also follows that introducing a natural enemy and taking the opposite approaches from the above are an effective method to control pest species.

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Conflict of interest

The authors declare no conflict of interest.

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