AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 6(11): 11894–11903.
DOI:10.3934/math.2021690
Received: 31 May 2021
Accepted: 03 August 2021
Published: 16 August 2021

Research article

Jacobi-type vector fields on $\mathbb{H}^3$ and $\mathbb{R}^3$

Yaning Wang* **and Yingdong Zhang***

School of Mathematics and Information Science, Henan Normal University, Xinxiang, Henan 453007, China

* **Correspondence:** Email: wyn051@163.com, zyd1090@163.com.

Abstract: In this paper, we determine the Jacobi-type vector fields on the hyperbolic 3-space $\mathbb{H}^3$ and the Euclidean 3-space $\mathbb{R}^3$, respectively. In terms of this, infinitely many non-trivial Jacobi-type vector fields are given.

Keywords: Jacobi-type vector field; Killing vector field; hyperbolic 3-space; Euclidean 3-space
Mathematics Subject Classification: 53C15, 53C25, 53B21

1. Introduction

On a Riemannian manifold (M, g), the Levi-Civita connection and the curvature tensor are denoted by ∇ and R, respectively. Let $\gamma : I \subset \mathbb{R} \to M$ be a geodesic with coordinate t in I. A vector field V along a geodesic γ is said to be a Jacobi field along γ if it satisfies the following Jacobi differential equation, i.e.,

$$\nabla_{\frac{d}{dt}}\nabla_{\frac{d}{dt}}V + R(V, \dot{\gamma})\dot{\gamma} = 0. \tag{1.1}$$

By a homothetic vector field V we refer to that the Lie derivative of the metric g of a Riemannian manifold M along V is a constant multiple of the metric g. When the constant vanishes, then a homothetic vector field becomes Killing. We remark that a Killing vector field on a Riemannian manifold is always a Jacobi field along each geodesic, but the converse is not necessarily true. By means of (1.1), S. Deshmukh in [4] defined the Jacobi-type vector fields on a Riemannian manifold M satisfying

$$\nabla_X\nabla_X V + R(V, X)X = 0 \tag{1.2}$$

for any vector field X. Such vector fields were studied in some characterizations of typical compact real hypersurfaces in non-flat complex space forms [4], and compact (or Hopf) real hypersurfaces in complex two-plane Grassmannians [7]. However, as pointed out in [2, Remark 1], those restrictions of Jacobi-type vector fields on the structure vector fields of compact real hypersurfaces [4, 7] are in fact redundant.

In 2012, S. Deshmukh in [5] defined another type of Jacobi-type vector fields satisfying

$$\nabla_X \nabla_X V - \nabla_{\nabla_X X} V + R(V, X)X = 0 \tag{1.3}$$

for any vector field X. From here to the ending of this present paper, when involving the Jacobi-type vector fields we always refer to Eq (1.3). Obviously, any Jacobi-type vector field is a Jacobi field along each geodesic. Also, the notion of the Jacobi-type vector fields is certainly an extension of the Killing ones [5, Proposition 2.1]. Recently, such a property has been generalized by A. M. Cherif in [3, Lemma 7] who proved that a homothetic vector field on a Riemannian manifold must be a Jacobi-type vector field. A Jacobi-type vector field is said to be trivial when it is Killing. Just like the case of the homothetic or Killing vector fields, Jacobi-type vector fields constrain the geometry as well as topology of a Riemannian manifold, and play important roles in differential geometry [1, 2, 5, 6].

It was proved in [2, Theorem 1] that a Jacobi-type vector field on a compact Riemannian manifold must be Killing. This arises a natural question [2]:
"Under what conditions is a Jacobi-type vector field on a non-compact Riemannian manifold a Killing vector field?"

The main motivation of the present paper is to investigate the above question on the most simplest non-compact real space forms, i.e., the hyperbolic 3-space $\mathbb{H}^3$ and the Euclidean 3-space $\mathbb{R}^3$. We determine all Jacobi-type vector fields on these two spaces and also present some sufficient and necessary conditions for those Jacobi-type vector fields becoming Killing ones. Applying this we obtain infinitely many non-Killing Jacobi-type vector fields.

2. Preliminaries

Now we introduce the well-known model for the hyperbolic 3-space $\mathbb{H}^3(-1)$. Let $\mathbb{H}^3 = \{(x, y, z) : (x, y, z) \in \mathbb{R}^3, z > 0\}$ and the metric g on it is

$$g = \frac{1}{z^2}(dx^2 + dy^2 + dz^2).$$

We adopt the global orthonormal frame $\{e_1 = z\frac{\partial}{\partial x}, e_2 = z\frac{\partial}{\partial y}, e_3 = z\frac{\partial}{\partial z}\}$ on $\mathbb{H}^3$. By a direct calculation, we give

$$[e_1, e_2] = 0,\ [e_2, e_3] = -e_2,\ [e_3, e_1] = e_1$$

and

$$\nabla_{e_i} e_j = \begin{pmatrix} e_3 & 0 & -e_1 \\ 0 & e_3 & -e_2 \\ 0 & 0 & 0 \end{pmatrix},\ i, j \in \{1, 2, 3\},$$

where ∇ denotes the Levi-Civita connection of the metric g. By applying these, some curvature tensors are given by

$$R_{121} = e_2,\ R_{122} = -e_1,\ R_{131} = e_3,\ R_{133} = -e_1,\ R_{232} = e_3,\ R_{233} = -e_2$$

and all others vanish, where $R_{ijk} := R(e_i, e_j)e_k$ for $i, j, k \in \{1, 2, 3\}$.

3. Jacobi-type vector fields on $\mathbb{H}^3$

Suppose that $V = \sum_{i=1}^{3} f_i e_i$ is a Jacobi-type vector field on $\mathbb{H}^3$, where f_i, $i = 1, 2, 3$, are smooth functions on (x, y, z). With the help of those preliminaries in Section two, we compute

$$\begin{aligned}
&\nabla_{e_1}\nabla_{e_1}V - \nabla_{\nabla_{e_1}e_1}V + R(V, e_1)e_1\\
=&(-f_1 + z^2\frac{\partial^2 f_1}{\partial x^2} - z\frac{\partial f_1}{\partial z} - 2z\frac{\partial f_3}{\partial x})e_1 + (-f_2 + z^2\frac{\partial^2 f_2}{\partial x^2} - z\frac{\partial f_2}{\partial z})e_2\\
&+ (-2f_3 + 2z\frac{\partial f_1}{\partial x} + z^2\frac{\partial^2 f_3}{\partial x^2} - z\frac{\partial f_3}{\partial z})e_3,
\end{aligned}$$

$$\begin{aligned}
&\nabla_{e_2}\nabla_{e_2}V - \nabla_{\nabla_{e_2}e_2}V + R(V, e_2)e_2\\
=&(-f_1 - z\frac{\partial f_1}{\partial z} + z^2\frac{\partial^2 f_1}{\partial y^2})e_1 + (-f_2 - z\frac{\partial f_2}{\partial z} + z^2\frac{\partial^2 f_2}{\partial y^2} - 2z\frac{\partial f_3}{\partial y})e_2\\
&+ (-2f_3 + 2z\frac{\partial f_2}{\partial y} - z\frac{\partial f_3}{\partial z} + z^2\frac{\partial^2 f_3}{\partial y^2})e_3
\end{aligned}$$

and

$$\begin{aligned}
&\nabla_{e_3}\nabla_{e_3}V - \nabla_{\nabla_{e_3}e_3}V + R(V, e_3)e_3\\
=&(-f_1 + z\frac{\partial f_1}{\partial z} + z^2\frac{\partial^2 f_1}{\partial z^2})e_1 + (-f_2 + z\frac{\partial f_2}{\partial z} + z^2\frac{\partial^2 f_2}{\partial z^2})e_2\\
&+ (z\frac{\partial f_3}{\partial z} + z^2\frac{\partial^2 f_3}{\partial z^2})e_3.
\end{aligned}$$

According to (1.3) and the above there relations, V is a Jacobi-type vector field if and only if the following nine partial differential equations hold:

$$\begin{cases}
-f_1 + z^2\frac{\partial^2 f_1}{\partial x^2} - z\frac{\partial f_1}{\partial z} - 2z\frac{\partial f_3}{\partial x} = 0,\\
-f_2 + z^2\frac{\partial^2 f_2}{\partial x^2} - z\frac{\partial f_2}{\partial z} = 0,\\
-2f_3 + 2z\frac{\partial f_1}{\partial x} + z^2\frac{\partial^2 f_3}{\partial x^2} - z\frac{\partial f_3}{\partial z} = 0,\\
-f_1 - z\frac{\partial f_1}{\partial z} + z^2\frac{\partial^2 f_1}{\partial y^2} = 0,\\
-f_2 - z\frac{\partial f_2}{\partial z} + z^2\frac{\partial^2 f_2}{\partial y^2} - 2z\frac{\partial f_3}{\partial y} = 0,\\
-2f_3 + 2z\frac{\partial f_2}{\partial y} - z\frac{\partial f_3}{\partial z} + z^2\frac{\partial^2 f_3}{\partial y^2} = 0,\\
-f_1 + z\frac{\partial f_1}{\partial z} + z^2\frac{\partial^2 f_1}{\partial z^2} = 0,\\
-f_2 + z\frac{\partial f_2}{\partial z} + z^2\frac{\partial^2 f_2}{\partial z^2} = 0,\\
z\frac{\partial f_3}{\partial z} + z^2\frac{\partial^2 f_3}{\partial z^2} = 0.
\end{cases} \tag{3.1}$$

The remaining of this section is to solve the above PDEs. First, notice that the ninth equation in (3.1) can be reduced to $\frac{\partial^2 f_3}{\partial z^2} + \frac{1}{z}\frac{\partial f_3}{\partial z} = 0$, and this is a linear equation. Solving this equation gives

$$f_3 = H(x, y)\ln z + K(x, y), \tag{3.2}$$

where both H and K are smooth functions varying only on (x, y). Substituting (3.2) into the third term in (3.1) yields

$$\frac{\partial f_1}{\partial x} = \frac{1}{2z}(2\ln z + 1)H + \frac{1}{z}K - \frac{1}{2}zlnz\frac{\partial^2 H}{\partial x^2} - \frac{2}{z}\frac{\partial^2 K}{\partial x^2}. \tag{3.3}$$

Taking derivative of the seventh term in (3.1) with respect to x yields

$$-\frac{\partial f_1}{\partial x} + z\frac{\partial^2 f_1}{\partial x\partial z} + z^2\frac{\partial^3 f_1}{\partial x\partial z^2} = 0,$$

which is simplified by applying (3.3) giving $2H + z^2\frac{\partial^2 H}{\partial x^2} = 0$. Recalling that this equation holds for any $z > 0$ and H varies only on (x, y). It follows immediately that H vanishes identically.

Taking derivative of the first equation in (3.1) with respect to x yields that

$$-\frac{\partial f_1}{\partial x} + z^2\frac{\partial^3 f_1}{\partial x^3} - z\frac{\partial^2 f_1}{\partial x\partial z} - 2z\frac{\partial^2 f_3}{\partial x^2} = 0,$$

which is substituted into (3.2), (3.3) and $H = 0$ giving $z^4\frac{\partial^4 K}{\partial x^4} = 0$. This equation reduces directly to $\frac{\partial^4 K}{\partial x^4} = 0$ in view of the arbitrary of z. Moreover, notice that with the help of $H = 0$, (3.2) and (3.3) become $f_3 = K$ and $\frac{\partial f_1}{\partial x} = \frac{1}{z}K - \frac{1}{z}\frac{\partial^2 K}{\partial x^2}$, respectively. Taking derivative of the forth term in (3.1) with respect to x yields

$$-\frac{\partial f_1}{\partial x} + z^2\frac{\partial^3 f_1}{\partial x\partial y^2} - z\frac{\partial^2 f_1}{\partial x\partial z} = 0,$$

which is substituted into $\frac{\partial f_1}{\partial x} = \frac{1}{z}K - \frac{1}{z}\frac{\partial^2 K}{\partial x^2}$ giving $\frac{\partial^2 K}{\partial x^2} + \frac{\partial^2 K}{\partial y^2} - \frac{1}{2}z^2\frac{\partial^4 K}{\partial x^2\partial y^2} = 0$. Applying again the arbitrary of z in this equation, we obtain

$$\frac{\partial^2 K}{\partial x^2} + \frac{\partial^2 K}{\partial y^2} = 0 \text{ and } \frac{\partial^4 K}{\partial x^2\partial y^2} = 0. \tag{3.4}$$

Recalling that we have already obtained $\frac{\partial^4 K}{\partial x^4} = 0$. Combining this with the second term in (3.4), in view of that K varies only on (x, y), we obtain

$$\frac{\partial^2 K}{\partial x^2} = k_1xy + k_2x + k_3y + k_4, \tag{3.5}$$

where k_1, k_2, k_3 and k_4 are all constants. Taking integral of (3.5) we also have

$$K(x, y) = \frac{1}{6}k_1x^3y + \frac{1}{6}k_2x^3 + \frac{1}{2}k_3x^2y + \frac{1}{2}k_4x^2 + \alpha_1(y)x + \alpha_2(y), \tag{3.6}$$

where both α_1 and α_2 are smooth functions varying only on y. Substituting the above relation into the first term in (3.4) gives an equation, and comparing the resulting equation with (3.5) we have

$$\begin{cases}\alpha_1(y) = -\frac{1}{6}k_1y^3 - \frac{1}{2}k_2y^2 + k_5y + k_6,\\ \alpha_2(y) = -\frac{1}{6}k_3y^3 - \frac{1}{2}k_4y^2 + k_7y + k_8,\end{cases} \tag{3.7}$$

where k_5, k_6, k_7 and k_8 are all constants. Putting (3.7) into (3.6) yields

$$\begin{aligned} K(x,y) =&\frac{1}{6}k_1xy(x^2-y^2)+\frac{1}{6}k_2x(x^2-3y^2)+\frac{1}{6}k_3y(3x^2-y^2)\\ &+\frac{1}{2}k_4(x^2-y^2)+k_5xy+k_6x+k_7y+k_8. \end{aligned} \tag{3.8}$$

Recalling that we have obtained $\frac{\partial f_1}{\partial x}=\frac{1}{z}K-\frac{1}{z}\frac{\partial^2 K}{\partial x^2}$, which is simplified by using (3.8) giving

$$\begin{aligned} f_1 =&\frac{1}{24z}k_1x^2y(x^2-2y^2)+\frac{1}{24z}k_2x^2(x^2-6y^2)+\frac{1}{6z}k_3xy(x^2-y^2)\\ &+\frac{1}{6z}k_4x(x^2-3y^2)+\frac{1}{2z}k_5x^2y+\frac{1}{2z}k_6x^2+\frac{1}{z}k_7xy+\frac{1}{z}k_8x\\ &-\frac{1}{4}xz(k_1xy+k_2x+2k_3y+2k_4)+M(y,z), \end{aligned} \tag{3.9}$$

where M is a smooth function varying only on (y,z). Substituting (3.9) into the first term in (3.1) we obtain

$$\frac{1}{6}z(k_1y^3+3k_2y^2-6k_5y-6k_6)-\frac{1}{2}z^3(k_1y+k_2)=M+z\frac{\partial M}{\partial z}.$$

Solving the above linear equation we obtain

$$M=\frac{1}{12}z(k_1y^3+3k_2y^2-6k_5y-6k_6)-\frac{1}{8}z^3(k_1y+k_2)+\frac{1}{z}\alpha_3(y), \tag{3.10}$$

where α_3 is a smooth function varying only on y. Similarly, substituting (3.9) into the seventh term in (3.1) we obtain

$$-M+z\frac{\partial M}{\partial z}+z^2\frac{\partial^2 M}{\partial z^2}=0, \tag{3.11}$$

which is substituted into (3.10) giving $k_1=k_2=0$. Now putting (3.9) into the forth term in (3.1) we obtain

$$-M-z\frac{\partial M}{\partial z}+z^2\frac{\partial^2 M}{\partial y^2}=0. \tag{3.12}$$

With the aid of $k_1=k_2=0$, substituting (3.10) into (3.12) yields

$$\alpha_3(y)=-\frac{1}{6}k_5y^3-\frac{1}{2}k_6y^2+k_9y+k_{10},$$

where both k_9 and k_{10} are constants. Therefore, f_1 can be expressed by means of the above equation, (3.9) and (3.10), i.e.,

$$\begin{aligned} f_1 =&\frac{1}{6z}k_3xy(x^2-y^2)+\frac{1}{6z}k_4x(x^2-3y^2)+\frac{1}{6z}k_5y(3x^2-y^2)\\ &+\frac{1}{2z}k_6(x^2-y^2)+\frac{1}{z}k_7xy+\frac{1}{z}k_8x+\frac{1}{z}k_9y\\ &+\frac{1}{z}k_{10}-\frac{1}{2}z(k_3xy+k_4x+k_5y+k_6). \end{aligned} \tag{3.13}$$

With the aid of (3.8) and $k_1 = k_2 = 0$, we have already obtained f_3 according to (3.2). Substituting this into the sixth term in (3.1) we obtain

$$\frac{\partial f_2}{\partial y} = \frac{1}{z}K - \frac{1}{2}z\frac{\partial^2 K}{\partial y^2}.$$

Substituting (3.8) and $k_1 = k_2 = 0$ into the above relation and taking integral we get

$$\begin{aligned} f_2 =& \frac{1}{24z}k_3y^2(6x^2 - y^2) + \frac{1}{6z}k_4y(3x^2 - y^2) + \frac{1}{2z}k_5xy^2 + \frac{1}{z}k_6xy \\ &+ \frac{1}{2z}k_7y^2 + \frac{1}{z}k_8y + \frac{1}{4}yz(k_3y + 2k_4) + N(x, z), \end{aligned} \tag{3.14}$$

where N is a smooth function varying only on (x, z). Substituting (3.14) into the fifth term in (3.1) gives

$$-\frac{1}{2}z(k_3x^2 + 2k_5x + 2k_7) + \frac{1}{2}z^3k_3 = N + z\frac{\partial N}{\partial z}.$$

Solving such a linear equation we obtain

$$N(x, z) = -\frac{1}{4}z(k_3x^2 + 2k_5x + 2k_7) + \frac{1}{8}z^3k_3 + \frac{1}{z}\beta(x), \tag{3.15}$$

where β is a smooth function varying on x. Putting (3.14) into the eighth term in (3.1) we get a new linear equation

$$-N + z\frac{\partial N}{\partial z} + z^2\frac{\partial^2 N}{\partial z^2} = 0.$$

Putting (3.15) into the above equation we obtain $k_3 = 0$. Finally, with the help of $k_3 = 0$, substituting (3.14) into the second term in (3.1) we acquire

$$-N - z\frac{\partial N}{\partial z} + z^2\frac{\partial^2 N}{\partial x^2} = 0. \tag{3.16}$$

Applying again $k_3 = 0$ and (3.15), according to (3.16) we obtain

$$\beta(x) = -\frac{1}{6}k_5x^3 - \frac{1}{2}k_7x^2 + k_{11}x + k_{12},$$

where k_{11} and k_{12} are two constants. Now, with the help of the above relation, from (3.15) we have

$$N(x, z) = -\frac{1}{2}z(k_5x + k_7) - \frac{1}{6z}(k_5x^3 + 3k_7x^2 - 6k_{11}x - 6k_{12}). \tag{3.17}$$

With the help of $k_3 = 0$, from (3.13), (3.14), (3.17), (3.2) and (3.8), the main theorem of this section is given as the following.

Theorem 3.1. *On the hyperbolic* 3*-space* $\mathbb{H}^3$*, a vector field* $V = \sum_{i=1}^{3} f_ie_i$ *is a Jacobi-type vector field if and only if*

$$\begin{aligned} f_1 =& \frac{1}{6z}k_4x(x^2 - 3y^2) + \frac{1}{6z}k_5y(3x^2 - y^2) + \frac{1}{2z}k_6(x^2 - y^2) + \frac{1}{z}k_7xy \\ &+ \frac{1}{z}k_8x + \frac{1}{z}k_9y + \frac{1}{z}k_{10} - \frac{1}{2}z(k_4x + k_5y + k_6), \end{aligned} \tag{3.18}$$

$$f_2 = \frac{1}{6z}k_4y(3x^2 - y^2) + \frac{1}{6z}k_5x(3y^2 - x^2) + \frac{1}{z}k_6xy + \frac{1}{2z}k_7(y^2 - x^2) + \frac{1}{z}k_8y + \frac{1}{z}k_{11}x + \frac{1}{z}k_{12} + \frac{1}{2}z(k_4y - k_5x - k_7), \tag{3.19}$$

$$f_3 = \frac{1}{2}k_4(x^2 - y^2) + k_5xy + k_6x + k_7y + k_8. \tag{3.20}$$

Proof. The "if" part of the proof is easy to check by applying (3.18)–(3.20). The "only if" part has been presented already. □

Considering a vector field $V = \sum_{i=1}^{3} f_ie_i$ on $\mathbb{H}^3$, and using those preliminaries in Section two we have

$$(\mathcal{L}_Vg)(e_1, e_1) = 2(z\frac{\partial f_1}{\partial x} - f_3).$$

$$(\mathcal{L}_Vg)(e_1, e_2) = z\frac{\partial f_1}{\partial y} + z\frac{\partial f_2}{\partial x}.$$

$$(\mathcal{L}_Vg)(e_1, e_3) = f_1 + z\frac{\partial f_1}{\partial z} + z\frac{\partial f_3}{\partial x}.$$

$$(\mathcal{L}_Vg)(e_2, e_2) = 2(z\frac{\partial f_2}{\partial y} - f_3).$$

$$(\mathcal{L}_Vg)(e_2, e_3) = f_2 + z\frac{\partial f_2}{\partial z} + z\frac{\partial f_3}{\partial y}.$$

$$(\mathcal{L}_Vg)(e_3, e_3) = 2z\frac{\partial f_3}{\partial z}.$$

With the help the above relations, V is a Killing vector field if and only if the following PDEs hold:

$$\begin{cases} z\frac{\partial f_1}{\partial x} = f_3, \ \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x} = 0, \ f_1 + z\frac{\partial f_1}{\partial z} + z\frac{\partial f_3}{\partial x} = 0, \\ z\frac{\partial f_2}{\partial y} = f_3, \ \frac{\partial f_3}{\partial z} = 0, \ f_2 + z\frac{\partial f_2}{\partial z} + z\frac{\partial f_3}{\partial y} = 0. \end{cases} \tag{3.21}$$

Applying (3.18) and (3.20) into the first term in (3.21) gives $k_4 = 0$. Moreover, applying (3.18) and (3.19) into the second term in (3.21) yields $k_5 = 0$ and $k_{11} = -k_9$. We remark that in this situation, all other equations in (3.21) are necessarily true. Therefore, the following theorem follows from Theorem 3.1.

Theorem 3.2. *On the hyperbolic* 3*-space* $\mathbb{H}^3$*, a Jacobi-type vector field* $V = \sum_{i=1}^{3} f_ie_i$ *is Killing if and only if*

$$f_1 = \frac{1}{2z}k_6(x^2 - y^2) + \frac{1}{z}k_7xy + \frac{1}{z}k_8x + \frac{1}{z}k_9y + \frac{1}{z}k_{10} - \frac{1}{2}zk_6, \tag{3.22}$$

$$f_2 = \frac{1}{z}k_6xy + \frac{1}{2z}k_7(y^2 - x^2) + \frac{1}{z}k_8y - \frac{1}{z}k_9x + \frac{1}{z}k_{12} - \frac{1}{2}zk_7, \tag{3.23}$$

$$f_3 = k_6x + k_7y + k_8, \tag{3.24}$$

where k_6*,* k_7*,* k_8*,* k_9*,* k_{10} *and* k_{12} *are all constants.*

Comparing the above Theorem 3.2 with Theorem 3.1, one finds infinitely many non-Killing Jacobi-type vector fields. By applying this theorem, we answer the question proposed in Section one on a special non-compact manifold $\mathbb{H}^3$.

4. Jacobi-type vector fields on $\mathbb{R}^3$

In this section, just like we have done in Section two, we determine all Jacobi-type vector fields on the Euclidean 3-space. Let (x, y, z) be the usual global coordinates on $\mathbb{R}^3$ and $V = \sum_{i=1}^{3} f_i e_i$ be a vector field, where $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$, $e_3 = \frac{\partial}{\partial z}$ are the global orthonormal frame. By a direct calculation, we have

$$\nabla_{e_1}\nabla_{e_1}V - \nabla_{\nabla_{e_1}e_1}V + R(V, e_1)e_1 = \frac{\partial^2 f_1}{\partial x^2}e_1 + \frac{\partial^2 f_2}{\partial x^2}e_2 + \frac{\partial^2 f_3}{\partial x^2}e_3.$$

$$\nabla_{e_2}\nabla_{e_2}V - \nabla_{\nabla_{e_2}e_2}V + R(V, e_2)e_2 = \frac{\partial^2 f_1}{\partial y^2}e_1 + \frac{\partial^2 f_2}{\partial y^2}e_2 + \frac{\partial^2 f_3}{\partial y^2}e_3.$$

$$\nabla_{e_3}\nabla_{e_3}V - \nabla_{\nabla_{e_3}e_3}V + R(V, e_3)e_3 = \frac{\partial^2 f_1}{\partial z^2}e_1 + \frac{\partial^2 f_2}{\partial z^2}e_2 + \frac{\partial^2 f_3}{\partial z^2}e_3.$$

According to (1.3), V is a Jacobi-type vector field if and only if the following PDEs hold:

$$\frac{\partial^2 f_i}{\partial x^2} = 0, \frac{\partial^2 f_i}{\partial y^2} = 0, \ \frac{\partial^2 f_i}{\partial z^2} = 0 \ , i = 1, 2, 3.$$

In view of $\frac{\partial^2 f_1}{\partial x^2} = 0$, we may write

$$f_1 = H(y, z)x + K(y, z), \tag{4.1}$$

where both H and K are smooth functions varying only on (y, z). Applying this on the fact $\frac{\partial^2 f_1}{\partial y^2} = 0$ we obtain $x\frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 K}{\partial y^2} = 0$, and this is equivalent to

$$\frac{\partial^2 H}{\partial y^2} = 0, \ \frac{\partial^2 K}{\partial y^2} = 0 \tag{4.2}$$

due to the arbitrary of x. Similarly, applying (4.1) on the fact $\frac{\partial^2 f_1}{\partial z^2} = 0$ we also obtain $x\frac{\partial^2 H}{\partial z^2} + \frac{\partial^2 K}{\partial z^2} = 0$, and this is equivalent to

$$\frac{\partial^2 H}{\partial z^2} = 0, \ \frac{\partial^2 K}{\partial z^2} = 0 \tag{4.3}$$

due to the arbitrary of x. According to (4.3) and (4.2), we may write

$$H(y, z) = k_1 yz + k_2 y + k_3 z + k_4, \ K(y, z) = k_5 yz + k_6 y + k_7 z + k_8,$$

where k_i, $i = 1, \cdots, 8$, are all constants. Similarly, f_2 and f_3 can be expressed according to $\frac{\partial^2 f_2}{\partial x^2} = \frac{\partial^2 f_2}{\partial y^2} = \frac{\partial^2 f_2}{\partial z^2} = 0$ and $\frac{\partial^2 f_3}{\partial x^2} = \frac{\partial^2 f_3}{\partial y^2} = \frac{\partial^2 f_3}{\partial z^2} = 0$, respectively.

Theorem 4.1. *On the Euclidean* 3*-space* $\mathbb{R}^3$*, a vector field* $V = \sum_{i=1}^{3} f_i e_i$ *is a Jacobi-type vector field if and only if*

$$f_1 = k_1 xyz + k_2 xy + k_3 xz + k_4 x + k_5 yz + k_6 y + k_7 z + k_8, \tag{4.4}$$

$$f_2 = l_1 xyz + l_2 xy + l_3 xz + l_4 x + l_5 yz + l_6 y + l_7 z + l_8, \tag{4.5}$$

$$f_3 = m_1 xyz + m_2 xy + m_3 xz + m_4 x + m_5 yz + m_6 y + m_7 z + m_8, \tag{4.6}$$

where k_i*,* l_i*,* m_i *for* $i = 1, \cdots, 8$ *are all constants.*

Proof. The "if" part of the proof is easy to check by applying (4.4)–(4.6). The "only if" part has been presented already. □

Remark 4.1. *According to Theorem 4.1, on the Euclidean* 3*-space* $\mathbb{R}^3$*, the position vector field* $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ *is a Jacobi-type vector field. Actually, it is a non-Killing homothetic vector field [5, Remark 2.1].*

Just like the case shown at the beginning of this Section, a vector field $V = \sum_{i=1}^{3} f_i e_i$ on the Euclidean 3-space $\mathbb{R}^3$ is Killing if and only if

$$\begin{cases} \frac{\partial f_1}{\partial x} = 0, \ \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x} = 0, \ \frac{\partial f_1}{\partial z} + \frac{\partial f_3}{\partial x} = 0, \\ \frac{\partial f_2}{\partial y} = 0, \ \frac{\partial f_3}{\partial z} = 0, \ \frac{\partial f_2}{\partial z} + \frac{\partial f_3}{\partial y} = 0. \end{cases} \tag{4.7}$$

Substituting (4.4) into the the first term in (4.7) gives $k_1 = k_2 = k_3 = k_4 = 0$. Similarly, substituting (4.5) and (4.6) into the forth and fifth terms in (4.7), respectively, we obtain $l_1 = l_2 = l_5 = l_6 = 0$ and $m_1 = m_3 = m_5 = m_7 = 0$. Also, with the help of these, putting (4.4) and (4.5) into the second term in (4.7) gives $k_5 + l_3 = 0$ and $k_6 + l_4 = 0$. Similarly, putting (4.4) and (4.6) into the third term in (4.7) gives $k_5 + m_2 = 0$ and $k_7 + m_4 = 0$. Putting (4.5) and (4.6) into the sixth term in (4.7) gives $l_3 + m_2 = 0$ and $l_7 + m_6 = 0$. Combining the above relations we also have $k_5 = l_3 = m_2 = 0$.

Theorem 4.2. *On the Euclidean* 3*-space* $\mathbb{R}^3$*, a Jacobi-type vector field* $V = \sum_{i=1}^{3} f_i e_i$ *is Killing if and only if*

$$f_1 = k_6 y + k_7 z + k_8, \tag{4.8}$$

$$f_2 = -k_6 x + l_7 z + l_8, \tag{4.9}$$

$$f_3 = -k_7 x - l_7 y + m_8, \tag{4.10}$$

where k_6*,* k_7*,* k_8*,* l_7*,* l_8 *and* m_8 *are all constants.*

Comparing Theorem 4.2 with 4.1, one obtains many non-Killing Jacobi-type vector fields on $\mathbb{R}^3$.

Acknowledgments

This work was supported by the Key Scientific Research Program in Universities of Henan Province (No. 20A110023) and the Fostering Foundation of National Foundation in Henan Normal University (No. 2019PL22). The authors would like to thank referees for their useful comments.

Conflict of interest

We declare no conflict of interest.

References

1. R. Al-Ghefari, Jacobi-type vector fields on Käehler manifold, *Pure Math. Sci.,* **2** (2013), 127–132.
2. B. Y. Chen, S. Deshmukh, A. A. Ishan, On Jacobi-type vector fields on Riemannian manifolds, *Math.,* **7** (2019), 1139.

3. A. M. Cherif, Some results on harmonic and bi-harmonic maps, *Int. J. Geom. Methods Mod. Phys.,* **14** (2017), 1750098.
4. S. Deshmukh, Real hypersurfaces of a complex projective space, *Proc. Indian Acad. Sci. Math. Sci.*, **121** (2011), 171–179.
5. S. Deshmukh, Jacobi-type vector fields on Ricci solitons, *Bull. Math. Soc. Sci. Math. Roumanie,* **55** (2012), 41–50.
6. S. Deshmukh, A. A. Ishan, A note on contact metric manifolds, *Hacet. J. Math. Stat.,* **49** (2020), 2007–2016.
7. C. J. G. Manchado, J. D. Perez, On the structure vector field of a real hypersurface in complex two-plane Grassmannians, *Cent. Eur. J. Math.,* **10** (2012), 451–455.

AIMS Press