Research article

On the strong *P*-regular splitting iterative methods for non-Hermitian linear systems

Junxiang Lu[1,*] **and Chengyi Zhang**[2]

[1] Department of Mathematics of School of Science, Xi'an Polytechnic University, Xi'an, Shaanxi 710048, China

[2] School of Economics and Finance, Xi'an Jiaotong University, Xi'an, Shaanxi 710049, China

* **Correspondence:** Email: jun-xianglu@163.com; Tel: +8613572112415; Fax: +8602982330234.

Abstract: The strong *P*-regular splitting is put forward and defined for iterative methods of non-Hermitian linear systems in the paper. The strong *P*-regular splitting combining SOR iterative methods and relaxed SOR iterative methods are established, and conditions guaranteeing the convergence are presented. Furthermore, two numerical experiments are done to illustrate the convergence and effectiveness of our iterative methods.

1. Introduction

Let $\mathbb{C}^n$ denote the complex *n*-dimensional vector space, and $\mathbb{C}^{n \times n}$ the complex $n \times n$ matrix space. Consider an iterative solution of the large sparse system of *n* linear equations in *n* unknowns,

$$Ax = b, \quad A = (a_{ij}) \in \mathbb{C}^{n \times n} \text{ nonsingular, and } b, x \in \mathbb{C}^n. \tag{1.1}$$

where *A* is a large, sparse non-Hermitian matrix. In this paper we consider the important case where *A* is *positive (semi-) definite*; i.e., the Hermitian part $H = (A + A^*)/2$ is Hermitian positive (semi-) definite, where A^* denotes the conjugate transpose of the matrix *A*. Such case of the matrices above including complex symmetric positive definite matrices [27], accretive-dissipative matrices [24], (generalized) saddle point matrices [11], complex Benzi-Golub matrices [28], symmetric quasi-definite matrices [36], and so on. Large, sparse systems of this type arise in many applications, including discretizations of convection-diffusion problems [22], regularized weighted least-squares problems [10], real-valued formulations of certain complex symmetric systems [9], and so forth.

In order to solve system (1.1) by iterative methods, it is useful to construct splitting of the coefficient matrix A. Such splittings are associated with stationary iterative methods, and are frequently used as pre-conditioners for Krylov subspace methods or as smoothers for multigrid or Schwarz-type schemes; see, e.g., [16–18, 25, 35, 42]. Recently, there has been considerable interest in the Hermitian and skew-Hermitian splitting (HSS) method introduced by Bai, Golub and Ng for solving non-Hermitian positive definite linear systems, see [3]. We further note the generalizations and extensions of this basic method proposed in [2, 4–7, 30, 45, 46]. Furthermore, these methods and their convergence theories have been shown to apply to (generalized) saddle point problems, either directly or indirectly (as a preconditioner), see [1, 2, 4–6, 12, 13, 30, 38, 39]. However, a potential difficulty with the HSS method is the need to solve two linear systems at each iteration (or at each application of the preconditioner), in which the shifted skew-Hermitian system can be much more problematic; in some cases its solution is as difficult as that of the original linear system $Ax = b$ [7].

It is well known that P-regular splitting methods for Hermitian (or symmetric) positive definite linear systems are convergent (see [8, 14, 19, 20, 23, 31, 40, 47], and so forth). But, such type of splitting methods for non-Hermitian (or nonsymmetric) positive definite linear systems is not necessarily convergent. There have been several studies on the convergence of such type of splitting methods for non-Hermitian positive definite linear systems. In reference [15], some convergence conditions for the splitting of non-Hermitian positive definite matrices have been established. More recently, references [21, 38, 39, 43] give some conditions for the convergence of P-splitting for this class of linear systems.

In general, the coefficient matrix $A \in \mathbb{C}^{n \times n}$ is split into

$$A = M - N, \tag{1.2}$$

where $M \in \mathbb{C}^{n \times n}$ is nonsingular and $N \in \mathbb{C}^{n \times n}$. Then, the general form of stationary iterative method and the corresponding relaxed form for (1.1) can be described as follows:

$$x^{(i+1)} = M^{-1}Nx^{(i)} + M^{-1}b, \quad i = 0, 1, 2, \dots \tag{1.3}$$

and

$$x^{(i+1)} = (1 - \tau)x^{(i)} + \tau M^{-1}Nx^{(i)} + \tau M^{-1}b, \ \tau \in (0, \ 1), \quad i = 0, 1, 2, \dots \tag{1.4}$$

The matrices $T = M^{-1}N$ and $T_\tau = (1-\tau)I + \tau M^{-1}N$ are called the iteration matrices of the methods (1.3) and (1.4), respectively. It is well known [37] that (1.3) converges for any given $x^{(0)}$ if and only if $\rho(T) < 1$, where $\rho(T)$ denotes the spectral radius of the matrix T. Thus, to establish convergence results for stationary iterative methods, we need to study the spectral radius of the iteration matrix in (1.3).

Continuing in this direction, in this paper we firstly establish new results on splitting methods for solving system (1.1) iteratively, focusing on a particular class of splittings. For a given matrix $A \in \mathbb{C}^{n \times n}$, a splitting $A = M - N$ with M nonsingular is called a *P-regular splitting* if the matrix $M^* + N$ is positive definite, i.e., the Hermitian part of $M^* + N$ is Hermitian positive definite [34]. It is a well known result [34, 41] that if A is Hermitian positive definite and $A = M - N$ is a P-regular splitting, then the splitting iterative method is convergent: $\rho(M^{-1}N) < 1$. Let $A \in \mathbb{C}^{n \times n}$ be non-Hermitian. Then the splitting $A = M - N$ with M nonsingular is called a *strong P-regular splitting* if the matrix $M + N$ is Hermitian positive definite. In this paper, we examine the spectral properties of the iteration

matrix induced by strong *P*-regular splitting of a non-Hermitian positive definite matrix. Based on these properties, we construct various SOR-type methods for non-Hermitian linear systems and prove their convergence under appropriate restrictions on the choice of the relaxation parameter.

For convenience, some of the terminology used in this paper will be given. The symbol $\mathbb{C}^{n \times n}$ will denote the set of all $n \times n$ complex matrices. Let $A, B \in \mathbb{C}^{n \times n}$. We use the notation $A > 0$ ($A \geq 0$) if A is Hermitian positive (semi-) definite. If A and B are both Hermitian, we write $A > B$ ($A \geq B$) if and only if $A - B > 0$ ($A - B \geq 0$). If A is Hermitian all of eigenvalues of A are real, and we denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ the smallest (i.e., leftmost) and largest (rightmost) eigenvalues, respectively. Let $A \in \mathbb{C}^{n \times n}$, $\sigma(A)$ denotes the spectra of A, i.e., the set of all eigenvalues of A, $\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$ the spectral radius of A and $\varrho(A) = \min_{0 \neq \lambda \in \sigma(A)} |\lambda|$. Further, $\|A\|_2 = \sqrt{\lambda_{\max}(A^*A)} = \sqrt{\rho(A^*A)}$. For a given matrices $B > 0$, $\|A\|_B = \|B^{1/2} A B^{-1/2}\|_2$. Let $A \in \mathbb{C}^{n \times n}$ with $H = (A + A^*)/2$ and $S = (A - A^*)/2$ its Hermitian and skew-Hermitian parts, respectively; then A is non-Hermitian positive (semi-) definite if and only if $H > 0$ ($H \geq 0$). Throughout the paper, I will denote the $n \times n$ identity matrix.

The paper is organized as follows: Some convergence results for strong *P*-regular splitting of non-Hermitian positive (semi) definite linear systems are given in section 2. In section 3 we construct SOR-type method and use the general theory of section 2 to study its convergence. Two numerical examples are given in section 4 to demonstrate the convergence results obtained in this paper. Some conclusions are given in section 5.

2. General convergence results for strong *P*-regular splittings

In this section some convergence results for strong *P*-regular splitting methods for non-Hermitian positive (semi) definite matrices are established. First, some lemmas will be presented to be used in the sequel.

Lemma 1. *[23] Let $A > 0$, and let $A = M - N$ be a P-regular splitting. Then $\rho(M^{-1}N) \leq \|M^{-1}N\|_A < 1$.*

Theorem 1. *Let $A \in \mathbb{C}^{n \times n}$ be non-Hermitian positive definite, and let $A = M - N$ be a strong P-regular splitting, i.e., $B := M + N > 0$. Then $\rho(M^{-1}N) \leq \|M^{-1}N\|_B < 1$.*

Proof. Since A is non-Hermitian positive definite and $A = M - N$ be a strong *P*-regular splitting, i.e., $B := M + N > 0$, $B = M - (-N)$ is a *P*-regular splitting. It follows from Lemma 1 that $\rho[M^{-1}(-N)] \leq \|M^{-1}(-N)\|_B < 1$ and hence, $\rho(M^{-1}N) \leq \|M^{-1}N\|_B < 1$. This completes the proof. □

Lemma 2. *[19, 40] Let $A \in \mathbb{C}^{n \times n}$ be an invertible Hermitian matrix, and let $A = M - N$ be a P-regular splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is positive definite.*

Theorem 2. *Let $A \in \mathbb{C}^{n \times n}$ be an invertible non-Hermitian matrix, and let $A = M - N$ be a strong P-regular splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is positive definite.*

Proof. Obviously, the proof can be obtained from Theorem 1 and Lemma 2. □

Lemma 3. *[19, 42] Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian positive definite matrix, and let $A = M - N$ be a splitting of A. Then $\|M^{-1}N\|_A < 1$ if and only if the splitting $A = M - N$ is a P-regular splitting.*

Theorem 3. *Let $A \in \mathbb{C}^{n \times n}$ be a non-Hermitian positive definite matrix, and let $A = M - N$ be a splitting of A. Then $\|M^{-1}N\|_A < 1$ if and only if the splitting $A = M - N$ be a strong P-regular splitting.*

Proof. It is obviously that we can obtain the proof immediately following from Theorem 1 and Lemma 3. □

Theorem 4. *Let $A \in \mathbb{C}^{n \times n}$ be non-Hermitian positive definite, and let $B \in \mathbb{C}^{n \times n}$ be any given Hermitian positive definite matrix. Then $\rho[(A + B)^{-1}(B - A)] \leq \|(A + B)^{-1}(B - A)\|_B < 1$.*

Proof. Let $M = (A + B)/2$ and $N = (B - A)/2$. Then $B = M + N > 0$ and consequently, $A = M - N$ is a strong P-regular splitting of A. Since A is non-Hermitian positive definite, it follows from Theorem 1 that $\rho(M^{-1}N) \leq \|M^{-1}N\|_B < 1$ and thus $\rho[(A + B)^{-1}(B - A)] \leq \|(A + B)^{-1}(B - A)\|_B < 1$ for $M^{-1}N = (A + B)^{-1}(B - A)$. This completes the proof. □

In what follows the spectral analysis of the iteration matrix induced by strong P-regular splitting of non-Hermitian positive semidefinite matrix A.

Theorem 5. *Let $A \in \mathbb{C}^{n \times n}$ be non-Hermitian positive semidefinite and nonsingular, and let $A = M - N$ be a strong P-regular splitting. Then $\rho(M^{-1}N) \leq 1$. Furthermore, assume that $\lambda \in \mathbb{C}$ is an eigenvalue of $M^{-1}N$ and $x \in \mathbb{C}^n$ is the corresponding eigenvector. Then $|\lambda| < 1$ if $x \notin N(H)$ and $|\lambda| = 1$ with $\mathrm{Im}(\lambda) \neq 0$ if $x \in N(H)$, where H, $N(H)$ and $\mathrm{Im}(\lambda)$ denotes the Hermitian part of A, the null space of H and the imaginary part of λ, respectively.*

Proof. Since the splitting $A = M - N$ is a strong P-regular splitting, $B := M + N > 0$. Furthermore, $M = (A + B)/2 = [(H + B) + S]/2$ and $N = (B - A)/2 = [(B - H) - S]/2$ with H and S Hermitian and skew-Hermitian parts of A, respectively. Let λ be an eigenvalue of $M^{-1}N$ satisfying $|\lambda| = \rho(M^{-1}N)$, and let $x \in \mathbb{C}^n$ be a corresponding eigenvector with $\|x\|_2 = 1$ (note that it must have $x \neq 0$). Then, one has $M^{-1}Nx = \lambda x$ and thus,

$$Nx = \lambda M x \tag{2.1}$$

and $x^* N x = \lambda x^* M x$. Since A is non-Hermitian positive semidefinite and $B > 0$, $H \geq 0$ and $M = (A + B)/2$ is non-Hermitian positive definite. As a result, $x^* M x \neq 0$, and consequently,

$$\lambda = \frac{x^* N x}{x^* M x} = \frac{x^*(B - H)x - x^* S x}{x^*(H + B)x + x^* S x}. \tag{2.2}$$

Noting $B > 0$ and $H \geq 0$, $x^*(H + B)x \geq |x^*(B - H)x|$. Consequently,

$$[x^*(B - H)x]^2 + |x^* S x|^2 \leq [x^*(H + B)x]^2 + |x^* S x|^2. \tag{2.3}$$

Therefore, it follows from (2.2) that

$$
\begin{aligned}
|\lambda| &= \frac{|x^*(B - H)x - x^* S x|}{|x^*(H + B)x + x^* S x|} \\
&= \sqrt{\frac{[x^*(B - H)x]^2 + |x^* S x|^2}{[x^*(B + H)x]^2 + |x^* S x|^2}} \\
&\leq 1,
\end{aligned}
\tag{2.4}
$$

which shows $\rho(M^{-1}N) \leq 1$. Furthermore, if $x \notin N(H)$, then $x^* H x > 0$ and thus $x^*(H + B)x > |x^*(B - H)x|$. As a result, (2.3) holds strictly. Following from (2.4), we have $|\lambda| < 1$. If $x \in N(H)$, then $x^* H x = 0$ and thus $x^*(H + B)x = x^*(B - H)x = x^* B x > 0$. Consequently, (2.2) becomes

$$\lambda = \frac{x^* B x - x^* S x}{x^* B x + x^* S x} = \frac{[(x^* B x)^2 - |x^* S x|^2] - 2 x^* B x \cdot x^* S x}{(x^* B x)^2 + |x^* S x|^2} \tag{2.5}$$

which shows that $|\lambda| = 1$ and $Im(\lambda) = 2i \cdot x^*Bx \cdot x^*Sx/[(x^*Bx)^2 + |x^*Sx|^2]$. Since $x^*Bx > 0$, $Im(\lambda) \neq 0$ only if $x^*Sx \neq 0$. Assume $x^*Sx = 0$. (2.5) indicates $\lambda = 1$. (2.1) yields $Nx = Mx$. As a result, $Ax = (M - N)x = 0$. Since $x \neq 0$, the matrix A is singular, which contradicts the nonsingularity of the matrix A. Thus $x^*Sx \neq 0$ which induces $Im(\lambda) \neq 0$. This completes the proof. □

Theorem 5 shows that the strong P-regular splitting method (1.4) for non-non-Hermitian positive semidefinite and nonsingular linear systems is not necessarily convergent. The following will present a convergence result for the relaxed strong P-regular splitting method (1.4). At first, the following lemmas will be used in this section.

Lemma 4. *[44] Let $A \in \mathbb{C}^{n \times n}$ be non-Hermitian positive semidefinite and let $H = (A + A^*)/2$ and $S = (A - A^*)/2$ be its Hermitian and skew-Hermitian parts, respectively. Then A is singular if and only if the set $K = N(H) \cap N(S) \neq \{0\}$, where $N(H)$ denotes the null space of the matrix H.*

Lemma 5. *Let $A, B \in \mathbb{C}^{n \times n}$ and $x \in \mathbb{C}^n$. If A is Hermitian positive definite and B is either Hermitian or skew-Hermitian with $x^*Bx \neq 0$, then*

$$\varrho(A^{-1}B) \leq \frac{|x^*Bx|}{x^*Ax} \leq \rho(A^{-1}B). \tag{2.6}$$

Proof. Since A is a Hermitian positive definite, $A^{1/2}$ and $A^{-1/2}$ exist. Furthermore, $x \in \mathbb{C}^n$ and $x^*Bx \neq 0$, so, $\dfrac{|x^*Bx|}{x^*Ax} \neq 0$. Let $y = A^{1/2}x$, so, $x = A^{-1/2}y$. Then,

$$
\begin{aligned}
\frac{|x^*Bx|}{x^*Ax} &= \frac{|y^*A^{-1/2}BA^{-1/2}y|}{y^*y} \\
&\leq \max \frac{|y^*A^{-1/2}BA^{-1/2}y|}{y^*y} \\
&= \rho(A^{-1/2}BA^{-1/2}) \\
&= \rho(A^{-1}B),
\end{aligned}
\tag{2.7}
$$

and

$$
\begin{aligned}
\frac{|x^*Bx|}{x^*Ax} &= \frac{|y^*A^{-1/2}BA^{-1/2}y|}{y^*y} \\
&\geq \min \frac{|y^*A^{-1/2}BA^{-1/2}y|}{y^*y} \\
&= \varrho(A^{-1/2}BA^{-1/2}) \\
&= \varrho(A^{-1}B).
\end{aligned}
\tag{2.8}
$$

□

Lemma 6. *Let $f(x) = \dfrac{x}{x^2 + \gamma^2}$, $x \in [a, b]$, where $b > a > 0$ and $\gamma > 0$. Then*

- $f(x) \in \left[\mu, \dfrac{1}{2\gamma}\right]$ *if $\gamma \in [a, b]$, where $\mu = \min\left\{\dfrac{a}{a^2 + \gamma^2}, \dfrac{b}{b^2 + \gamma^2}\right\}$;*
- $f(x) \in \left[\dfrac{a}{a^2 + \gamma^2}, \dfrac{b}{b^2 + \gamma^2}\right]$ *if $b < \gamma$;*
- $f(x) \in \left[\dfrac{b}{b^2 + \gamma^2}, \dfrac{a}{a^2 + \gamma^2}\right]$ *if $a > \gamma$.*

Proof. It is obviously that we can obtain the proof immediately following from the monotonicity of function. □

Theorem 6. *Let $A \in \mathbb{C}^{n \times n}$ be non-Hermitian positive semidefinite and nonsingular with $H = (A + A^*)/2$ and $S = (A - A^*)/2$ its Hermitian and skew-Hermitian parts, respectively. Assume that $A = M - N$ is strong P-regular, i.e., $B := M + N > 0$. Then the relaxed method (1.4) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$, provided $\tau \in (0, 1)$, i.e., $\rho(T_\tau) < 1$ for all $\tau \in (0, 1)$, where T_τ is defined in (1.4). Furthermore,*

$$\tau^* = arg\ min_{\tau \in (0,1)}\{\rho(T_\tau)\} = \frac{1}{2} \quad and$$

$$min_{\tau \in (0,1)}\{\rho(T_\tau)\} = \rho(T_{1/2}) \in [\ \sqrt{a},\ \ \sqrt{b}\ ], \tag{2.9}$$

where

$$a = \min\left\{\frac{1}{1+\zeta^2}, \frac{1+\varphi}{(1+\varphi)^2 + \zeta^2}, \frac{1+\psi}{(1+\psi)^2 + \zeta^2}\right\}, \ b = \max\left\{\frac{1}{1+\varepsilon^2}, \frac{1}{1+\varphi}\right\}, \tag{2.10}$$

$\varepsilon = \varrho(B^{-1}S), \ \zeta = \rho(B^{-1}S), \ \varphi = \varrho(B^{-1}H) \ and \ \psi = \rho(B^{-1}H)$.

Proof. Let λ be an eigenvalue of $M^{-1}N$, and let $x \in \mathbb{C}^n$ be a corresponding eigenvector with $\|x\|_2 = 1$ (note that it must have $x \neq 0$). Since $T_\tau = (1 - \tau)I + \tau M^{-1}N$, $\mu_\tau = (1 - \tau) + \tau\lambda$ is an eigenvalue of T_τ and $x \in \mathbb{C}^n$ is its corresponding eigenvector. Further assume that $|\mu_\tau| = \rho(T_\tau)$. If $x \notin N(H)$, Theorem 5 shows $|\lambda| < 1$, and consequently, $\rho(T_\tau) = |\mu_\tau| \leq (1 - \tau) + \tau|\lambda| < 1$ for all $\tau \in (0, 1)$. Conversely, if $x \in N(H)$, Theorem 5 yields $|\lambda| = 1$ with $Im(\lambda) \neq 0$, where $Im(\lambda)$ is the imaginary part of λ. Let $Re(\lambda)$ is the real part of λ. Then $Re^2(\lambda) + Im^2(\lambda) = 1$ and $|Re(\lambda)| < 1$ for $Im(\lambda) \neq 0$. Therefore,

$$\begin{aligned}
\rho(T_\tau) &= (1 - \tau) + \tau[Re(\lambda) + i \cdot Im(\lambda)] \\
&= \sqrt{[1 + \tau(Re(\lambda) - 1)]^2 + \tau^2 Im^2(\lambda)} \\
&= \sqrt{(1 - \tau)^2 + 2(1 - \tau)\tau Re(\lambda) + \tau^2} \\
&\leq \sqrt{(1 - \tau)^2 + 2(1 - \tau)\tau|Re(\lambda)| + \tau^2} \\
&< \sqrt{(1 - \tau)^2 + 2(1 - \tau)\tau + \tau^2} \\
&= \sqrt{(1 - \tau + \tau)^2} \\
&= 1,
\end{aligned} \tag{2.11}$$

for all $\tau \in (0, 1)$, i.e., the relaxed method (1.4) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$, provided $\tau \in (0, 1)$.

Again, it follows from the third equality in (2.11) that

$$\begin{aligned}
\rho^2(T_\tau) &= (1 - \tau)^2 + 2(1 - \tau)\tau Re(\lambda) + \tau^2 \\
&= 2(1 - Re(\lambda))(\tau^2 - \tau) + 1 \\
&= 2(1 - Re(\lambda))(\tau - \frac{1}{2})^2 + \frac{1 + Re(\lambda)}{2}.
\end{aligned} \tag{2.12}$$

Since $1 - Re(\lambda) > 0$ for $|\lambda| < 1$ if $x \notin N(H)$ and $|\lambda| = 1$ with $Im(\lambda) \neq 0$ if $x \notin N(H)$, it is easy to see from the last equality in (2.12) that

$$\tau^* = arg\ min_{\tau \in (0,1)}\{\rho(T_\tau)\} = \frac{1}{2} \tag{2.13}$$

and

$$\min_{\tau \in (0,1)}\{\rho(T_\tau)\} = \rho(T_{1/2}) = \sqrt{\frac{1 + Re(\lambda)}{2}}. \tag{2.14}$$

Following (2.2),

$$Re(\lambda) = \frac{x^*(H + B)x \cdot x^*(B - H)x - |x^*Sx|^2}{[x^*(H + B)x]^2 + |x^*Sx|^2}, \tag{2.15}$$

where $B = M + N > 0$, and H and S are Hermitian and skew-Hermitian parts of A, respectively. Then (2.14) and (2.15) yield

$$\min_{\tau \in (0,1)}\{\rho(T_\tau)\} = \rho(T_{1/2}) = \sqrt{\frac{x^*(H + B)x \cdot x^*Bx}{[x^*(H + B)x]^2 + |x^*Sx|^2}}. \tag{2.16}$$

Let $y = x^*Hx/x^*Bx \in [\varphi, \psi]$ and $t = |x^*Sx|/x^*Bx \in [\varepsilon, \zeta]$. Then

$$\min_{\tau \in (0,1)}\{\rho(T_\tau)\} = \rho(T_{1/2}) = \sqrt{\frac{1 + y}{(1 + y)^2 + t^2}}. \tag{2.17}$$

Lemma 5 shows that $y \in [\varphi, \psi]$ and $t \in [\varepsilon, \zeta]$ with $\varepsilon = \varrho(B^{-1}S)$, $\zeta = \rho(B^{-1}S)$, $\varphi = \varrho(B^{-1}H)$ and $\psi = \rho(B^{-1}H)$. In what follows, we distinguish between the following three cases.

(i) If $x \in N(H)$, then $x^*Hx = 0$. Now, we assert $x^*Sx \neq 0$. Otherwise, it follows from (2.16) that $\rho(T_{1/2}) = 1$ which contradicts (2.11). Thus, $x^*Sx \neq 0$. Consequently, Lemma 5 shows that

$$\min_{\tau \in (0,1)}\{\rho(T_\tau)\} = \rho(T_{1/2}) = \sqrt{\frac{1}{1 + t^2}} \in \left[\sqrt{\frac{1}{1 + \zeta^2}}, \sqrt{\frac{1}{1 + \varepsilon^2}}\right]. \tag{2.18}$$

(ii) If $x \in N(S)$, it follows from Lemma 4 $x \notin N(H)$. As a result, $x^*Hx \neq 0$ and $x^*Sx = 0$. Hence,

$$\min_{\tau \in (0,1)}\{\rho(T_\tau)\} = \rho(T_{1/2}) = \sqrt{\frac{1}{1 + y}} \in \left[\sqrt{\frac{1}{1 + \psi}}, \sqrt{\frac{1}{1 + \varphi}}\right]. \tag{2.19}$$

(iii) If $x \notin N(S)$ and $x \notin N(H)$, then $x^*Hx \neq 0$ and $x^*Sx \neq 0$. Hence, (2.17) holds. Since

$$\frac{1 + y}{(1 + y)^2 + t^2} < \frac{1}{1 + y} \leq \frac{1}{1 + \varphi} \tag{2.20}$$

and Lemma 6 yields

$$\frac{1 + y}{(1 + y)^2 + t^2} \geq \frac{1 + y}{(1 + y)^2 + \zeta^2} \geq \phi := \min\{\frac{1 + \varphi}{(1 + \varphi)^2 + \zeta^2}, \frac{1 + \psi}{(1 + \psi)^2 + \zeta^2}\}, \tag{2.21}$$

$$\min_{\tau \in (0,1)}\{\rho(T_\tau)\} = \frac{1 + y}{(1 + y)^2 + t^2} \in \left[\phi, \sqrt{\frac{1}{1 + \varphi}}\right). \tag{2.22}$$

It follows from the three cases above that

$$\min_{\tau \in (0,1)}\{\rho(T_\tau)\} = \rho(T_{1/2}) \in [\sqrt{a}, \sqrt{b}], \tag{2.23}$$

where a and b is defined in (2.10). This completes the proof. $\qquad\square$

Corollary 1. *Let $A \in \mathbb{C}^{n \times n}$ be non-Hermitian positive definite with $H = (A + A^*)/2$ and $S = (A - A^*)/2$ its Hermitian and skew-Hermitian parts, respectively. Assume that $A = M - N$ is strong P-regular, i.e., $B := M + N > 0$. Then the relaxed method (1.4) converges to the unique solution of (1.1) for any choice of the initial guess $x^{(0)}$, provided $\tau \in (0, 1)$, i.e., $\rho(T_\tau) < 1$ for all $\tau \in (0, 1)$, where T_τ is defined in (1.4). Furthermore,*

$$\tau^* = arg\ min_{\tau \in (0,1)}\{\rho(T_\tau)\} = \frac{1}{2} \tag{2.24}$$

and

$$min_{\tau \in (0,1)}\{\rho(T_\tau)\} = \rho(T_{1/2}) \in \left[\ \sqrt{\phi},\ \ \sqrt{\frac{1}{1+\varphi}}\ \right],$$

where $\phi = \min\left\{\dfrac{1+\varphi}{(1+\varphi)^2+\zeta^2}, \dfrac{1+\psi}{(1+\psi)^2+\zeta^2}\right\}$, $\zeta = \rho(B^{-1}S)$, $\varphi = \varrho(B^{-1}H)$ and $\psi = \rho(B^{-1}H)$.

3. The SOR method and its relaxed version for non-Hermitian linear systems

While convergence results on the classic SOR methods have been known for many years for Hermitian positive definite matrices, monotone matrices and H-matrices (see, e.g., [15, 25, 26, 34, 35, 37, 42]), very little appears to be known in the non-Hermitian positive definite case. Among the few studies known to us we mention [15, 29, 32, 33]. Recently, Zhang and Benzi in [43] proposed some new SOR methods and obtained their convergence. In this section new SOR methods and their relaxed version are constructed and the general theory developed in the previous section is applied to establish the convergence of such iteration methods for non-Hermitian positive (semi-) definite systems. Our results are more general and complete than the few results found in literature.

Without loss of generality, we write

$$\begin{aligned} A &= I - L - U \\ &= I - \frac{L - U^*}{2} - \frac{2U + U^* + L}{2} = I - \frac{U - L^*}{2} - \frac{2L + L^* + U}{2}, \end{aligned} \tag{3.1}$$

where L and U are lower and upper triangular matrices, respectively, with their diagonal entries either imaginary numbers or zero. The forward and backward successive over-relaxation methods (forward and backward SOR methods) are defined by the iteration matrices

$$\mathscr{L}_\omega = \left[I - \frac{\omega(L - U^*)}{2}\right]^{-1}\left[\frac{\omega(2U + U^* + L)}{2} + (1 - \omega)I\right] \tag{3.2}$$

and

$$\mathscr{U}_\omega = \left[I - \frac{\omega(U - L^*)}{2}\right]^{-1}\left[\frac{\omega(2L + L^* + U)}{2} + (1 - \omega)I\right], \tag{3.3}$$

respectively, while their relaxed version are given by the iteration matrices

$$\mathscr{S}_{\tau,\omega} = (1 - \tau)I + \tau\mathscr{L}_\omega, \tag{3.4}$$

$$\mathscr{T}_{\tau,\omega} = (1 - \tau)I + \tau\mathscr{U}_\omega, \tag{3.5}$$

respectively.

Theorem 7. *Let $A \in \mathbb{C}^{n \times n}$ be non-Hermitian positive definite, and let $A = I - L - U$ be defined by* (3.1). *Also, let $\eta = \lambda_{\min}(B)$ be the smallest eigenvalue of $B := U + U^*$. Then the forward SOR method* (3.2) *converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$, provided $\omega \in (0, \frac{2}{1-\eta})$, i.e., $\rho(\mathscr{L}_\omega) < 1$ for all $\omega \in (0, \frac{2}{1-\eta})$, where $\mathscr{L}_\omega$ is defined in* (3.2).

Proof. Let $M = \dfrac{1}{\omega} I - \dfrac{L - U^*}{2}$ and $N = \left(\dfrac{1}{\omega} - 1\right) I + \dfrac{2U + U^* + L}{2}$. Then $\mathscr{L}_\omega = M^{-1} N$ and

$$M + N = \left(\frac{2}{\omega} - 1\right) I + (U + U^*) \tag{3.6}$$

Since $\omega \in (0, \frac{2}{1-\eta})$, $\dfrac{2}{\omega} - 1 + \eta > 0$, and consequently,

$$M + N = \left(\frac{2}{\omega} - 1\right) I + (U + U^*) \geq \left(\frac{2}{\omega} - 1 + \eta\right) I > 0,$$

which shows that $A = M - N$ be strong *P*-regular. Since A is non-Hermitian positive definite, it follows again from Theorem 1 that $\rho(\mathscr{L}_\omega) = \rho(M^{-1}N) < 1$, i.e., the SOR method is convergent for all $\omega \in (0, \frac{2}{1-\eta})$. This completes the proof. $\square$

In the same method of proof, we can obtain the following conclusion.

Theorem 8. *Let $A \in \mathbb{C}^{n \times n}$ be non-Hermitian positive definite, and let $A = I - L - U$ be defined by* (3.1). *Also, let $\phi = \lambda_{\min}(C)$ be the smallest eigenvalue of $C := L + L^*$. Then the backward SOR method* (3.3) *converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$, provided $\omega \in (0, \frac{2}{1-\phi})$.*

Remark 1. *Theorem 7 becomes Theorem 1 in* [32] *if $A = I - L + L^T \in \mathbb{R}^{n \times n}$; hence, Theorem 7 generalizes the convergence result of Niethammer and Schade.*

Theorem 9. *Let $A \in \mathbb{C}^{n \times n}$ be non-Hermitian positive semidefinite and nonsingular, and let $A = I - L - U$ be defined by* (3.1). *Also, let $\eta = \lambda_{\min}(B)$ and $\phi = \lambda_{\min}(C)$ be the smallest eigenvalues of $B := U + U^*$ and $C := L + L^*$, respectively. Then,*

(i) *the relaxed forward SOR method converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$, provided $\omega \in (0, \frac{2}{1-\eta})$; and*

(ii) *the relaxed backward SOR method converges to the unique solution of* (1.1) *for any choice of the initial guess $x^{(0)}$, provided $\omega \in (0, \frac{2}{1-\phi})$.*

Proof. The proof can be immediately obtained from Theorem 6. $\square$

4. Numerical experiments

In this section we obtain the results of some numerical experiments with the SOR method and relaxed SOR method on linear systems.

Example 1. Consider the following three-dimensional convection-diffusion equations,

$$- (u_{xx} + u_{yy} + u_{zz}) + q(u_x + u_y + u_z) = f(x, y, z) \tag{4.1}$$

on the unite cube $\Omega = [0, 1] \times [0, 1] \times [0, 1]$ with the first boundary conditions, where q is coefficient constant. We get the coefficient matrix A by using the centered differences to the diffusive terms and convective terms

$$A = A_x \otimes I \otimes I + I \otimes A_y \otimes I + I \otimes I \otimes A_z, \tag{4.2}$$

where $\otimes$ denotes the Kronecker product, A_x, A_y and A_z are tri-diagonal matrices given by

$$A_x = tridiag(t_2, t_1, t_3), \quad A_y = tridiag(t_2, 0, t_3), A_z = tridiag(t_2, 0, t_3), \tag{4.3}$$

with $t_1 = 6, t_2 = -1 - r, t_3 = -1 + r$. where $r = \frac{qh}{2}$, the step length $h = \frac{1}{n+1}$, n is the number of intervals along axes.

Obviously, A satisfies the conditions of Theorem 3.4, following we do experiments to illustrate the efficiency of results. For simplicity, The iterative scheme (1.4) is written as A1 (stationary iterative method) given in [37], the scheme (3.4) is written as A2 (SOR forward iterative method), the scheme (3.5) is written as A3 (SOR backward iterative method) respectively. These four algorithms were coded in Matlab, and all computations were performed on a DESKTOP-GBSME13 (Intel(R) Core(TM) i5-8250U, CPU 1.80 GHz, RAM 8.00 GB) with Matlab R2020a.

The stopping criterion is defined as

$$RE = \frac{\|x^{k+1} - x^k\|^2}{\max\{1, \|x^k\|_2\}} \leq 10^{-6}.$$

where x^{k+1} is the $k + 1$−th step's iterative value, x^k is the k−th step's iterative value.

Numerical results compared are presented in Table 1. In particular, we report in Figure 1 the change of RE of A1–A3, when $q = 1$ with the iteration number increasing, where k is the convergent steps.

From Table 1, we observe that SOR iterative method generally has much less iteration number than stationary iterative method at the same accuracy, when $n = 8, n = 16, n = 32, n = 64$, and SOR iterative method is basically stable as the number n increasing. Thus, SOR iterative method is generally superior to stationary iterative method in terms of iteration number and computation times.

Figure 1 shows that RE generated by SOR iterative method quickly converges to 0 with the interval number increasing when $q = 1$.

Table 1. Performance of A1–A3 with different n.

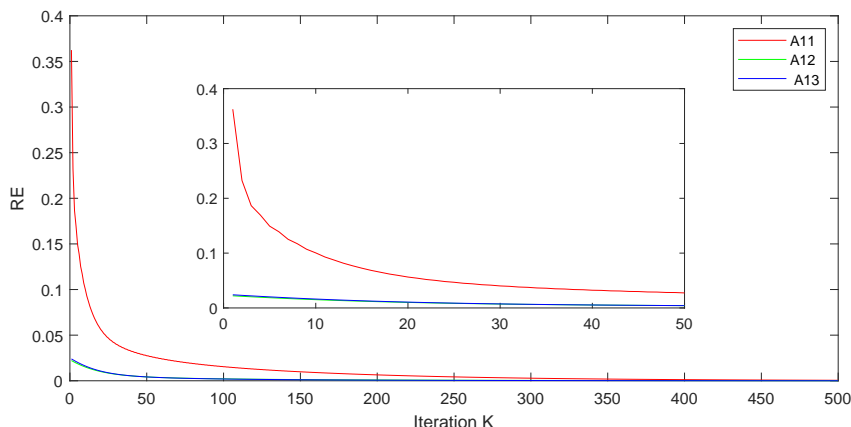| n	Algorithm	k	RE	times(s)
8	A1	3044	$9.9089e^{-07}$	0.746
	A2	441	$9.8110e^{-07}$	0.078
	A3	276	$9.7385e^{-07}$	0.056
16	A1	1534	$9.9581e^{-07}$	0.134
	A2	404	$9.9714e^{-07}$	0.072
	A3	321	$9.7366e^{-07}$	0.058
32	A1	927	$9.9549e^{-07}$	0.178
	A2	309	$9.8860e^{-07}$	0.054
	A3	293	$9.9925e^{-07}$	0.050
64	A1	884	$9.9495e^{-07}$	0.153
	A2	327	$9.8474e^{-07}$	0.054
	A3	325	$9.9430e^{-07}$	0.055

Figure 1. When $n = 64, q = 1$ the change of RE of A1–A3 with the iteration number increasing.

Example 2. Consider the linear system $Ax = b$ with the matrix $A = I_n - L - U$, and $b = ones(n, 1)$, where n is the size of A, L is the lower triangular matrices with diagonal entries zero, secondary diagonal entries is -0.25 and others entries are 0.25, and U is the upper triangular matrices with diagonal entries zero and others entries are 0.25.

Obviously, A satisfies the conditions of Theorems 3.1 and 3.2, so we do experiments to illustrate the efficiency of results. For simplicity, The iterative scheme (1.3) is written as A21 (relaxed stationary iterative method) given in [37], the scheme (3.1) is written as A22 (relaxed SOR forward iterative method), the scheme (3.2) is written as A23 (relaxed SOR backward iterative method) respectively. These four algorithms were coded in Matlab, and all computations were performed on a DESKTOP-GBSME13 (Intel(R) Core(TM) i5-8250U, CPU 1.80 GHz, RAM 8.00 GB) with Matlab R2020a.

The stopping criterion is defined as

$$RE = \frac{\|x^{k+1} - x^k\|^2}{\max\{1, \|x^k\|_2\}} =< 10^{-6}.$$

where x^{k+1} is the $k + 1$−th step's iterative value, x^k is the k−th step's iterative value.

Numerical results compared are presented in Table 2. In particular, we report in Figure 2 the change of RE of A1–A3, when n=100 with the iteration number increasing.

From Table 2, we observe that SOR iterative method generally has much less iteration number than relaxed stationary iterative method at the same accuracy, when the matrices dimension $n = 100, n = 200$ and $n = 500$. Thus, relaxed SOR iterative method is generally superior to relaxed stationary iterative method in terms of iteration number and computation times.

Figure 2 shows that RE generated by relaxed SOR iterative method quickly converges to 0 with the iteration number increasing when $n = 100$.

Table 2. Performance of A1–A3 with different n.

Algorithm	n	RE	k	times(s)
A21	100	$9.9801e^{-07}$	2940	3.458
	200	$9.9878e^{-07}$	5529	17.284
	500	$9.9983e^{-07}$	12565	378.391
A22	100	$9.9650e^{-07}$	577	0.462
	200	$9.96795e^{-07}$	1325	2.268
	500	$9.9902e^{-07}$	5789	56.365
A23	100	$9.8396e^{-07}$	581	0.484
	200	$9.9962e^{-07}$	1302	2.278
	500	$9.9860e^{-07}$	5782	57.126
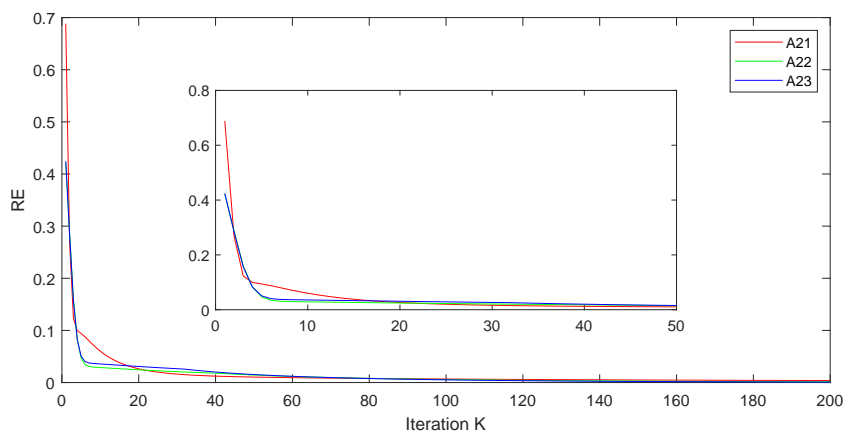
Figure 2. When $n = 100$ the change of RE of A1–A3 with the iteration number increasing.

5. Conclusions

The strong P-regular splitting firstly has been defined for non-Hermitian positive (semi-) definite linear systems, and the spectral radius of the iteration matrix obtained by strong P-regular splitting for the non-Hermitian positive (semi-) definite matrix has been analyzed in this paper, and we have studied the convergence of strong P-regular splitting methods for the solution of non-Hermitian positive definite linear systems. Some of our results can be regarded as generalizations of analogous results for the Hermitian positive definite case. As an application of our theory, we obtain new convergence conditions for SOR-like methods in the non-Hermitian case.

Acknowledgments

Conflict of interest

The authors declared that they have no conflicts of interest to this work.

References

1. Z. Z. Bai, G. H. Golub, Accelerated Hermitian and skew-Hermitian splitting iteration methods for saddle-point problems, *IMA J. Numer. Anal.*, **27** (2007), 1–23.

2. Z. Z. Bai, G. H. Golub, L. Z. Lu, J. F. Yin, Block triangular and skew-Hermitian splitting methods for positive-definite linear systems, *SIAM J. Sci. Comput.*, **26** (2005), 844–863.

3. Z. Z. Bai, G. H. Golub, M. K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, *SIAM J. Matrix Anal. Appl.*, **24** (2003), 603–626.

4. Z. Z. Bai, G. H. Golub, M. K. Ng, On successive-overrelaxation acceleration of the Hermitian and skew-Hermitian splitting iterations, *Numer. Linear Algebra Appl.*, **17** (2007), 319–335.

5. Z. Z. Bai, G. H. Golub, M. K. Ng, On inexact Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, *Linear Algebra Appl.*, **428** (2008), 413–440.

6. Z. Z. Bai, G. H. Golub, J. Y. Pan, Preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite linear systems, *Numer. Math.*, **98** (2004), 1–32.

7. M. Benzi, A generalization of the Hermitian and skew-Hermitian splitting iteration, *SIAM J. Matrix. Anal. Appl.*, **31** (2009), 360–374.

8. M. Benzi, Splittings of symmetric matrices and a question of Ortega, *Linear Algebra Appl.*, **429** (2008), 2340–2343.

9. M. Benzi, D. Bertaccini, Block preconditioning of real-valued iterative algorithms for complex linear systems, *IMA J. Numer. Anal.*, **28** (2008), 598–618.

10. M. Benzi, M. K. Ng, Preconditioned iterative methods for weighted Toeplitz least squares problems, *SIAM J. Matrix Anal. Appl.*, **27** (2006), 1106–1124.

11. M. Benzi, G. H. Golub, J. Liesen, Numerical solution of saddle point problems, *Acta Numer.*, (2005), 1–137.

12. M. Benzi, G. H. Golub, A preconditioner for generalized saddle point problems, *SIAM J. Matrix Anal. Appl.*, **26** (2004), 20–41.

13. M. Benzi, M. Gander, G. H. Golub, Optimization of the Hermitian and skew-Hermitian splitting iteration for saddle-point problems, *BIT Numer. Math.*, **43** (2003), 881–900.

14. M. Benzi, D. B. Szyld, Existence and uniqueness of splittings for stationary iterative methods with applications to alternating methods, *Numer. Math.*, **76** (1997), 309–321.

15. A. Berman, R. J. Plemmons, *Nonnegative matrices in the mathematical sciences*, Philadelphia: Society for Industrial and Applied Mathematics, 1994.

16. M. Masoudi, D. K. Salkuyeh, An extension of the positive-definite and skew-Hermitian splitting method for preconditioning of generalized saddle point problems, *Comput. Math. Appl.*, **79** (2020), 2304–2321.

17. M. Dehghan, A. Shirilord, A generalized modified Hermitian and skew-Hermitian splitting (GMHSS) method for solving complex Sylvester matrix equation, *Appl. Math. Comput.*, **348** (2019), 632–651.

18. Z. Q. Wang, J. F. Yin, Q. Y. Dou, Preconditioned modified Hermitian and skew-Hermitian splitting iteration methods for fractional nonlinear Schrdinger equations, *J. Comput. Appl. Math.*, **367** (2020), 112420.

19. Z. H. Cao, A note on *P*-regular splitting of Hermitian matrix, *SIAM J. Matrix Anal. Appl.*, **21** (2000), 1392–1393.

20. Z. H. Cao, Convegence of nested iterative methods for symmetric *P*-regular splitting, *SIAM J. Matrix Anal. Appl.*, **22** (2000), 20–32.

21. J. Chen, W. Li, Equivalent conditions for convergence of splittings of non-Hermitian indefinite matrices, *J. Sci. Comput.*, **30** (2006), 117–130.

22. H. Elman, D. Silvester, A. Wathen, *Finite elements and fast iterative solvers: With applications in incompressible fluid dynamics*, Oxford University Press, 2014.

23. A. Frommer, D. B. Szyld, Weighted max norms, splitting, and overlapping additive Schwarz iterations, *Numer. Math.*, **83** (1999), 259–278.

24. A. George, Kh. D. Ikramov, On the properties of accretive-dissipative matrices, *Math. Notes*, **77** (2005), 767–776.

25. G. H. Golub , C. F. Van Loan, *Matrix computations*, 3 Eds., Baltimore: Johns Hopkins University Press, 1996.

26. A. Hadjidimos, Accelerated overrelaxation method, *Math. Compt.*, **32** (1978), 149–157.

27. N. J. Higham, Factorizing complex symmetric matrices with positive definite real and imaginary parts, *Math. Compt.*, **67** (1998), 1591–1599.

28. Kh. D. Ikramov, On complex Benzi-Golub matrices, *Doklady Math.*, **79** (2009), 342–344.

29. D. R. Kincaid, A. C. Elster, *Iterative methods in scientific computation IV*, IMACS, 1999.

30. L. Li, T. Z. Huang, X. P. Liu, Modified Hermitian and skew-Hermitian splitting methods for non-Hermitian positive-definite linear systems, *Numer. Linear Algebra Appl.*, **14** (2007), 217–235.

31. R. Nabben, A note on comparison theormes for splittings and multisplittings of Hermitian positive definite matrices, *Linear Algebra Appl.*, **233** (1996), 67–80.

32. W. Niethammer, J. Schade, On a relaxed SOR-method applied to nonsymmetric linear systems, *J. Comput. Appl. Math.*, **1** (1975), 133–136.

33. W. Niethammer, R. S. Varga, Relaxation methods for non-Hermitian linear systems, *Results Math.*, **16** (1989), 308–320.

34. J. M. Ortega, *Numerical analysis: A second course*, Philadelphia: Society for Industrial and Applied Mathematics, 1990.

35. Y. Saad, *Iterative methods for sparse linear systems*, 2 Eds., Philadelphia: Society for Industrial and Applied Mathematics, 2003.

36. R. J. Vanderbei, Symmetric quasidefinite matrices, *SIAM J. Optim.*, **5** (1995), 100–113.

37. R. S. Varga, *Matrix iterative analysis*, 2 Eds., Berlin/Heidelberg: Spriger-Verlag, 2000.

38. C. L. Wang, Z. Z. Bai, Sufficient conditions for the convergent splitting of non-Hermitian positive definite matrices, *Linear Algebra Appl.*, **330** (2001), 215–218.

39. L. Wang, Z. Z. Bai, Convergence conditions for splitting iteration methods for non-Hermitian linear systems, *Linear Algebra Appl.*, **428** (2008), 453–468.

40. F. B. Weissler, Some remarks concerning iterative methods for linear systems, *SIAM J. Matrix Anal. Appl.*, **16** (1995), 448–461.

41. J. Weissinger, Verallgemainerungen des Seidelschen Iterationsverfahrens, *Z. Angew. Math. Mech.*, **33** (1953), 155–162.

42. D. M. Young, *Iterative solution of large linear systems*, New York: Academic Press, 1971.

43. C. Y. Zhang, M. Benzi, *P*-regular splitting iterative methods for non-Hermitian linear systems, *Electron. Trans. Numer. Anal.*, **36** (2009), 39–53.

44. C. Y. Zhang, On convergence of double splitting methods for non-Hermitian positive semidefinite linear systems, *Calcolo*, **47** (2010), 103–112.

45. S. L. Wu, Several variants of the Hermitian and skew-Hermitian splitting method for a class of complex symmetric linear systems, *Numer. Linear Algebra Appl.*, **22** (2015), 338–356.

46. Z. Z. Bai, M. Rozložnĺk, On the numerical behavior of matrix splitting iteration methods for solving linear systems, *SIAM J. Numer. Anal.*, **53** (2015), 1716–1737.

47. J. Y. Yuan, *Applied iterative analysis*, Science Press, 2014.