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Research article

Weighted generalized Quasi Lindley distribution: Different methods of estimation, applications for Covid-19 and engineering data

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Abstract: Recently, a new lifetime distribution known as a generalized Quasi Lindley distribution (GQLD) is suggested. In this paper, we modified the GQLD and suggested a two parameters lifetime distribution called as a weighted generalized Quasi Lindley distribution (WGQLD). The main mathematical properties of the WGQLD including the moments, coefficient of variation, coefficient of skewness, coefficient of kurtosis, stochastic ordering, median deviation, harmonic mean, and reliability functions are derived. The model parameters are estimated by using the ordinary least squares, weighted least squares, maximum likelihood, maximum product of spacing's, Anderson-Darling and Cramer-von-Mises methods. The performances of the proposed estimators are compared based on numerical calculations for various values of the distribution parameters and sample sizes in terms of the mean squared error (MSE) and estimated values (Es). To demonstrate the applicability of the new model, four applications of various real data sets consist of the infected cases in Covid-19 in Algeria and Saudi Arabia, carbon fibers and rain fall are analyzed for illustration. It turns out that the WGQLD is empirically better than the other competing distributions considered in this study.

Keywords: generalized Quasi Lindley distribution; weighted distribution; methods of least squares; maximum likelihood method; methods of minimum distances; lifetime distribution
Mathematics Subject Classification: 62E15, 60E05, 62F10

1. Introduction

Recently, some researchers in statistics have been interested in generating new flexible statistical models based on different techniques such as the weight distributions which are commonly used in many fields of life situations such as sciences, ecology, biostatistics, medicine, engineering, pharmacy and environment and so on.

The concept of weighted distribution is suggested by [15] to study how verification methods can affect the form of the distribution of recorded observations. Later, the weighted distributions are unified and formulated by [31] in general terms in connection with modeling statistical data.

Suppose X is a non-negative random variable with a probability density function (pdf) $f(x)$. Let $W(x)$ be a non negative weight function, then the probability density function of the weighted random variable X_w is given by:

$$f_w(x) = \frac{W(x)f(x)}{\mathbb{E}(W(x))}. \tag{1.1}$$

If the weight function has the form $W(x) = x^\lambda$, the resulting distribution is known as a size biased distribution of order λ with pdf given by:

$$f_\lambda(x) = \frac{x^\lambda f(x)}{\mathbb{E}(X^\lambda)}. \tag{1.2}$$

If λ =1 or 2, the yielded distributions are known as the length biased and area biased distributions, respectively.

For example, the length-biased Suja distribution is proposed by [5] as a generalization of the Suja distribution. The size biased Ishita distribution is offered by [6] as a new modification of the Ishita distribution. The Marshall-Olkin length-biased exponential distribution is suggested by [18]. The length-biased weighted generalized Rayleigh distribution with its properties is proposed by [2]. The weighted Lomax distribution is introduced by [23]. Other types of distributions are also suggested based on other procedures as [8] for the exponentiated new Weibull-Pareto distribution. The Topp-Leone Mukherjee-Islam distribution is offered by [7]. Also, see [27, 16, 17, 26].

To the best of our knowledge, the use of the weighted method to extend the generalized Quasi Lindley distribution introduced by [9] is still unexplored in the literature. In this study, the weighted generalized Quasi Lindley distribution is proposed. Indeed, the importance of the suggestion of the WGQLD arises from the fact it is a modification of the well known Quasi Lindley distribution which is considered by many researchers in different life situations.

The layout of this paper is organized as follows. Section 2 concerns with the pdf and cumulative distribution function (cdf) of the WGQLD and its shapes. Moment generating function and moments includes the rth moment, variance, the coefficient of skewness, kurtosis, and variation are presented in Section 3 theoretically and supported by some simulations. The distribution of order statistics, median deviations and harmonic mean are presented in Section 4. The stochastic ordering, reliability, hazard, reversed hazard rate and odds functions are given in Section 5. Bonferroni and Lorenz curves as well as the Gini index are provided in Section 6. In Section 7, different methods of estimation for the distribution parameters are discussed including maximum likelihood, maximum product of spacing's, ordinary least squares, weighted least squares, Cramer-von-Mises, and Anderson-Darling methods. A simulation study is conducted to compare the performance of the proposed estimators in Section 8.

Illustrative examples of real data applications are given in Section 9. The paper is concluded with some suggestions for future works in Section 10.

2. The WGQLD distribution

This section introduces the pdf and cdf of the WGQLD along with the shapes of the model. The probability density function of the generalized Quasi Lindley distribution is given by:

$$f_{GQLD}(x;\theta,\alpha) = \frac{\theta^2\left(\frac{\theta^2 x^3}{6} + \alpha\theta x^2 + \alpha^2 x\right)}{(\alpha+1)^2} \mathrm{e}^{-\theta x};\ x \geq 0,\ \alpha > -1,\ \theta \geq 0, \tag{2.1}$$

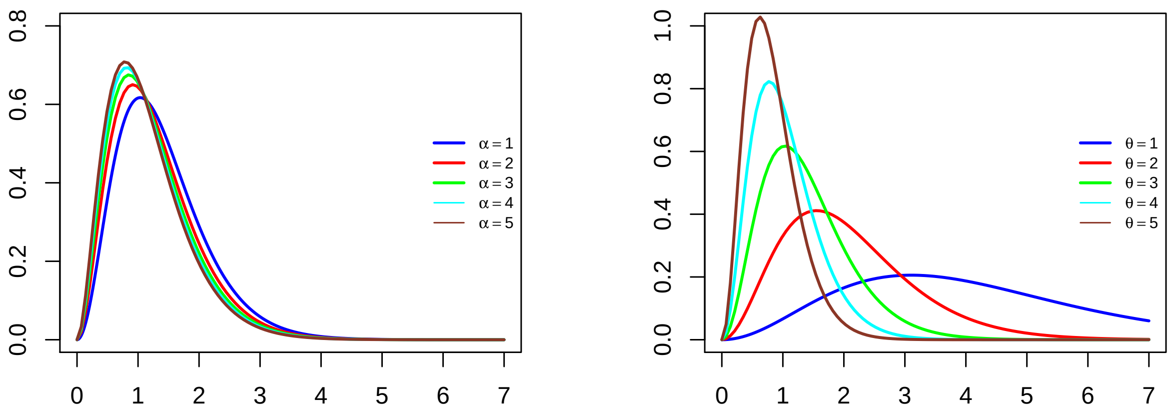
and the corresponding cdf is:

$$F_{GQLD}(x;\theta,\alpha) = 1 - \frac{\left(\theta^3 x^3 + 3\,(2\alpha+1)\,\theta^2 x^2 + 6\,(\alpha+1)^2\,(\theta x+1)\right)}{6\,(\alpha+1)^2} \mathrm{e}^{-\theta x}. \tag{2.2}$$

With reference to Eq 1.1, and in this work without loss of generality we considered the weight function as $W(x) = x$ given that the mean of the GQLD is $\mathbb{E}(X) = \frac{2(2+\alpha)}{\theta(1+\alpha)}$. Hence, the weighted generalized Quasi Lindley distribution can be obtained by substituting $f_{GQLD}(x;\theta,\alpha)$, $E(X)$ and $W(x)$ in Eq 1.1 to get:

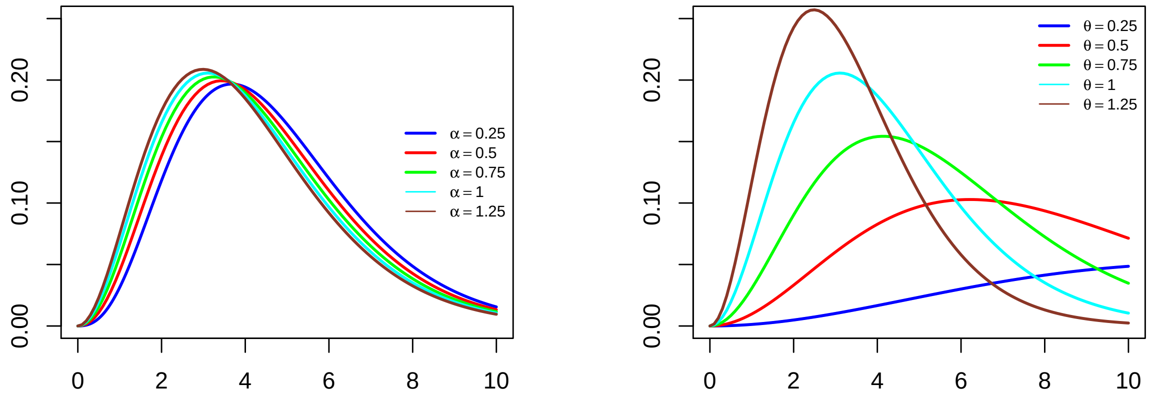
$$f_{WGQLD}(x;\theta,\alpha) = \frac{\theta^3}{2\,(\alpha^2+3\alpha+2)}\left(\frac{\theta^2 x^4}{6} + \alpha\theta x^3 + \alpha^2 x^2\right)\mathrm{e}^{-\theta x};\ x \geq 0,\ \alpha > -1,\ \theta \geq 0. \tag{2.3}$$

Figures 1 and 2 demonstrate the graphs of the pdf of the WGQLD for different values of θ and α. Also, Figures 3 and 4 show various curves of the cdf of WGQLD for selected distribution parameters.



(a) $\theta = 3$ (b) $\alpha = 1$

Figure 1. Plots of the WGQLD probability density function with different parameters values.



(a) $\theta = 1$ (b) $\alpha = 1$

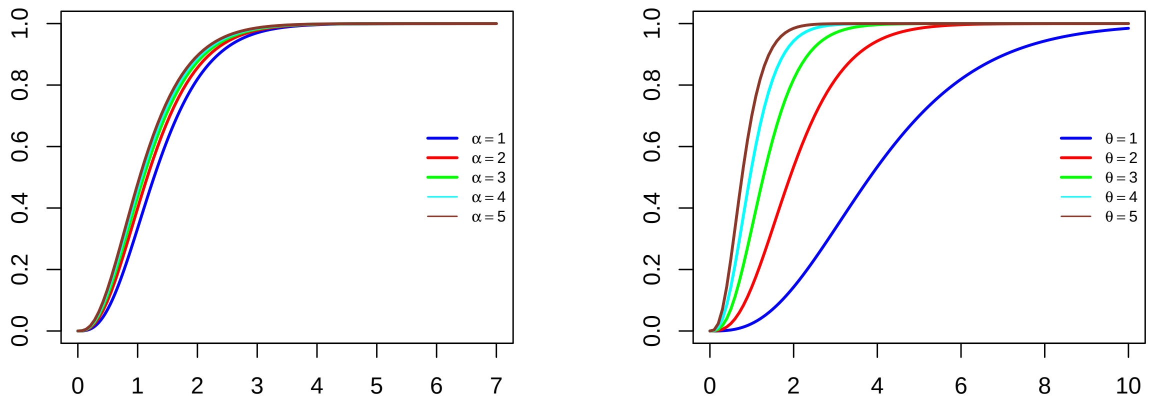
Figure 2. Plots of the WGQLD probability density function with different parameters values.

Based on Figures 1 and 2, it can be observed that the WGQLD is a unimodel and positively skewed. Also, it is approximately symmetric as in Figure 2 when $\theta = 1$ for some parameters. The curve of the distribution function is more flat for $\alpha = 1, \theta = 0.25$ and when $\alpha = \theta = 1$. than other values.

The corresponding cdf equation of the WGQLD is given by

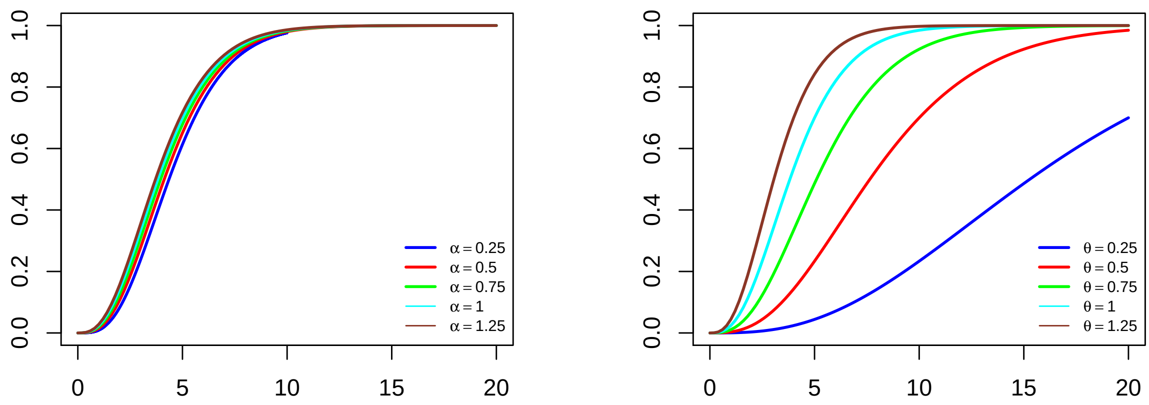
$$F_{WGQLD}(x;\theta,\alpha) = 1 - \frac{\begin{array}{c}24 + 6\alpha^2[2 + x\theta(2 + x\theta)] + 6\alpha[6 + x\theta(6 + x\theta(3 + x\theta))] \\ +x\theta[24 + x\theta(12 + x\theta(4 + x\theta))]\end{array}}{12(1 + \alpha)(2 + \alpha)} e^{-\theta x}. \tag{2.4}$$

Figure 3 reveals that the cdf plots are approach 1 when $\theta = 3$ (a) faster than that of $\alpha = 1$ (b). In comparing Figure 3(b) with Figure 4(b), it can be observed that the cdf plots are more spread with smaller values of θ. However, the same thing can be concluded in comparing Figure 3(a) with Figure 4(a).



(a) $\theta = 3$ (b) $\alpha = 1$

Figure 3. The cdf of the WGQLD with different parameters values.



(a) $\theta = 1$ (b) $\alpha = 1$

Figure 4. The cdf of the WGQLD with different parameters values.

3. Moments and some related measures

Theorem 1. *: Let $X \sim f_{WGQLD}(x, \theta, \alpha)$, then the rth moment of X about the origin is:*

$$E(X^r) = \frac{\left(r^2 + 6(1+\alpha)(2+\alpha) + r(7+6\alpha)\right)\Gamma(3+r)}{12\theta^r(2+3\alpha+\alpha^2)}, \alpha > -1, \theta > 0,\ r = 1, 2, 3... \tag{3.1}$$

Proof. : The proof is direct by using $E(X^r) = \int_0^\infty x^r f(x, \theta, \alpha)dx$ as

$$\begin{aligned}
E\left(X^r\right) &= \int_0^\infty x^r \frac{\theta^3}{2\left(\alpha^2+3\alpha+2\right)}\left(\alpha^2 x^2 + \alpha\theta x^3 + \theta^2\frac{x^4}{6}\right)\mathrm{e}^{-\theta x}dx \\
&= \frac{\theta^3}{2\left(\alpha^2+3\alpha+2\right)}\int_0^\infty \left(x^r\alpha^2x^2\mathrm{e}^{-\theta x} + x^r\alpha\theta x^3\mathrm{e}^{-\theta x} + x^r\theta^2\frac{x^4}{6}\mathrm{e}^{-\theta x}\right)dx \\
&= \frac{\theta^3}{2\left(\alpha^2+3\alpha+2\right)}\left(\int_0^\infty x^r\alpha^2x^2\mathrm{e}^{-\theta x}dx + \int_0^\infty x^r\alpha\theta x^3\mathrm{e}^{-\theta x}dx + \int_0^\infty x^r\theta^2\frac{x^4}{6}\mathrm{e}^{-\theta x}dx\right) \\
&= \frac{\theta^3}{2\left(\alpha^2+3\alpha+2\right)}\left(\int_0^\infty x^{r+2}\alpha^2\mathrm{e}^{-\theta x}dx + \int_0^\infty x^{r+3}\alpha\theta\mathrm{e}^{-\theta x}dx + \int_0^\infty \frac{1}{6}x^{r+4}\theta^2\mathrm{e}^{-\theta x}dx\right) \\
&= \frac{\theta^3}{2\left(\alpha^2+3\alpha+2\right)}\left(\frac{\alpha^2(r+2)!}{\theta^{r+3}} + \frac{\alpha\theta(r+3)!}{\theta^{r+4}} + \frac{\theta^2(r+3)!}{6\,\theta^{r+5}}\right) \\
&= \frac{\theta^3}{2\left(\alpha^2+3\alpha+2\right)}\left(\frac{\alpha^2(r+2)!}{\theta^{r+3}} + \frac{\alpha(r+3)(r+2)!}{\theta^{r+3}} + \frac{(r+4)(r+3)(r+2)!}{6\,\theta^{r+3}}\right) \\
&= \frac{(r+2)!}{12\theta^r\left(\alpha^2+3\alpha+2\right)}\left(6\alpha^2 + 6\alpha(r+3) + (r+4)(r+3)\right) \\
&= \frac{\left[r^2 + 6\left(1+\alpha\right)\left(2+\alpha\right) + r\left(7+6\alpha\right)\right]\Gamma\left[3+r\right]}{12\theta^r\left(2+3\alpha+\alpha^2\right)}.
\end{aligned}$$

□

Based on Eq 7, it is simple to have the first, second, third and fourth moments of the WGQLD, respectively, as

$$E(X) = \frac{6\left(\alpha+1\right)\left(\alpha+2\right)+6\alpha+8}{2\left(\alpha^2+3\alpha+2\right)\theta} = \frac{3\alpha(\alpha+4)+10}{(\alpha+1)(\alpha+2)\theta}, \tag{3.2}$$

$$E(X^2) = \frac{2\left(2\left(6\alpha+7\right)+6\left(\alpha+1\right)\left(\alpha+2\right)+4\right)}{\left(\alpha^2+3\alpha+2\right)\theta^2} = \frac{12(\alpha(\alpha+5)+5)}{(\alpha+1)(\alpha+2)\theta^2}, \tag{3.3}$$

$$E(X^3) = \frac{10\left(3\left(6\alpha+7\right)+6\left(\alpha+1\right)\left(\alpha+2\right)+9\right)}{\left(\alpha^2+3\alpha+2\right)\theta^3} = \frac{60(\alpha(\alpha+6)+7)}{(\alpha+1)(\alpha+2)\theta^3}, \tag{3.4}$$

$$E(X^4) = \frac{60\left(4\left(6\alpha+7\right)+6\left(\alpha+1\right)\left(\alpha+2\right)+16\right)}{\left(\alpha^2+3\alpha+2\right)\theta^4} = \frac{120(3\alpha(\alpha+7)+28)}{(\alpha+1)(\alpha+2)\theta^4}. \tag{3.5}$$

Therefore, the variance of the WGQLD can be obtained as

$$V(X) = E(X^2) - (E(X))^2 = \frac{3\alpha^4 + 24\alpha^3 + 60\alpha^2 + 60\alpha + 20}{(\alpha+1)^2(\alpha+2)^2\theta^2}. \tag{3.6}$$

The distribution shape analysis can be performed by studying the coefficient of skewness, coefficient of kurtosis, and coefficient of variation. For the WGQLD, these coefficients, respectively, are given by:

$$SK_{WGQLD} = \frac{2\left(3\alpha^6 + 36\alpha^5 + 150\alpha^4 + 306\alpha^3 + 330\alpha^2 + 180\alpha + 40\right)(\alpha+1)(\alpha+2)}{(\alpha+1)(\alpha+2)(3\alpha^4 + 24\alpha^3 + 60\alpha^2 + 60\alpha + 20)^{\frac{3}{2}}}, \tag{3.7}$$

$$Ku_{WGQLD} = 3\frac{\begin{array}{c}15\alpha^8 + 240\alpha^7 + 1496\alpha^6 + 4968\alpha^5 + 9776\alpha^4\\ +11760\alpha^3 + 8480\alpha^2 + 3360\alpha + 560\end{array}}{(3\alpha^4 + 24\alpha^3 + 60\alpha^2 + 60\alpha + 20)^2}, \tag{3.8}$$

$$CV_{WGQLD} = \frac{(\alpha+1)(\alpha+2)\sqrt{3\alpha^4 + 24\alpha^3 + 60\alpha^2 + 60\alpha + 20}}{(3\alpha^2 + 12\alpha + 10)(\alpha+1)(\alpha+2)}. \tag{3.9}$$

It is of interest to note here that the coefficient of skewness, coefficient of variation and coefficient of kurtosis are free of θ. Table (1) presents some values of the mean, standard deviation, coefficient of variation, coefficient of skewness and coefficient of kurtosis of WGQLD for various values of the parameters α and θ.

Table 1. The mean, standard deviation, coefficients of variation, skewness and kurtosis for the WGQLD(θ,α) with different values of α and θ.

	μ_{WGQLD}	σ_{WGQLD}	SK_{WGQLD}	Ku_{WGQLD}	CV_{WGQLD}
α			$\theta = 1.25$		
0.1	3.889177	1.786969	0.897106	4.205054	0.459472
0.2	3.793939	1.782395	0.903213	4.217285	0.469801
0.3	3.711037	1.776240	0.910968	4.233681	0.478637
0.4	3.638095	1.769174	0.919440	4.252427	0.486291
0.5	3.573333	1.761615	0.928126	4.272400	0.492989
0.6	3.515385	1.753829	0.936749	4.292896	0.498901
0.7	3.463181	1.745987	0.945154	4.313464	0.504157
0.8	3.415873	1.738199	0.953260	4.333820	0.508859
0.9	3.372777	1.730537	0.961027	4.353780	0.513090
1	3.333333	1.723046	0.968437	4.373230	0.516914
1.1	3.297081	1.715755	0.975492	4.392101	0.520386
1.2	3.263636	1.708680	0.982196	4.410356	0.523551
1.3	3.232675	1.701828	0.988563	4.427977	0.526446
1.4	3.203922	1.695202	0.994608	4.444961	0.529102
1.5	3.177143	1.688801	1.000346	4.461313	0.531547
θ			$\alpha = 2$		
0.1	38.333333	20.749833	1.025012	4.534152	0.5413
0.2	19.166667	10.374916	1.025012	4.534152	0.5413
0.3	12.777778	6.916611	1.025012	4.534152	0.5413
0.4	9.583333	5.187458	1.025012	4.534152	0.5413
0.5	7.666667	4.149967	1.025012	4.534152	0.5413
0.6	6.388889	3.458305	1.025012	4.534152	0.5413
0.7	5.476190	2.964262	1.025012	4.534152	0.5413
0.8	4.791667	2.593729	1.025012	4.534152	0.5413
0.9	4.259259	2.305537	1.025012	4.534152	0.5413
1	3.833333	2.074983	1.025012	4.534152	0.5413
1.1	3.484848	1.886348	1.025012	4.534152	0.5413
1.2	3.194444	1.729153	1.025012	4.534152	0.5413
1.3	2.948718	1.596141	1.025012	4.534152	0.5413
1.4	2.738095	1.482131	1.025012	4.534152	0.5413
1.5	2.555556	1.383322	1.025012	4.534152	0.5413

It can be noted that the mean and the standard deviation are decreasing when values of α and θ are increasing for fixed values of θ and α, respectively. The results of simulation emphasize that the coefficient of skewness, coefficient of variation and coefficient of kurtosis don't depend on θ and increasing when α values are increasing.

The moment generating function (MGF) of the WGQLD is given in the following theorem.

Theorem 2. *: Let $X \sim f_{WGQLD}(x, \theta, \alpha)$, then the MGF of X about the origin is:*

$$M(t) = \frac{\theta^3(\theta + \alpha\theta - t\alpha)((2 + \alpha)\theta - t\alpha)}{(2 + 3\alpha + \alpha^2)(\theta - t)^5}. \tag{3.10}$$

Proof. : To prove the MGF of the WGQLD, let

$$\begin{aligned}
E\left(e^{tx}\right) &= \int_0^\infty e^{tx} \frac{\theta^3}{2\left(\alpha^2 + 3\alpha + 2\right)} \left(\alpha^2 x^2 + \alpha\theta x^3 + \theta^2 \frac{x^4}{6}\right) \mathrm{e}^{-\theta x} dx \\
&= \frac{\theta^3}{2\left(\alpha^2 + 3\alpha + 2\right)} \left(\int_0^\infty \alpha^2 x^2 \mathrm{e}^{-(\theta - t)x} dx + \int_0^\infty \alpha\theta x^3 \mathrm{e}^{-(\theta - t)x} dx + \int_0^\infty \theta^2 \frac{x^4}{6} \mathrm{e}^{-(\theta - t)x} dx\right) \\
&= \frac{\theta^3}{2\left(\alpha^2 + 3\alpha + 2\right)} \left(\frac{\alpha^2 2!}{(\theta - t)^3} + \frac{\alpha\theta 3!}{(\theta - t)^4} + \frac{\theta^2 4!}{6\,(\theta - t)^5}\right) \\
&= \frac{2\theta^3}{2\left(\alpha^2 + 3\alpha + 2\right)} \left(\frac{\alpha^2}{(\theta - t)^3} + \frac{3\alpha\theta}{(\theta - t)^4} + \frac{2\theta^2}{(\theta - t)^5}\right) \\
&= \frac{\theta^3}{\left(\alpha^2 + 3\alpha + 2\right)(\theta - t)^3} \left(\alpha^2 + \frac{3\alpha\theta}{(\theta - t)} + \frac{2\theta^2}{(\theta - t)^2}\right) \\
&= \frac{\theta^3}{\left(\alpha^2 + 3\alpha + 2\right)(\theta - t)^3} \left(\frac{t^2\alpha^2 - 2t\theta\alpha^2 + \theta^2\alpha^2 - 3t\alpha\theta + 3\alpha\theta^2 + 2\theta^2}{(\theta - t)^2}\right) \\
&= \frac{\theta^3(\alpha\theta - \alpha t + \theta)((\alpha + 2)\theta - \alpha t)}{\left(\alpha^2 + 3\alpha + 2\right)(\theta - t)^5}.
\end{aligned}$$

□

4. Order statistics, median deviations and harmonic mean

4.1. Order statistics

Let $X_1, X_2, \ldots, X_m$ be a random sample of size m from a distribution with pdf $f(x)$ and cdf $F(x)$. The pdf of the ith order statistic $X_{(j:m)}$ for $j = 1, 2, \ldots, m$ are defined by [14] as

$$f_{(j:m)}(x) = \frac{m!}{(j - 1)(m - j)} F(x)]^j [1 - F(x)]^{m-j} f(x). \tag{4.1}$$

Based on Eq (4.1), the pdf of the ith order statistic, $X_{(i:m)}$, from the WGQLD, will be

$$f_{(j:m)}(x) = \frac{\theta^3(C + 1)^{j-1} 2^{2j-2m-1} 3^{j-m} \Gamma(m + 1) e^{-\theta x} A^{m-j} \left(\frac{\theta^2 x^4}{6} + \alpha\theta x^3 + \alpha^2 x^2\right)}{(\alpha + 1)(\alpha + 2)\Gamma(j)\Gamma(-j + m + 1)}, \tag{4.2}$$

where

$$A = \frac{e^{-\theta x}\left(12(\alpha + 1)(\alpha + 2) + \theta^4 x^4 + 2(3\alpha + 2)\theta^3 x^3 + 6(\alpha + 1)(\alpha + 2)\theta^2 x^2 + 12(\alpha + 1)(\alpha + 2)\theta x\right)}{(\alpha + 1)(\alpha + 2)},$$

and

$$C = \frac{e^{-\theta x}\left(-6\alpha^2(\theta x(\theta x+2)+2)-6\alpha(\theta x(\theta x(\theta x+3)+6)+6)-\theta x(\theta x(\theta x(\theta x+4)+12)+24)-24\right)}{12(\alpha+1)(\alpha+2)}.$$

Based on Eq (18), the pdf of the minimum and maximum order statistics, respectively, are given by

$$f_{(1:m)}(x) = \frac{\theta^3 2^{1-2m} 3^{1-m} m e^{-\theta m x}\left(\frac{\theta^2 x^4}{6}+\alpha\theta x^3+\alpha^2 x^2\right)\left[-\left(-6\alpha^2(\theta x(\theta x+2)+2)-B\right)\right]^{m-1}}{(\alpha^2+3\alpha+2)^m},$$

where $B = \{6\alpha(\theta x(\theta x(\theta x+3)+6)+6)+\theta x(\theta x(\theta x(\theta x+4)+12)+24)+24\}$, and

$$f_{(m:m)}(x) = \frac{\theta^3 m e^{-\theta x}\left(\frac{\theta^2 x^4}{6}+\alpha\theta x^3+\alpha^2 x^2\right)\left(\frac{e^{-\theta x}D}{12(\alpha+1)(\alpha+2)}+1\right)^{m-1}}{2\left(\alpha^2+3\alpha+2\right)},$$

where

$$D = -6\alpha^2(\theta x(\theta x+2)+2)-6\alpha(\theta x(\theta x(\theta x+3)+6)+6) \\ -\theta x(\theta x(\theta x(\theta x+4)+12)+24)-24$$

.

4.2. Median deviations

To measure the scatter in the population, the mean deviation about the median denoted by $\mathbb{MD}(x)$ can be used, where

$$\mathbb{MD}(x) = \int_0^\infty |x-M| f(x)dx = \mu - 2\int_0^M x f(x)dx, \tag{4.3}$$

where M is the population median. The mean deviation about the median for the WGQL distribution is defined as:

$$\mathbb{MD} = \frac{\begin{pmatrix} 6\alpha^2(\theta M(\theta M(\theta M+3)+6)+6)-6(3\alpha(\alpha+4)+10)\mathrm{e}^{\theta M} \\ +6\alpha(\theta M(\theta M(\theta M(\theta M+4)+12)+24)+24) \\ +\theta M(\theta M(\theta M(\theta M(\theta M+5)+20)+60)+120)+120 \end{pmatrix}}{6(\alpha+1)(\alpha+2)\theta}\mathrm{e}^{-\theta M}. \tag{4.4}$$

4.3. Harmonic mean

Let $X \sim f_{WGQLD}(x,\theta,\alpha)$, then

$$\mathbb{E}\left(\frac{1}{X}\right) = \frac{(\alpha+1)^2\theta}{2(\alpha^2+3\alpha+2)}.$$

Therefore, the harmonic mean of the WGQLD is given by

$$HM(\theta,\alpha) = \frac{2(\alpha^2+3\alpha+2)}{(\alpha+1)^2\theta}. \tag{4.5}$$

Table 2 presents some values of the harmonic mean of WGQLD for various values of the parameters θ and α.

Table 2. Harmonic mean of the WGQLD for selected values of (α, θ).

α	$HM(5,\alpha)$	α	$HM(5,\alpha)$	α	$HM(5,\alpha)$	θ	$HM(\theta,2)$	θ	$HM(\theta,2)$	θ	$HM(\theta,2)$
1	0.6000	16	0.4235	31	0.4125	1	2.6666	16	0.1666	31	0.0860
2	0.5333	17	0.4222	32	0.4121	2	1.3333	17	0.1568	32	0.0833
3	0.5000	18	0.4210	33	0.4117	3	0.8888	18	0.1481	33	0.0808
4	0.4800	19	0.4200	34	0.4114	4	0.6666	19	0.1403	34	0.0784
5	0.4666	20	0.4190	35	0.4111	5	0.5333	20	0.1333	35	0.0761
6	0.4571	21	0.4181	36	0.4108	6	0.4444	21	0.1269	36	0.0740
7	0.4500	22	0.4173	37	0.4105	7	0.3809	22	0.1212	37	0.0720
8	0.4444	23	0.4166	38	0.4102	8	0.3333	23	0.1159	38	0.0701
9	0.4400	24	0.4160	39	0.4100	9	0.2962	24	0.1111	39	0.0683
10	0.4363	25	0.4153	40	0.4097	10	0.2666	25	0.1066	40	0.0666
11	0.4333	26	0.4148	41	0.4095	11	0.2424	26	0.1025	41	0.0650
12	0.4307	27	0.4142	42	0.4093	12	0.2222	27	0.0987	42	0.0634
13	0.4285	28	0.4137	43	0.4090	13	0.2051	28	0.0952	43	0.0620
14	0.4266	29	0.4133	44	0.4088	14	0.1904	29	0.0919	44	0.0606
15	0.4250	30	0.4129	45	0.4086	15	0.1777	30	0.0888	45	0.0592

Based on Table 2, we can conclude that: the harmonic mean values are decreasing in α when $\theta = 5$ and are decreasing in θ when $\alpha = 2$. Also, in general the harmonic mean values when $\theta < \alpha$ are larger than the case of $\theta > \alpha$.

5. Stochastic ordering and reliability analysis

5.1. Stochastic ordering

The stochastic ordering is an important tool in finance and reliability theory to evaluate the comparative behaviour of the models or systems. Let the random variables X and Y having the probability density functions, cumulative distribution functions and survival functions $f(x), f(y), F(x), F(y), \overline{F}(x) = 1 - F(x)$, and $\overline{F}(y) = 1 - F(y)$, respectively, then $X \le Y$ in:

1. Mean residual life order denoted by $X \le_{MRLO} Y$, if $m_X(x) \le m_Y(x)$ for all x,
2. Hazard rate order denoted by $X \le_{HRO} Y$, if $\overline{F}_X(x)/\overline{F}_Y(x)$ is decreasing in $x \ge 0$,
3. Stochastic order denoted by $X \le_{SO} Y$, if $\overline{F}_X(x) \le\le \overline{F}_Y(x)$ for all x,
4. Likelihood ratio order denoted by $X \le_{LRO} Y$, if $\dfrac{fx(x)}{f_Y(x)}$ is decreasing in $x \ge 0$.

All these stochastic orders defined above are related to each other and [29] showed the following relation is hold.

$$\begin{array}{ccccc} X \le_{LRO} Y \Rightarrow & X \le_{HRO} Y \Rightarrow & X \le_{MRLO} Y. \\ & \Downarrow & \\ & X \le_{SO} Y. & \end{array}$$

Theorem 3. *: Let the random variables X and Y be independent follow the pdf $f_X(x,\theta,\alpha)$ and $f_Y(x,\mu,\omega)$, respectively. If $\theta > \mu$ and $\alpha > \omega$, then $X \leq_{LRO} Y, X \leq_{HRO} Y, X \leq_{MRLO} Y$ and $X \leq_{SO} Y$.*

Proof: Let $X \sim f_X(x,\theta,\alpha)$, $Y \sim f_Y(x,\mu,\omega)$, then

$$\frac{f_X(x,\theta,\alpha)}{f_Y(x,\mu,\omega)} = \frac{\frac{\theta^3\left(\frac{\theta^2x^4}{6}+\alpha\theta x^3+\alpha^2x^2\right)e^{-\theta x}}{2\left(\alpha^2+3\alpha+2\right)}}{\frac{\mu^3\left(\frac{\mu^2x^4}{6}+\omega\mu x^3+\omega^2x^2\right)e^{-\mu x}}{2\left(\omega^2+3\omega+2\right)}},$$

and

$$\begin{aligned}\log\left(\frac{f_X(x,\theta,\alpha)}{f_Y(x,\mu,\omega)}\right) &= \log\left(\frac{\frac{\theta^3\left(\frac{\theta^2x^4}{6}+\alpha\theta x^3+\alpha^2x^2\right)e^{-\theta x}}{2\left(\alpha^2+3\alpha+2\right)}}{\frac{\mu^3\left(\frac{\mu^2x^4}{6}+\omega\mu x^3+\omega^2x^2\right)e^{-\mu x}}{2\left(\omega^2+3\omega+2\right)}}\right),\\ &= \log\left(\frac{\theta^3(\omega^2+3\omega+2)}{\mu^3(\alpha^2+3\alpha+2)}\left(\frac{\frac{\theta^2x^4}{6}+\alpha\theta x^3+\alpha^2x^2}{\frac{\mu^2x^4}{6}+\omega\mu x^3+\omega^2x^2}\right)\exp((\mu-\theta)x)\right),\\ &= \log\left(\frac{\theta^3(\omega^2+3\omega+2)}{\mu^3(\alpha^2+3\alpha+2)}\right)+\log\left(\frac{\frac{\theta^2x^4}{6}+\alpha\theta x^3+\alpha^2x^2}{\frac{\mu^2x^4}{6}+\omega\mu x^3+\omega^2x^2}\right)+(\mu-\theta)x.\end{aligned}$$

Taking the derivative of the last equation with respect to x yields

$$\frac{d}{dx}\log\left(\frac{f_X(x,\theta,\alpha)}{f_Y(x,\mu,\omega)}\right) = \frac{2\theta^2x+6\alpha\theta}{\theta^2x^2+6\alpha\theta x+6\alpha^2}-\frac{2\mu^2x+6\mu\omega}{\mu^2x^2+6\mu\omega x+6\omega^2}+\mu-\theta.$$

Hence, if $\theta > \mu$, $\alpha > \omega$, then $\frac{d}{dx}\log\left(\frac{f_X(x,\theta,\alpha)}{f_Y(x,\mu,\omega)}\right) < 0$. Therefore, $X \leq_{LRO} Y$, $X \leq_{HRO} Y$, $X \leq_{MRLO} Y$ and $X \leq_{SO} Y$.

5.2. Reliability analysis

The corresponding reliability and hazard functions of the WGQLD distribution are given, respectively by:

$$\begin{aligned}R_{WGQLD}(x;\theta,\ \alpha) &= 1-F_{WGQLD}(x;\theta,\ \alpha)\\ &= \frac{\left(\begin{array}{c}24+6\alpha^2[2+x\theta(2+x\theta)]\\ +6\alpha[6+x\theta(6+x\theta(3+x\theta))]+x\theta[24+x\theta(12+x\theta(4+x\theta))]\end{array}\right)}{12(1+\alpha)(2+\alpha)}e^{-\theta x},\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}H_{WGQLD}(x;\theta,\alpha) &= \frac{f_{WGQLD}(x;\theta,\alpha)}{1-F_{WGQLD}(x;\theta,\alpha)}\\ &= \frac{\theta^3\left(\frac{\theta^2x^4}{6}+\alpha\theta x^3+\alpha^2x^2\right)}{\begin{array}{c}24+6\alpha^2[2+x\theta(2+x\theta)]+6\alpha[6+x\theta(6+x\theta(3+x\theta))]\\ +x\theta[24+x\theta(12+x\theta(4+x\theta))]\end{array}}.\end{aligned} \tag{5.2}$$

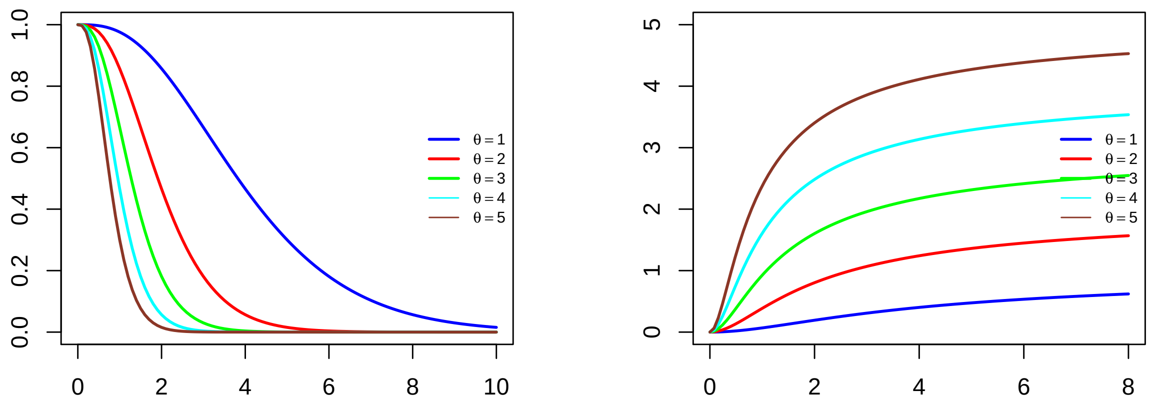
Figure 5, reveals that the reliability of the WGQLD are decreasing while the hazard rate function is increasing for $\alpha = 1, 2, \ldots, 5$. The reversed hazard rate and odds functions for the WGQLD distribution, respectively, are defined as

$$\begin{aligned} RH_{WGQLD}(x;\theta,\alpha) &= \frac{f_{WGQLD}(x;\theta,\alpha)}{F_{WGQLD}(x;\theta,\alpha)} \\ &= \frac{\theta^3 x^2 \left(\theta^2 x^2 + 6\alpha\theta x + 6\alpha^2\right)}{\begin{array}{c} 12\,(\alpha+1)\,(\alpha+2)\,\mathrm{e}^{\theta x} - 12\,(\alpha+1)\,(\alpha+2) \\ -\theta x\,(\theta x\,(\theta x\,(\theta x + 6\alpha + 4) + 6\,(\alpha+1)\,(\alpha+2)) + 12\,(\alpha+1)\,(\alpha+2)) \end{array}}, \end{aligned}$$

and

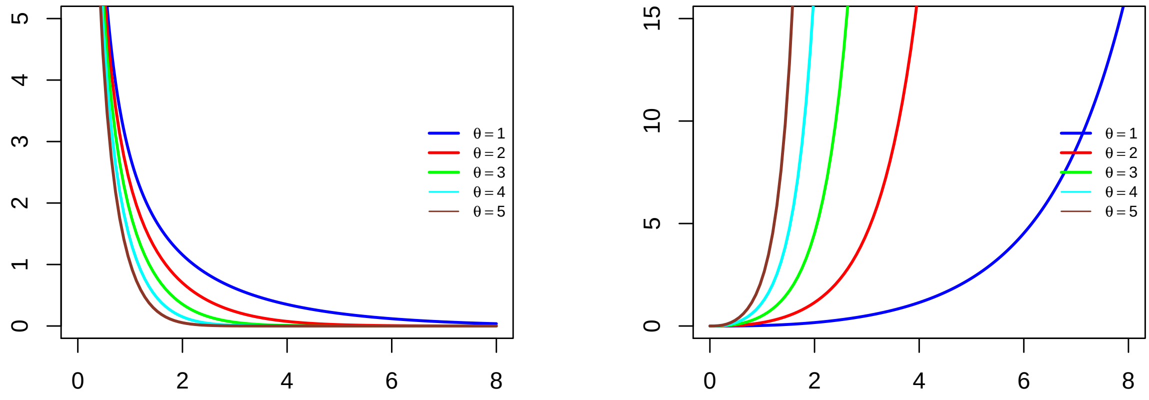
$$\begin{aligned} O_{WGQLD}(x;\theta,\alpha) &= \frac{F_{WGQLD}(x;\theta,\alpha)}{1 - F_{WGQLD}(x;\theta,\alpha)} \\ &= \frac{12\,(\alpha+1)\,(\alpha+2)\,\mathrm{e}^{\theta x}}{\begin{array}{c} \theta^4 x^{+}4\theta^3 x^3 + 12\theta^2 x^2 + 24\theta x + 6\alpha\theta^3 x^3 + 18\alpha\theta^2 x^2 \\ +36\alpha\theta x + 36\alpha + 6\alpha^2\theta^2 x^2 + 12\alpha^2\theta x + 12\alpha^2 + 24 \end{array}} - 1. \end{aligned}$$

As it is shown in Figure 6, the reversed hazard function is decreasing taking inverse J shape while the odds functions is increasing with J shape.



(a) (b)

Figure 5. Reliability function (a) and hazard rate function (b) of the WGQLD for various values of θ when $\alpha = 1$.



(a) (b)

Figure 6. The reversed hazard rate (a) and the odds functions (b) of the WGQLD for various values of θ when $\alpha = 1$.

6. Bonferroni and Lorenz curves and Gini index

Assume that the random variable X is a non-negative with continuous and twice differentiable cumulative distribution function $F(x)$. The Bonferroni curve of the random variable X is defined as

$$BC = \frac{1}{p\mu}\left(\int_0^\infty xf(x)dx - \int_q^\infty xf(x)dx\right) = \frac{1}{p\mu}\left(\mu - \int_q^\infty xf(x)dx\right), \tag{6.1}$$

where $q = F^{-1}(p)$ and $p \in (0, 1]$. The Lorenz curve is defined as

$$LC = \frac{1}{\mu}\left(\int_0^\infty xf(x)dx - \int_q^\infty xf(x)dx\right) = \frac{1}{\mu}\left(\mu - \int_q^\infty xf(x)dx\right). \tag{6.2}$$

The Gini index is given by

$$GI = 1 - \frac{1}{\mu}\int_0^\infty (1 - F(x))^2 dx = \frac{1}{\mu}\int_0^\infty F(x)(1 - F(x))dx. \tag{6.3}$$

Now, for the WGQL distribution, the Bonferroni curve, Lorenz curve and Gini index are given, respectively, as

$$BC = \frac{\begin{pmatrix} 12e^{q\theta}(10 + 3\alpha(4 + \alpha)) - 120 - 6\alpha^2(6 + q\theta[6 + q\theta(3 + q\theta)]) \\ -6\alpha(24 + q\theta(24 + q\theta[12 + q\theta(4 + q\theta)])) \\ -q\theta(120 + q\theta[60 + q\theta(20 + q\theta(5 + q\theta))]) \end{pmatrix}}{12p(10 + 3\alpha(4 + \alpha))} e^{-q\theta}, \tag{6.4}$$

$$LC = \frac{\begin{pmatrix} -120 + 12e^{q\theta}(10 + 3\alpha(4 + \alpha)) - 6\alpha^2(6 + q\theta(6 + q\theta(3 + q\theta))) \\ -6\alpha(24 + q\theta(24 + q\theta(12 + q\theta(4 + q\theta)))) \\ -q\theta(120 + q\theta(60 + q\theta(20 + q\theta(5 + q\theta)))) \end{pmatrix}}{12(10 + 3\alpha(4 + \alpha))} e^{-q\theta}, \tag{6.5}$$

$$GI = \frac{5(63 + \alpha[189 + 4\alpha(49 + 3\alpha(7 + \alpha))])}{64(1 + \alpha)(2 + \alpha)[10 + 3\alpha(4 + \alpha)]}. \tag{6.6}$$

Table 3 presents some values of the Gini Index for WGQLD for different values of α

Table 3. Gini index values for the WGQLD for selected values of α.

α	$GI(\alpha)$	α	$GI(\alpha)$	α	$GI(\alpha)$
1	0.2833333	16	0.3115732	31	0.3122193
2	0.2956578	17	0.3116666	32	0.3122354
3	0.3014769	18	0.3117464	33	0.3122502
4	0.3047121	19	0.3118154	34	0.3122637
5	0.3067016	20	0.3118753	35	0.3122762
6	0.3080137	21	0.3119276	36	0.3122877
7	0.3089251	22	0.3119736	37	0.3122984
8	0.3095841	23	0.3120143	38	0.3123082
9	0.3100762	24	0.3120505	39	0.3123174
10	0.3104533	25	0.3120828	40	0.3123259
11	0.3107487	26	0.3121117	41	0.3123339
12	0.3109844	27	0.3121377	42	0.3123413
13	0.3111755	28	0.3121611	43	0.3123482
14	0.3113326	29	0.3121824	44	0.3123547
15	0.3114633	30	0.3122017	45	0.3123607

Table 3 explains that the Gini index values for the WGQL distribution are increasing as α values are increasing, and on the average its value is about 0.31.

7. Methods of estimation

In this section, we consider six methods of estimation for estimating the unknowns parameters α and θ of the WGQLD distribution. These methods include the (1) maximum likelihood (ML) method, (2) method of maximum product of spacings, (3) ordinary least square method, (4) weight least square method, (5) method of Cramer-Von-Mises, and (6) method of Anderson-Darling.

7.1. Maximum likelihood estimation

First of all, we investigate the ML estimates (MLEs) of θ and α. Let $X_1, X_2 \ldots, X_n$ be a random sample of size n selected from the WGQLD. The likelihood function is given by:

$$L(x;\theta,\ \alpha) = \prod_{i=1}^{n} f(x_i,\ \theta,\ \alpha) = \left(\frac{\theta^3}{2\left(\alpha^2 + 3\alpha + 2\right)}\right)^n \prod_{i=1}^{n} \left(\alpha^2 x_i^2 + \alpha\theta x_i^3 + \theta^2 \frac{x_i^4}{6}\right) e^{-\theta x_i}, \tag{7.1}$$

and the log-likelihood function $\Xi = \ln L\,(x;\theta,\ \alpha)$ is:

$$\Xi = n\ln\left(\frac{\theta^3}{2\,(\alpha^2+3\alpha+2)}\right)+\sum_{i=1}^{n}\ln\left(\alpha^2x_i^2+\alpha\theta x_i^3+\theta^2\frac{x_i^4}{6}\right)-\theta\sum_{i=1}^{n}x_i. \tag{7.2}$$

The derivatives of Ξ with respect to θ and α are:

$$\frac{d\Xi}{d\theta}=\frac{3n}{\theta}+\sum_{i=1}^{n}\frac{2x_i\,(x_i\theta+3\alpha)}{x_i^2\theta^2+6\alpha x_i\theta+6\alpha^2}-\sum_{i=1}^{n}x_i, \tag{7.3}$$

$$\frac{d\Xi}{d\alpha}=-\frac{n\,(2\alpha+3)}{\alpha^2+3\alpha+2}+\sum_{i=1}^{n}\frac{12\alpha+6\theta x_i}{6\alpha^2+6\theta x_i\alpha+\theta^2x_i^2}. \tag{7.4}$$

Since there is no closed form for these equations, then the MLEs $\hat{\theta}$ and $\hat{\alpha}$ of θ and α, respectively, can be solved simultaneously using a numerical method as Newton Raphson method.

7.2. Method of maximum product of spacings

The maximum product of spacing (MPS) method is proposed by [11, 12] as an alternative to the maximum likelihood method. The MPS method requires a maximization of the geometric mean of the spacings in the data with respect to the parameters. Consider a random sample of size n, $X_1, X_2, \ldots, X_n$ from WGQLD distribution, then uniform spacings is given as:

$$\Upsilon_i(\theta,\alpha)=F(x_{i:n}|\theta,\alpha)-F(x_{i-1:n}|\theta,\alpha),\, i=1,\ldots,n,$$

where $F(x_{0:n}|\theta,\alpha)=0$ and $F(x_{n+1:n}|\theta,\alpha)=1$. Clearly $\sum_{i=1}^{n+1}\Upsilon_i(\theta,\alpha)=1$.

The MPSs, $\hat{\alpha}_{MPS}$ and $\hat{\theta}_{MPS}$, are the values of α and θ, which maximize the geometric mean of the spacing:

$$Z(\theta,\alpha|\mathrm{x})=\left[\prod_{i=1}^{n+1}\Upsilon_i(\theta,\alpha)\right]^{\frac{1}{n+1}}. \tag{7.5}$$

The natural logarithm of (7.5) is:

$$H(\theta,\alpha|\mathrm{x})=\frac{1}{n+1}\sum_{i=1}^{n+1}\log\Upsilon_i(\theta,\alpha).$$

The MPSs estimators $\hat{\alpha}_{MPS}$ and $\hat{\theta}_{MPS}$ of the parameters α and θ, respectively, can also be obtained by solving the nonlinear equations:

$$\frac{\partial}{\partial\theta}H(\theta,\alpha)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{\Upsilon_i(\theta,\alpha)}\left[\Delta_1(x_{i:n}|\theta,\alpha)-\Delta_1(x_{i-1:n}|\theta,\alpha)\right]=0,$$

$$\frac{\partial}{\partial\alpha}H(\theta,\alpha)=\frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{\Upsilon_i(\theta,\alpha)}\left[\Delta_2(x_{i:n}|\theta,\alpha)-\Delta_2(x_{i-1:n}|\theta,\alpha)\right]=0,$$

where

$$\Delta_1(x_{i:n}|\theta,\alpha) = \frac{\partial}{\partial\theta}F(x_{i:n}|\theta,\alpha) = \frac{\theta^3 x_{i:n}^3 \mathrm{e}^{-\theta x_{i:n}}\left(6\alpha^2 + (2\theta x_{i:n} + 8)\,\alpha + 3\theta x_{i:n}\right)}{12\,(\alpha+1)^2\,(\alpha+2)^2}, \tag{7.6}$$

$$\Delta_2(x_{i:n}|\theta,\alpha) = \frac{\partial}{\partial\alpha}F(x_{i:n}|\theta,\alpha) = \frac{x_{i:n}^3\theta^2\left(x_{i:n}^2\theta^2 + 6\alpha x_{i:n}\theta + 6\alpha^2\right)\mathrm{e}^{-x_{i:n}\theta}}{12\,(\alpha+1)\,(\alpha+2)}, \tag{7.7}$$

which can be obtained numerically.

7.3. Methods of least squares

The least square methods are introduced by [28] to estimate the parameters of beta distribution. Let $X_{i:n}$ be the ith order statistic of the random sample $X_1, X_2, \ldots, X_n$ with distribution function $F(x)$, then a main result in probability theory indicates that $F(X_{i:n}) \sim Beta(i, n-i+1)$. Moreover, we have $E[F(X_{i:n})] = \dfrac{i}{n+1}$ and $\mathrm{Var}[F(X_{i:n})] = \dfrac{i(n-i+1)}{(n+1)^2(n+2)}$.
Using the expectations and variances, we obtain two variants of the least squares methods.

7.3.1. Ordinary least squares

For the WGQLD distribution parameters estimation, the ordinary least square estimators $\hat{\theta}_{OLS}$ and $\hat{\alpha}_{OLS}$ of the parameters θ and α, respectively can be obtained by minimizing the function:

$$\begin{aligned}
\Omega(\theta,\alpha|\mathrm{x}) &= \sum_{i=1}^{n}\left[F(x_{i:n}|\theta,\alpha) - \frac{i}{n+1}\right]^2 \\
&= \sum_{i=1}^{n}\left[1 - \frac{\begin{pmatrix} 24 + 6\alpha^2[2 + x_{i:n}\theta(2 + x_{i:n}\theta)] \\ +6\alpha[6 + x_{i:n}\theta(6 + x_{i:n}\theta(3 + x_{i:n}\theta))] \\ +x\theta[24 + x_{i:n}\theta(12 + x_{i:n}\theta(4 + x_{i:n}\theta))] \end{pmatrix}}{12(1+\alpha)(2+\alpha)}\mathrm{e}^{-\theta x_{i:n}} - \frac{i}{n+1}\right]^2,
\end{aligned}$$

with respect to θ and α. Alternatively, these estimates can also be obtained by solving the following nonlinear equations:

$$\sum_{i=1}^{n}\left[F(x_{i:n}|\theta,\alpha) - \frac{i}{n+1}\right]\Delta_1(x_{i:n}|\theta,\alpha) = 0,$$

$$\sum_{i=1}^{n}\left[F(x_{i:n}|\theta,\alpha) - \frac{i}{n+1}\right]\Delta_2(x_{i:n}|\theta,\alpha) = 0,$$

where $\Delta_1(x_{i:n}|\theta,\alpha)$ and $\Delta_2(x_{i:n}|\theta,\alpha)$ are defined as in 7.6 and 7.7, respectively.

7.3.2. Weighted least squares

For the WGQLD distribution, the weighted least square estimators of θ and α say, $\hat{\theta}_{WLS}$ and $\hat{\alpha}_{WLS}$, respectively can be obtained by minimizing the function:

$$
\begin{aligned}
W(\theta, \alpha|\mathrm{x}) &= \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}|\theta,\alpha) - \frac{i}{n+1}\right]^2 \\
&= \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \\
&\left[1 - \frac{\begin{pmatrix} 24 + 6\alpha^2[2 + x_{i:n}\theta(2 + x_{i:n}\theta)] \\ +6\alpha[6 + x_{i:n}\theta(6 + x_{i:n}\theta(3 + x_{i:n}\theta))] \\ +x\theta[24 + x_{i:n}\theta(12 + x_{i:n}\theta(4 + x_{i:n}\theta))] \end{pmatrix}}{12(1+\alpha)(2+\alpha)} \mathrm{e}^{-\theta x_{i:n}} - \frac{i}{n+1}\right]^2,
\end{aligned}
$$

with respect to θ and α. Equivalently, these estimators are the solution of the following nonlinear equations:

$$
\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}|\theta,\alpha) - \frac{i}{n+1}\right] \Delta_1(x_{i:n}|\theta,\alpha) = 0,
$$

$$
\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}|\theta,\alpha) - \frac{i}{n+1}\right] \Delta_2(x_{i:n}|\theta,\alpha) = 0,
$$

where $\Delta_1(x_{i:n}|\theta,\alpha)$ and $\Delta_2(x_{i:n}|\theta,\alpha)$ are specified as in 7.6 and 7.7, respectively.

7.4. Methods of minimum distances

Here, we use two popular methods based on the minimization of test statistics between the theoretical and empirical cumulative distribution functions. The methods are Cramer-von-Mises method and the method of Anderson-Darling (for more details see [13] and [24]).

7.4.1. Cramer-von-Mises method

The Cramer-von-Mises estimators (CVEs) $\hat{\theta}$ and $\hat{\alpha}$ of θ and α respectively, are obtained by minimizing the following function:

$$
\begin{aligned}
CV(\theta, \alpha) &= \frac{1}{12n} + \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{(i:n)};\theta,\alpha) - \frac{2i-1}{2n}\right]^2 \\
&= \frac{1}{12n} + \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \\
&\left[1 - \frac{\begin{pmatrix} 24 + 6\alpha^2[2 + x_{i:n}\theta(2 + x_{i:n}\theta)] \\ +6\alpha[6 + x_{i:n}\theta(6 + x_{i:n}\theta(3 + x_{i:n}\theta))] \\ +x\theta[24 + x_{i:n}\theta(12 + x_{i:n}\theta(4 + x_{i:n}\theta))] \end{pmatrix}}{12(1+\alpha)(2+\alpha)} \mathrm{e}^{-\theta x_{i:n}} - \frac{2i-1}{2n}\right]^2,
\end{aligned}
$$

with respect to θ and α. Equivalently, these estimators are the solution of the following nonlinear equations:

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}|\theta, \alpha) - \frac{2i-1}{2n} \right] \Delta_1(x_{i:n}|\theta, \alpha) = 0,$$

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}|\theta, \alpha) - \frac{2i-1}{2n} \right] \Delta_2(x_{i:n}|\theta, \alpha) = 0,$$

where $\Delta_1(x_{i:n}|\theta, \alpha)$ and $\Delta_2(x_{i:n}|\theta, \alpha)$ are given in 7.6 and 7.7, respectively.

7.4.2. Method of Anderson-Darling

The Anderson-Darling (AD) estimates of the WGQLD distribution parameters θ and α denoted by $\hat{\theta}_{AD}$ and $\hat{\alpha}_{AD}$ can be obtained by minimizing the following function

$$A(\alpha,\ \theta) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \{ \log\ F(x_{i:n}|\alpha,\ \theta) + \log \overline{F}(x_{n-i+1:n}|\alpha,\ \theta) \},$$

with respect to θ and α, or by solving the following two equations

$$\frac{\partial A(\alpha, \lambda)}{\partial \lambda} = \sum_{i=1}^{n} (2i-1) \left\{ \frac{\Delta_1(x_{i:n}|\alpha, \lambda)}{F(x_{i:n}|\alpha, \lambda)} - \frac{\Delta_1(x_{n-i+1:n}|\alpha, \lambda)}{\overline{F}(x_{n-i+1:n}|\alpha, \lambda)} \right\} = 0,$$

and

$$\frac{\partial A(\alpha, \lambda)}{\partial \alpha} = \sum_{i=1}^{n} (2i-1) \left\{ \frac{\Delta_2(x_{i:n}|\alpha, \lambda)}{F(x_{i:n}|\alpha, \lambda)} - \frac{\Delta_2(x_{n-i+1:n}|\alpha, \lambda)}{\overline{F}(x_{n-i+1:n}|\alpha, \lambda)} \right\} = 0,$$

where $\Delta_1(x_{i:n}|\theta, \alpha)$ and $\Delta_2(x_{i:n}|\theta, \alpha)$ are specified in 7.6 and 7.7, respectively.

8. Simulation

This section compares the performances of the proposed estimators of the WGQLD parameters α and θ. This comparison is carried out by taking random samples of different sizes (n = 20, 40, 60, 80, 100 and 200) with various pairs of parameters values (θ, α) = (0.25,1), (0.5,1.5), (0.75,2), (1,1), (0.3,2), (0.2,3), (0.8,1), (1,3). The estimators are compared in terms of there mean square errors (MSE) and the estimated (Es) values of the parameters. The results are summarized in the Tables 4–7.

Table 4. Estimates and MSEs using the ML, MPS, OLS, WLS, AD and CV methods for the WGQLD model for n=20.

Parameters	MLEs		MPS		OLS		WLS		CVEs		AD	
	Es	MSE	Es	MSE	Es	MSE	Es	MSE	Es	MSE	Es	MSE
α=0.25	0.426	0.3075	0.377	0.2919	0.486	0.4311	0.472	0.3890	0.456	0.3717	0.463	0.3680
θ=1	1.010	0.0118	1.026	0.0129	1.006	0.0134	1.005	0.0127	1.008	0.0127	1.005	0.0121
α=0.5	0.781	1.0029	0.718	1.0197	0.835	1.1394	0.822	1.0732	0.804	1.0553	0.811	1.0250
θ=1.5	1.523	0.0288	1.550	0.0325	1.516	0.0335	1.515	0.0319	1.521	0.0321	1.515	0.0304
α=0.75	1.133	1.8082	1.075	1.9053	1.196	2.0488	1.177	1.9076	1.157	1.8866	1.167	1.8744
θ=2	2.031	0.0562	2.068	0.0626	2.018	0.0639	2.017	0.0604	2.025	0.0607	2.017	0.0583
α=1	1.485	2.5509	1.429	2.6727	1.552	2.7457	1.530	2.6005	1.511	2.5964	1.514	2.528
θ=1	1.016	0.0140	1.036	0.0159	1.012	0.0165	1.010	0.0156	1.015	0.0156	1.010	0.0148
α=0.3	0.496	0.4830	0.441	0.4607	0.546	0.5756	0.534	0.5275	0.518	0.5111	0.527	0.5156
θ=2	2.021	0.0487	2.053	0.0538	2.010	0.0551	2.008	0.0527	2.015	0.0528	2.009	0.0509
α=0.2	0.412	0.3789	0.368	0.3773	0.464	0.5029	0.455	0.4562	0.440	0.4340	0.445	0.4371
θ=3	3.046	0.1080	3.094	0.1213	3.034	0.1251	3.033	0.1202	3.044	0.1202	3.033	0.1134
α=0.8	1.202	2.0044	1.148	2.1251	1.266	2.2173	1.248	2.0845	1.228	2.0673	1.237	2.0439
θ=1	1.014	0.0141	1.034	0.0161	1.009	0.0166	1.009	0.0157	1.013	0.0157	1.009	0.0151
α=1	1.480	2.6725	1.429	2.8627	1.546	2.9849	1.519	2.8085	1.502	2.7973	1.511	2.7333
θ=3	3.043	0.1308	3.103	0.1497	3.027	0.154	3.023	0.1462	3.036	0.1464	3.024	0.1392

Table 5. Estimates and MSEs using the ML, MPS, OLS, WLS, AD and CV methods for the WGQLD model for n=50.

Parameters	MLEs		MPS		OLS		WLS		CVEs		AD	
	Es	MSE	Es	MSE	Es	MSE	Es	MSE	Es	MSE	Es	MSE
α=0.25	0.315	0.0771	0.283	0.0674	0.335	0.0949	0.331	0.0878	0.323	0.0849	0.329	0.0868
θ=1	1.001	0.0047	1.010	0.0049	0.999	0.0053	0.999	0.0050	1.001	0.0050	0.999	0.0049
α=0.5	0.576	0.1816	0.534	0.1741	0.604	0.2142	0.596	0.1981	0.586	0.1943	0.595	0.1965
θ=1.5	1.505	0.0113	1.520	0.0120	1.504	0.0130	1.503	0.0123	1.506	0.0123	1.503	0.0121
α=0.75	0.875	0.4170	0.829	0.4199	0.902	0.4771	0.894	0.4451	0.884	0.4409	0.892	0.4370
θ=2	2.012	0.0217	2.033	0.0231	2.007	0.0256	2.007	0.0238	2.011	0.0239	2.007	0.0234
α=1	1.145	0.5971	1.101	0.6059	1.174	0.6490	1.163	0.6149	1.153	0.6132	1.162	0.6064
θ=1	1.004	0.0051	1.014	0.0054	1.002	0.0058	1.002	0.0055	1.004	0.0055	1.002	0.0054
α=0.3	0.363	0.0849	0.328	0.0754	0.389	0.1047	0.383	0.0972	0.375	0.0941	0.381	0.0957
θ=2	2.007	0.0180	2.025	0.0192	2.003	0.0211	2.003	0.0198	2.007	0.0198	2.003	0.0195
α=0.2	0.281	0.0654	0.252	0.0560	0.306	0.0862	0.300	0.0780	0.293	0.0748	0.298	0.0777
θ=3	3.015	0.0415	3.040	0.0441	3.010	0.0479	3.010	0.0452	3.015	0.0453	3.009	0.0444
α=0.8	0.935	0.4789	0.891	0.4883	0.958	0.5383	0.953	0.5067	0.943	0.5026	0.950	0.5011
θ=1	1.005	0.0054	1.016	0.0058	1.003	0.0062	1.003	0.0059	1.005	0.0059	1.003	0.0058
α=1	1.157	0.7315	1.115	0.7467	1.187	0.8219	1.174	0.7597	1.164	0.7562	1.175	0.7566
θ=3	3.010	0.0528	3.042	0.0561	3.004	0.0615	3.003	0.0577	3.009	0.0578	3.003	0.0570

Table 6. Estimates and MSEs using the ML, MPS, OLS, WLS, AD and CV methods for the WGQLD model for n=100.

Parameters	MLEs		MPS		OLS		WLS		CVEs		AD	
	Es	MSE	Es	MSE	Es	MSE	Es	MSE	Es	MSE	Es	MSE
α=0.25	0.287	0.0343	0.265	0.0312	0.3042	0.0433	0.298	0.0393	0.294	0.0383	0.299	0.0395
θ=1	1.003	0.0023	1.008	0.0024	1.003	0.0026	1.002	0.0025	1.003	0.0025	1.002	0.0024
α=0.5	0.540	0.0819	0.511	0.0797	0.551	0.0967	0.547	0.0892	0.542	0.0882	0.548	0.0892
θ=1.5	1.504	0.0057	1.513	0.0059	1.503	0.0066	1.503	0.0062	1.504	0.0062	1.503	0.0061
α=0.75	0.797	0.1301	0.768	0.1295	0.811	0.1509	0.806	0.1397	0.801	0.1387	0.807	0.1392
θ=2	2.006	0.0099	2.018	0.0104	2.003	0.0116	2.003	0.0107	2.005	0.0107	2.003	0.0106
α=1	1.096	0.2412	1.069	0.2411	1.107	0.2800	1.104	0.2593	1.098	0.2578	1.104	0.2577
θ=1	1.005	0.0028	1.011	0.0029	1.003	0.0033	1.004	0.0030	1.005	0.0031	1.003	0.0030
α=0.3	0.326	0.0369	0.302	0.0344	0.336	0.0442	0.333	0.0412	0.329	0.0404	0.334	0.0411
θ=2	2.002	0.0092	2.012	0.0094	1.999	0.0102	1.999	0.0096	2.001	0.0096	1.999	0.0096
α=0.2	0.240	0.0262	0.220	0.0228	0.252	0.0320	0.248	0.0297	0.244	0.0288	0.249	0.0296
θ=3	3.009	0.0194	3.024	0.0201	3.005	0.0220	3.005	0.0207	3.008	0.0207	3.005	0.0205
α=0.8	0.846	0.1625	0.818	0.1621	0.854	0.1858	0.852	0.1748	0.847	0.1739	0.852	0.1725
θ=1	1.000	0.0027	1.007	0.0028	0.999	0.0032	0.999	0.0030	1.000	0.0030	0.999	0.0029
α=1	1.071	0.2538	1.044	0.2564	1.087	0.2868	1.080	0.2664	1.074	0.2651	1.081	0.2669
θ=3	3.006	0.0248	3.025	0.0258	3.003	0.0298	3.002	0.0274	3.006	0.0274	3.002	0.0273

Table 7. Estimates and MSEs using the ML, MPS, OLS, WLS, AD and CV methods for the WGQLD model for n=200.

Parameters	MLEs		MPS		OLS		WLS		CVEs		AD	
	Es	MSE	Es	MSE	Es	MSE	Es	MSE	Es	MSE	Es	MSE
α=0.25	0.263	0.0162	0.249	0.0154	0.271	0.0204	0.268	0.0185	0.266	0.0183	0.269	0.0185
θ=1	1.000	0.0011	1.003	0.0011	1.000	0.0013	1.000	0.0012	1.001	0.0012	1.000	0.0012
α=0.5	0.512	0.0364	0.495	0.0362	0.519	0.0430	0.517	0.0397	0.514	0.0395	0.517	0.0397
θ=1.5	1.501	0.0028	1.506	0.0028	1.500	0.0032	1.500	0.0030	1.501	0.0030	1.500	0.0030
α=0.75	0.762	0.0563	0.745	0.0564	0.770	0.0637	0.767	0.0594	0.764	0.0592	0.767	0.0592
θ=2	2.000	0.0048	2.007	0.0049	1.999	0.0054	1.999	0.00511	2.000	0.0051	1.998	0.0050
α=1	1.026	0.0976	1.011	0.0979	1.034	0.1101	1.031	0.1031	1.028	0.1028	1.031	0.1033
θ=1	1.000	0.0013	1.004	0.0013	0.999	0.0015	1.000	0.0014	1.000	0.0014	1.000	0.0014
α=0.3	0.307	0.0191	0.291	0.0186	0.315	0.0229	0.313	0.0212	0.310	0.0210	0.313	0.0212
θ=2	2.000	0.0046	2.007	0.0047	1.999	0.0051	1.999	0.0048	2.000	0.0048	1.999	0.0048
α=0.2	0.215	0.0125	0.202	0.0114	0.222	0.0149	0.220	0.0138	0.217	0.0136	0.220	0.0138
θ=3	2.999	0.0100	3.008	0.0102	2.998	0.0113	2.998	0.0107	2.999	0.0107	2.998	0.0106
α=0.8	0.837	0.0749	0.821	0.0744	0.842	0.0846	0.841	0.0793	0.838	0.0790	0.842	0.0792
θ=1	1.002	0.0013	1.006	0.0014	1.001	0.0015	1.002	0.0014	1.002	0.0014	1.002	0.0014
α=1	1.030	0.0935	1.015	0.0935	1.040	0.1044	1.036	0.0978	1.033	0.0975	1.036	0.0976
θ=3	3.003	0.0116	3.014	0.0119	3.002	0.0133	3.002	0.0125	3.004	0.0125	3.001	0.0124

Based on Tables 4–7 it is clear that:

- The MSEs values are decreasing as the sample sizes values are increasing for all cases considered in this section. As an example, for $\alpha = 1, \theta = 3$, with $n = 50$ based on the AD, the MSEs are 0.7566 and 0.0570 compared to 0.2669 and 0.0273 for $n = 100$, respectively.
- The bias values of the suggested estimators are decreasing as the sample sizes are increasing, and approaches zero for all cases for large n. For illustration, for $\alpha = 0.3, \theta = 2$, with $n = 100$ the Es values are 0.302 and 2.012 using the MPS as compared to 0.291 and 2.007 for $n = 200$, respectively.
- It can be observed that for most of the cases, the MLEs method has the smallest values of the MSEs among all methods of estimation.

9. Application on real data

In this section, we use data sets to illustrate the usefulness of the WGQL model, where four different data sets are used related to the environment, engineering and two medical data sets are considered. We compare the WGQL distribution to some well known distributions of two parameters as the generalized Quasi Lindley distribution, Quasi Lindley distribution, two-parameter Sujatha distribution, and the Pareto distribution.

The first data set is taken from [22] represents the 100 annual maximum precipitation (inches) for one rain gauge in Fort Collins, Colorado, from 1900 through 1999. The data are given below:

Data Set 1: 239, 232, 434, 85, 302, 174, 170, 121, 193, 168, 148, 116, 132, 132, 144, 183, 223, 96, 298, 97, 116, 146, 84, 230, 138, 170, 117, 115, 132, 125, 156, 124, 189, 193, 71, 176, 105, 93, 354, 60, 151, 160, 219, 142, 117, 87, 223, 215, 108, 354, 213, 306, 169, 184, 71, 98, 96, 218, 176, 121, 161, 321, 102, 269, 98, 271, 95, 212, 151, 136, 240, 162, 71, 110, 285, 215, 103, 443, 185, 199, 115, 134, 297, 187, 203, 146, 94, 129, 162, 112, 348, 95, 249, 103, 181, 152, 135, 463, 183, 241 5.

The second data set from [25] consists of 100 observations on breaking stress of carbon fibers (in Gba). The data are as follows:

Data Set 2: 3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11,4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53,2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59,2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19,1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69,1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12,1.89, 2.88, 2.82, 2.05, 3.655.

Due to the importance of the studies about the Covid-19 in the last years, we considered two sets of Covid-19 related to Algeria and Saudi Arabia in various times. The third data set is the Covid-19 data for the daily new cases in Algeria from 12 August 2020 to 09 November 2020 and it is available on the following electronic address https://sehhty.com/dz-covid/. It is given as follows

Data Set 3: 642, 670, 581, 631, 642, 548, 405, 302, 330, 291, 319, 306, 320, 287, 276, 263, 250, 273, 276, 252, 213, 214, 199, 205, 221, 193, 185, 174, 153, 132, 136, 146, 138, 121, 129, 134, 141, 148, 157, 160, 162, 155, 146, 153, 160, 175, 179, 186, 191, 197, 203, 210, 219, 228, 232, 238, 242, 247, 255, 264, 272, 278, 285, 289, 293, 298, 304, 311, 325, 339, 348, 365, 378, 387, 397, 391, 370, 398, 392, 401, 409, 411, 403, 419, 442, 450, 469, 477, 488, 495 .

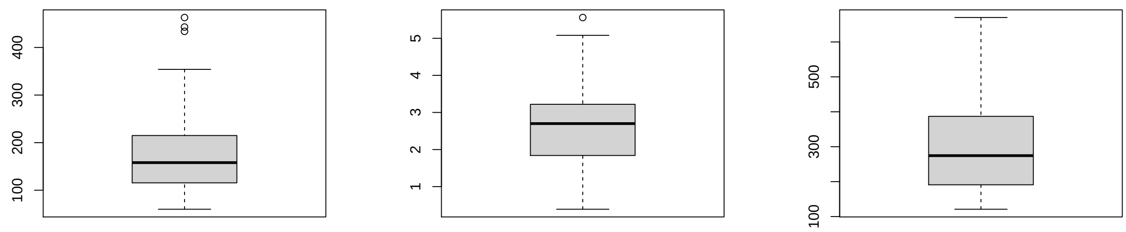
The box and TTT plots for the above data are given in Figures 7 and 8, respectively.

The fourth data set is calling the Covid-19 data which present the daily new cases in Saudi Arabia from 24 March 2020 to 24 April 2020 and it is given by

Data Set 4: 1, 1, 1, 0, 1, 4, 0, 2, 6, 5, 4, 4, 5, 4, 3, 0, 3, 3, 5, 7, 6, 8, 6, 4, 4, 5, 5, 6, 6, 5, 7, 6. The descriptive statistics for the data is given in Table 8.

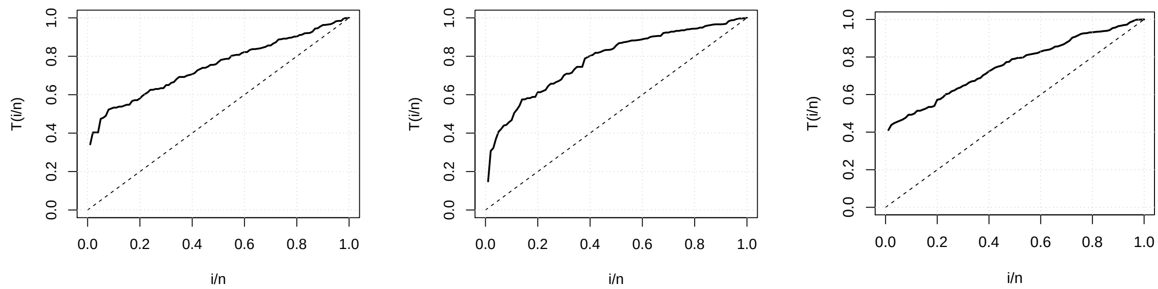
Table 8. Statistical properties of the data sets 1,2 and 3.

	n	Min	Max	Mean	Median	St derivation	Kurtosis	Skweness
Data Set 1:	100	60.000	463.000	175.670	158.000	83.166	1.713	1.316
Data Set 2:	100	0.390	5.560	2.621	2.700	1.013	0.043	0.362
Data Set 3:	90	121.000	670.000	294.322	274.500	131.618	0.369	0.927
Data Set 4:	32	0	8	3.97	4	2.24	-0.98	-3.35



(a) data set 1 (b) data set 2 (c) data set 3

Figure 7. Box plot for data sets 1, 2 and 3.



(a) data set 1 (b) data set 2 (c) data set 3

Figure 8. TTT plot for data sets 1, 2 and 3.

The WGQLD distribution is fitted to these two real data sets and compared with the following models:

- The generalized Quasi Lindley distribution: $f(x) = \frac{\theta^2\left(\frac{\theta^2 x^3}{6}+\alpha\theta x^2+\alpha^2 x\right)e^{-\theta x}}{(\alpha+1)^2}$.
- Quasi Lindley distribution: $f(x) = \frac{\theta(\alpha+x\theta)}{\alpha+1}e^{-\theta x}$.
- The Pareto distribution: $f(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}$.
- Two-parameter Sujatha distribution: $f(x) = \frac{\theta^3\left(x^2+\alpha x+1\right)e^{-\theta x}}{\theta^2+\alpha\theta+2}$.

To choose the best model fitting , we considered Akaike information criterion (AIC) introduced by [1], Baysian information criterion (BIC) proposed by [30], Hannan Quinn Information Criterion (HQIC) suggested by [21], Consistent Akaike Information Criterion (CAIC) by [10], Kolmogorov-Smirnov (KS), where AIC=$-2L + k$,, CAIC=$-2L + 2\frac{2kn}{n-k-1}$, HQIC=$2\log\log(n)[k-2L]$, BIC=$-2L + k\log(n)$, KS=$\sup|F_n(x) - F(x)|$, $F_n(x) = \frac{1}{n}\sum_i^n \mathrm{I}_{x_i\leq x}$, where k is the number of parameters and n is the sample size and L is the value of maximum log-likelihood function.

Based on the results reported in Tables 9, 10, 11 and 12, we observe that the WGQLD provides the better fit with the smallest values of AIC, AICc, BIC, HQIC and K-S with maximum P-values as compared to its competitive models considered in this study. Figures 9, 10, 11 and 12 support this claim.

Table 9. The goodness of fit tests for data set 1.

Model	AIC	CAIC	BIC	HQIC	K-S	p-value
WGQLD	1142.145	1142.268	1147.355	1144.253	0.059902	0.865573
GQLD	1145.844	1145.968	1151.055	1147.953	0.095349	0.323229
QLD	1180.179	1180.303	1185.389	1182.288	0.216170	0.000174
PD	1237.721	1237.845	1242.932	1239.83	0.340971	1.59e-10
TSPD	1156.301	1156.425	1161.511	1158.41	0.143704	0.032159

Table 10. The goodness of fit tests for data set 2.

Model	AIC	AICc	BIC	HQIC	K-S	p-value
WGQLD	295.1091	295.2328	300.3194	297.2178	0.105898	0.212049
GQLD	306.1634	306.2871	311.3737	308.2721	0.123234	0.095915
QLD	346.108	346.2317	351.3183	348.2167	0.223871	8.86e-05
PD	396.7418	396.8655	401.9522	398.8505	0.320272	2.46-09
TSPD	350.3233	350.447	355.5336	352.432	0.220935	0.000115

Table 11. The goodness of fit tests for data set 3.

Model	AIC	AICc	BIC	HQIC	K-S	p-value
WGQLD	1118.42	1118.558	1123.420	1120.436	0.0606885	0.8946827
GQLD	1122.41	1122.548	1127.410	1124.426	0.0900125	0.4593636
QLD	1154.697	1154.835	1159.696	1156.713	0.2081433	0.0008209
PD	1207.242	1207.38	1212.241	1209.258	0.3435296	1.190e-09
TSPD	1132.559	1132.697	1137.559	1134.575	0.1352032	0.0744752

Table 12. The goodness of fit tests for data set 4.

Model	AIC	AICc	BIC	HQIC	K-S	p-value
WGQLD	161.0522	161.4966	163.8546	161.9487	0.3077698	0.006804395
GQLD	166.9975	167.4419	169.7999	167.894	0.3318577	0.00269967

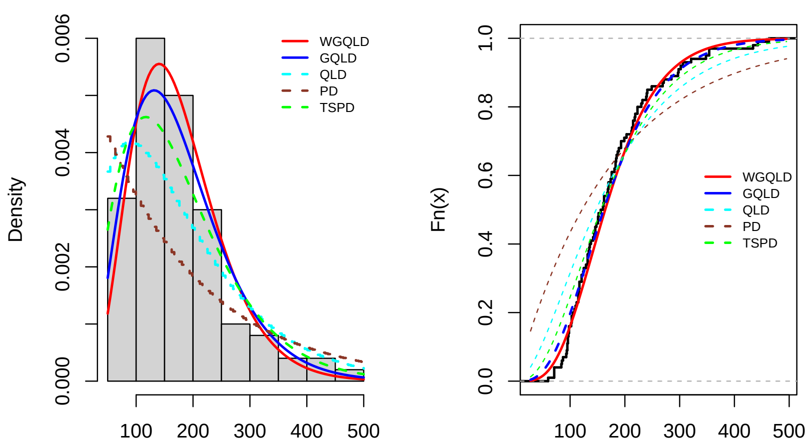


Figure 9. Plots of estimated probability density functions and cumulative distribution functions for data set 1.

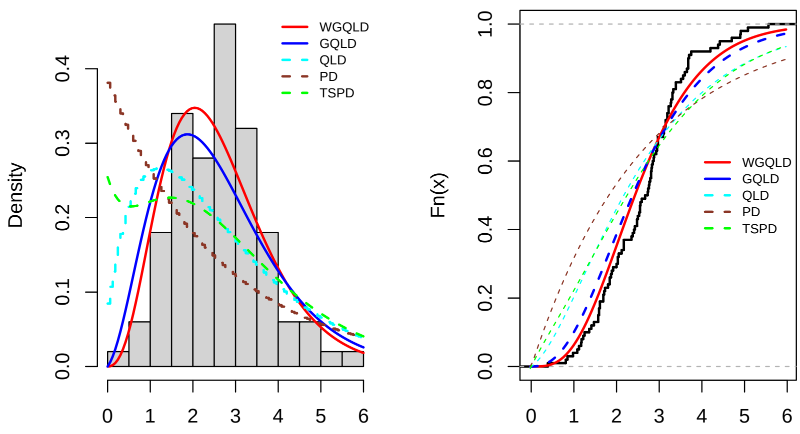


Figure 10. Plots of estimated probability density functions and cumulative distribution functions for data set 2.

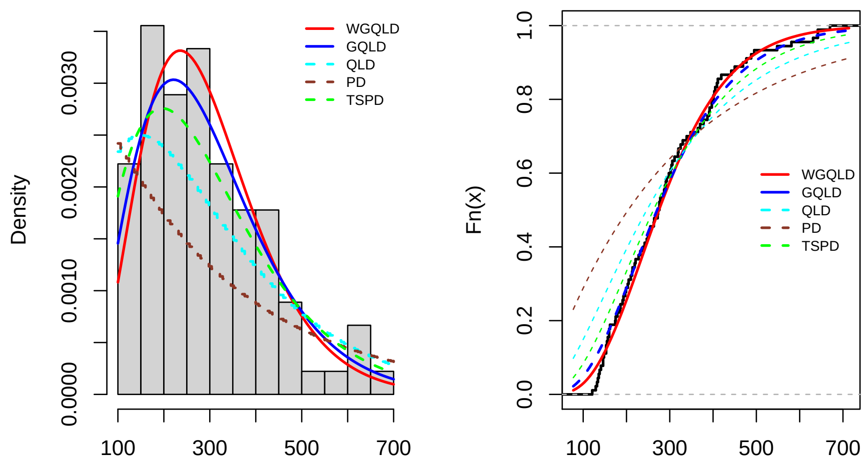


Figure 11. Plots of estimated probability density functions and cumulative distribution functions for data set 3.

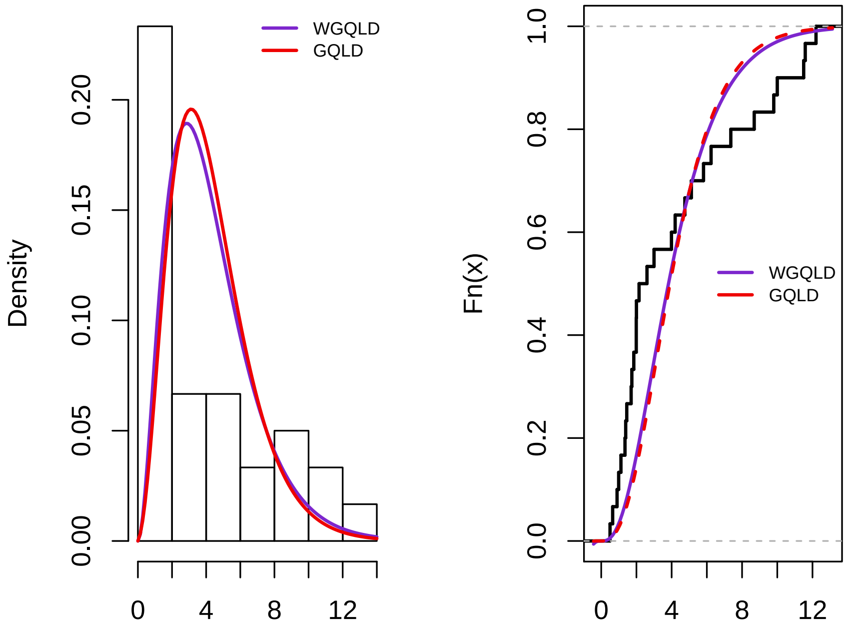


Figure 12. Plots of estimated probability density functions and cumulative distribution functions for data set 4.

10. Conclusions

In this article, we proposed the WGQLD distribution along with some of its properties such as, stochastic ordering, Median deviation, Harmonic mean, some plots of the pdf and cdf, Bonferroni and Lorenz curves and Gini index moments, coefficient of variation, coefficient of skewness and coefficient of kurtosis. Also, the hazard rate function, reliability function, reversed hazard rate and odds functions are presented. The maximum likelihood estimates is computed as well as the maximum product of spacing's, ordinary least squares, weighted least squares, Cramer-von-Mises, and Anderson-Darling methods are obtained. The results show that the best method of estimation is the MLE method. Applications of various real data sets are analyzed for illustration. It is proved that the WGQLD is empirically better than other competitors models considered in this study including the base GQLD. Therefore, in the future, the authors intend to investigate the performance of different estimators of the WGQLD based on ranked set sampling method and its modifications, see [3, 4, 19, 20, 32].

Conflict of interest

The author declare that they have no conflict of interest

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References

1. H. Akaike, A new look at the statistical model identification, *IEEE Trans. Autom. Control,* **19** (1974), 716–723.
2. A. K. Al-Khadim, A. N. Hussein, New proposed length biased weighted exponential and Rayleigh distribution with application, *Math. Theo. Mod.,* **4** (2014), 2224–2235.
3. A. I. Al-Omari, Estimation of mean based on modified robust extreme ranked set sampling, *J. Stat. Comput. Sim.,* **81** (2011), 1055–1066.
4. A. I. Al-Omari, Ratio estimation of population mean using auxiliary information in simple random sampling and median ranked set sampling, *Stat. Probab. Lett.,* **82** (2012), 1883–1990.
5. A. I. Al-Omari, I. K. Alsmairan, Length-biased Suja distribution: Properties and application, *J. Appl. Prob. Stat.,* **14** (2019), 95–116.
6. A. I. Al-Omari, A. Al-Nasser, E. Ciavolino, A size-biased Ishita distribution: Application to real data, *Qual. Quant.,* **53** (2019), 493–512.
7. A. I. Al-Omari, M. Gharaibeh, Topp-Leone Mukherjee-Islam distribution: Properties and applications, *Pakistan J. Stat.,* **34** (2018), 479–494.
8. A. I. Al-Omari, A. Al-khazaleh, M. Al-khazaleh, Exponentiated new Weibull-Pareto distribution, *Rev. Invest. Operacional,* **40** (2019), 165–175.
9. S. Benchiha, A. I. Al-Omari, Generalized Quasi Lindley distribution: Theoretical properties, estimation methods, and applications, *Electron. J. Appl. Stat. Anal.,* **14** (2021), 167–196.
10. H. Bozdogan, Model selection and Akaike's information criterion (AIC): The general theory and its analytical extensions, *Psychometrika,* **52** (1987), 345–370.
11. R. C. H. Cheng, N. A. K. Amin, Maximum product-of-spacings estimation with applications to the log-normal distribution, *Tech. rep., Department of Mathematics, University of Wales.* 1979.
12. R. C. H. Cheng, N. A. K. Amin, Estimating parameters in continuous univariate distributions with a shifted origin, *J. Royal Stat. Soc.,* **45** (1983), 394–403.
13. R. B. D'Agostino, M. A. Stephens, Goodness-of-Fit Techniques, *Marcel Dekker: New York, NY, USA,* 1986.
14. H. David, H. Nagaraja, *Order Statistics*, John Wiley and Sons, New York, 2003.
15. R. A. Fisher, The efects of methods of ascertainment upon the estimation of frequencies, *Ann. Eugen.,* **6** (1934), 13–25.
16. R. C. Gupta, J. P. Keating, Relations for reliability measures under length biased sampling, *Scan. J. Statist,* **13** (1985), 49–56.
17. R. C. Gupta, S. N. U. A. Kirmani, The role of weighted distributions in stochastic modeling, *Commun. Stat.,* **19** (1990), 3147–3162.

18. M. Haq, R. Usman, S. Hashmi, A. I. Al-Omari, The Marshall-Olkin length-biased exponential distribution and its applications, *J. King Saud Univ. Sci.,* **31** (2019), 246–251.

19. A. Haq, J. Brown, E. Moltchanova, A. I. Al-Omari, Ordered double ranked set samples and applications to inference, *Am. J. Math. Manage. Sci.,* **33** (2014), 239–260.

20. A. Haq, J. Brown, E. Moltchanova, A. I. Al-Omari, Varied L ranked set sampling scheme, *J. Stat. Theory Pract.,* **9** (2015), 741–767.

21. E. J. Hannan, B. G. Quinn, The determination of the order of an autoregression, *J. Royal Stat. Soc.,* **41** (1979), 190–195.

22. R. W. Katz, M. B. Parlange, P. Naveau, Statistics of extremes in hydrology, *Adv. Water Res,* **25** (2002), 1287–1304.

23. M. Kilany, Weighted Lomax Distribution, *Springer Plus,* **5** (2016).

24. A. Luceno, Fitting the generalized Pareto distribution to data using maximum goodness-of-fit estimators, *Comput. Stat. Data Anal.,* **51** (2006), 904–917.

25. Y. Benoist, P. Foulon, F. Labourie, A bootstrap control chart for Weibull percentiles, *Qual. Reliab. Eng. Int.,* **22** (2006), 141–151

26. B. O. Oluyede, On inequalities and selection of experiments for length-biased distributions, *Probab. Eng. Inform. Sci.,* **13** (1999), 169–185.

27. G. P. Patil, G. R. Rao, Weighted distributions and size biased sampling with applications to wildlife populations and human families, *Biometrics,* **34** (1978), 179–189.

28. J. Swain, S. Venkatraman, J. Wilson, Least squares estimation of distribution function in Johnson's translation system, *J. Stat. Comput. Simul,* **29** (1988), 271–297.

29. M. Shaked, J. Shanthikumar, Stochastic orders and their applications, distribution and associated inference, *Academic Press New York,* (1994).

30. G. Schwarz, Estimating the dimension of a model, *Ann. Stat.,* **6** (1978), 461–464.

31. C. R. Rao, On discrete distributions arising out of methods of ascertainment, In: classical and contagious discrete distribution, *Pergamon Press and Statistical Publishing Society, Calcutta,* (1965), 320–332.

32. E. Zamanzade, A. I. Al-Omari, New ranked set sampling for estimating the population mean and variance, *Hacet. J. Math. Stat.,* **45** (2016), 1891–1905.

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