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Research article

Stability of the 3D incompressible MHD equations with horizontal dissipation in periodic domain

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Abstract: The stability problem on the magnetohydrodynamics (MHD) equations with partial or no dissipation is not well-understood. This paper focuses on the 3D incompressible MHD equations with mixed partial dissipation and magnetic diffusion. Our main result assesses the stability of perturbations near the steady solution given by a background magnetic field in periodic domain. The new stability result presented here is among few stability conclusions currently available for ideal or partially dissipated MHD equations.

Keywords: MHD equations; hydrostatic equilibrium; partial dissipation; stability
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1. Introduction

In recent few years, there have been substantial developments concerning the MHD equations, especially there is only partial or fractional dissipation. The MHD equations govern the motion of electrically conducting fluids such as plasmas, liquid metals, and electrolytes. The fundamental concept behind MHD is that magnetic fields can induce currents in a moving conductive fluid, which in turn polarizes the fluid and reciprocally changes the magnetic field itself. The set of equations that describe MHD are a combination of the Navier-Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism. Since their initial derivation by the Nobel Laureate H. Alfvén [1] in 1924, the MHD equations have played vital roles in the study of many phenomena in geophysics, astrophysics, cosmology and engineering (see, e.g., [2, 3]).

This paper establishes the stability of perturbations near a background magnetic field of the 3D

MHD equations with mixed partial dissipation and magnetic diffusion in periodic domain.

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla P + \nu \, \Delta_h u + B \cdot \nabla B, \quad x \in \Omega, \ t > 0, \\ \partial_t B + u \cdot \nabla B = \eta \Delta_h B + B \cdot \nabla u, \quad x \in \Omega, \ t > 0, \\ \nabla \cdot u = \nabla \cdot B = 0, \quad x \in \Omega, \ t > 0, \end{cases} \tag{1.1}$$

where u denotes the velocity field of the fluid, P the total pressure, B the magnetic field, $\nu > 0$ and $\eta > 0$ are the viscosity and the magnetic diffusivity. We define the 3D periodic space domain $\Omega = [0, L]^2 \times \mathbb{R}$, the periodic solution means $u(x + e_i, t) = u(x, t)$ $(i = 1, 2, 3)$, for all x and $t \geq 0$, where e_i are the standard basis vectors, $e_1 = (1, 0, 0)^t$. We know that (1.1) admits the following steady state solution

$$u^{(0)} = (0, 0, 0), \ B^{(0)} = (1, 0, 0), \ P^{(0)} = 0.$$

It is clear that a special solution of (1.1) is given by the zero velocity field and the background magnetic fields $B^{(0)} = (1, 0, 0)$. The perturbation (u, b) with $b = B - B^{(0)}$ obeys,

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla P + \nu \, \Delta_h u + b \cdot \nabla b + \partial_1 b, \quad x \in \Omega, \ t > 0, \\ \partial_t b + u \cdot \nabla b = \eta \Delta_h b + b \cdot \nabla u + \partial_1 u, \quad x \in \Omega, \ t > 0, \\ \nabla \cdot u = \nabla \cdot b = 0, \quad x \in \Omega, \ t > 0, \end{cases} \tag{1.2}$$

where, for notational convenience, we write

$$\partial_i = \partial_{x_i}, \ \ \nabla_h = (\partial_1, \partial_2), \ \ \triangle_h = \partial_1^2 + \partial_2^2.$$

In addition, for convenience, we define the norm for the $L^p(\Omega)$ space, for $p \in [1, \infty]$, is denoted by $\|f\|_p$. The inner product of f and g in the $L^p(\Omega)$ space is denoted by $(f, g) = \iiint_\Omega fg \, dx_1 dx_2 dx_3 := \int_\Omega fg \, dx$. Respectively, the horizontal flow is defined in Ω with $\int_{[0,L]^2} u \, dx = 0$ and $\int_{[0,L]^2} b \, dx = 0$.

This paper aims at the stability problem on the perturbation of (1.1) near $(u^{(0)}, B^{(0)})$. Equivalently, we establish a small data global well-posedness result for (1.2) supplemented with the initial condition

$$u(x, 0) = u_0(x), \ \ b(x, 0) = b_0(x).$$

Our main result can be stated as follows.

Theorem 1.1. *Consider (1.2) with initial data $(u_0, b_0) \in H^2(\Omega)$ satisfies $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$, $\int_{[0,L]^2} u_0 dx = 0$ and $\int_{[0,L]^2} b_0 dx = 0$. Then there exists a constant $\delta = \delta(\nu, \eta) > 0$ such that, if*

$$\|(u_0, b_0)\|_{H^2} \leq \delta, \tag{1.3}$$

then (1.2) has a unique global solution

$$(u, b) \in L^\infty(0, \infty; H^2(\Omega)), \quad \nabla_h u, \nabla_h b \in L^2(0, \infty; H^2(\Omega)), \tag{1.4}$$

satisfying

$$\sup_{\tau \in [0,t]} (\|u(\tau)\|_{H^2}^2 + \|b(\tau)\|_{H^2}^2) + 2\nu \int_0^t \|\nabla_h u(\tau)\|_{H^2}^2 \, d\tau + 2\eta \int_0^t \|\nabla_h b(\tau)\|_{H^2}^2 \, d\tau \leq C\delta^2, \tag{1.5}$$

for any $t > 0$ and $C = C(\nu, \eta)$ is a constant.

The MHD equation, especially those with partial dissipation have recently attracted considerable interests. There are substantial developments on two fundamental problems, the global regularity and stability problems, which have been successfully established by many authors via different approaches [4–7]. In particular, it is also worth mentioning the beautiful work of [8], which made further progress by providing the stability of perturbations near a background magnetic field of the 3D incompressible MHD equation with mixed partial dissipation and deal with the H^3-estimate. To give a more complete views of current studies on the stability, we also mention some of exciting results in [9–12]. In this paper, we mainly deal with the H^2-estimate for the solution of (1.2). The stability of the incompressible MHD equation with mixed partial dissipation is not well-solved, except in the periodic case. Our study of the stability problem on (1.2) is inspired by the recent important result in [13], which is different with the whole region is that helps to solve the periodic problem.

We employ the bootstrapping argument to prove the desired H^2-stability. And we define the H^2-energy E(t) by

$$E(t) = \sup_{\tau\in[0,t]}(\|u(\tau)\|_{H^2}^2 + \|b(\tau)\|_{H^2}^2) + 2\nu\int_0^t \|\nabla_h u(\tau)\|_{H^2}^2\,d\tau + 2\eta\int_0^t \|\nabla_h b(\tau)\|_{H^2}^2\,d\tau,$$

and prove that, for a constant $C > 0$ and any $t \geq 0$,

$$E(t) \leq E(0) + CE(t)^{\frac{3}{2}}. \tag{1.6}$$

Once (1.6) is established, an application of the bootstrapping argument would imply the desired global stability. The details are given in section 2. Due to the presence of the anisotropic dissipation, we make use of anisotropic estimates for triple products (see Lemma 2.1 in section 2).

The proof of Theorem 1.1 is not trivial. A natural starting point is to bound $\|u\|_{H^3} + \|b\|_{H^3}$ via the energy estimates. However, due to the lack of the vertical dissipation, some of the nonlinear terms can not be controlled in terms of $\|u\|_{H^3} + \|b\|_{H^3}$ or the dissipation parts $\|\nabla_h u\|_{H^3}$ and $\|\nabla_h b\|_{H^3}$. Thus, we show the stability of equations (1.1) by bootstrapping argument which will be shown in section 3, and we also show the uniqueness in that section.

2. Proof of Theorem 1.1 and anisotropic estimates

This section applies the bootstrapping argument to prove Theorem 1.1. In addition, we provide the anisotropic inequality to be used in the proof of (2.1) in the subsequent section.

2.1. Proof of Theorem 1.1

Roughly speaking, the bootstrapping argument starts with an ansatz that $E(t)$ is bounded, say

$$E(t) \leq M,$$

and show that $E(t)$ actually admits a smaller bound, say

$$E(t) \leq \frac{1}{2}M,$$

when the initial condition is sufficiently small. A rigorous statement of the abstract bootstrapping principle can be found in T. Tao's book (see [14]). To apply the bootstrapping argument to (2.1), we assume that

$$E(t) \le M = \frac{1}{4C^2}, \tag{2.1}$$

When (2.1) holds, we have

$$CE(t)^{\frac{1}{2}} \le \frac{1}{2}.$$

It then follows from (1.6) that

$$E(t) \le E(0) + \frac{1}{2}E(t) \ \ or \ \ E(t) \le 2E(0), \tag{2.2}$$

if we choose $\delta > 0$ sufficiently small such that

$$\delta^2 \le \frac{M}{4},$$

then (1.3) and (2.2) imply that

$$E(t) \le \frac{1}{2}M,$$

the bootstrapping argument then leads to the desired global bound

$$E(t) \le M,$$

this completes the proof of Theorem 1.1.

As usual, the Sobolev space $H^1(\Omega) = \{f \in L^2(\Omega) : \nabla f \in L^2(\Omega)\}$. In addition, we define the following Hilbert space,

$$H_h^1(\Omega) = \{f \in L^2(\Omega) : \nabla_h f \in L^2(\Omega)\},$$

that features the inner product $(f, g)_{H_h^1(\Omega)} = (f, g)_{L^2(\Omega)} + (\nabla_h f, \nabla_h g)_{L^2(\Omega)}$.

The rest of this section provides the anisotropic inequality. The MHD system examined in this paper involves the estimates of quite a few triple terms. Anisotropic inequality appears to be necessary to deal with such partially dissipated system.

Lemma 2.1. *Let $f \in H^1(\Omega)$, $g \in H_h^1(\Omega)$, $h \in L^2(\Omega)$. Then,*

$$\int_\Omega |fgh| dx \le C(\|f\|_2 + \|\nabla_h f\|_2)^{\frac{1}{2}}(\|f\|_2 + \|\partial_3 f\|_2)^{\frac{1}{2}}\|g\|_2^{\frac{1}{2}}(\|g\|_2 + \|\nabla_h g\|_2)^{\frac{1}{2}}\|h\|_2.$$

The proof of Lemma 2.1 can be found in [9].

3. Proof of kernal part of bootstrapping

This section proves the major estimate in (1.6), namely

$$E(t) \le E(0) + CE(t)^{\frac{3}{2}}.$$

where $E(t)$ is defined in (1.5). The core of the proof is to bound the H^2-norm of (u, b) suitably. For the sake of clarity, the proof is divided to two main parts, the first one is devoted to the H^2-stability and the second one is to the uniqueness. The local existence can be obtained by a standard approach of Friedrichs' method of cutoff in Fourier space (see, e.g., [15]), we omit the details here.

3.1. The H^2-stability

Due to the equivalence of $\|(u,b)\|_{H^2}$ with $\|(u,b)\|_{L^2} + \|(u,b)\|_{\dot{H}^2}$, it suffices to bound the L^2-norm and the $\dot{H}^2$-norm of (u,b). By a simple energy estimate and $\nabla \cdot u = \nabla \cdot b = 0$, we find that the L^2-norm of (u,b) obeys

$$\|u(t)\|_2^2 + \|b(t)\|_2^2 + 2\nu \int_0^t \|\nabla_h u(\tau)\|_2^2 \, d\tau + 2\eta \int_0^t \|\nabla_h b(\tau)\|_2^2 \, d\tau = \|u(0)\|_2^2 + \|b(0)\|_2^2. \tag{3.1}$$

The rest of the proof focuses on the $\dot{H}^2$-norm, applying $\partial_i^2 (i = 1,2,3)$ to (1.2) and then dotting by $(\partial_i^2 u, \partial_i^2 b)$, we find

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{3}(\|\partial_i^2 u\|_2^2 + \|\partial_i^2 b\|_2^2) + \nu\|\partial_i^2 \nabla_h u\|_2^2 + \eta\|\partial_i^2 \nabla_h b\|_2^2 = I_1 + I_2 + I_3 + I_4 + I_5, \tag{3.2}$$

where

$$\begin{aligned}
I_1 &= \sum_{i=1}^{3} \int_\Omega \partial_i^2 \partial_1 b \cdot \partial_i^2 u + \partial_i^2 \partial_1 u \cdot \partial_i^2 b \, dx, \\
I_2 &= -\sum_{i=1}^{3} \int_\Omega \partial_i^2 (u \cdot \nabla u) \cdot \partial_i^2 u \, dx, \\
I_3 &= \sum_{i=1}^{3} \int_\Omega [\partial_i^2 (b \cdot \nabla b) - b \cdot \nabla \partial_i^2 b] \cdot \partial_i^2 u \, dx, \\
I_4 &= -\sum_{i=1}^{3} \int_\Omega \partial_i^2 (u \cdot \nabla b) \cdot \partial_i^2 b \, dx, \\
I_5 &= \sum_{i=1}^{3} \int_\Omega [\partial_i^2 (b \cdot \nabla u) - b \cdot \nabla \partial_i^2 u] \cdot \partial_i^2 b \, dx.
\end{aligned}$$

Note that

$$\int_\Omega b \cdot \nabla \partial_i^2 b \cdot \partial_i^2 u \, dx + \int_\Omega b \cdot \nabla \partial_i^2 u \cdot \partial_i^2 b \, dx = 0.$$

Integrating by parts and $u(x + e_i, t) = u(x,t)$ $(i = 1,2,3)$, $I_1 = 0$. To bound I_2, we decompose it into two pieces

$$I_2 = -\sum_{i=1}^{3} \int_\Omega \partial_i^2 (u \cdot \nabla u) \cdot \partial_i^2 u \, dx = I_{21} + I_{22}.$$

I_{21} involves the favorable partial derivatives in x_1 and x_2, respectively. Its handling is not difficult. In contrast, I_{22} has partial in terms of x_3 and the control of I_{22} is delicate.

By Lemma 2.1 with $f = \partial_i^k u$, $g = \partial_i^{2-k} \nabla u$, $h = \partial_i^2 u$ and Poincaré's inequality, we obtain

$$I_{21} = -\sum_{i=1}^{2} \sum_{k=1}^{2} C_2^k \int_\Omega \partial_i^k u \cdot \partial_i^{2-k} \nabla u \cdot \partial_i^2 u \, dx$$

$$
\begin{aligned}
&\leq C\sum_{i=1}^{2}\sum_{k=1}^{2}(\|\partial_i^k u\|_2+\|\partial_i^k\nabla_h u\|_2)^{\frac{1}{2}}(\|\partial_i^k u\|_2+\|\partial_3\partial_i^k u\|_2)^{\frac{1}{2}}(\|\partial_i^{2-k}\nabla u\|_2+\|\partial_i^{2-k}\nabla\nabla_h u\|_2)^{\frac{1}{2}}\\
&\quad\|\partial_i^{2-k}\nabla u\|_2^{\frac{1}{2}}\|\partial_i^2 u\|_2\\
&\leq C(\|u\|_{H^2}+\|\nabla_h u\|_{H^2})^{\frac{1}{2}}(\|u\|_{H^2}+\|\nabla_h u\|_{H^2})^{\frac{1}{2}}(\|u\|_{H^2}+\|\nabla_h u\|_{H^2})^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|u\|_{H^2}\\
&\leq C\|\nabla_h u\|_{H^2}^{\frac{3}{2}}\|u\|_{H^2}^{\frac{1}{2}}\|u\|_{H^2}\\
&\leq C\|\nabla_h u\|_{H^2}^{2}\|u\|_{H^2},
\end{aligned}\tag{3.3}
$$

where we have used the Poincaré's inequality

$$\|u\|_{H^2}\leq C\|\nabla_h u\|_{H^2}.$$

We further decomposed I_{22} into two terms

$$I_{22}=-\int_\Omega\partial_3^2(u\cdot\nabla u)\cdot\partial_3^2 u\,dx=-\int_\Omega\partial_3^2(u_h\cdot\nabla_h u+u_3\cdot\partial_3 u)\cdot\partial_3^2 u\,dx=I_{221}+I_{222}.$$

By Lemma 2.1,

$$
\begin{aligned}
I_{221}&=-\sum_{k=1}^{2}C_2^k\int_\Omega\partial_3^k u_h\cdot\partial_3^{2-k}\nabla_h u\cdot\partial_3^2 u\,dx\\
&\leq C\sum_{k=1}^{2}(\|\partial_3^{2-k}\nabla_h u\|_2+\|\nabla_h\partial_3^{2-k}\nabla_h u\|_2)^{\frac{1}{2}}(\|\partial_3^{2-k}\nabla_h u\|_2+\|\partial_3\partial_3^{2-k}\nabla_h u\|_2)^{\frac{1}{2}}\|\partial_3^k u_h\|_2^{\frac{1}{2}}\\
&\quad(\|\partial_3^k u_h\|_2+\|\nabla_h\partial_3^k u_h\|_2)^{\frac{1}{2}}\|\partial_3^2 u\|_2\\
&\leq C(\|u\|_{H^2}+\|\nabla_h u\|_{H^2})^{\frac{1}{2}}(\|u\|_{H^2}+\|\nabla_h u\|_{H^2})^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}}(\|u\|_{H^2}+\|\nabla_h u\|_{H^2})^{\frac{1}{2}}\|u\|_{H^2}\\
&\leq C\|u\|_{H^2}\|\nabla_h u\|_{H^2}^2.
\end{aligned}\tag{3.4}
$$

Using Lemma 2.1 and $\nabla\cdot u=0$, we obtain

$$
\begin{aligned}
I_{222}&=-\sum_{k=1}^{2}C_2^k\int_\Omega\partial_3^k u_3\cdot\partial_3^{2-k}\partial_3 u\cdot\partial_3^2 u\,dx\\
&=\sum_{k=1}^{2}C_2^k\int_\Omega\partial_3^{k-1}\nabla_h u_h\cdot\partial_3^{2-k}\partial_3 u\cdot\partial_3^2 u\,dx\\
&\leq C\sum_{k=1}^{2}(\|\partial_3^{k-1}\nabla_h u_h\|_2+\|\nabla_h\partial_3^{k-1}\nabla_h u_h\|_2)^{\frac{1}{2}}(\|\partial_3^{k-1}\nabla_h u_h\|_2+\|\partial_3\partial_3^{k-1}\nabla_h u_h\|_2)^{\frac{1}{2}}\|\partial_3^{3-k}u\|_2^{\frac{1}{2}}\\
&\quad(\|\partial_3^{3-k}u\|_2+\|\nabla_h\partial_3^{3-k}u\|_2)^{\frac{1}{2}}\|\partial_3^2 u\|_2\\
&\leq C(\|u\|_{H^2}+\|\nabla_h u\|_{H^2})^{\frac{1}{2}}(\|u\|_{H^2}+\|\nabla_h u\|_{H^2})^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}}(\|u\|_{H^2}+\|\nabla_h u\|_{H^2})^{\frac{1}{2}}\|u\|_{H^2}\\
&\leq C\|u\|_{H^2}\|\nabla_h u\|_{H^2}^2.
\end{aligned}\tag{3.5}
$$

Combining (3.3)–(3.5), we find

$$I_2\leq C\|u\|_{H^2}\|\nabla_h u\|_{H^2}^2.$$

We now turn to the estimates of I_3,

$$I_3 = \sum_{i=1}^{3}\sum_{k=1}^{2} C_2^k \int_\Omega \partial_i^k b \cdot \nabla \partial_i^{2-k} b \cdot \partial_i^2 u \, dx = I_{31} + I_{32}.$$

By Lemma 2.1,

$$\begin{aligned} I_{31} =& \sum_{i=1}^{2}\sum_{k=1}^{2} C_2^k \int_\Omega \partial_i^k b \cdot \nabla \partial_i^{2-k} b \cdot \partial_i^2 u \, dx \\ \leq& C \sum_{i=1}^{2}\sum_{k=1}^{2} (\|\partial_i^k b\|_2 + \|\nabla_h \partial_i^k b\|_2)^{\frac{1}{2}} (\|\partial_i^k b\|_2 + \|\partial_3 \partial_i^k b\|_2)^{\frac{1}{2}} \|\nabla \partial_i^{2-k} b\|_2^{\frac{1}{2}} (\|\nabla \partial_i^{2-k} b\|_2 \\ &+ \|\nabla_h \nabla \partial_i^{2-k} b\|_2)^{\frac{1}{2}} \|\partial_i^2 u\|_2 \\ \leq& C(\|b\|_{H^2} + \|\nabla_h b\|_{H^2})^{\frac{1}{2}} (\|b\|_{H^2} + \|\nabla_h b\|_{H^2})^{\frac{1}{2}} \|b\|_{H^2}^{\frac{1}{2}} (\|b\|_{H^2} + \|\nabla_h b\|_{H^2})^{\frac{1}{2}} \|u\|_{H^2} \\ \leq& C\|u\|_{H^2} \|\nabla_h b\|_{H^2}^2. \end{aligned} \tag{3.6}$$

Similar to I_{22}, I_{32} is naturally split into two terms

$$I_{32} = \sum_{k=1}^{2} C_2^k \int_\Omega \partial_3^k b \cdot \nabla \partial_3^{2-k} b \cdot \partial_3^2 u \, dx = I_{321} + I_{322}.$$

By Lemma 2.1,

$$\begin{aligned} I_{321} =& 2 \int_\Omega \partial_3 b \cdot \nabla \partial_3 b \cdot \partial_3^2 u \, dx \\ \leq& C(\|\partial_3 b\|_2 + \|\nabla_h \partial_3 b\|_2)^{\frac{1}{2}} (\|\partial_3 b\|_2 + \|\partial_3 \partial_3 b\|_2)^{\frac{1}{2}} \|\nabla \partial_3 b\|_2^{\frac{1}{2}} (\|\nabla \partial_3 b\|_2 + \|\nabla_h \nabla \partial_3 b\|_2)^{\frac{1}{2}} \|\partial_3^2 u\|_2 \\ \leq& C(\|b\|_{H^2} + \|\nabla_h b\|_{H^2})^{\frac{1}{2}} (\|b\|_{H^2} + \|b\|_{H^2})^{\frac{1}{2}} \|b\|_{H^2}^{\frac{1}{2}} (\|b\|_{H^2} + \|\nabla_h b\|_{H^2})^{\frac{1}{2}} \|u\|_{H^2} \\ \leq& C\|u\|_{H^2} \|\nabla_h b\|_{H^2}^2. \end{aligned} \tag{3.7}$$

Also

$$\begin{aligned} I_{322} =& \int_\Omega \partial_3^2 b \cdot \nabla b \cdot \partial_3^2 u \, dx \\ \leq& C(\|\nabla b\|_2 + \|\nabla_h \nabla b\|_2)^{\frac{1}{2}} (\|\nabla b\|_2 + \|\partial_3 \nabla b\|_2)^{\frac{1}{2}} \|\partial_3^2 b\|_2^{\frac{1}{2}} (\|\partial_3^2 b\|_2 + \|\nabla_h \partial_3^2 b\|_2)^{\frac{1}{2}} \|\partial_3^2 u\|_2 \\ \leq& C(\|b\|_{H^2} + \|\nabla_h b\|_{H^2})^{\frac{1}{2}} (\|b\|_{H^2} + \|b\|_{H^2})^{\frac{1}{2}} \|b\|_{H^2}^{\frac{1}{2}} (\|b\|_{H^2} + \|\nabla_h b\|_{H^2})^{\frac{1}{2}} \|u\|_{H^2} \\ \leq& C\|u\|_{H^2} \|\nabla_h b\|_{H^2}^2. \end{aligned} \tag{3.8}$$

Combining(3.6)–(3.8) yields

$$I_3 \leq C\|u\|_{H^2} \|\nabla_h b\|_{H^2}^2.$$

For I_4,

$$I_4 = -\sum_{i=1}^{3}\sum_{k=1}^{2} C_2^k \int_\Omega \partial_i^k u \cdot \nabla \partial_i^{2-k} b \cdot \partial_i^2 b \, dx = I_{41} + I_{42}.$$

By Lemma 2.1,

$$\begin{aligned}
I_{41} =& -\sum_{i=1}^{2}\sum_{k=1}^{2} C_2^k \int_\Omega \partial_i^k u \cdot \nabla \partial_i^{2-k} b \cdot \partial_i^2 b \, dx \\
\leq& C \sum_{i=1}^{2}\sum_{k=1}^{2} (\|\partial_i^k u\|_2 + \|\nabla_h \partial_i^k u\|_2)^{\frac{1}{2}} (\|\partial_i^k u\|_2 + \|\partial_3 \partial_i^k u\|_2)^{\frac{1}{2}} \|\nabla \partial_i^{2-k} b\|_2^{\frac{1}{2}} (\|\nabla \partial_i^{2-k} b\|_2 \\
&+ \|\nabla_h \nabla \partial_i^{2-k} b\|_2)^{\frac{1}{2}} \|\partial_i^2 b\|_2 \\
\leq& C(\|u\|_{H^2} + \|\nabla_h u\|_{H^2})^{\frac{1}{2}} (\|u\|_{H^2} + \|\nabla_h u\|_{H^2})^{\frac{1}{2}} \|b\|_{H^2}^{\frac{1}{2}} (\|b\|_{H^2} + \|\nabla_h b\|_{H^2})^{\frac{1}{2}} \|b\|_{H^2} \\
\leq& C\|b\|_{H^2} \|\nabla_h b\|_{H^2} \|\nabla_h u\|_{H^2}.
\end{aligned} \tag{3.9}$$

We decompose I_{42} into two terms

$$I_{42} = -\sum_{k=1}^{2} C_2^k \int_\Omega \partial_3^k u \cdot \nabla \partial_3^{2-k} b \cdot \partial_3^2 b \, dx = I_{421} + I_{422}.$$

Using Lemma 2.1,

$$\begin{aligned}
I_{421} =& 2 \int_\Omega \partial_3 u \cdot \nabla \partial_3 b \cdot \partial_3^2 b \, dx \\
\leq& C(\|\partial_3 u\|_2 + \|\nabla_h \partial_3 u\|_2)^{\frac{1}{2}} (\|\partial_3 u\|_2 + \|\partial_3 \partial_3 u\|_2)^{\frac{1}{2}} \|\nabla \partial_3 b\|_2^{\frac{1}{2}} (\|\nabla \partial_3 b\|_2 + \|\nabla_h \nabla \partial_3 b\|_2)^{\frac{1}{2}} \|\partial_3^2 b\|_2. \\
\leq& C(\|u\|_{H^2} + \|\nabla_h u\|_{H^2})^{\frac{1}{2}} (\|u\|_{H^2} + \|u\|_{H^2})^{\frac{1}{2}} \|b\|_{H^2}^{\frac{1}{2}} (\|b\|_{H^2} + \|\nabla_h b\|_{H^2})^{\frac{1}{2}} \|b\|_{H^2} \\
\leq& C\|b\|_{H^2} \|\nabla_h b\|_{H^2} \|\nabla_h u\|_{H^2}.
\end{aligned} \tag{3.10}$$

Similarly

$$\begin{aligned}
I_{422} =& \int_\Omega \partial_3^2 u \cdot \nabla b \cdot \partial_3^2 b \, dx \\
\leq& C(\|\nabla b\|_2 + \|\nabla_h \nabla b\|_2)^{\frac{1}{2}} (\|\nabla b\|_2 + \|\partial_3 \nabla b\|_2)^{\frac{1}{2}} \|\partial_3^2 u\|_2^{\frac{1}{2}} (\|\partial_3^2 u\|_2 + \|\nabla_h \partial_3^2 u\|_2)^{\frac{1}{2}} \|\partial_3^2 b\|_2 \\
\leq& C(\|b\|_{H^2} + \|\nabla_h b\|_{H^2})^{\frac{1}{2}} (\|b\|_{H^2} + \|b\|_{H^2})^{\frac{1}{2}} \|u\|_{H^2}^{\frac{1}{2}} (\|u\|_{H^2} + \|\nabla_h u\|_{H^2})^{\frac{1}{2}} \|b\|_{H^2} \\
\leq& C\|b\|_{H^2} \|\nabla_h b\|_{H^2} \|\nabla_h u\|_{H^2}.
\end{aligned} \tag{3.11}$$

Combining all the estimates(3.9) through (3.11) yields

$$I_4 \leq C\|b\|_{H^2} \|\nabla_h b\|_{H^2} \|\nabla_h u\|_{H^2}.$$

It remains to estimate I_5,

$$I_5 = \sum_{i=1}^{3}\sum_{k=1}^{2} C_2^k \int_\Omega \partial_i^k b \cdot \nabla \partial_i^{2-k} u \cdot \partial_i^2 b \, dx = I_{51} + I_{52}.$$

By Lemma 2.1,

$$I_{51} = \sum_{i=1}^{2}\sum_{k=1}^{2} C_2^k \int_\Omega \partial_i^k b \cdot \nabla \partial_i^{2-k} u \cdot \partial_i^2 b \, dx$$

$$
\begin{aligned}
\leq & C \sum_{i=1}^{2} \sum_{k=1}^{2} (\|\partial_i^k b\|_2 + \|\nabla_h \partial_i^k b\|_2)^{\frac{1}{2}} (\|\partial_i^k b\|_2 + \|\partial_3 \partial_i^k b\|_2)^{\frac{1}{2}} \|\nabla \partial_i^{2-k} u\|_2^{\frac{1}{2}} (\|\nabla \partial_i^{2-k} u\|_2 \\
& + \|\nabla_h \nabla \partial_i^{2-k} u\|_2)^{\frac{1}{2}} \|\partial_i^2 b\|_2 \\
\leq & C(\|b\|_{H^2} + \|\nabla_h b\|_{H^2})^{\frac{1}{2}} (\|b\|_{H^2} + \|\nabla_h b\|_{H^2})^{\frac{1}{2}} \|u\|_{H^2}^{\frac{1}{2}} (\|u\|_{H^2} + \|\nabla_h u\|_{H^2})^{\frac{1}{2}} \|b\|_{H^2} \\
\leq & C\|b\|_{H^2} \|\nabla_h u\|_{H^2} \|\nabla_h b\|_{H^2}. \qquad (3.12)
\end{aligned}
$$

The difficult term is I_{52}, which is further decomposed into two terms

$$
I_{52} = \sum_{k=1}^{2} C_2^k \int_\Omega \partial_3^k b \cdot \nabla \partial_3^{2-k} u \cdot \partial_3^2 b \, dx = I_{521} + I_{522}.
$$

By Lemma 2.1,

$$
\begin{aligned}
I_{521} = & 2 \int_\Omega \partial_3 b \cdot \nabla \partial_3 u \cdot \partial_3^2 b \, dx \\
\leq & C(\|\partial_3 b\|_2 + \|\nabla_h \partial_3 b\|_2)^{\frac{1}{2}} (\|\partial_3 b\|_2 + \|\partial_3 \partial_3 b\|_2)^{\frac{1}{2}} \|\nabla \partial_3 u\|_2^{\frac{1}{2}} (\|\nabla \partial_3 u\|_2 + \|\nabla_h \nabla \partial_3 u\|_2)^{\frac{1}{2}} \|\partial_3^2 b\|_2. \\
\leq & C(\|b\|_{H^2} + \|\nabla_h b\|_{H^2})^{\frac{1}{2}} (\|b\|_{H^2} + \|b\|_{H^2})^{\frac{1}{2}} \|u\|_{H^2}^{\frac{1}{2}} (\|u\|_{H^2} + \|\nabla_h u\|_{H^2})^{\frac{1}{2}} \|b\|_{H^2} \\
\leq & C\|b\|_{H^2} \|\nabla_h u\|_{H^2} \|\nabla_h b\|_{H^2}. \qquad (3.13)
\end{aligned}
$$

Similarly, I_{522} can be estimated as follows

$$
\begin{aligned}
I_{522} = & \int_\Omega \partial_3^2 b \cdot \nabla u \cdot \partial_3^2 b \, dx \\
\leq & C(\|\nabla u\|_2 + \|\nabla_h \nabla u\|_2)^{\frac{1}{2}} (\|\nabla u\|_2 + \|\partial_3 \nabla u\|_2)^{\frac{1}{2}} \|\partial_3^2 b\|_2^{\frac{1}{2}} (\|\partial_3^2 b\|_2 + \|\nabla_h \partial_3^2 b\|_2)^{\frac{1}{2}} \|\partial_3^2 b\|_2 \\
\leq & C(\|u\|_{H^2} + \|\nabla_h u\|_{H^2})^{\frac{1}{2}} (\|u\|_{H^2} + \|u\|_{H^2})^{\frac{1}{2}} \|b\|_{H^2}^{\frac{1}{2}} (\|b\|_{H^2} + \|\nabla_h b\|_{H^2})^{\frac{1}{2}} \|b\|_{H^2} \\
\leq & C\|b\|_{H^2} \|\nabla_h u\|_{H^2} \|\nabla_h b\|_{H^2}. \qquad (3.14)
\end{aligned}
$$

Combining (3.12)–(3.14), we have

$$
I_5 \leq C\|b\|_{H^2} \|\nabla_h u\|_{H^2} \|\nabla_h b\|_{H^2}.
$$

Therefore, if we set

$$
E(t) = \sup_{\tau \in [0,t]} (\|u(\tau)\|_{H^2}^2 + \|b(\tau)\|_{H^2}^2) + 2\nu \int_0^t \|\nabla_h u(\tau)\|_{H^2}^2 \, d\tau + 2\eta \int_0^t \|\nabla_h b(\tau)\|_{H^2}^2 \, d\tau.
$$

By Hölder's inequality, the time integral of the bounds for I_2, I_3, I_4 and I_5 can be estimated as follows

$$
\begin{aligned}
\int_0^t |I_2| d\tau \leq & C \int_0^t \|u(\tau)\|_{H^2} \|\nabla_h u(\tau)\|_{H^2}^2 \, d\tau \\
\leq & C \sup_{\tau \in [0,t]} \|u(\tau)\|_{H^2} \int_0^t \|\nabla_h u(\tau)\|_{H^2}^2 \, d\tau \leq CE(t)^{\frac{3}{2}},
\end{aligned}
$$

$$\int_0^t |I_3| d\tau \leq C \int_0^t \|u(\tau)\|_{H^2} \|\nabla_h b(\tau)\|_{H^2}^2 \, d\tau$$
$$\leq C \sup_{\tau\in[0,t]} \|u(\tau)\|_{H^2} \int_0^t \|\nabla_h b(\tau)\|_{H^2}^2 \, d\tau \leq CE(t)^{\frac{3}{2}},$$

$$\int_0^t |I_4| d\tau \leq C \int_0^t \|b(\tau)\|_{H^2} \|\nabla_h u(\tau)\|_{H^2} \|\nabla_h b(\tau)\|_{H^2} \, d\tau \leq CE(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}} = CE(t)^{\frac{3}{2}},$$
$$\int_0^t |I_5| d\tau \leq C \int_0^t \|b(\tau)\|_{H^2} \|\nabla_h u(\tau)\|_{H^2} \|\nabla_h b(\tau)\|_{H^2} \, d\tau \leq CE(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}} = CE(t)^{\frac{3}{2}}.$$

Integrating (3.2) in time and combining with (3.1), we find

$$E(t) \leq E(0) + CE(t)^{\frac{3}{2}}.$$

A bootstrapping argument implies that, there is $\delta > 0$, such that, if $E(0) < \delta^2$, then

$$E(t) \leq C\delta^2$$

for a pure constant C and for all $t > 0$, which implies H^2-stability.

3.2. Uniqueness

This subsection proves the uniqueness part of Theorem 1.1. We show that two solutions $(u^{(1)}, P^{(1)}, b^{(1)})$ and $(u^{(2)}, P^{(2)}, b^{(2)})$ of (1.2) in the regularity class (1.4) must coincide. Their difference $(\widetilde{u}, \widetilde{P}, \widetilde{b})$ with

$$\widetilde{u} = u^{(1)} - u^{(2)}, \ \widetilde{P} = P^{(1)} - P^{(2)}, \ \widetilde{b} = b^{(1)} - b^{(2)}$$

satisfies, according to (1.2)

$$\begin{cases} \partial_t \widetilde{u} + u^{(1)} \cdot \nabla \widetilde{u} + \widetilde{u} \cdot \nabla u^{(2)} = -\nabla \widetilde{P} + \nu \, \Delta_h \widetilde{u} + b^{(1)} \cdot \nabla \widetilde{b} + \widetilde{b} \cdot \nabla b^{(2)} + \partial_1 \widetilde{b}, \\ \partial_t \widetilde{b} + u^{(1)} \cdot \nabla \widetilde{b} + \widetilde{u} \cdot \nabla b^{(2)} = \eta \, \Delta_h \widetilde{b} + b^{(1)} \cdot \nabla \widetilde{u} + \widetilde{b} \cdot \nabla u^{(2)} + \partial_1 \widetilde{u}, \\ \nabla \cdot u = \nabla \cdot b = 0. \end{cases} \tag{3.15}$$

Basic energy estimates show that

$$\frac{1}{2} \frac{d}{dt} (\|\widetilde{u}\|_2^2 + \|\widetilde{b}\|_2^2) + \nu \|\nabla_h \widetilde{u}\|_2^2 + \eta \|\nabla_h \widetilde{b}\|_2^2 = K_1 + K_2 + K_3 + K_4,$$

where

$$K_1 = -\int_\Omega \widetilde{u} \cdot \nabla u^{(2)} \cdot \widetilde{u} \, dx,$$
$$K_2 = \int_\Omega \widetilde{b} \cdot \nabla b^{(2)} \cdot \widetilde{u} \, dx,$$
$$K_3 = -\int_\Omega \widetilde{u} \cdot \nabla b^{(2)} \cdot \widetilde{b} \, dx,$$
$$K_4 = \int_\Omega \widetilde{b} \cdot \nabla u^{(2)} \cdot \widetilde{b} \, dx.$$

By Lemma 2.1, K_1, K_2, K_3, K_4 can be bounded as follows

$$\begin{aligned} K_1 \leq & C(\|\nabla u^{(2)}\|_2 + \|\nabla_h \nabla u^{(2)}\|_2)^{\frac{1}{2}}(\|\nabla u^{(2)}\|_2 + \|\partial_3 \nabla u^{(2)}\|_2)^{\frac{1}{2}}\|\widetilde{u}\|_2^{\frac{1}{2}}(\|\widetilde{u}\|_2 + \|\nabla_h \widetilde{u}\|_2)^{\frac{1}{2}}\|\widetilde{u}\|_2 \\ \leq & C\|\nabla_h \nabla u^{(2)}\|_2^{\frac{1}{2}}(\|\nabla u^{(2)}\|_2 + \|\partial_3 \nabla u^{(2)}\|_2)^{\frac{1}{2}}\|\widetilde{u}\|_2\|\nabla_h \widetilde{u}\|_2 \\ \leq & \frac{\nu}{6}\|\nabla_h \widetilde{u}\|_2^2 + C\|\widetilde{u}\|_2^2\|\nabla_h \nabla u^{(2)}\|_2(\|\nabla u^{(2)}\|_2 + \|\partial_3 \nabla u^{(2)}\|_2). \end{aligned} \tag{3.16}$$

$$\begin{aligned} K_2 \leq & C(\|\nabla b^{(2)}\|_2 + \|\nabla_h \nabla b^{(2)}\|_2)^{\frac{1}{2}}(\|\nabla b^{(2)}\|_2 + \|\partial_3 \nabla b^{(2)}\|_2)^{\frac{1}{2}}\|\widetilde{b}\|_{L^2}^{\frac{1}{2}}(\|\widetilde{b}\|_2 + \|\nabla_h \widetilde{b}\|_2)^{\frac{1}{2}}\|\widetilde{u}\|_2 \\ \leq & C\|\nabla_h \nabla b^{(2)}\|_2^{\frac{1}{2}}(\|\nabla b^{(2)}\|_2 + \|\partial_3 \nabla b^{(2)}\|_2)^{\frac{1}{2}}\|\widetilde{b}\|_2^{\frac{1}{2}}\|\nabla_h \widetilde{b}\|_2^{\frac{1}{2}}\|\nabla_h \widetilde{u}\|_2 \\ \leq & \frac{\nu}{6}\|\nabla_h \widetilde{u}\|_2^2 + \frac{\eta}{6}\|\nabla_h \widetilde{b}\|_2^2 + C\|\widetilde{b}\|_2^2\|\nabla_h \nabla b^{(2)}\|_2^2(\|\nabla b^{(2)}\|_2 + \|\partial_3 \nabla b^{(2)}\|_2)^2. \end{aligned} \tag{3.17}$$

$$\begin{aligned} K_3 \leq & C(\|\nabla b^{(2)}\|_2 + \|\nabla_h \nabla b^{(2)}\|_2)^{\frac{1}{2}}(\|\nabla b^{(2)}\|_2 + \|\partial_3 \nabla b^{(2)}\|_2)^{\frac{1}{2}}\|\widetilde{u}\|_2^{\frac{1}{2}}(\|\widetilde{u}\|_{L^2} + \|\nabla_h \widetilde{u}\|_2)^{\frac{1}{2}}\|\widetilde{b}\|_{L^2} \\ \leq & C\|\nabla_h \nabla b^{(2)}\|_2^{\frac{1}{2}}(\|\nabla b^{(2)}\|_2 + \|\partial_3 \nabla b^{(2)}\|_2)^{\frac{1}{2}}\|\widetilde{u}\|_2^{\frac{1}{2}}\|\nabla_h \widetilde{u}\|_2^{\frac{1}{2}}\|\nabla_h \widetilde{b}\|_2 \\ \leq & \frac{\nu}{6}\|\nabla_h \widetilde{u}\|_2^2 + \frac{\eta}{6}\|\nabla_h \widetilde{b}\|_2^2 + C\|\widetilde{u}\|_2^2\|\nabla_h \nabla b^{(2)}\|_2^2(\|\nabla b^{(2)}\|_2 + \|\partial_3 \nabla b^{(2)}\|_2)^2. \end{aligned} \tag{3.18}$$

$$\begin{aligned} K_4 \leq & C(\|\nabla u^{(2)}\|_2 + \|\nabla_h \nabla u^{(2)}\|_2)^{\frac{1}{2}}(\|\nabla u^{(2)}\|_2 + \|\partial_3 \nabla u^{(2)}\|_2)^{\frac{1}{2}}\|\widetilde{b}\|_2^{\frac{1}{2}}(\|\widetilde{b}\|_2 + \|\nabla_h \widetilde{b}\|_2)^{\frac{1}{2}}\|\widetilde{b}\|_2 \\ \leq & C\|\nabla_h \nabla u^{(2)}\|_2^{\frac{1}{2}}(\|\nabla u^{(2)}\|_2 + \|\partial_3 \nabla u^{(2)}\|_2)^{\frac{1}{2}}\|\widetilde{b}\|_2\|\nabla_h \widetilde{b}\|_2 \\ \leq & \frac{\nu}{6}\|\nabla_h \widetilde{b}\|_2^2 + C\|\widetilde{b}\|_2^2\|\nabla_h \nabla u^{(2)}\|_2(\|\nabla u^{(2)}\|_2 + \|\partial_3 \nabla u^{(2)}\|_2). \end{aligned} \tag{3.19}$$

Combining (3.16)–(3.19), we set $Y(t) = (\|\widetilde{u}(t)\|_2^2 + \|\widetilde{b}(t)\|_2^2)$,

$$\frac{d}{dt}Y(t) + \nu\|\nabla_h \widetilde{u}\|_2^2 + \eta\|\nabla_h \widetilde{b}\|_2^2 \leq a(t)Y(t), \tag{3.20}$$

where

$$a(t) = C\|\nabla_h \nabla u^{(2)}\|_2(\|\nabla u^{(2)}\|_2 + \|\partial_3 \nabla u^{(2)}\|_2) + C\|\nabla_h \nabla b^{(2)}\|_2^2(\|\nabla b^{(2)}\|_2 + \|\partial_3 \nabla b^{(2)}\|_2)^2.$$

Since $(u^{(2)}, b^{(2)})$ is in the regularity class (1.4). For any $T > 0$, we have

$$\begin{aligned} \int_0^T a(t)\,dt \leq & C\int_0^T \|\nabla_h \nabla u^{(2)}\|_2(\|\nabla u^{(2)}\|_2 + \|\partial_3 \nabla u^{(2)}\|_2) \\ & + \|\nabla_h \nabla b^{(2)}\|_2^2(\|\nabla b^{(2)}\|_2 + \|\partial_3 \nabla b^{(2)}\|_2)^2\,dt \\ \leq & C\int_0^T \|\nabla_h u^{(2)}\|_{H^2}^2 + \|\nabla_h b^{(2)}\|_{H^2}^2\|b^{(2)}\|_{H^2}^2\,dt \\ \leq & C\int_0^T \|\nabla_h u^{(2)}\|_{H^2}^2\,dt + C\sup_{\tau\in[0,t]}\|b^{(2)}\|_{H^2}^2\int_0^T \|\nabla_h b^{(2)}\|_{H^2}^2\,dt \leq C(T) < +\infty. \end{aligned}$$

Gronwall's inequality applied to (3.20) implies that, for any $T > 0$,

$$\begin{aligned} \|\widetilde{u}(t)\|_2^2 + \|\widetilde{b}(t)\|_2^2 \leq & (\|\widetilde{u}(0)\|_2^2 + \|\widetilde{b}(0)\|_2^2)e^{C\int_0^T a(t)\,dt} \\ \leq & C(\|\widetilde{u}(0)\|_2^2 + \|\widetilde{b}(0)\|_2^2). \end{aligned} \tag{3.21}$$

In particular, the initial values of the two solutions in the regularity class (1.4), then (3.21) implies $Y(t) = \|\widetilde{u}(t)\|_2^2 + \|\widetilde{b}(t)\|_2^2 \equiv 0$ for any $T > 0$. This completes the proof of the uniqueness.

4. Conclusions

In this paper, we gave the stability of the 3D incompressible MHD equations near a background magnetic field with horizontal dissipation in periodic domain by bootstrapping argument. The main part of bootstrapping argument relies on proof of inequality (2.1). We get through it by the anisotropic inequality, and Poincaré's inequality helps a lot in periodic domain.

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Conflict of interest

The authors declare no conflict of interest.

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